THERE EXIST NO MINIMALLY KNOTTED PLANAR SPATIAL GRAPHS ON THE TORUS

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Abstract. We show that all nontrivial embeddings of planar graphs on the torus contain a nontrivial knot or a nonsplit link. This is equivalent to showing that no minimally knotted planar spatial graphs on the torus exist that contain neither a nontrivial knot nor a nonsplit link all of whose components are unknots.

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1. Introduction

All considered graphs are undirected finite graphs and we will work in the piecewise linear category. A **graph embedding** is an embedding $f : G \to S^3$ of a graph G in S^3 up to ambient isotopy and the corresponding **spatial graph** $\mathcal{G}$ is the image of this embedding. A graph G is **planar** if there exists an embedding $f : G \to S^2$. An embedding $f : G \to S^3$ is **trivial** if $\mathcal{G}$ is contained in a 2-sphere embedded in S^3. Its image $\mathcal{G}$ is a **trivial spatial graph**. A spatial graph $\mathcal{G}$ is **minimally knotted** if $\mathcal{G}$ is nontrivial but $\mathcal{G} - e$ is trivial for every edge e. Some authors call minimally knotted spatial graphs **almost trivial**, **almost unknotted** or **Brunnian**. In this paper, a nontrivial **link** is a nonsplit link with at least two components.
Previous research on minimally knotted spatial graphs has been undertaken: The first example of a minimally knotted spatial graph was an embedding of a handcuff graph given by Suzuki [1]. Kawauchi [2], Wu [3] and Inaba and Soma [4] showed that every planar graph has a minimally knotted embedding. Ozawa and Tsutsumi [5] proved that minimally knotted embeddings of planar graphs are totally knotted. Especially minimally knotted θ_n-graphs have generated some interest. Kinoshita [6] gave the first example of a minimally knotted θ_3-graph (see Fig. 1) which Suzuki [7] generalised to give examples of minimally knotted θ_n-graphs for all $n \geq 3$. Closely related are **ravels** which are nontrivial embeddings of θ_n-graphs that contain no nontrivially knotted subgraph; this definition is equivalent to the one given by Farkas, Flapan and Sullivan [8]. The concept of ravels has been introduced by Castle, Evans and Hyde [9] as local entanglements that are not caused by knots or links and may lead to new topological structures in coordination polymers. A ravel in a molecule has been synthesized by Lindoy *et al* [10]. Castle, Evans and Hyde [11] conjectured the following:

Conjecture (Castle, Evans, Hyde [11])**.** All nontrivial embeddings of planar graphs on the torus include a nontrivial knot or a nonsplit link.

With Theorem 1 we prove that their conjecture is true. With **torus** we refer to an embedded torus in the 3-sphere S^3 which may be nonstandardly embedded. A **standardly embedded torus** is a torus that bounds two solid tori in S^3. A nonstandardly embedded torus still bounds a solid torus in S^3 by the Solid Torus Theorem [12].

Theorem 1 (Knots and links existence)**.** *Let G be a planar graph and $f : G \to S^3$ be an embedding of G with image $\mathcal{G}$. If $\mathcal{G}$ is contained in the torus T^2 and contains neither a nontrivial knot nor a nonsplit link, then f is trivial.*

Since θ_n-graphs are planar, it follows from Theorem 1 that on the torus there exist no minimally knotted embeddings of θ_n-graphs with $n > 2$. This gives us the following:

Corollary 1 (Ravels do not embed on the torus)**.** Every nontrivial embedding of θ_n-graphs on the torus contains a nontrivial knot.

We conclude by showing that all assumptions of Theorem 1 are necessary. Explicit ambient isotopies that transform spatial graphs that fulfil the assumptions of Theorem 1 into the plane $\mathbb{R}^2$, are given in [13]. Another consequence of Theorem 1 that is stated in the remark has been shown in [11] together with [14]: Nontrivial 3-connected and simple planar spatial graphs that are embedded on a torus are chiral. A graph is **simple** if it contains no loops and no multi-edges. It is **3-connected** if at least three vertices and their incident edges have to be deleted to decompose the graph or to reduce it to a single vertex. A spatial graph is **chiral** if it is not ambient isotopic to its mirror image.

2. Proof of Theorem 1

2.1. **Outline of the proof.** The proof uses two theorems of Scharlemann, Thompson [15] and Ozawa, Tsutsumi [5]. We assume that the spatial graph $\mathcal{G}$ we consider is given by an embedding $f : G \to T^2$ of a planar graph G and furthermore that $\mathcal{G}$ contains no nontrivially knotted or linked subgraph. We conclude that $\mathcal{G}$ must be trivial. During the proof, we need the following two definitions:

Definition 1. *An embedding $f : G \to S^3$ of a graph G is **primitive**, if for each component G_i of G and any spanning tree T_i of G_i, the bouquet graph $f(G_i)/f(T_i)$ obtained from $f(G_i)$ by contracting all edges of $f(T_i)$ in S^3 is trivial.*

Definition 2. *An embedding $f : G \to S^3$ of a graph G is **free**, if the fundamental group of $S^3 - f(G)$ is free.*

The argument of the proof is as follows: We start showing that the statement is true for nonstandardly embedded tori in Lemma 1. With Lemma 2 we argue that it is sufficient to consider connected graphs. Then we show in Lemma 3 that a bouquet graph on T^2 either contains a nontrivial knot or is trivial. Since any connected spatial graph $\mathcal{G}$ on T^2 contracts to a bouquet graph on T^2, it follows that $\mathcal{G}$ is primitive if it contains no nontrivial knot. By Theorem 2 we know that the restriction $f|_{G'}$ is free for all connected subgraphs G' of G. Applying Lemma 2 to the subgraphs G'' of G that are not connected, we see that $f|_{G_s}$ is free for all subgraphs G_s of G. Using Theorem 3 we conclude that $\mathcal{G}$ is trivial.

2.2. **Preparations for the proof.**

Lemma 1 (Nonstandardly embedded torus)**.** Let $\mathfrak{T}^2$ be a torus that is not standardly embedded. Any spatial graph $\mathcal{G}$ that is embedded in $\mathfrak{T}^2$ and that contains no nontrivial knot is trivial.

Proof. If the spatial graph $\mathcal{G}$ contains a cycle that follows a longitude of the torus $\mathfrak{T}^2$, this cycle is knotted since $\mathfrak{T}^2$ itself is knotted. Therefore, no such subgraph of $\mathcal{G}$ can exist and we find a meridian m of $\mathfrak{T}^2$ that has no intersection with $\mathcal{G}$. This shows that $\mathcal{G}$ in embedded in the twice punctured sphere $\mathfrak{T}^2 - m \simeq S^2 - \{p_1, p_2\}$. Therefore, $\mathcal{G}$ is trivial. □

It follows from Lemma 1 that the statement of Theorem 1 is true for nonstandardly embedded tori. Therefore, we only consider the standardly embedded torus T^2 from now on which saves us from considering different cases.

Lemma 2 (Connectivity Lemma)**.** The image $\mathcal{G}$ of an embedding $f : G \to T^2 \subset S^3$ of a graph G with $n > 1$ connected components on the standard torus T^2 contains either a nonsplit link, or contains no nonsplit link and decomposes into n disjoint components of which at least $n - 1$ components are trivial.

Proof. Take any connected component $f(G_i)$ of the embedding $f(G)$ on the torus T^2. The complement of $f(G_i)$ *in the torus* (without considering the rest of the spatial graph $f(G - G_i)$) is a collection of pieces that can be the punctured torus, discs, and essential annuli without boundaries. (An essential annulus contains a simple closed curve that does not bound a disc in the torus.)

In the case that the complement of $f(G_i)$ in T^2 includes the punctured torus, $f(G_i)$ is trivial and splits from the other components.

If the complement of $f(G_i)$ in T^2 is only a collection of discs, then all other components of $f(G)$ lie in one of those discs and therefore are trivial and the graph is split. ($f(G_i)$ might or might not contain a nonsplit link.)

In the case that the complement of $f(G_i)$ in T^2 includes an essential annulus A, it is possible that other components of G are embedded in this annulus. A component G_j might be embedded in the annulus in two ways: Either the complement of $f(G_j)$ in A includes a punctured annulus and therefore $f(G_j)$ is trivial and splits from the rest of the spatial graph $f(G - G_j)$. Or $A - f(G_j)$ contains two annuli. The annulus A has one type of an essential curve c running inside it; c is parallel to the boundary curves of A. In the case that $A - f(G_j)$ contains two annuli, a subgraph of $f(G_j)$ must be deformable to be parallel to c. If c is a meridian or a prefered longitude of T^2, both components $f(G_i)$ and $f(G_j)$ are split and trivial since the torus is a standard torus. If c is neither a meridian nor a longitude of T^2, $f(G_i)$ and $f(G_j)$ are nonsplittably linked. □

Lemma 3 (Bouquet Lemma)**.** The image $\mathcal{B}$ of an embedding $f : B \to T^2 \subset S^3$ of a connected bouquet graph B on the torus T^2 either contains a nontrivial knot or is trivial.

Proof. A bouquet graph $\mathcal{B}$ on T^2 that contains no nontrivial knot contains only cycles which all are the unknot by assumption. The unknot on the torus can take the following forms:

(1) $T(0,0)$ loop that bound a disc in T^2 (trivial elements in $\pi_1(T^2)$),
(2) $T(0,1)$ meridional loop,
(3) $T(1,0)$ longitudinal loop,
(4) $T(1,n)$ loop or alternatively $T(n,1)$ loop, $n \geq 1$

Loops of type (1) do not contribute to the nontriviality of $\mathcal{B}$.
If $\mathcal{B}$ has loops of the types (1), (2) and (3) only, it is trivial.
If $\mathcal{B}$ has loops of type (4), there are – beside the loops $T(0,0)$ – only three types of loops simultaneously embeddable on the torus without self-intersections: $T(0,1), T(1,n)$ and $T(1,n+1)$ (respectively $T(1,0), T(n,1)$ and $T(n+1,1)$). This can easily be confirmed by applying the formula of Rolfsen's exercise 2.7 [16]: If two torus knots $T(p,q)$ and $T(p',q')$ intersect in one point transversally, then $pq' - qp' = \pm 1$. Such a bouquet is trivial. □

Theorem 2 (Ozawa and Tsutsumi's freeness criterion [5])**.** *An embedding $f : G \to S^3$ of a graph G is primitive if and only if the restriction $f|_{G'}$ is free for all connected subgraphs G' of G.*

Theorem 3 (Scharlemann and Thompson's planarity criterion [15])**.** *An embedding $f : G \to S^3$ of a graph G is trivial if and only if*
(a) G is planar and
(b) for every subgraph $G_s \subset G$, the restriction $f|_{G_s}$ is free.

2.3. **The proof.** We are now ready to prove Theorem 1 and Corollary 1:

Proof. (of Theorem 1). It follows from Lemma 1 that the statement of Theorem 1 is true for nonstandardly embedded tori. Therefore, we assume that $\mathcal{G}$ is embedded in the standard torus T^2. Since $\mathcal{G}$ contains no nonsplit link by assumption, we can assume by Lemma 2 that G is connected. Any connected spatial graph contracts to a spatial bouquet graph $\mathcal{B}$ if a spanning tree T is contracted in S^3. If the spatial graph is embedded in a surface, edge contractions can be realised in the surface. It follows that contracting a spanning tree of a connected spatial graph that is embedded in T^2 results in a bouquet graph that is embedded in T^2 itself. Since $\mathcal{G}$ contains no nontrivial knot by assumption, $\mathcal{B}$ also contains no nontrivial knot. We know from Lemma 3 that a bouquet graph that is embedded in the torus T^2 and that contains no nontrivial knot is trivial. Therefore it follows that, for any chosen spanning tree T of G, the bouquet graph $\mathcal{B} = f(G)/f(T)$ which is obtained from $f(G)$ by contracting all edges of $f(T)$ in S^3 is trivial. Consequently f is primitive by definition. By Theorem 2, the restriction $f|_{G'}$ is free for all connected subgraphs G' of G. Let G'' be a subgraph of G that is not connected. Since G'' is a subgraph of G, it does neither contain nontrivial links nor nontrivial knots by assumption. Applying Lemma 2 to G'' shows that the connected components of $f|_{G''}$ are split and at most one connected component $f|_{G''_1}$ of $f|_{G''}$ is not trivial. Therefore, the restriction $f|_{G''}$ is free if and only if $f|_{G''_1}$ is free. Since G''_1 is a connected subgraph of G, we know already that $f|_{G''_1}$ is free. Therefore, the restriction $f|_{G_s}$ is free for all subgraphs G_s of G. As G is planar by assumption, it follows from Theorem 3 that f is trivial. □

Proof. (of Corollary 1). As there exists no pair of disjoint cycles in a θ_n-graph, such a graph does not contain a nontrivial link. Since θ_n-graphs are planar, the statement of the corollary follows directly from Theorem 1. □

It has been shown in [11] together with [14] that every nontrivial embedding of a simple 3-connected spatial graph on the torus that contains a nontrivial knot or a nonsplit link is chiral. The following remark is therefore a consequence of Theorem 1.

Remark (Chirality)**.** Nontrivial embeddings of simple 3-connected planar graphs in the torus are chiral.

2.4. **All assumptions that have been made are necessary.**

This can be seen by considering the following examples:

- There exist nontrivial embeddedings on T^2 that contain neither a nontrivial knot nor a nonsplit link.
  These are embeddings of nonplanar graphs.
  Examples: $K_{3,3}$ and K_5 embedded as shown left in Fig. 1.

- There exist nontrivial embeddings of planar graphs that contain neither a nontrivial knot nor a nonsplit link.
  These embeddings are not embedded on the torus.
  Examples: Kinoshita-theta curve (middle in Fig. 1) and every ravel.
- There exist nontrivial embeddings of planar graphs on T^2.
  Examples: Spatial graphs that are subdivisions of nontrivial torus knots with $n > 0$ vertices and n edges (right in Fig. 1).

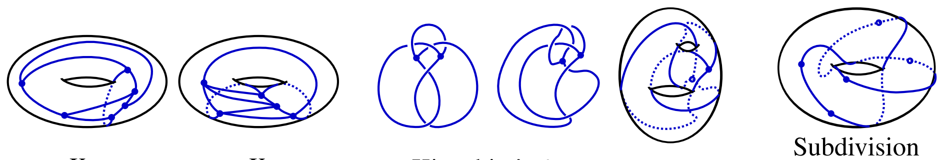

FIGURE 1. All assumptions are necessary.

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