

# Supergravity one-loop corrections on $AdS_7$ and $AdS_3$ , higher spins and AdS/CFT

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Matteo Beccaria,<sup>a</sup> Guido Macorini,<sup>a</sup> Arkady A. Tseytlin<sup>1b</sup>

<sup>a</sup>*Dipartimento di Matematica e Fisica Ennio De Giorgi,  
Università del Salento & INFN, Via Arnesano, 73100 Lecce, Italy*

<sup>b</sup>*The Blackett Laboratory, Imperial College, London SW7 2AZ, U.K.*

*E-mail:* [matteo.beccaria@le.infn.it](mailto:matteo.beccaria@le.infn.it), [macorini@nbi.ku.dk](mailto:macorini@nbi.ku.dk),  
[tseytlin@imperial.ac.uk](mailto:tseytlin@imperial.ac.uk)

ABSTRACT: As was shown earlier, the one-loop correction in 10d supergravity on  $AdS_5 \times S^5$  corresponds to the contributions to the vacuum energy and 4d boundary conformal anomaly which are minus the values for one  $\mathcal{N} = 4$  Maxwell supermultiplet, thus reproducing the subleading term in the  $N^2 - 1$  coefficient in the dual  $SU(N)$  SYM theory. We perform similar one-loop computations in 11d supergravity on  $AdS_7 \times S^4$  and 10d supergravity on  $AdS_3 \times S^3 \times T^4$ . In the  $AdS_7$  case we find that the corrections to the 6d conformal anomaly a-coefficient and the vacuum energy are again minus the ones for one (2,0) tensor multiplet, suggesting that the total a-anomaly coefficient for the dual (2,0) theory is  $4N^3 - 9/4N - 7/4$  and thus vanishes for  $N = 1$ . In the  $AdS_3$  case the one-loop correction to the vacuum energy or 2d central charge turns out to be equal to that of one free (4,4) scalar multiplet, i.e. is  $c = +6$ . This reproduces the subleading term in the central charge  $c = 6(Q_1 Q_5 + 1)$  of the dual 2d CFT describing decoupling limit of D5-D1 system. We also present the expressions for the 6d a-anomaly coefficient and vacuum energy contributions of general-symmetry higher spin field in  $AdS_7$  and consider their application to tests of vectorial AdS/CFT with the boundary conformal 6d theory represented by free scalars, spinors or rank 2 antisymmetric tensors.

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<sup>1</sup>Also at Lebedev Institute, Moscow

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## 1 Introduction

One of the key probes of the AdS/CFT correspondence [1–3] is the boundary theory conformal anomaly which is closely related to the simplest correlators of the stress tensor [4–6]. In the case of the duality between  $\mathcal{N} = 4$   $SU(N)$  SYM theory and string theory in  $AdS_5 \times S^5$  the gauge-theory result for the Weyl anomaly is  $\mathcal{A}_4 = -a\mathcal{E}_4 + c\mathcal{W}_4$ ,  $a = c = (N^2 - 1)k_1$  ( $k_1 = \frac{1}{4}$  is the contribution of a single  $\mathcal{N} = 4$  vector multiplet). It is determined by the 2- and 3-point correlators of stress tensor and should thus be exact. The  $N^2$  term is indeed reproduced at strong coupling by the classical supergravity action [5].

It was suggested in [7, 8]<sup>1</sup> that the -1 term in  $N^2 - 1$  coefficient should come from the one-loop 10d supergravity correction (the contribution of all massive string mode multiplets

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<sup>1</sup>This follows the analogy with what happens in the case of the R-symmetry anomaly [9].

should vanish). This was recently confirmed in [10] where it was found that the contributions of the massless 5d supergravity modes and the massive  $S^5$  KK modes to the boundary conformal anomaly can be universally described by a simple formula:  $a_p = c_p = pk_1$ , where  $p = 1$  for a vector multiplet (or boundary doubleton to be omitted),  $p = 2$  for the massless 5d supergravity modes, and  $p = 3, 4, \dots$  for the massive KK levels. Summing over  $p$  using a special regularization prescription  $\sum_{p=1}^{\infty} p = 0$  (which is, in fact, required for consistency with the standard  $\zeta$ -function regularization for the Casimir energy in 10d) gives indeed  $(a = c)_{1\text{-loop sugra}} = -1$ .

Below will perform a similar one-loop computation of the boundary a-anomaly in the case of 11d supergravity on  $AdS_7 \times S^4$  (correcting an earlier attempt in [11]). This will determine the subleading  $N^0$  term in the a-coefficient of conformal anomaly of the 6d (2,0) theory describing  $N$  coincident M5-branes which should be dual to M-theory on  $AdS_7 \times S^4$ .

In addition to the duality examples based on  $AdS_5 \times S^5$  and  $AdS_7 \times S^4$  supergravity backgrounds there is also the duality [1, 12] between string theory in  $AdS_3 \times S^3 \times T^4$  space supported by RR 3-form flux and 2d CFT corresponding to gauge theory describing low-energy limit D5-D1 system. The central charge of this CFT is  $c = 6(Q_1 Q_5 + 1)$  [13, 12] ( $Q_i$  are the number of branes). The leading  $6Q_1 Q_5$  can be reproduced from the classical action of 10d supergravity on  $S^3 \times T^4$  [5, 14]. Here we shall demonstrate that the subleading +6 term is reproduced by the one-loop 10d supergravity contribution. This provides a non-trivial test of this  $AdS_3/CFT_2$  duality.

### 1.1 $AdS_7/CFT_6$

The conformal anomaly of a classical Weyl invariant theory in 6d has the following general form [15–17]

$$\mathcal{A}_6 = a \mathcal{E}_6 + W_6 + D_6, \quad W_6 = c_1 I_1 + c_2 I_2 + c_3 I_3, \quad (1.1)$$

where  $\mathcal{E}_6$  is the Euler density in six dimensions,  $W_6$  is a combination of three independent Weyl invariants and  $D_6$  is a total derivative term (which can be changed by adding a local counterterm and thus depends on a scheme). Omitting the derivative  $D_6$  term, the conformal anomaly corresponding to a single 6d tensor multiplet [17] and the 6d conformal anomaly contribution coming from the classical 11d supergravity action on  $S^7$  [5] (that should be representing the large  $N$  limit of the (2,0) theory result) may be written as<sup>2</sup>

$$\mathcal{A}_6 = a \mathcal{E}_6 + c \mathcal{W}_6, \quad \mathcal{W}_6 \equiv 96I_1 + 24I_2 - 8I_3, \quad (1.2)$$

$$a_{\text{tens}} = \frac{7}{4}, \quad c_{\text{tens}} = 1, \quad a_{(2,0)} = 4N^3 + \dots, \quad c_{(2,0)} = 4N^3 + \dots. \quad (1.3)$$

The fact that the anomaly in these two cases contains the same Weyl-invariant combination  $\mathcal{W}_6$  (so that its Weyl-tensor or B-anomaly part is effectively parametrized by just one overall coefficient  $c$ ) is related to non-renormalization of the ratio of the 2- and 3- point correlation functions of the corresponding stress tensor [19].<sup>3</sup>

<sup>2</sup>Here  $\mathcal{A}_6$  stands for the integrand of the Seeley coefficient  $b_6$  up to the overall factor  $-\frac{1}{3^2 \times 2^5}$  required to reconcile the normalized values of a coefficient below with the ones in eqs. (3.3) and (A.2); see also [18] for other notation.

<sup>3</sup>The a-coefficient in 6d is related to 4-point stress tensor correlator and may thus receive a more non-trivial renormalization.

By analogy with a subleading order- $N$  term in the R-symmetry anomaly of (2,0) theory [20] it was suggested in [18] that there should be also order  $N$  contributions to  $a_{(2,0)}$  and  $c_{(2,0)}$  coming from the  $R^4$  term in the M-theory 11d effective action,

$$a_{(2,0)} = 4N^3 - \frac{9}{4}N + a_1, \quad c_{(2,0)} = 4N^3 - 3N + c_1. \quad (1.4)$$

In [18] the further  $N^0$  corrections  $a_1, c_1$  were ignored, while the coefficients of order  $N$  terms were fixed so that the resulting  $N^3 + N$  terms interpolated to  $N = 1$  match the single tensor-multiplet anomalies in (1.3). As in the case of 10d supergravity on  $S^5$ , one may expect that  $a_1$  and  $c_1$  should be determined by the one-loop 11d supergravity correction [11].

Following the example of the D3-brane-based  $AdS_5 \times S^5$  duality where the full anomaly coefficient  $N^2 - 1$  vanishes for  $N = 1$  it is natural to expect that here too the boundary singleton (single M5-brane tensor multiplet) should decouple and thus the full 6d anomaly of the (2,0) theory should vanish for  $N = 1$ . This suggests that  $a_1$  and  $c_1$  should be non-zero and given by minus the values for a single tensor multiplet in (1.3)

$$a_1 = -a_{\text{tens}} = -\frac{7}{4}, \quad c_1 = -c_{\text{tens}} = -1. \quad (1.5)$$

It was noted in [21] that the expression  $c_{(2,0)} = 4N^3 - 3N - 1 = (N-1)(2N+1)^2$  is exactly the same as the central charge of the  $A_{N-1}$  Toda theory at the “symmetric” coupling point (cf. also [22, 23]).<sup>4</sup>

Here we shall provide support for (1.5) by showing that the one-loop 11d supergravity correction indeed produces the value  $a_1 = -a_{\text{tens}}$ . Then the expected exact value of  $a_{(2,0)}$  is<sup>5</sup>

$$a_{(2,0)} = 4N^3 - \frac{9}{4}N - \frac{7}{4} = (N-1)(4N^2 + 4N + \frac{7}{4}). \quad (1.6)$$

Below we shall consider the one-loop 11d supergravity on  $S^7$  supergravity contributions in the case when the 6d boundary of  $AdS_7$  is either  $S^6$  (determining the a-anomaly part of  $\mathcal{A}_6$ ) or  $R \times S^5$  (finding the vacuum or Casimir energy  $E_c$ ). We will find that in both cases the result is minus that of a single tensor multiplet

$$a_{1\text{-loop sugra}} = -a_{\text{tens.}}, \quad E_{c\ 1\text{-loop sugra}} = -E_{c\ \text{tens.}}. \quad (1.7)$$

We shall use similar methods as in the  $AdS_5 \times S^5$  case in [10], i.e. first determine the contributions to  $a$  and  $E_c$  coming from a generic  $AdS_7$  higher spin field in representation

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<sup>4</sup>6d CFT with (2,0) supersymmetry possess a protected sector of operators and observables related to a 2d chiral algebra [21] which is  $\mathcal{W}$ -algebra labelled by a simply-laced Lie algebra  $\mathfrak{g}$  for a specific value of the central charge. In the  $\mathfrak{g} = A_{N-1}$  case this leads to  $c_1 = -1$ .

<sup>5</sup>The non-vanishing 1-loop supergravity correction to the conformal anomaly implies that there should be also a similar correction also to the corresponding R-symmetry anomaly (i.e.  $N \rightarrow N - 1$  in the  $I_8$  term in the anomaly [24]) implying its vanishing for  $N = 1$ . The chiral anomaly of the boundary theory is accounted for by the Chern-Simons terms in the supergravity action. In the case of  $AdS_5 \times S^5$  the 1-loop supergravity correction shifts the Chern-Simons coefficient  $N^2 \rightarrow N^2 - 1$  [9]. A similar shift is then expected in the  $AdS_7 \times S^7$  case where the CS term reproduces the leading  $N^3$  anomaly and also the  $\mathcal{O}(N)$  correction [20].

$(\Delta; h_1, h_2, h_3)$  of  $SO(2, 6)$  and then sum up the contributions of the relevant fields appearing in the supergravity spectrum.

We shall also apply our general expressions for  $a(\Delta; h_1, h_2, h_3)$  and  $E_c(\Delta; h_1, h_2, h_3)$  to provide tests of the vectorial AdS/CFT duality [25–27] in the case when the boundary theory is represented by a free scalar, spinor or tensor singleton.

## 1.2 AdS<sub>3</sub>/CFT<sub>2</sub>

The 2d CFT dual to superstring in  $AdS_3 \times S^3 \times T^4$  with RR charges  $Q_5, Q_1$  is described by a coupled system of three (4,4) supersymmetric multiplets (see [13, 12] and [28] for a recent review):  $U(Q_1)$  adjoint vector multiplet,  $U(Q_1)$  adjoint hypermultiplet, and  $U(Q_1) \times U(Q_5)$  bi-fundamental hypermultiplet. The contribution to 2d conformal anomaly of a single free (4,4) hypermultiplet (with 4 real scalars and 4 real fermions) is  $c = 4 + 4 \times \frac{1}{2} = 6$ .<sup>6</sup> The 2d vector multiplet has an irrelevant kinetic term and thus contributes to anomaly only through measure (or ghost) factor, with single  $U(1)$  vector giving negative contribution  $c = -1$ .<sup>7</sup> The  $U(1)$  part of the vector multiplet is decoupled (representing the c.o.m. of the bound D5-D1 system) and thus the total central charge count is<sup>8</sup>

$$c = 6Q_1Q_5 + Q_1^2 - 6(Q_1^2 - 1) = 6Q_1Q_5 + 6, \quad (1.8)$$

where the first term is the contribution of bi-fundamental hypers, the second – of adjoint hypers and the third one of the vectors (with the  $U(1)$  part subtracted).<sup>9</sup>

A peculiarity of the 2d case is that here the subleading (for large  $Q_5$ ) term in the central charge which is responsible for subtraction of the decoupled c.o.m. modes enters with plus rather than minus sign (as was in 4d and 6d examples). Still, we shall demonstrate below that as in the  $AdS_5$  and  $AdS_7$  cases this extra +6 term (which should be protected and thus receive contributions only from the BPS modes) is also reproduced on the dual AdS theory side by the corresponding one-loop correction in 10d supergravity on  $AdS_3 \times S^3 \times M^4$  with  $M^4 = T^4$  or  $K3$ .

More precisely, instead of computing directly the correction to the central  $c$  we shall determine the one-loop correction to the  $AdS_3$  vacuum energy or  $S^1$  Casimir energy in 2d; the latter should be directly related to the central charge [31]

$$E_c = -\frac{1}{12}c, \quad \text{i.e.} \quad c = 6 \quad \leftrightarrow \quad E_c = -\frac{1}{2}. \quad (1.9)$$

<sup>6</sup>In 2d the conformal anomaly is  $\mathcal{A}_2 = 4\pi b_2 = aR$ ,  $a = \frac{1}{6}c$ , so that  $c = 1$  for one real scalar.

<sup>7</sup>The contribution of “non-dynamical” 2d vector gauge field to the central charge is negative (-1) [29] just like that of non-dynamical 2d gravity (-26) [30]. The reason for this -1 contribution can be understood also by giving vector a mass by coupling it to a complex scalar so that it will not contribute to  $c$ ; then the central charge of the scalar part is reduced by 1 as one scalar component is absorbed by the vector.

<sup>8</sup>The same result is found by counting the  $SU(2)$  chiral anomaly of the (4,4) superconformal algebra [13, 12].

<sup>9</sup>The (4,4) vector multiplet contains one 2d vector  $A_m$ , 4 scalars  $\phi_i$ , 4 real spinors  $\psi_k$  and 3 auxiliary fields  $D_r$ , all having canonical dimensions (i.e. 1 for  $A_m$  and  $\phi_i$ ,  $\frac{1}{2}$  for  $\psi_k$  and 2 for  $D_r$ ). With these dimension assignments the corresponding 2d conformal anomaly can be found from the following dimensionless action (same as the standard one but with each kinetic term containing an extra  $\partial^{-2}$  factor)  $\int d^2x [(\mathcal{A}_m^\perp)^2 + \phi_i^2 + D_r \partial^{-2} D_r + \psi_k \partial^{-1} \psi_k]$ . As a result, the total central charge contribution is  $c = -1 + 0 + 3 \times (-1) + 4 \times (-\frac{1}{2}) = -6$ .

We shall find that the one-loop supergravity contribution gives indeed  $E_c = -\frac{1}{2}$  after summing over the contributions of the KK modes of 10d supergravity on  $S^3 \times M^4$ .

The rest of this paper is organized as follows. In section 2 we shall present the expressions for the a-anomaly coefficient and the vacuum energy of a higher-spin field in  $AdS_7$  corresponding to an arbitrary (massive or massless) representation of  $SO(2,6)$ , generalising earlier results for symmetric tensors to mixed symmetry case.

In section 3 we shall apply these results to compute the one-loop corrections to the 6d boundary a-anomaly and vacuum energy in 11d supergravity compactified on  $S^7$  obtaining eq. (1.7). As another application, in section 4 we shall perform checks of vectorial  $AdS_7/CFT_4$  duality in the cases when the boundary 6d theory is represented by free scalars, spinors or (self-dual) rank 2 tensors. We shall find that matching of both a-anomaly and Casimir energy requires particular shifts of the inverse coupling of the  $AdS_7$  higher spin theory.

In section 5 we shall turn to the case of 10d supergravity in  $AdS_3 \times S^3 \times M^4$  and compute the corresponding one-loop correction to the vacuum energy, demonstrating that it is equal to  $-\frac{1}{2}$  as in (1.8), thus deriving the subleading term in the central charge (1.9) on the dual string theory side.

There are several technical appendices. In appendix A we present the expressions for the Casimir energy, a-anomaly and partition function for the fields of the free  $(2,0)$  multiplet in 6d. In Appendix B we derive the 6d boundary a-anomaly coefficient corresponding to a generic higher spin field on  $AdS_7$  using spectral  $\zeta$ -function method. Appendix C collects decompositions of tensor products of two  $SO(2,6)$  singleton representations with spin  $0, \frac{1}{2}, 1$  into infinite sums of other representations and the corresponding relations for the characters. These Flato-Fronsdal like relations are used in the discussion of applications to vectorial AdS/CFT duality in section 4. Appendix D contains discussion of some properties of the Casimir energy of spin  $0, \frac{1}{2}, 1$  singletons in  $AdS_{d+1}$  for general  $d$ . They are useful in comparing the 6d results to the previously studied 4d case. In appendix E we list the explicit field content of the  $SU(2,2|1) \times SU(2,2|1)$  building blocks appearing in the Kaluza-Klein towers of 6d supergravity compactified on  $S^3$ . Appendix F contains the discussion of the relation between the expression for the 2d Casimir energy in section 5 and the 2d central charge derived [32] using  $AdS_3$  method for short  $SU(2,2|1) \times SU(2,2|1)$  multiplets.

## 2 Casimir energy and a-anomaly for generic higher spin fields in $AdS_7$

Given a generic conformal field in 6d we may associate to it a field in  $AdS_7$  corresponding to the same representation of  $SO(2,6)$ . That allows to interpret the one-loop contributions for a field in  $AdS_7$  in terms of Casimir energy and conformal anomaly of the boundary field (see [10] and refs. there).

The  $SO(2,6)$  conformal group representations will be denoted as  $(\Delta; \mathbf{h})$  where  $\mathbf{h} = (h_1, h_2, h_3)$  are the  $SO(6)$  highest weights or Young tableau labels ( $h_i$  are all integers or all half-integers with  $h_1 \geq h_2 \geq |h_3|$ ).<sup>10</sup>

<sup>10</sup>An alternative is to use  $SO(6)$  Dynkin labels  $[r_1, r_2, r_3] = (h_2 - h_3, h_1 - h_2, h_2 + h_3)$ .

The unitary irreducible representations of  $SO(2, 6)$  have (see, e.g., [33, 34])

$$\begin{aligned}
(i) \quad & \Delta \geq \Delta = h_1 + 4, & \text{for } h_1 > h_2 \geq |h_3|, \\
(ii) \quad & \Delta \geq \Delta = h_1 + 3, & \text{for } h_1 = h_2 > |h_3|, \\
(iii) \quad & \Delta \geq \Delta = h_1 + 2, & \text{for } h_1 = h_2 = \pm h_3, \\
(iv) \quad & \Delta \geq 2 \text{ or } \Delta = 0 & \text{for } h_1 = h_2 = h_3 = 0.
\end{aligned} \tag{2.1}$$

If  $\Delta$  does not saturate the above inequalities then the character of the corresponding *massive* representation is<sup>11</sup>

$$\widehat{\mathcal{Z}}^+(\Delta; \mathbf{h}) = d(\mathbf{h}) \frac{q^\Delta}{(1-q)^6}, \tag{2.2}$$

where  $d(\mathbf{h})$  is the multiplicity of the representation

$$d(\mathbf{h}) = \frac{1}{12}(1+h_1-h_2)(1+h_2-h_3)(1+h_2+h_3)(2+h_1-h_3)(2+h_1+h_3)(3+h_1+h_2). \tag{2.3}$$

If  $\Delta$  is at one of the unitarity bounds the corresponding representation is short or *massless* (i.e. corresponds to a massless field in  $AdS_7$  space)<sup>12</sup> and its character requires a proper subtraction of null states and their descendants. For the  $\Delta = h_1 + 4$  case in (i) in (2.1) we have the following massless representation character

$$\mathcal{Z}^+(h_1 + 4; h_1, h_2, h_3) = \widehat{\mathcal{Z}}^+(h_1 + 4; h_1, h_2, h_3) - \widehat{\mathcal{Z}}^+(h_1 + 5; h_1 - 1, h_2, h_3), \tag{2.4}$$

where  $\mathcal{Z}^+$  is given in (2.2). For the massless  $\Delta = h_1 + 3$  case with  $h_1 = h_2 = h > |h_3|$  in (ii) we get

$$\begin{aligned}
\mathcal{Z}^+(h + 3; h, h, h_3) &= \widehat{\mathcal{Z}}^+(h_1 + 3; h, h, h_3) - \widehat{\mathcal{Z}}^+(h + 4; h, h - 1, h_3) \\
&\quad + \widehat{\mathcal{Z}}^+(h + 5; h - 1, h - 1, h_3).
\end{aligned} \tag{2.5}$$

In the massless case of (iii) with  $\Delta = h + 2$  and  $\mathbf{h} = (h, h, \pm h)$  which corresponds to the *singleton* representation the character is

$$\begin{aligned}
\mathcal{Z}^+(h + 2; h, h, \pm h) &= \widehat{\mathcal{Z}}^+(h + 2; h, h, \pm h) - \widehat{\mathcal{Z}}^+(h + 3; h, h, \pm(h - 1)) \\
&\quad + \widehat{\mathcal{Z}}^+(h + 4; h, h - 1, \pm(h - 1)) - \widehat{\mathcal{Z}}^+(h + 5; h - 1, h - 1, \pm(h - 1)).
\end{aligned} \tag{2.6}$$

In particular, it is possible to view the  $(2, 0)$  tensor multiplet as supersingleton [35] which is a combination of 6d singletons with  $h = 0, \frac{1}{2}, 1$ : the one-particle partition functions for a scalar  $\phi$ , Majorana-Weyl fermion  $\psi$  and self-dual tensor  $T$  are the characters of the corresponding singleton representations (see also Appendices A and C)

$$\mathcal{Z}_\phi = \mathcal{Z}_{\{0\}} = \mathcal{Z}^+(2; 0, 0, 0), \quad \mathcal{Z}_\psi = \mathcal{Z}_{\{\frac{1}{2}\}} = \mathcal{Z}^+(\frac{5}{2}; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}),$$

<sup>11</sup>The label  $^+$  indicates that this will represent the partition function of the corresponding  $AdS_7$  field with standard (Dirichlet) boundary conditions. Same quantity without  $^+$  corresponds to associated conformal field in boundary theory (see [10] for details).  $\widehat{\mathcal{Z}}$  indicates massive representation character.

<sup>12</sup>In general [33], given a field in  $AdS_{d+1}$  (with even  $d$ ) corresponding to  $SO(2, d)$  representation  $(\Delta; h_1, h_2, \dots, h_{\frac{d}{2}})$  where first  $k = 0, 1, 2, \dots$  rows of the  $SO(d)$  Young tableau may be equal, i.e.  $h_1 = \dots = h_k > h_{k+1} \geq h_{k+2} \geq \dots \geq h_{\frac{d}{2}}$ , this field is massless if  $\Delta = h_k - k + d - 2$ . In the case of (2.1) where  $d = 6$  the lower bounds in (i),(ii) and (iii) correspond to  $k = 0, 1, 2$ .

$$\mathcal{Z}_T = \mathcal{Z}_{\{1\}} = \mathcal{Z}^+(3; 1, 1, 1) . \quad (2.7)$$

From one-particle partition function  $\mathcal{Z}(q)$  given by the corresponding  $SO(2, 6)$  character one can extract the expression for the Casimir energy  $E_c$  as [36]

$$E_c = \frac{1}{2} (-1)^F \sum_n d_n \omega_n = \frac{1}{2} (-1)^F \zeta_E(-1) , \quad (2.8)$$

$$\zeta_E(z) = \sum_n \frac{d_n}{\omega_n^z} = \frac{1}{\Gamma(z)} \int_0^\infty d\beta \beta^{z-1} \mathcal{Z}(e^{-\beta}) . \quad (2.9)$$

For a generic massive representation  $(\Delta; \mathbf{h})$  with the character (2.2) the corresponding Casimir energy is found to be ( $\bar{h} \equiv h_1 + h_2 + h_3$ )

$$\widehat{E}_c^+(\Delta; \mathbf{h}) = \frac{(-1)^{2\bar{h}} d(\mathbf{h})}{120960} (\Delta - 3) \left[ 12 (\Delta - 3)^6 - 126 (\Delta - 3)^4 + 336 (\Delta - 3)^2 - 191 \right] . \quad (2.10)$$

The expression for the a-anomaly can be found from the one-loop partition function on euclidean  $AdS_7$  as explained in appendix B

$$\begin{aligned} \widehat{a}^+(\Delta; \mathbf{h}) &= \frac{(-1)^{2\bar{h}} d(\mathbf{h})}{2 \times 96 \times 37800} (\Delta - 3) \left[ 15 (\Delta - 3)^6 \right. \\ &\quad - 21 (\Delta - 3)^4 [h_3^2 + h_1 (h_1 + 4) + h_2 (h_2 + 2) + 5] \\ &\quad + 35 (\Delta - 3)^2 [(h_1 + 2)^2 (h_2 + 1)^2 + (h_1 (h_1 + 4) + h_2 (h_2 + 2) + 5) h_3^2] \\ &\quad \left. - 105 (h_1 + 2)^2 (h_2 + 1)^2 h_3^2 \right] . \end{aligned} \quad (2.11)$$

In the case of short representations saturating a unitarity bound one needs to combine the massive representation expression as in (2.4),(2.5),(2.6).

In the special case of the totally symmetric massive spin  $s$  tensor representation with  $\mathbf{h} = (s, 0, 0)$ , the expression (2.11) can be written in the following alternative form

$$\begin{aligned} \widehat{a}^+(\Delta; s, 0, 0) &= \frac{5 (s + 2)(s + 3)!}{8 (6!)^2 \pi s!} \\ &\quad \times \int_3^\Delta dx (x - 3)(x + s - 1)(x - s - 5) \Gamma(x - 1) \Gamma(5 - x) \sin(\pi x), \end{aligned} \quad (2.12)$$

which is in agreement with the earlier result in [32, 26].

### 3 One-loop correction to vacuum energy and a-anomaly in 11d supergravity on $AdS_7 \times S^4$

Let us now apply the above results (2.10) and (2.11) to compute the corresponding total contribution of the fields in the spectrum of 11d supergravity compactified on  $S^4$ . The corresponding KK spectrum [37, 35, 38] is given in Table 1 (see also [39]). The massless level  $p = 2$  correspond to the fields of maximal gauged 7d supergravity with  $AdS_7$  vacuum.



	$(\Delta; h_1, h_2, h_3)$	$USp(4)$		$(\Delta; h_1, h_2, h_3)$	$USp(4)$
$p \geq 2$	$(2p; 0, 0, 0)$	$[0, p]$	$p \geq 4$	$(2p+2; 0, 0, 0)$	$[4, p-4]$
	$(2p + \frac{1}{2}; \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$	$[1, p-1]$		$(2p + \frac{5}{2}; \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$	$[3, p-4]$
	$(2p+1; 1, 1, -1)$	$[0, p-1]$		$(2p+3; 1, 0, 0)$	$[2, p-4]$
	$(2p+1; 1, 0, 0)$	$[2, p-2]$		$(2p + \frac{7}{2}; \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	$[1, p-4]$
	$(2p + \frac{3}{2}; \frac{3}{2}, \frac{1}{2}, -\frac{1}{2})$	$[1, p-2]$		$(2p+4; 0, 0, 0)$	$[0, p-4]$
$p \geq 3$	$(2p + \frac{3}{2}; \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	$[3, p-3]$			
	$(2p+2; 1, 1, 0)$	$[2, p-3]$			
	$(2p + \frac{5}{2}; \frac{3}{2}, \frac{1}{2}, \frac{1}{2})$	$[1, p-3]$			
	$(2p+3; 1, 1, 1)$	$[0, p-3]$			

**Table 1.**  $SO(2, 6) \times USp(4)$  representations of the fields of 11d supergravity on  $AdS_7 \times S^4$ .

Contributions of the  $AdS_7$  fields should be summed with multiplicities corresponding to their  $USp(4) = SO(5)$  representations.<sup>13</sup>

Using (2.10) to sum of the vacuum energy contributions at each level  $p$  we find

$$E_{c,p=2}^+ = -\frac{325}{384}, \quad E_{c,p=3}^+ = -\frac{925}{384}, \quad E_{c,p \geq 4}^+ = -\frac{25}{384} (6p^2 - 6p + 1). \quad (3.1)$$

The value for the massless multiplet  $p = 2$  is in agreement with [36]. The expressions for the a-anomaly are similar

$$a_{p=2}^+ = -\frac{91}{1152}, \quad a_{p=3}^+ = -\frac{259}{1152}, \quad a_{p \geq 4}^+ = -\frac{7}{1152} (6p^2 - 6p + 1). \quad (3.2)$$

Recalling that for one (2,0) tensor multiplet (see appendix A)

$$E_{c, \text{tens.}} = E_{c,1}^+ = -\frac{25}{384}, \quad a_{\text{tens.}} = a_1^+ = -\frac{7}{1152}. \quad (3.3)$$

we observe that, remarkably, both the vacuum energy and a-anomaly has the following universal expressions for any value of  $p = 1, 2, 3, \dots$

$$E_{c,p}^+ = (6p^2 - 6p + 1) E_{c, \text{tens.}}, \quad a_p^+ = (6p^2 - 6p + 1) a_{\text{tens.}}. \quad (3.4)$$

This is the direct analog to what was found in the case of 10d supergravity on  $AdS_5 \times S^5$  in [10] where the role of tensor multiplet was played by  $\mathcal{N} = 4$  vector one (or superdoubleton) and instead of the coefficient  $6p^2 - 6p + 1$  we had simply  $p$ .<sup>14</sup>

To sum over  $p$  we shall use the same prescription as in [10], i.e. introducing a sharp cutoff and dropping all divergent terms<sup>15</sup>

$$\sum_{p=1}^{\infty} (6p^2 - 6p + 1) = 0. \quad (3.5)$$

<sup>13</sup>The dimension of the  $USp(4)$  representation  $[a, b]$  ( $a, b$  are Dynkin labels) is  $\dim(a, b) = \frac{1}{6} (a+1)(b+1)(a+b+2)(a+2b+3)$ .

<sup>14</sup>For comparison, in the case of 11d supergravity on  $AdS_4 \times S^7$  one finds [40, 41] that the contributions to the  $AdS_4$  vacuum energy sum up to zero at each level  $p$  separately, i.e.  $E_{c,p}^+ = 0$ . The boundary conformal anomaly also vanishes as the boundary is 3-dimensional.

<sup>15</sup>Explicitly,  $\sum_{p=1}^P (6p^2 - 6p + 1) = 2P^3 - P \rightarrow 0$ .

This prescription can be justified by using the spectral  $\zeta$ -function regularization directly in 11d, i.e. before explicitly expanding in modes of  $S^4$  (see below); it is such a regularization that should be consistent with diffeomorphism symmetry of 11d theory.

Assuming (3.5) we conclude that if the boundary (2,0) singleton were included in the spectrum of 11d supergravity, the total vacuum energy and a-anomaly would vanish. However, it should be left out representing gauge degrees of freedom. Thus we conclude that the total one-loop supergravity contributions are exactly minus the tensor multiplet ones

$$\sum_{p=2}^{\infty} E_{c,p}^+ = -E_{c,1}^+ = -E_{c,\text{tens.}} , \quad \sum_{p=2}^{\infty} a_p^+ = -a_1^+ = -a_{\text{tens.}} , \quad (3.6)$$

as claimed in (1.7).

Let us now demonstrate that the prescription (3.5) is indeed equivalent to the use of spectral  $\zeta$ -function directly in 11d theory. We shall consider the case of the Casimir energy (for a similar discussion on 10d case see [10]). For a massive 7d field in representation  $(\Delta; \mathbf{h})$  the vacuum energy can be extracted from the partition function (2.2) that we may write in the form

$$\mathcal{Z}^+(\Delta; \mathbf{h}) = d(\mathbf{h}) \sum_{n=0}^{\infty} \binom{n+5}{5} q^{\Delta+n} . \quad (3.7)$$

Then from (2.8),(2.9) we obtain a formal (divergent) expression for  $E_c$

$$\widehat{E}_c^+(\Delta; \mathbf{h}) = \sum_{n=0}^{\infty} e_n(\Delta; \mathbf{h}), \quad e_n(\Delta; \mathbf{h}) = \frac{1}{2} (-1)^{2\bar{h}} d(\mathbf{h}) \binom{n+5}{5} (\Delta + n) . \quad (3.8)$$

This sum can be computed using the  $\zeta$ -function regularization applied to the full effective energy eigenvalue  $\Delta + n$ , or, equivalently, by introducing an exponential cutoff via  $e_n \rightarrow e_n e^{-\epsilon(\Delta+n)}$ , doing the sum, expanding in  $\epsilon \rightarrow 0$ , and finally dropping all singular terms. Keeping  $\epsilon$  finite we may find the contribution to the sum (3.8) from all KK states (taking into account that  $p = 2$  states are massless, cf. (2.4),(2.5)). Denoting the total summand from level  $p$  as  $e_n(p; \epsilon)$  and, summing over both  $n$  and  $p = 1, 2, \dots$ , we obtain

$$\begin{aligned} \sum_{p=1}^{\infty} \sum_{n=0}^{\infty} e_n(p; \epsilon) &= \frac{e^{2\epsilon}}{(e^{\epsilon/2} - 1)^3 (e^{\epsilon/2} + 1)^{11} (e^{\epsilon} + 1)^5} \left( 20e^{\epsilon/2} + 50e^{\epsilon} + 100e^{3\epsilon/2} + 178e^{2\epsilon} \right. \\ &\quad + 260e^{5\epsilon/2} + 343e^{3\epsilon} + 400e^{7\epsilon/2} + 428e^{4\epsilon} + 400e^{9\epsilon/2} + 343e^{5\epsilon} + 260e^{11\epsilon/2} \\ &\quad \left. + 178e^{6\epsilon} + 100e^{13\epsilon/2} + 50e^{7\epsilon} + 20e^{15\epsilon/2} + 5e^{8\epsilon} + 5 \right) = \frac{785}{2048\epsilon^3} + \mathcal{O}(\epsilon) . \end{aligned} \quad (3.9)$$

Thus the finite part of the sum over  $p \geq 1$  vanishes in agreement with (3.5). Equivalently,

$$\sum_{p=2}^{\infty} \sum_{n=0}^{\infty} e_n(p; \epsilon) = - \sum_{n=0}^{\infty} e_n(1; \epsilon) = -\frac{5}{16\epsilon^2} + \frac{25}{384} + \dots , \quad (3.10)$$

in agreement with (3.3),(3.6).

## 4 Vectorial $AdS_7/CFT_6$ duality

As in lower dimensions, we may start with a free CFT in 6d described, e.g., by  $N$  (complex or real) scalars, spinors or rank 2 antisymmetric tensors and consider the duality between its singlet sector represented by the corresponding bilinear conserved currents and higher spin theory in  $AdS_7$  (see, e.g., [26]). The representation content of the 7d theory is determined from the Flato-Fronsdal type decomposition of the product of 2 singleton representations into sum of higher-spin  $SO(2,6)$  representations described in appendix C (see also [10]). Then using the general expressions for the Casimir energy (2.10) and a-anomaly coefficient (2.11) given in section 2 we may study the matching of these quantities on the two sides of the duality. In what follows we shall denote by  $K^+$  the two quantities  $a^+$  and  $E_c^+$  corresponding to  $AdS_7$  field in a generic massless  $SO(2,6)$  representation and also use  $K = -2K^+$  for the associated boundary conformal field values.

Starting with the case of a free conformal scalar boundary 6d theory, the corresponding fields of the dual  $AdS_7$  theory (“type A” theory) are massless totally symmetric tensors with spin  $s$ , for which we find from (2.10),(2.11)

$$E_c^+(s+4; s, 0, 0) = -\frac{1}{483840} \nu^2 (12\nu^3 - 58\nu^2 - 6\nu + 117) \quad , \quad \nu \equiv (s+1)(s+2) \quad (4.1)$$

$$a^+(s+4; s, 0, 0) = -\frac{1}{29030400} \nu^2 (22\nu^3 - 55\nu^2 - 4\nu + 2) \quad . \quad (4.2)$$

The Casimir energy (4.1) is a simple extension of the results in [27]. The a-anomaly expression (4.2) is the same as found in [26]. To sum over spins we shall follow the spectral  $\zeta$ -function prescription of [26] which is equivalent to introducing the cutoff  $e^{-\epsilon(s+\frac{d-3}{2})} = e^{-\epsilon(s+\frac{3}{2})}$  and dropping all singular terms in the limit  $\epsilon \rightarrow 0$ , i.e.

$$\sum_{s=1}^{\infty} K(s) \equiv \sum_{s=1}^{\infty} e^{-\epsilon(s+\frac{3}{2})} K(s) \Big|_{\text{finite part, } \epsilon \rightarrow 0} \quad (4.3)$$

Below we shall use the same prescription also for mixed representations with  $s \equiv \Delta - 4$ .

One can then verify the following relations

$$K^+(4; 0, 0, 0) + \sum_{s=1}^{\infty} K^+(4+s; s, 0, 0) = 0 \quad , \quad (4.4)$$

$$K^+(4; 0, 0, 0) + \sum_{s=2,4,\dots}^{\infty} K^+(4+s; s, 0, 0) = K_\phi \quad , \quad (4.5)$$

where  $K_\phi = (a_\phi, E_{c\phi})$  are the real scalar values from (A.1) and (A.5), As discussed in appendix C, the l.h.s. of (4.4) corresponds to the representation content of the tensor product of two scalar singletons and the associated sum of characters is equal to the partition function of the singlet sector of the 6d  $U(N)$  invariant theory of  $N$  free complex scalars, see (C.5). The vanishing to the r.h.s. of (4.5) is consistent with the expectation that the a-anomaly and Casimir energy of the  $U(N)$  6d CFT which are proportional to  $N$  should be exactly reproduced by the classical action of “non-minimal” type A higher spin theory in  $AdS_7$  with the inverse coupling  $G_{\text{non-min}}^{-1} \sim N$ , so that the one-loop HS correction should vanish [25, 26].

The l.h.s. of (4.5) corresponds the field content of the “minimal” type A theory in  $AdS_7$  which should be dual to singlet sector of  $O(N)$  invariant free real scalar 6d theory, with the partition function relation given by (C.8) (for similar relations in the case of 3d and 4d cases see [27, 10]). Here the non-vanishing r.h.s. may be cancelled against part of the classical contribution of non-minimal type A theory if one assumed that in this case  $G_{\min}^{-1} \sim N - 1$  [25, 27].

Similarly, in the case when the boundary 6d theory is the  $U(N)$  invariant free complex (Weyl) fermion theory or  $O(N)$  invariant free Majorana-Weyl fermion theory (with the dual theory being non-minimal or minimal type B theory in  $AdS_7$ ) we get

$$\sum_{s=1}^{\infty} \left[ K^+(4+s; s, 1, 1) + K^+(4+s; s, 0, 0) \right] = 0, \quad (4.6)$$

$$\sum_{s=2,4,\dots}^{\infty} K^+(4+s; s, 1, 1) + \sum_{s=1,3,\dots}^{\infty} K^+(4+s; s, 0, 0) = K_{\psi}, \quad (4.7)$$

where the field content corresponds to the one in the r.h.s. of (C.3),(C.6) and (C.9) and  $K_{\psi}$  is given in (A.1),(A.5). Here we have also other representations than totally symmetric tensors and thus require general expressions in (2.10),(2.11). As in the scalar case, the non-vanishing r.h.s. of (4.7) may be compensated by assuming that the coupling constant of minimal type B theory is  $G_{\min}^{-1} \sim N - 1$ .

When the 6d boundary theory is described by  $N$  real or complex self-dual 2-tensors with dual theory being non-minimal or minimal “type C” theory in  $AdS_7$  we find (see (C.4),(C.7), (C.10) and (A.1),(A.5))

$$\sum_{s=2}^{\infty} \left[ K^+(4+s; s, 2, 2) + K^+(4+s; s, 1, 1) + K^+(4+s; s, 0, 0) \right] = -K_T, \quad (4.8)$$

$$\sum_{s=2,4,\dots}^{\infty} \left[ K^+(4+s; s, 2, 2) + K^+(4+s; s, 0, 0) \right] + \sum_{s=3,5,\dots}^{\infty} K^+(4+s; s, 1, 1) = \frac{1}{2} K_T. \quad (4.9)$$

Here the non-vanishing result is found in both non-minimal and minimal cases. This is similar to what was found in the case of the  $AdS_5/CFT_4$  duality with the boundary theory represented by  $N$  complex or real Maxwell vectors [10, 42]. The (real) vector corresponds to the parity invariant singleton combination  $\{1\}_c = (2; 1, 0) + (2; 0, 1)$  in the  $SO(2, 4)$  notation.<sup>16</sup> There the r.h.s. of the analogs of eqs. (4.8) and (4.9) for the non-minimal and minimal type C theories was the same  $2K_V$ , i.e. twice a single 4d real vector contribution, implying the same -2 shift of couplings, i.e.  $G_{\text{non-min}}^{-1} \sim 2N - 2$  and  $G_{\min}^{-1} \sim N - 2$ .

In the present case of the 6d self-dual tensor multiplet theory corresponding to chiral  $\{1\}$  singleton eqs. (4.8) and (4.9) imply instead  $G_{\text{non-min}}^{-1} \sim 2N + 1$  and  $G_{\min}^{-1} \sim N - \frac{1}{2}$ . Considering instead the full (self-dual + anti self-dual) tensor represented by  $\{1\}_c = (3; 1, 1, 1) + (3; 1, 1, -1)$  (see (C.1),(C.12)) one finds that the r.h.s. of the analogs of (4.8) and (4.9) become  $-2K_T$  and 0 respectively (for the values of  $E_c$  see (D.6),(D.10)). This

<sup>16</sup>Here we follow [10] and use the  $SU(2) \times SU(2)$  weight notation for  $SO(2, 4)$  representation:  $(\Delta; j_1, j_2)$ , where  $h_1 = j_1 + j_2$ ,  $h_2 = j_1 - j_2$ .

implies that in the  $AdS_7$  theory dual to the 6d theory theory of  $N$  complex 6d tensors  $G_{\text{non-min}}^{-1} \sim 2N - 1$  and  $G_{\text{min}}^{-1} \sim 2N$ .

The l.h.s. of the above relations (4.4),(4.6) and (4.8) correspond to  $K$  of the products of singletons  $\{0\} \times \{0\}$ ,  $\{\frac{1}{2}\} \times \{\frac{1}{2}\}$ , and  $\{1\} \times \{1\}$  (see (C.2),(C.3),(C.4)). One can also consider a generalization when each factor in the product is a linear combination of the singletons, i.e.  $n_\phi \{0\} + n_\psi \{\frac{1}{2}\} + n_T \{1\}$ . Then (4.4),(4.6),(4.8) are generalized to

$$K^+ \left[ (n_\phi \{0\} + n_\psi \{\frac{1}{2}\} + n_T \{1\}) \times (n_\phi \{0\} + n_\psi \{\frac{1}{2}\} + n_T \{1\}) \right] = -n_T (n_\phi K_\phi + n_\psi K_\psi + n_T K_T), \quad (4.10)$$

where the l.h.s. is computed for the representation content appearing in the character relation in (C.11). For example, in the case when the boundary theory is described by  $N$  complex (2,0) tensor multiplets we have  $n_\phi = 5$ ,  $n_\psi = 4$ ,  $n_T = 1$  we get

$$K^+(\{\text{tens.}\} \times \{\text{tens.}\}) = -K_{\text{tens.}}, \quad \{\text{tens.}\} = \{1\} + 4\{\frac{1}{2}\} + 5\{0\}, \quad (4.11)$$

where the tensor multiplet values of  $K_{\text{tens.}}$  are given in (3.3). This may be compared with the relation found in the case of  $\mathcal{N} = 4$  vector multiplet in 4d [10, 42]:  $K^+(\{\text{vect.}\} \times \{\text{vect.}\}) = 2K_{\text{vect.}}$ .

## 5 One-loop vacuum energy in 10d supergravity on $AdS_3 \times S^3 \times M^4$

As discussed in the Introduction, one may also perform a similar one-loop computations in the supergravity sector of type IIB superstring on  $AdS_3 \times S^3 \times T^4$  to determine the subleading term in the central charge (1.8) or the vacuum energy (1.9).<sup>17</sup>

The one-loop  $AdS_3$  vacuum energy can be computed by starting with the spectrum of 6d supergravity on  $AdS_3 \times S^3$  as massive KK multiplets on  $M^4 = T^4$  should not contribute due to supersymmetric cancellation. More generally, we may consider in parallel the cases of IIA or IIB supergravities on  $M^4 = T^4$  or  $K3$ . The results for the one-loop vacuum energy are expected to be the same.<sup>18</sup>

The list of relevant 6d supergravities with  $\mathcal{N} = (n_L, n_R)$  supersymmetry was given in [43], where an algorithm for construction of the corresponding KK spectrum on  $S^3$  was presented. Below we shall consider the following cases:

10d	$M^4$	$(n_L, n_R)$	
IIB	$K3$	$(2, 0)$	(5.1)
IIA	$K3$	$(1, 1)$	
IIA or IIB	$T^4$	$(2, 2)$	

<sup>17</sup>String modes corresponding to massive unprotected multiplets are expected not to contribute to  $c$ .

<sup>18</sup>For example, type IIB theory on  $AdS_3 \times S^3 \times T^4$  with RR 3-form flux is S-dual to type IIB theory with NSNS flux and as the supergravity theory is S-duality invariant the same should be true for the value of  $E_c$ . Since NS-NS sector is common to IIB and IIA theories, the same result should be found also in the corresponding IIA theory.

### 5.1 KK towers of states on $S^3$

The 6d supergravity fields transform in representations  $(j_1, j_2)$  of the 6d little group  $SO'(4) \simeq SU(2) \times SU(2)$  (of  $SO(1, 5)$  in the tangent space). This gives a set  $\Phi$  of representations of the diagonal subgroup  $SO(3) \simeq SU(2)$  of  $SO(4)$ . Considering compactification on  $S^3$ , the above  $SO(3)$  can be identified with the factor in  $S^3 = SO(4)/SO(3)$ . Each representation  $R \in \Phi$  is associated with a tower of KK states with  $SO(4)$  representations containing  $R$  under restriction to their diagonal  $SO(3)$ .

These KK fields carry also representation of the  $AdS_3$  isometry group  $SO(2, 2)$  (or global part of 2d conformal group) which are are labelled by scaling dimension and spin  $(\Delta, s)$ , with  $\Delta \geq |s|$ . The values of  $(\Delta, s)$  can be determined by re-organizing the KK towers in short supermultiplets of  $SU(2, 2|1) \times SU(2, 2|1)$  since its generators include the dilatation (Virasoro  $L_0$ ) and spin operators. The relevant short representations  $(J)_s$  of  $SU(2, 2|1)$  have the following content

$$(J)_s : \begin{array}{c|cc} \text{states} & j & L_0 \\ \hline |0\rangle & J & J \\ Q_{\pm}|0\rangle & J - \frac{1}{2} & J + \frac{1}{2} \\ Q_+Q_-|0\rangle & J - 1 & J + 1 \end{array} \quad (5.2)$$

where  $|0\rangle$  is the lowest weight of the representation in the usual oscillator construction [44],  $Q_{\pm}$  are the supercharges, and  $j$  is  $SU(2)$  spin. Thus, in general, each short  $(J)_s$  representation contains four  $SO(2, 2)$  representations. Using (5.2) and that  $\Delta = L_0 + \bar{L}_0$ ,  $s = L_0 - \bar{L}_0$  one obtains the quantum numbers of representations in the tensor products  $(\bar{J}, J)_s$ .

Let us now list the KK towers that appear in the theories in (5.1). For (2,0) 6d supergravity, or IIB theory dimensionally reduced on K3 the field content is a graviton, five self-dual two-forms, four gravitinos, and  $n_T = 21$  tensor multiplet of one anti self-dual two-form, four fermions and five scalars (see also [45, 46])<sup>19</sup>

$$\Phi^{(2,0)} = (1, 1) + 4 \left(\frac{1}{2}, 1\right) + 5 (0, 1) + n_T \left[ (1, 0) + 4 \left(\frac{1}{2}, 0\right) + 5 (0, 0) \right], \quad (5.3)$$

$$g_{\mu\nu} \quad \psi_{\mu} \quad B \quad \tilde{B} \quad \psi \quad \varphi$$

Reorganising KK towers in short multiplets of  $SU(2, 2|1) \times SU(2, 2|1)$ , we find

$$\Phi_{\text{KK}}^{(2,0)} = \sum_{\ell=0}^{\infty} \Phi_2(\ell) + (n_T + 1) \sum_{\ell=0}^{\infty} \Phi_1(\ell) + n_T \left(\frac{1}{2}, \frac{1}{2}\right)_s, \quad (5.4)$$

where

$$\Phi_2(\ell) = \left(\frac{\ell+1}{2}, \frac{\ell+3}{2}\right)_s + \left(\frac{\ell+3}{2}, \frac{\ell+1}{2}\right)_s, \quad \Phi_1(\ell) = \left(\frac{\ell+2}{2}, \frac{\ell+2}{2}\right)_s. \quad (5.5)$$

The towers in the first and second sums are called spin-2 and spin-1 towers because of the maximum spin of their bottom floor  $\ell = 0$ . The explicit field content is collected in Appendix E and their 6d origin is discussed in [45].

<sup>19</sup>We shall keep  $n_T$  generic because this will be useful in comparing with IIA case.

For (1, 1) 6d supergravity, or 10d IIA supergravity reduced on K3, the field content is the sum of 6d graviton multiplet and  $n_V = 20$  vector multiplets [47]. The  $SO(4)$  little group representations are<sup>20</sup>

$$\begin{aligned} \Phi^{(1,1)} = & (1, 1) + 4\left(\frac{1}{2}, 1\right) + 2(0, 1) + 4\left(\frac{1}{2}, \frac{1}{2}\right) + 4\left(\frac{1}{2}, 0\right) + (0, 0) \\ & + n_V \left[ \left(\frac{1}{2}, \frac{1}{2}\right) + 4\left(\frac{1}{2}, 0\right) + 4(0, 0) \right], \end{aligned} \quad (5.6)$$

and the KK towers are

$$\Phi_{\text{KK}}^{(1,1)} = \sum_{\ell=0}^{\infty} \Phi_2(\ell) + (n_V + 2) \sum_{\ell=0}^{\infty} \Phi_1(\ell) + (n_V + 1) \left(\frac{1}{2}, \frac{1}{2}\right)_s. \quad (5.7)$$

Comparing (5.4) and (5.7), we see that they are equal under the identification  $n_V + 1 = n_T$  that is indeed true for the physical values. Thus we should find that  $E_c(\text{IIB on K3}) = E_c(\text{IIA on K3})$  (as was already mentioned above, this is implied by S-duality of IIB theory and NS-NS sector being common for IIA and IIB theories).

Finally, for (2, 2) 6d supergravity, or IIA or IIB theory dimensionally reduced on  $T^4$  the field content is a graviton, five self-dual and five anti self-dual two-forms, eight gravitinos, 16 gauge fields, 40 fermions and 25 scalars:

$$\begin{aligned} \Phi^{(2,2)} = & (1, 1) + 8\left(\frac{1}{2}, 1\right) + 5(0, 1) + 5(1, 0) + 16\left(\frac{1}{2}, \frac{1}{2}\right) + 40\left(\frac{1}{2}, 0\right) + 25(0, 0) \\ & \begin{matrix} g_{\mu\nu} & \psi_\mu & B & \tilde{B} & V_\mu & \psi & \varphi \end{matrix} \end{aligned} \quad (5.8)$$

The KK towers here are

$$\Phi_{\text{KK}}^{(2,2)} = \sum_{\ell=0}^{\infty} \Phi_2(\ell) + 4 \sum_{\ell=0}^{\infty} \Phi_{\frac{3}{2}}(\ell) + 6 \sum_{\ell=0}^{\infty} \Phi_1(\ell) + 5 \left(\frac{1}{2}, \frac{1}{2}\right)_s, \quad (5.9)$$

where

$$\Phi_{\frac{3}{2}}(\ell) = \left(\frac{\ell+1}{2}, \frac{\ell+2}{2}\right)_s + \left(\frac{\ell+2}{2}, \frac{\ell+1}{2}\right)_s \quad (5.10)$$

is a fermionic spin- $\frac{3}{2}$  tower (see Appendix E).

## 5.2 Vacuum energy

The  $AdS_3$  vacuum energy contributions of the above KK towers can be computed using the expressions for the characters or one-particle partition functions of the corresponding  $SO(2, 2)$  representations which we shall first recall.

$SO(2, 2)$  viewed as global conformal group in 2d is generated by the  $L_0, L_{\pm 1}$  and  $\bar{L}_0, \bar{L}_{\pm 1}$  Virasoro generators. Unitary irreducible representations of  $SO(2, 2)$  are massive for  $\Delta > |s|$  and massless for  $\Delta = |s|$ . A massive representation is built on a ground state  $|h, \bar{h}\rangle$  with  $h\bar{h} > 0$ . Thus, both  $L_{-1}$  and  $\bar{L}_{-1}$  give a non zero result and the resulting character is (see, e.g., [44])<sup>21</sup>

$$\Delta > |s| : \quad \widehat{\mathcal{Z}}^+(\Delta; s) = \frac{q^\Delta}{(1-q)^2}, \quad (5.11)$$

<sup>20</sup>Here we combine representations related by conjugation  $(j_1, j_2) \rightarrow (j_2, j_1)$  since they give same contribution to KK spectrum.

<sup>21</sup>The double factor of  $1/(1-q)$  takes into account multiple applications of both  $L_{-1}$  and  $\bar{L}_{-1}$ .

A massless representation with  $\Delta = |s| > 0$  has conformal weights  $(h, 0)$  or  $(0, \bar{h})$ . Acting with the lowering operators  $L_{-1}$  and  $\bar{L}_{-1}$  on  $|h, \bar{h}\rangle$  only one of them gives a non-zero result. As a consequence, here

$$\Delta = |s| : \quad \mathcal{Z}^+(|s|; s) = \frac{q^\Delta}{1-q} = \frac{q^\Delta - q^{\Delta+1}}{(1-q)^2} = \widehat{\mathcal{Z}}^+(\Delta; s) - \widehat{\mathcal{Z}}^+(\Delta+1; s), \quad (5.12)$$

Finally, for  $\Delta = s = 0$ , we have only the ground state  $|0, 0\rangle$  and  $\mathcal{Z}^+(0; 0) = 1$ . The expressions (5.11) and (5.12) can be used to prove that  $SU(1, 1|2)$  short multiplets obey the important relation  $E_c = -\frac{1}{12}c$ , see (1.9). We discuss this in details in Appendix F.

The contribution from a particular  $SO(2, 2)$  representation to the  $AdS_3$  vacuum or  $S^1$  2d Casimir energy  $E_c$  can then be computed using (2.8), (2.9). Explicitly, for a massive field in  $AdS_3$ , we may write the partition function (5.11) as

$$\widehat{\mathcal{Z}}^+(\Delta; s) = \sum_{n=0}^{\infty} (n+1) q^{\Delta+n}. \quad (5.13)$$

We then obtain a formal (divergent) expression for the corresponding  $E_c$  as (cf. (3.7), (3.8))

$$\widehat{E}_c^+(\Delta; s) = \sum_{n=0}^{\infty} e_n(\Delta; s), \quad e_n(\Delta; s) = \frac{1}{2} (-1)^{2s} (n+1) (\Delta+n). \quad (5.14)$$

In addition, we then need to sum over the KK states.

There will be divergences coming from the sum over  $n$ , but also from the sum over the KK level  $\ell$ . Like in  $AdS_5 \times S^5$  case [10] and  $AdS_7 \times S^4$  case in section 3 the total sum may be again computed using the  $\zeta$ -function regularization applied to the full effective 6d energy eigenvalue  $\Delta+n$ , or, equivalently, by introducing the cutoff  $e_n \rightarrow e_n e^{-\epsilon(\Delta+n)}$ , doing the sum, expanding in  $\epsilon \rightarrow 0$ , and dropping all singular terms. Applying this procedure to the KK towers appearing in (5.4), (5.7) and (5.9), we obtain

$$E_{c,2} = E_c \left[ \sum_{\ell=0}^{\infty} \Phi_2(\ell) \right] = -\frac{89}{192}, \quad E_{c, \frac{3}{2}} = E_c \left[ \sum_{\ell=0}^{\infty} \Phi_{\frac{3}{2}}(\ell) \right] = \frac{19}{96}, \quad (5.15)$$

$$E_{c,1} = E_c \left[ \sum_{\ell=0}^{\infty} \Phi_1(\ell) \right] = -\frac{101}{384}, \quad E_{c,\text{extra}} = E_c \left[ \left( \frac{1}{2}, \frac{1}{2} \right)_s \right] = \frac{1}{4}, \quad (5.16)$$

where  $E_{c,\text{extra}}$  is the contribution from the  $(\frac{1}{2}, \frac{1}{2})_s$  representation appearing in (5.4), (5.7), and (5.9) in the bottom part of the KK towers.

The above are the contributions from the massive  $SO(2, 2)$  representations. As discussed in [45, 43], the resolution of the missing states puzzle raised in [48] amounts to the re-introduction of the massless representations ( $\ell = -1$  states in the spin 2 and  $\frac{3}{2}$  towers). These are massless multiplets in  $AdS_3$  that do not carry propagating degrees of freedom. Their structure is presented in Appendix E. For these multiplets we find

$$E_{c,2}^{\text{massless}} = E_c \left[ (0, 1)_s + (1, 0)_s \right] = \frac{1}{2}, \quad E_{c, \frac{3}{2}}^{\text{massless}} = E_c \left[ \left( 0, \frac{1}{2} \right)_s + \left( \frac{1}{2}, 0 \right)_s \right] = -\frac{1}{4}. \quad (5.17)$$



Collecting all contributions of states in (5.4), we find in the case of for IIB theory on K3

$$\begin{aligned} E_c^{(2,0)} &= E_{c,2}^{\text{massless}} + E_{c,2} + (n_T + 1) E_{c,1} + n_T E_{c,\text{extra}} \\ &= \frac{1}{2} - \frac{89}{192} - (n_T + 1) \frac{101}{384} + n_T \frac{1}{4} = -\frac{29}{128} - \frac{5}{384} n_T \xrightarrow{n_T=21} -\frac{1}{2}. \end{aligned} \quad (5.18)$$

This is also the result for IIA theory on K3, as follows from (5.7). From (5.9) we also get exactly the same result for IIA or IIB theory on  $T^4$ ,

$$\begin{aligned} E_c^{(2,2)} &= E_{c,2}^{\text{massless}} + E_{c,2} + 4 \left( E_{c,\frac{3}{2}}^{\text{massless}} + E_{c,\frac{3}{2}} \right) + 6 E_{c,1} + 5 E_{c,\text{extra}} \\ &= \frac{1}{2} - \frac{89}{192} + 4 \left( -\frac{1}{4} + \frac{19}{96} \right) - 6 \frac{101}{384} + 5 \frac{1}{4} = -\frac{1}{2}, \end{aligned} \quad (5.19)$$

in agreement with the claim in (1.9).

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## A Free (2,0) multiplet in 6d

The field content of the (2,0) tensor multiplet is composed of five scalars fields  $\phi^a$ , two complex Weyl fermions  $\psi_L^I$  or 4 Majorana-Weyl fermions (each with 4 real components), and an antisymmetric tensor  $T_{ij}$  with (anti) selfdual strength. It represents a free 6d CFT invariant under superconformal  $\mathcal{N} = (2,0)$  group [49, 50] containing the conformal group  $SO(2,6)$  and the R-symmetry group  $SO(5) \simeq USp(4)$ .

The Weyl anomaly of the (2,0) multiplet was discussed in [17]. The values of the a-anomaly coefficients for the individual fields are (here  $\psi$  stands for one 6d Majorana-Weyl fermion)

$$a_\phi = -\frac{1}{72576}, \quad a_\psi = -\frac{191}{1451520}, \quad a_T = -\frac{221}{40320}. \quad (A.1)$$

The total a-anomaly of one free (2,0) tensor multiplet is thus

$$a_{\text{tens.}} = 5 a_\phi + 4 a_\psi + a_T = -\frac{7}{1152}. \quad (A.2)$$

Considering (2,0) multiplet on  $S^1 \times S^5$  one may compute the corresponding thermal partition function. The canonical (or one-particle) partition function of a free CFT in  $S^1 \times S^{d-1}$  can be computed by direct evaluation of the free QFT path-integral in terms of the eigenmodes of the quadratic kinetic operator. An alternative approach is the operator counting method [51–53]. From the spectrum of eigenvalues of the Hamiltonian or dilatation operator  $\omega_n = \Delta_n$  and their degeneracies  $d_n$  one gets

$$\mathcal{Z}(q) = \text{Tr} e^{-\beta H} = \sum_n d_n e^{-\beta \omega_n} = \sum_n d_n q^{\Delta_n}, \quad q \equiv e^{-\beta}. \quad (A.3)$$

In the approach based on counting of states one needs to consider the contribution of off-shell components (and their derivative descendants) of a suitable gauge invariant field strength modulo non-trivial gauge identities and then subtract the components of the equations of motion for the field strength (and their derivatives). The single particle partition functions for the 5 scalars, 4 Majorana-Weyl fermions, and self-dual tensor in  $S^1 \times S^5$  are [52]

$$\begin{aligned}\mathcal{Z}_\phi(q) &= \frac{1}{12} \sum_{n=0}^{\infty} (n+1)(n+2)^2(n+3) q^{n+2} = \frac{q^2 - q^4}{(1-q)^6}, \\ \mathcal{Z}_\psi(q) &= \frac{1}{6} \sum_{n=0}^{\infty} (n+1)(n+2)(n+3)(n+4) q^{n+\frac{5}{2}} = \frac{4q^{\frac{5}{2}} - 4q^{\frac{7}{2}}}{(1-q)^6}, \\ \mathcal{Z}_T(q) &= \frac{1}{4} \sum_{n=0}^{\infty} (n+1)(n+2)(n+4)(n+5) q^{n+3} = \frac{10q^3 - 15q^4 + 6q^5 - q^6}{(1-q)^6}.\end{aligned}\tag{A.4}$$

These expressions are in agreement with (2.7),(2.6),(2.2).

The related Casimir energy on  $S^5$  can be computed from the one-particle partition function  $\mathcal{Z}(q)$  using (2.8),(2.9):

$$E_{c,\phi} = -\frac{31}{60480}, \quad E_{c,\psi} = -\frac{367}{96768}, \quad E_{c,T} = -\frac{191}{4032}.\tag{A.5}$$

Then the total Casimir energy for the free (2,0) tensor multiplet is

$$E_{c,\text{tens.}} = 5 E_{c,\phi} + 4 E_{c,\psi} + E_{c,T} = -\frac{25}{384}.\tag{A.6}$$

This agrees with the value found in [36].<sup>22</sup>

Let us note that the expressions in (A.4) admit also  $AdS_7$  interpretation. In general, given a conformal 6d field, the corresponding one-particle partition function  $\mathcal{Z}(q)$  may be expressed as [53]

$$\mathcal{Z}(q) = \mathcal{Z}^-(q) - \mathcal{Z}^+(q),\tag{A.7}$$

where  $\mathcal{Z}^\pm(q)$  are the one-particle partition functions for the one-loop partition function  $Z^\pm$  of the associated higher spin field in (thermal quotient of)  $AdS_7$  computed with the standard (“Dirichlet”) or alternative (“Neumann”) boundary conditions. The canonical dimension of the conformal 6d field is equal to  $\Delta_- = 6 - \Delta$ ,  $\Delta = \Delta_+$ . For generic representation (A.7) may be written as

$$\mathcal{Z}(q) = \mathcal{Z}^+(\Delta; \mathbf{h})(q^{-1}) - \mathcal{Z}^+(\Delta; \mathbf{h})(q) + \sigma(q),\tag{A.8}$$

where  $\sigma(q)$  can be interpreted as a Killing tensor character associated with missing gauge invariances [53]. This term is a polynomial in  $q$  and  $1/q$  which is symmetric under  $q \rightarrow 1/q$ . For a 6d conformal scalar with canonical dimension 2 we find that (see (2.2),(2.6),(2.7),(A.4))

$$\mathcal{Z}_\phi(q) = \mathcal{Z}^+(4; 0, 0, 0)(q^{-1}) - \mathcal{Z}^+(4; 0, 0, 0)(q).\tag{A.9}$$

---

<sup>22</sup>Note that the ratio of the vacuum energy (A.6) and the a-anomaly (A.2), i.e.  $E_{c,\text{tens.}}/a_{\text{tens.}} = \frac{75}{7}$ , differs from the expression in [54]. The reason is that the Casimir energy is computed in the standard  $\zeta$ -function regularization scheme in which derivative terms  $D_6$  in the conformal anomaly (1.1) do not vanish [17] while ref. [54] assumed an abstract scheme where there are no derivative terms in the anomaly (see also a related discussion in [10]).

For the Majorana-Weyl 6d fermion with canonical dimension  $\frac{5}{2}$  we get

$$\mathcal{Z}_\psi(q) = \mathcal{Z}^+(\frac{7}{2}; \frac{1}{2}, \frac{1}{2}, \frac{1}{2})(q^{-1}) - \mathcal{Z}^+(\frac{7}{2}; \frac{1}{2}, \frac{1}{2}, \frac{1}{2})(q) . \quad (\text{A.10})$$

In the case of rank 2 tensor of dimension 2 let us define (see (2.2),(2.5))

$$\mathcal{Z}_T^+ \equiv \mathcal{Z}^+(4; 1, 1, 0) = \widehat{\mathcal{Z}}^+(4; 1, 1, 0) - \widehat{\mathcal{Z}}^+(5; 1, 0, 0) + \widehat{\mathcal{Z}}^+(6; 0, 0, 0) . \quad (\text{A.11})$$

This is a  $SO(2, 6)$  character corresponding to the massless case (unitarity bound) in (ii) in (2.1).<sup>23</sup> One observes then that

$$2 \mathcal{Z}_T(q) = \mathcal{Z}_T^+(q^{-1}) - \mathcal{Z}_T^+(q) - 1 , \quad (\text{A.12})$$

where  $\mathcal{Z}_T$  is the self-dual tensor partition function in (2.7),(A.4). The -1 term should be interpreted as a subtraction of a non-normalizable gauge transformation.

Note that in the case of the (2,0) tensor multiplet corresponding formally to the  $p = 1$  singleton level of KK tower in Table 1 we find

$$\mathcal{Z}_{\text{tens.}}(q) = 5 \mathcal{Z}_\phi + 4 \mathcal{Z}_\psi + \mathcal{Z}_T = \frac{5q^2 + 16q^{\frac{5}{2}} + 15q^3 - 5q^4 + q^5}{(1-q)^5} , \quad (\text{A.13})$$

which satisfies the relation

$$\mathcal{Z}_{\text{tens.}}(q) + \mathcal{Z}_{\text{tens.}}(q^{-1}) + 1 = 0 . \quad (\text{A.14})$$

The general relations for the boundary conformal anomaly and Casimir energy are [10]

$$\mathfrak{a} = -2\mathfrak{a}^+ , \quad E_c = -2E_c^+ . \quad (\text{A.15})$$

Denoting  $K = (E_c, \mathfrak{a})$  and  $K^+ = (E_c^+, \mathfrak{a}^+)$ , we have<sup>24</sup>

$$K^+(4; 0, 0, 0) = -\frac{1}{2} K_\phi, \quad K^+(\frac{7}{2}; \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = -\frac{1}{2} K_\psi, \quad K^+(4; 1, 1, 0) = -K_T, \quad (\text{A.16})$$

where  $K_\phi, K_\psi, K_T$  are given by (A.1) and (A.5).

## B a-anomaly from spectral $\zeta$ -function in $AdS_7$

The a-coefficient of the boundary conformal anomaly can be determined from the logarithmic IR singular part of the one-loop partition function in Euclidean  $AdS_7$  with boundary  $S^6$ , i.e. hyperboloid  $\mathbb{H}^7$  (see, e.g., [55, 32])

$$\log Z^+ = -\frac{1}{2} \log \det_+ \mathcal{O} = \frac{1}{2} \zeta'(0) = -96 \mathfrak{a}^+ \log R + \dots . \quad (\text{B.1})$$

Here  $\zeta(z)$  is the spectral zeta function found by evaluating the trace of the  $\mathbb{H}^7$  heat kernel [56] associated with the 7d operator  $\mathcal{O}$  and  $R$  is an IR cutoff regularising the volume of  $\mathbb{H}^7$ .

<sup>23</sup>Here the two additional terms are related to gauge freedom in the rank 2 tensor potential.

<sup>24</sup>In the tensor case, the factor  $\frac{1}{2}$  is absent due to the self-duality condition.

Below we shall consider the operator  $\mathcal{O}$  corresponding to a generic massive (or massless) higher spin field in representation  $(\Delta; \mathbf{h})$  generalizing the expression in [32] found in the totally symmetric tensor case  $h_1 = s, h_2 = h_3 = 0$  <sup>25</sup>

$$\mathcal{O} = -D^2 + X, \quad X = \Delta(\Delta - 6) - h_1 - h_2 - |h_3|. \quad (\text{B.2})$$

Here  $D^2$  is the standard Laplacian in  $AdS_7$  defined on transverse field. The discussion will be parallel to the one in  $AdS_5$  case in [10].

The spectral  $\zeta$ -function of the operator  $\mathcal{O}$  can be expressed in terms of the heat kernel

$$\zeta(z) = \frac{1}{\Gamma(z)} \int_0^\infty dt t^{z-1} \text{Tr} K, \quad K(x, y; t) = \langle x | e^{-t\mathcal{O}} | y \rangle. \quad (\text{B.3})$$

Since  $\mathbb{H}^7$  is homogeneous, the trace over the position  $x$  gives a factor of (regularized) volume, i.e.

$$\zeta(z) = \text{Vol}(\mathbb{H}^7) \zeta(z; x), \quad \zeta(z; x) \equiv \frac{1}{\Gamma(z)} \int_0^\infty dt t^{z-1} \text{tr} K(x, x; t), \quad (\text{B.4})$$

where  $\text{tr}$  is the trace over the representation indices of the operator and  $\zeta(z; x)$  does not actually depend on  $x$ .

One can use the results for the heat kernel of the Laplacian in  $AdS_{2n+1}$  with even  $n$  derived in [56, 59] applying them to the case of  $n = 3$ . It is convenient to start with heat-kernel for the sphere  $S^7$  and then analytically continue to  $AdS_7$ . Let us consider a field on  $S^7$  transforming under the tangent space rotations in a representation  $\mathbf{G}$  of  $SO(7)$ . Since  $S^7 = SO(8)/SO(7)$ , the heat kernel receives contributions from each representation  $\mathbf{R}$  of  $SO(8)$  that contains  $\mathbf{G}$  when restricted to  $SO(7)$ . Let us denote  $\mathbf{R}$  and  $\mathbf{G}$  by the corresponding weights as

$$\mathbf{R} = (\ell_1, \ell_2, \ell_3, \ell_4), \quad \ell_1 \geq \ell_2 \geq \ell_3 \geq |\ell_4|, \quad \mathbf{G} = (g_1, g_2, g_3), \quad g_1 \geq g_2 \geq g_3 \geq 0, \quad (\text{B.5})$$

were all labels are integer or half integer. The branching condition on the representation  $\mathbf{R}$  is

$$\ell_1 \geq g_1 \geq \ell_2 \geq g_2 \geq \ell_3 \geq g_3 \geq |\ell_4|, \quad (\text{B.6})$$

with the additional requirement that  $\ell_i - g_i \in \mathbb{Z}$ . The heat kernel at the coincident points, traced over representation indices, can be written as

$$\text{tr} K(x, x; t) = \frac{3}{\pi^4} \sum_{\ell_i} d_{\mathbf{R}} e^{-t E_{\mathbf{R}}^{(\mathbf{G})}}, \quad (\text{B.7})$$

where  $E_{\mathbf{R}}^{(\mathbf{H})}$  are the eigenvalues of the Laplacian  $-D^2$  on  $S^7$  expressed in terms of the second Casimir values for the two representations and  $d_{\mathbf{R}}$  is the dimension of  $\mathbf{R}$

$$-D^2|_{S^7} \rightarrow E_{\mathbf{R}}^{(\mathbf{G})} = C_2(\mathbf{R}) - C_2(\mathbf{G}), \quad (\text{B.8})$$

$$C_2(\mathbf{R}) = \ell_4^2 + \ell_1(\ell_1 + 6) + \ell_2(\ell_2 + 4) + \ell_3(\ell_3 + 2), \quad (\text{B.9})$$

$$C_2(\mathbf{G}) = g_3^2 + g_3 + g_1(g_1 + 5) + g_2(g_2 + 3), \quad (\text{B.10})$$

---

<sup>25</sup>For the general form of  $X$  see [57, 33, 58].

$$d_R = \frac{1}{4320} [(\ell_1 + 3)^2 - (\ell_2 + 2)^2] [(\ell_1 + 3)^2 - (\ell_3 + 1)^2] \\ \times [(\ell_2 + 2)^2 - (\ell_3 + 1)^2] [(\ell_1 + 3)^2 - \ell_4^2] [(\ell_2 + 2)^2 - \ell_4^2] [(\ell_3 + 1)^2 - \ell_4^2] . \quad (\text{B.11})$$

The analytic continuation from  $S^7$  to  $AdS_7$  amounts to [56, 59]

$$\ell_1 \rightarrow i\lambda - 3 , \quad (\text{B.12})$$

with the sum over  $\ell_1$  becoming an integral over  $\lambda \geq 0$ . Finally, considering states saturating the inequalities (B.6) and identifying  $(\ell_2, \ell_3, \ell_4) = \mathbf{h} = (h_1, h_2, h_3)$ , we find that the eigenvalues of the operator (B.2) for the representation  $(\Delta; \mathbf{h})$  are

$$(-D^2 + X)|_{AdS_7} \rightarrow \lambda^2 + (\Delta - 3)^2 . \quad (\text{B.13})$$

The regularised volume may be written as  $\text{Vol}(\mathbb{H}^7) = \frac{1}{3}\pi^3 \log R + \dots$  where the IR cutoff  $R$  is the radius of  $S^6$  measured in 7d metric  $d\rho^2 + \sinh^2 \rho d\Omega_6^2$  at large  $\rho$ . Doing the analytic continuation (B.12) in the dimension  $d_R$  in (B.11) we finally obtain

$$\zeta(z) = \text{Vol}(\mathbb{H}^7) \zeta(z; x) \\ \rightarrow -\frac{\log R}{4320\pi} (h_1 - h_2 + 1) (h_1 + h_2 + 3) [(h_1 + 2)^2 - h_3^2] ((h_2 + 1)^2 - h_3^2) \\ \times \int_0^\infty d\lambda \frac{[h_1(h_1 + 4) + \lambda^2 + 4] [h_2(h_2 + 2) + \lambda^2 + 1] (h_3^2 + \lambda^2)}{[\lambda^2 + (\Delta - 3)^2]^z} . \quad (\text{B.14})$$

Integrating over  $\lambda$  and taking the  $z$ -derivative at  $z = 0$  we may then use (B.1) to find the expression for  $\hat{a}^+$  in (2.11).

## C Tensor products of $SO(2, 6)$ singleton representations and associated character relations

Let us introduce the following notation for the spin  $j = 0, \frac{1}{2}, 1, \dots$  singleton representations of  $SO(2, 6)$

$$\{j\} = (2 + j; j, j, j) . \quad (\text{C.1})$$

Here  $\{0\}$  corresponds to a real scalar  $\phi$ ,  $\{\frac{1}{2}\}$  to MW fermion  $\psi$ , and  $\{1\}$  to self-dual tensor  $T$ . We shall also use the notation  $(\Delta; h_1, h_2, h_3)_c \equiv (\Delta; h_1, h_2, h_3) + (\Delta; h_1, h_2, -h_3)$ , so that  $\{j\}_c = (2 + j; j, j, j) + (2 + j; j, j, -j)$ .

From the general Flato-Fronsdal relations in [60, 34] we get in the present 6d case

$$\{0\} \times \{0\} = (4; 0, 0, 0) + \bigoplus_{s=1}^{\infty} (4 + s; s, 0, 0) , \quad (\text{C.2})$$

$$\{\frac{1}{2}\} \times \{\frac{1}{2}\} = \bigoplus_{s=1}^{\infty} [(4 + s; s, 1, 1) + (4 + s; s, 0, 0)] , \quad (\text{C.3})$$

$$\{1\} \times \{1\} = \bigoplus_{s=2}^{\infty} [(4 + s; s, 2, 2) + (4 + s; s, 1, 1) + (4 + s; s, 0, 0)] . \quad (\text{C.4})$$

The above relations imply analogous relations for the characters or one-particle partition functions

$$[\mathcal{Z}_\phi(q)]^2 = \mathcal{Z}^+(4; 0, 0, 0) + \sum_{s=1}^{\infty} \mathcal{Z}^+(4+s; s, 0, 0), \quad (\text{C.5})$$

$$[\mathcal{Z}_\psi(q)]^2 = \sum_{s=1}^{\infty} \left[ \mathcal{Z}^+(4+s; s, 1, 1) + \mathcal{Z}^+(4+s; s, 0, 0) \right], \quad (\text{C.6})$$

$$[\mathcal{Z}_T(q)]^2 = \sum_{s=2}^{\infty} \left[ \mathcal{Z}^+(4+s; s, 2, 2) + \mathcal{Z}^+(4+s; s, 1, 1) + \mathcal{Z}^+(4+s; s, 0, 0) \right]. \quad (\text{C.7})$$

Here the l.h.s. may be interpreted as the one-particle partition functions corresponding to the single sector of the  $U(N)$  boundary theory, with  $\mathcal{Z}_\phi = \mathcal{Z}_{\{0\}}$ ,  $\mathcal{Z}_\psi = \mathcal{Z}_{\{\frac{1}{2}\}}$ ,  $\mathcal{Z}_T = \mathcal{Z}_{\{1\}}$  given in (2.7),(A.4).

The case of the real  $O(N)$  invariant theory is found by an appropriate  $\mathbb{Z}_2$  projection. The corresponding sums then represent the partition functions of the singlet sector of  $O(N)$  invariant free real scalar, Majorana fermion, and real self-dual tensor theories in 6d:

$$\frac{1}{2} [\mathcal{Z}_\phi(q)]^2 + \frac{1}{2} \mathcal{Z}_\phi(q^2) = \mathcal{Z}^+(4; \mathbf{0}) + \sum_{s=2,4,\dots}^{\infty} \mathcal{Z}^+(4+s; s, 0, 0), \quad (\text{C.8})$$

$$\frac{1}{2} [\mathcal{Z}_\psi(q)]^2 - \frac{1}{2} \mathcal{Z}_\psi(q^2) = \sum_{s=2,4,\dots}^{\infty} \mathcal{Z}^+(4+s; s, 1, 1) + \sum_{s=1,3,\dots}^{\infty} \mathcal{Z}^+(4+s; s, 0, 0), \quad (\text{C.9})$$

$$\begin{aligned} \frac{1}{2} [\mathcal{Z}_T(q)]^2 + \frac{1}{2} \mathcal{Z}_T(q^2) &= \sum_{s=2,4,\dots}^{\infty} \left[ \mathcal{Z}^+(4+s; s, 2, 2) + \mathcal{Z}^+(4+s; s, 0, 0) \right] \\ &+ \sum_{s=3,5,\dots}^{\infty} \mathcal{Z}^+(4+s; s, 1, 1). \end{aligned} \quad (\text{C.10})$$

These relations (C.5)–(C.7) can be generalized by considering a tensor product of the linear combination of singletons:  $[n_\phi \{0\} + n_\psi \{\frac{1}{2}\} + n_T \{1\}] \times [n_\phi \{0\} + n_\psi \{\frac{1}{2}\} + n_T \{1\}]$ . This gives for the corresponding characters

$$\begin{aligned} &\left[ n_\phi \mathcal{Z}_\phi(q) + n_\psi \mathcal{Z}_\psi(q) + n_T \mathcal{Z}_T(q) \right]^2 \\ &= n_\phi^2 \sum_{s=0}^{\infty} \mathcal{Z}^+(s+4; s, 0, 0) + n_\psi^2 \sum_{s=1}^{\infty} \left[ \mathcal{Z}^+(s+4; s, 0, 0) + \mathcal{Z}^+(s+4; s, 1, 1) \right] \\ &+ n_T^2 \sum_{s=2}^{\infty} \left[ \mathcal{Z}^+(s+4; s, 0, 0) + \mathcal{Z}^+(s+4; s, 1, 1) + \mathcal{Z}^+(s+4; s, 2, 2) \right] \\ &+ 2 n_\phi n_\psi \sum_{s=0}^{\infty} \mathcal{Z}^+\left(\frac{9}{2} + s, \frac{1}{2} + s, \frac{1}{2}, \frac{1}{2}\right) + 2 n_\phi n_T \sum_{s=1}^{\infty} \mathcal{Z}^+(s+4; s, 1, 1) \\ &+ 2 n_\psi n_T \sum_{s=1}^{\infty} \left[ \mathcal{Z}^+\left(\frac{9}{2} + s, \frac{1}{2} + s, \frac{1}{2}, \frac{1}{2}\right) + \mathcal{Z}^+\left(\frac{9}{2} + s, \frac{1}{2} + s, \frac{3}{2}, \frac{3}{2}\right) \right]. \end{aligned} \quad (\text{C.11})$$

It is of interest to consider also the reducible case when the boundary theory is represented by an unrestricted 2-tensor, i.e. the parity-invariant combination of self-dual and anti self-dual tensors, i.e.  $\{1\}_c = (3; 1, 1, 1) + (3; 1, 1, -1)$ . Then the corresponding Flato-Fronsdal type relation becomes (cf. (C.4))

$$\begin{aligned} \{1\}_c \times \{1\}_c &= 2 \left[ (6; 2, 2, 0) + (6; 1, 1, 0) + (6; 0, 0, 0) \right] + 2 \bigoplus_{s=3}^{\infty} (4 + s; s, 2, 0) \\ &+ \bigoplus_{s=2}^{\infty} \left[ (4 + s; s, 2, 2)_c + (4 + s; s, 1, 1)_c + 2(4 + s; s, 0, 0) \right]. \end{aligned} \quad (\text{C.12})$$

Then  $\mathcal{Z}_{\{1\}_c} = 2 \mathcal{Z}_T$  and one finds that (C.7) is replaced by

$$\begin{aligned} [2\mathcal{Z}_T(q)]^2 &= 2 \left[ \mathcal{Z}^+(6; 2, 2, 0) + \mathcal{Z}^+(6; 1, 1, 0) + \mathcal{Z}^+(6; 0, 0, 0) \right] + 2 \bigoplus_{s=3}^{\infty} \mathcal{Z}^+(4 + s; s, 2, 0) \\ &+ \bigoplus_{s=2}^{\infty} \left[ \mathcal{Z}^+(4 + s; s, 2, 2)_c + \mathcal{Z}^+(4 + s; s, 1, 1)_c + 2\mathcal{Z}^+(4 + s; s, 0, 0) \right]. \end{aligned} \quad (\text{C.13})$$

Also, the analog of (C.10) is

$$\begin{aligned} \frac{1}{2} [2\mathcal{Z}_T(q)]^2 + \frac{1}{2} [2\mathcal{Z}_T(q^2)] &= \mathcal{Z}^+(6; 2, 2, 0) + \mathcal{Z}^+(6; 1, 1, 0) + \mathcal{Z}^+(6; 0, 0, 0) \\ &+ \sum_{s=2,4,\dots}^{\infty} \left[ \mathcal{Z}^+(4 + s; s, 2, 2)_c + 2\mathcal{Z}^+(4 + s; s, 0, 0) \right] \\ &+ \sum_{s=3,5,\dots}^{\infty} \mathcal{Z}^+(4 + s; s, 1, 1)_c + \sum_{s=3}^{\infty} \mathcal{Z}^+(4 + s; s, 2, 0). \end{aligned} \quad (\text{C.14})$$

## D Casimir energy for spin $0, \frac{1}{2}, 1$ singletons in $\text{AdS}_{d+1}$

It is useful to derive the general expressions for the Casimir energy for spin  $j = 0, \frac{1}{2}, 1$   $SO(2, d)$  singletons in the general case of even dimension  $d = 2, 4, 6, \dots$  of the boundary.

For  $j = 0, \frac{1}{2}$  the corresponding character or one-particle partition functions are readily found, e.g., by counting states of free scalar or fermion in  $d$  dimensions <sup>26</sup>

$$\mathcal{Z}_0(q) = \frac{q^{\frac{d-2}{2}} (1 - q^2)}{(1 - q)^d}, \quad \mathcal{Z}_{\frac{1}{2}}(q) = 2^{\frac{d}{2}} \frac{q^{\frac{d-1}{2}} (1 - q)}{(1 - q)^d} \quad (\text{D.1})$$

These satisfy

$$\mathcal{Z}_0(q) + \mathcal{Z}_0(q^{-1}) = 0, \quad \mathcal{Z}_{\frac{1}{2}}(q) + \mathcal{Z}_{\frac{1}{2}}(q^{-1}) = 0. \quad (\text{D.2})$$

As a consequence, in any even  $d$  the Casimir energy associated with the  $U(N)$  singlet partition functions  $[\mathcal{Z}_0]^2$  and  $[\mathcal{Z}_{\frac{1}{2}}]^2$  vanishes because these functions are invariant under  $q \rightarrow q^{-1}$ .

<sup>26</sup>The singleton with spin  $j$  occurs with two possible chiralities. Here we consider one of them.

The character of the  $j = 1$  singleton representation is [34]<sup>27</sup>

$$\mathcal{Z}_1(q) = \frac{1}{[(\frac{d}{2} - 1)!]^2} \sum_{n=0}^{\infty} \frac{(n+d-1)!}{n!(n+\frac{d}{2})} q^{n+\frac{d}{2}} = \frac{d!}{2(\frac{d}{2})!^2} \frac{q^{\frac{d}{2}}}{(1-q)^{d-1}} {}_2F_1(1, 1 - \frac{d}{2}; 1 + \frac{d}{2}; q) . \quad (\text{D.3})$$

One can check that

$$\mathcal{Z}_1(q) + \mathcal{Z}_1(q^{-1}) = (-1)^{d/2} , \quad (\text{D.4})$$

so that

$$[\mathcal{Z}_1(q)]^2 = \frac{1}{2} \left( [\mathcal{Z}_1(q)]^2 + [\mathcal{Z}_1(q^{-1})]^2 - 1 \right) + (-1)^{\frac{d}{2}} \mathcal{Z}_1(q) . \quad (\text{D.5})$$

The first term in the r.h.s. is symmetric under  $q \rightarrow q^{-1}$  and thus it does not contribute the Casimir energy. As a result, we find for the Casimir energy of the product of two spin 1 singletons (see (2.9),(A.5))

$$E_c(\{1\} \times \{1\}) = (-1)^{\frac{d}{2}} E_c(\{1\}) . \quad (\text{D.6})$$

For example, for the boundary  $U(N)$  theory described by the  $d = 4$  vector corresponding to  $\{1\}_c$  (self-dual and anti self-dual strength) we get [10]

$$d = 4 : \quad E_c(\{1\}_c \times \{1\}_c) = 4 E_c(\{1\} \times \{1\}) = 4 E_c(\{1\}) = 2 E_c(\{1\}_c), \quad (\text{D.7})$$

while for chiral singleton in 6d, i.e. self-dual (or anti self-dual) 6d tensor

$$d = 6 : \quad E_c(\{1\} \times \{1\}) = -E_c(\{1\}), \quad (\text{D.8})$$

in agreement with (4.8),(4.10).

Similar results can be obtained in the  $O(N)$  case of real boundary singleton theory, i.e. for the singlet partition functions [27, 42]

$$\mathcal{Z}_{j,\text{real}}(q) = \frac{1}{2} [\mathcal{Z}_j(q)]^2 + \frac{1}{2} (-1)^{2s} \mathcal{Z}_j(q^2) . \quad (\text{D.9})$$

In this case<sup>28</sup>

$$\begin{aligned} E_c(\{0\} \times \{0\})_{\text{real}} &= E_c(\{0\}), & E_c(n \cdot \{\frac{1}{2}\} \times n \cdot \{\frac{1}{2}\})_{\text{real}} &= n E_c(\{\frac{1}{2}\}), \\ E_c(n \cdot \{1\} \times n \cdot \{1\})_{\text{real}} &= \frac{2+(-1)^{d/2}n}{2} E_c(\{1\}). \end{aligned} \quad (\text{D.10})$$

Then in 4d for the scalar, Dirac fermion and the vector we recover the results from [27, 42]

$$d = 4 : \quad E_c(\{0\} \times \{0\})_{\text{real}} = E_c(\{0\}), \quad E_c(\{\frac{1}{2}\}_c \times \{\frac{1}{2}\}_c)_{\text{real}} = 2 E_c(\{\frac{1}{2}\}) = E_c(\{\frac{1}{2}\}_c),$$

<sup>27</sup>Explicitly, we find  $\mathcal{Z}_1(q) = P_d(q)/(1-q)^d$  where

$$\begin{aligned} P_4(q) &= 3q^2 - 4q^3 + q^4, & P_8(q) &= 35q^4 - 56q^5 + 28q^6 - 8q^7 + q^8, \\ P_6(q) &= 10q^3 - 15q^4 + 6q^5 - q^6, & P_{10}(q) &= 126q^5 - 210q^6 + 120q^7 - 45q^8 + 10q^9 - q^{10}. \end{aligned}$$

<sup>28</sup>Here  $n \cdot \{j\}$  denotes  $n$  copies of the singleton, with partition function  $n \mathcal{Z}_j$ . Note also that  $E_c(\{j\}_c) = 2E_c(\{j\})$ .



$$E_c(\{1\}_c \times \{1\}_c)_{\text{real}} = 4 E_c(\{1\}) = 2 E_c(\{1\}_c). \quad (\text{D.11})$$

In 6d we get instead

$$\begin{aligned} d = 6 : \quad E_c(\{0\} \times \{0\})_{\text{real}} &= E_c(\{0\}), & E_c(\{\tfrac{1}{2}\} \times \{\tfrac{1}{2}\})_{\text{real}} &= E_c(\{\tfrac{1}{2}\}), \\ E_c(\{1\} \times \{1\})_{\text{real}} &= \tfrac{1}{2} E_c(\{1\}), \end{aligned} \quad (\text{D.12})$$

in agreement with (4.5),(4.7),(4.9).

## E Field content of KK towers in 6d supergravity on $S^3$

Here we collect the field content of the KK towers discussed in section 5. Let us list the representations of fields transforming in the  $(\Delta; s) \times (j_1, j_2)$  representations of  $SO(2, 2) \times SO(4)$  as a formal sum of the form

$$\sum n_{\Delta, s; j_1, j_2} q^\Delta x^s R_{j_1, j_2}. \quad (\text{E.1})$$

For states of the spin-2 tower in (5.5) with  $\ell \geq 0$  we get

$$\begin{aligned} \Phi_2(\ell) &= \left(\tfrac{\ell+1}{2}, \tfrac{\ell+3}{2}\right)_s + \left(\tfrac{\ell+3}{2}, \tfrac{\ell+1}{2}\right)_s \\ &= q^\ell \left[ q^2 (x R_{\tfrac{\ell+1}{2}, \tfrac{\ell+3}{2}} + x^{-1} R_{\tfrac{\ell+3}{2}, \tfrac{\ell+1}{2}}) \right. \\ &\quad + q^{5/2} (2x^{3/2} R_{\tfrac{\ell+1}{2}, \tfrac{\ell+2}{2}} + 2x^{-3/2} R_{\tfrac{\ell+2}{2}, \tfrac{\ell+1}{2}} + 2x^{1/2} R_{\tfrac{\ell}{2}, \tfrac{\ell+3}{2}} + 2x^{-1/2} R_{\tfrac{\ell+3}{2}, \tfrac{\ell}{2}}) \\ &\quad + q^3 (x^2 R_{\tfrac{\ell+1}{2}, \tfrac{\ell+1}{2}} + x^{-2} R_{\tfrac{\ell+1}{2}, \tfrac{\ell+1}{2}} + 4x R_{\tfrac{\ell}{2}, \tfrac{\ell+1}{2}} + 4x^{-1} R_{\tfrac{\ell}{2}+1, \tfrac{\ell}{2}} + R_{\tfrac{\ell-1}{2}, \tfrac{\ell+3}{2}} + R_{\tfrac{\ell+3}{2}, \tfrac{\ell-1}{2}}) \\ &\quad + q^{7/2} (2x^{3/2} R_{\tfrac{\ell}{2}, \tfrac{\ell+1}{2}} + 2x^{-3/2} R_{\tfrac{\ell+1}{2}, \tfrac{\ell}{2}} + 2x^{1/2} R_{\tfrac{\ell-1}{2}, \tfrac{\ell+2}{2}} + 2x^{-1/2} R_{\tfrac{\ell+2}{2}, \tfrac{\ell-1}{2}}) \\ &\quad \left. + q^4 (x R_{\tfrac{\ell-1}{2}, \tfrac{\ell+1}{2}} + x^{-1} R_{\tfrac{\ell+1}{2}, \tfrac{\ell-1}{2}}) \right]. \end{aligned} \quad (\text{E.2})$$

The massless states at  $\ell = -1$  are

$$\begin{aligned} &(0, 1)_s + (1, 0)_s \\ &= q(x R_{0,1} + x^{-1} R_{1,0}) + q^{3/2} (2x^{3/2} R_{0, \tfrac{1}{2}} + 2x^{-3/2} R_{\tfrac{1}{2}, 0}) + q^2 (x^2 R_{0,0} + x^{-2} R_{0,0}). \end{aligned} \quad (\text{E.3})$$

For the spin  $\frac{3}{2}$  tower (5.10) we get

$$\begin{aligned} \Phi_{\frac{3}{2}}(\ell) &= \left(\tfrac{\ell+1}{2}, \tfrac{\ell+2}{2}\right)_s + \left(\tfrac{\ell+2}{2}, \tfrac{\ell+1}{2}\right)_s \\ &= q^\ell \left[ q^{3/2} (\sqrt{x} R_{\tfrac{\ell+1}{2}, \tfrac{\ell+2}{2}} + x^{-1/2} R_{\tfrac{\ell+2}{2}, \tfrac{\ell+1}{2}}) \right. \\ &\quad + q^2 (2x R_{\tfrac{\ell+1}{2}, \tfrac{\ell+1}{2}} + 2x^{-1} R_{\tfrac{\ell+1}{2}, \tfrac{\ell+1}{2}} + 2(R_{\tfrac{\ell}{2}+1, \tfrac{\ell}{2}} + R_{\tfrac{\ell}{2}, \tfrac{\ell}{2}+1})) \\ &\quad + q^{5/2} (x^{3/2} R_{\tfrac{\ell+1}{2}, \tfrac{\ell}{2}} + x^{-3/2} R_{\tfrac{\ell}{2}, \tfrac{\ell+1}{2}} + \sqrt{x} (4R_{\tfrac{\ell}{2}, \tfrac{\ell+1}{2}} + R_{\tfrac{\ell+2}{2}, \tfrac{\ell-1}{2}}) + x^{-1/2} (R_{\tfrac{\ell-1}{2}, \tfrac{\ell+2}{2}} + 4R_{\tfrac{\ell+1}{2}, \tfrac{\ell}{2}})) \\ &\quad \left. + q^3 (2x R_{\tfrac{\ell}{2}, \tfrac{\ell}{2}} + 2x^{-1} R_{\tfrac{\ell}{2}, \tfrac{\ell}{2}} + 2(R_{\tfrac{\ell-1}{2}, \tfrac{\ell+1}{2}} + R_{\tfrac{\ell+1}{2}, \tfrac{\ell-1}{2}})) + q^{7/2} (\sqrt{x} R_{\tfrac{\ell-1}{2}, \tfrac{\ell}{2}} + x^{-1/2} R_{\tfrac{\ell}{2}, \tfrac{\ell-1}{2}}) \right], \end{aligned} \quad (\text{E.4})$$

and its massless part at  $\ell = -1$  is

$$(0, \tfrac{1}{2})_s + (\tfrac{1}{2}, 0)_s = \sqrt{q} (x^{1/2} R_{0, \tfrac{1}{2}} + x^{-1/2} R_{\tfrac{1}{2}, 0}) + q (2x R_{0,0} + 2x^{-1} R_{0,0}). \quad (\text{E.5})$$

For the spin-1 tower in (5.5) we have for  $\ell \geq 0$

$$\begin{aligned}
\Phi_1(\ell) &= \left(\frac{\ell+2}{2}, \frac{\ell+2}{2}\right)_s \\
&= q^\ell \left[ q^2 R_{\frac{\ell+2}{2}, \frac{\ell+2}{2}} + q^{5/2} (2x^{-1/2} R_{\frac{\ell+1}{2}, \frac{\ell+2}{2}} + 2x^{1/2} R_{\frac{\ell+2}{2}, \frac{\ell+1}{2}}) \right. \\
&\quad + q^3 (x R_{\frac{\ell}{2}+1, \frac{\ell}{2}} + x^{-1} R_{\frac{\ell}{2}, \frac{\ell}{2}+1} + 4R_{\frac{\ell+1}{2}, \frac{\ell+1}{2}}) \\
&\quad \left. + q^{7/2} (2x^{-1/2} R_{\frac{\ell}{2}, \frac{\ell+1}{2}} + 2x^{1/2} R_{\frac{\ell+1}{2}, \frac{\ell}{2}}) + q^4 R_{\frac{\ell}{2}, \frac{\ell}{2}} \right].
\end{aligned} \tag{E.6}$$

Finally, the extra term in (4.5), (5.7), and (5.9) is

$$\left(\frac{1}{2}, \frac{1}{2}\right)_s = q R_{\frac{1}{2}, \frac{1}{2}} + q^{3/2} (2x^{-1/2} R_{0, \frac{1}{2}} + 2x^{1/2} R_{\frac{1}{2}, 0}) + 4q^2 R_{0,0}. \tag{E.7}$$

## F Relation between Casimir energy and 2d central charge computed from $AdS_3$ for short $SU(2, 2 | 1) \times SU(2, 2 | 1)$ multiplets

The short multiplet  $(J_1, J_2)_s$  of  $SU(2, 2 | 1) \times SU(2, 2 | 1)$  contains, for generic  $j_1, j_2$ , the following representations  $(\Delta; s)_{(j_1, j_2)}$  of  $SO(2, 2) \times SO(4)$  (see (5.2)):

$$\begin{aligned}
(J_1, J_2)_s &= (J_1 + J_2; J_1 - J_2)_{(J_1, J_2)} + 2 (J_1 + J_2 + \frac{1}{2}; J_1 - J_2 - \frac{1}{2})_{(J_1, J_2 - \frac{1}{2})} \\
&\quad + 2 (J_1 + J_2 + \frac{1}{2}; J_1 - J_2 + \frac{1}{2})_{(J_1 - \frac{1}{2}, J_2)} + (J_1 + J_2 + 1; J_1 - J_2 - 1)_{(J_1, J_2 - 1)} \\
&\quad + 4 (J_1 + J_2 + 1; J_1 - J_2)_{(J_1 - \frac{1}{2}, J_2 - \frac{1}{2})} + (J_1 + J_2 + 1; J_1 - J_2 + 1)_{(J_1 - 1, J_2)} \\
&\quad + 2 (J_1 + J_2 + \frac{3}{2}; J_1 - J_2 - \frac{1}{2})_{(J_1 - \frac{1}{2}, J_2 - 1)} + 2 (J_1 + J_2 + \frac{3}{2}; J_1 - J_2 + \frac{1}{2})_{(J_1 - 1, J_2 - \frac{1}{2})} \\
&\quad + (J_1 + J_2 + 2; J_1 - J_2)_{(J_1 - 1, J_2 - 1)}.
\end{aligned} \tag{F.1}$$

The  $S^1$  Casimir energy for a 2d conformal field in the  $SO(2, 2)$  representation  $(\Delta; s)$  can be found from the partition functions  $\mathcal{Z}^+$  in (5.11) and (5.12) and using  $E_c = -2 E_c^+$ :

$$E_c(\Delta; s) = -\frac{1}{12} (-1)^{2s} (\Delta - 1) [2(\Delta - 1)^2 - 1]. \tag{F.2}$$

At the same time, the 2d central charge can be computed via “dual” route as the coefficient of the logarithmic IR divergence of 1-loop partition function of the corresponding higher spin field in  $AdS_3$  [32, 25, 26]; for a single chiral spin  $s$  component it reads

$$c_{AdS_3}(\Delta; s) = (-1)^{2s} (\Delta - 1) [(\Delta - 1)^2 - 3s^2]. \tag{F.3}$$

Comparing (F.2) and (F.3) we observe that the 2d relation  $E_c = -\frac{1}{12} c$  in (1.9) does not hold for a single massive field. Nevertheless, this relation holds for a massless field with  $\Delta = s$ , because

$$\begin{aligned}
E_c(s; s) - E_c(s + 1; s - 1) &= -\frac{1}{12} [c_{AdS_3}(s; s) - c_{AdS_3}(s + 1, s - 1)] \\
&= \frac{1}{12} (-1)^{2s} [1 - 6s(1 - s)].
\end{aligned} \tag{F.4}$$

It also holds if we evaluate the total  $E_c$  and  $c = c_{AdS_3}$  for a short multiplet  $(J_1, J_2)_s$ , i.e.

$$E_c[(J_1, J_2)_s] = -\frac{1}{12} c[(J_1, J_2)_s] = -\frac{1}{2} (-1)^{2(J_1 + J_2)} (J_1 + J_2). \tag{F.5}$$

Different expressions are found when additional shortening occur due to particular low values of  $J_1$  or  $J_2$ , but we checked that the relation  $E_c = -\frac{1}{12} c$  always holds.

## References

- [1] J. M. Maldacena, *The Large  $N$  limit of superconformal field theories and supergravity*, *Int.J.Theor.Phys.* **38** (1999) 1113–1133, [[hep-th/9711200](#)].
- [2] S. Gubser, I. R. Klebanov, and A. M. Polyakov, *Gauge theory correlators from noncritical string theory*, *Phys.Lett.* **B428** (1998) 105–114, [[hep-th/9802109](#)].
- [3] E. Witten, *Anti-de Sitter space and holography*, *Adv.Theor.Math.Phys.* **2** (1998) 253–291, [[hep-th/9802150](#)].
- [4] H. Liu and A. A. Tseytlin,  *$D = 4$  superYang-Mills,  $D = 5$  gauged supergravity, and  $D = 4$  conformal supergravity*, *Nucl.Phys.* **B533** (1998) 88–108, [[hep-th/9804083](#)].
- [5] M. Henningson and K. Skenderis, *The Holographic Weyl anomaly*, *JHEP* **9807** (1998) 023, [[hep-th/9806087](#)].
- [6] M. Blau, K. Narain, and E. Gava, *On subleading contributions to the AdS / CFT trace anomaly*, *JHEP* **9909** (1999) 018, [[hep-th/9904179](#)].
- [7] P. Mansfield and D. Nolland, *Order  $1/N^2$  test of the Maldacena conjecture: Cancellation of the one loop Weyl anomaly*, *Phys.Lett.* **B495** (2000) 435–439, [[hep-th/0005224](#)].
- [8] P. Mansfield, D. Nolland, and T. Ueno, *The Boundary Weyl anomaly in the  $\mathcal{N} = 4$  SYM / type IIB supergravity correspondence*, *JHEP* **0401** (2004) 013, [[hep-th/0311021](#)].
- [9] A. Bilal and C.-S. Chu, *A Note on the chiral anomaly in the AdS / CFT correspondence and  $1 / N^2$  correction*, *Nucl.Phys.* **B562** (1999) 181–190, [[hep-th/9907106](#)].
- [10] M. Beccaria and A. Tseytlin, *Higher spins in AdS<sub>5</sub> at one loop: vacuum energy, boundary conformal anomalies and AdS/CFT*, [arXiv:1410.3273](#).
- [11] P. Mansfield, D. Nolland, and T. Ueno, *Order  $1 / N^3$  corrections to the conformal anomaly of the  $(2,0)$  theory in six-dimensions*, *Phys.Lett.* **B566** (2003) 157–163, [[hep-th/0305015](#)].
- [12] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri, and Y. Oz, *Large  $N$  field theories, string theory and gravity*, *Phys.Rept.* **323** (2000) 183–386, [[hep-th/9905111](#)].
- [13] E. Witten, *On the conformal field theory of the Higgs branch*, *JHEP* **9707** (1997) 003, [[hep-th/9707093](#)].
- [14] V. Balasubramanian and P. Kraus, *A Stress tensor for Anti-de Sitter gravity*, *Commun.Math.Phys.* **208** (1999) 413–428, [[hep-th/9902121](#)].
- [15] L. Bonora, P. Pasti, and M. Bregola, *WEYL COCYCLES*, *Class.Quant.Grav.* **3** (1986) 635.
- [16] S. Deser and A. Schwimmer, *Geometric classification of conformal anomalies in arbitrary dimensions*, *Phys.Lett.* **B309** (1993) 279–284, [[hep-th/9302047](#)].
- [17] F. Bastianelli, S. Frolov, and A. A. Tseytlin, *Conformal anomaly of  $(2,0)$  tensor multiplet in six-dimensions and AdS / CFT correspondence*, *JHEP* **0002** (2000) 013, [[hep-th/0001041](#)].
- [18] A. A. Tseytlin,  *$R^4$  terms in 11 dimensions and conformal anomaly of  $(2,0)$  theory*, *Nucl.Phys.* **B584** (2000) 233–250, [[hep-th/0005072](#)].
- [19] F. Bastianelli, S. Frolov, and A. A. Tseytlin, *Three point correlators of stress tensors in maximally supersymmetric conformal theories in  $D = 3$  and  $D = 6$* , *Nucl.Phys.* **B578** (2000) 139–152, [[hep-th/9911135](#)].
- [20] J. A. Harvey, R. Minasian, and G. W. Moore, *NonAbelian tensor multiplet anomalies*, *JHEP* **9809** (1998) 004, [[hep-th/9808060](#)].

- [21] C. Beem, L. Rastelli, and B. C. van Rees, *W Symmetry in six dimensions*, [arXiv:1404.1079](#).
- [22] L. F. Alday, D. Gaiotto, and Y. Tachikawa, *Liouville Correlation Functions from Four-dimensional Gauge Theories*, *Lett.Math.Phys.* **91** (2010) 167–197, [[arXiv:0906.3219](#)].
- [23] G. Bonelli and A. Tanzini, *Hitchin systems, N=2 gauge theories and W-gravity*, *Phys.Lett.* **B691** (2010) 111–115, [[arXiv:0909.4031](#)].
- [24] E. Witten, *Five-brane effective action in M theory*, *J.Geom.Phys.* **22** (1997) 103–133, [[hep-th/9610234](#)].
- [25] S. Giombi and I. R. Klebanov, *One Loop Tests of Higher Spin AdS/CFT*, *JHEP* **1312** (2013) 068, [[arXiv:1308.2337](#)].
- [26] S. Giombi, I. R. Klebanov, and B. R. Safdi, *Higher Spin AdS<sub>d+1</sub>/CFT<sub>d</sub> at One Loop*, *Phys.Rev.* **D89** (2014) 084004, [[arXiv:1401.0825](#)].
- [27] S. Giombi, I. R. Klebanov, and A. A. Tseytlin, *Partition Functions and Casimir Energies in Higher Spin AdS<sub>d+1</sub>/CFT<sub>d</sub>*, [arXiv:1402.5396](#).
- [28] O. O. Sax, A. Sfondrini, and B. Stefanski, *Integrability and the Conformal Field Theory of the Higgs branch*, [arXiv:1411.3676](#).
- [29] E. Fradkin and A. A. Tseytlin, *Quantization of Two-Dimensional Supergravity and Critical Dimensions for String Models*, *Phys.Lett.* **B106** (1981) 63–69.
- [30] A. M. Polyakov, *Quantum Geometry of Bosonic Strings*, *Phys.Lett.* **B103** (1981) 207–210.
- [31] H. Bloete, J. L. Cardy, and M. Nightingale, *Conformal Invariance, the Central Charge, and Universal Finite Size Amplitudes at Criticality*, *Phys.Rev.Lett.* **56** (1986) 742–745.
- [32] S. Giombi, I. R. Klebanov, S. S. Pufu, B. R. Safdi, and G. Tarnopolsky, *AdS Description of Induced Higher-Spin Gauge Theory*, *JHEP* **1310** (2013) 016, [[arXiv:1306.5242](#)].
- [33] R. R. Metsaev, *Massless mixed symmetry bosonic free fields in d-dimensional anti-de Sitter space-time*, *Phys.Lett.* **B354** (1995) 78–84.
- [34] F. Dolan, *Character formulae and partition functions in higher dimensional conformal field theory*, *J.Math.Phys.* **47** (2006) 062303, [[hep-th/0508031](#)].
- [35] M. Gunaydin, P. van Nieuwenhuizen, and N. Warner, *General Construction of the Unitary Representations of Anti-de Sitter Superalgebras and the Spectrum of the S<sub>4</sub> Compactification of Eleven-dimensional Supergravity*, *Nucl.Phys.* **B255** (1985) 63.
- [36] G. Gibbons, M. Perry, and C. Pope, *Partition functions, the Bekenstein bound and temperature inversion in anti-de Sitter space and its conformal boundary*, *Phys.Rev.* **D74** (2006) 084009, [[hep-th/0606186](#)].
- [37] A. Casher, F. Englert, H. Nicolai, and M. Rooman, *The Mass Spectrum of Supergravity on the Round Seven Sphere*, *Nucl.Phys.* **B243** (1984) 173.
- [38] P. van Nieuwenhuizen, *The Complete Mass Spectrum of d = 11 Supergravity Compactified on S(4) and a General Mass Formula for Arbitrary Cosets M(4)*, *Class.Quant.Grav.* **2** (1985) 1.
- [39] E. D’Hoker and B. Pioline, *Near extremal correlators and generalized consistent truncation for AdS(4/7) x S(7/4)*, *JHEP* **0007** (2000) 021, [[hep-th/0006103](#)].
- [40] G. Gibbons and H. Nicolai, *One Loop Effects on the Round Seven Sphere*, *Phys.Lett.* **B143** (1984) 108–114.

- [41] T. Inami and K. Yamagishi, *Vanishing Quantum Vacuum Energy in Eleven-dimensional Supergravity on the Round Seven Sphere*, *Phys.Lett.* **B143** (1984) 115–120.
- [42] M. Beccaria and A. Tseytlin, *Vectorial  $AdS_5/CFT_4$  duality for spin-one boundary theory*, *J.Phys.* **A47** (2014) 492001, [[arXiv:1410.4457](#)].
- [43] J. de Boer, *Six-dimensional supergravity on  $S^{**3} \times AdS(3)$  and 2-D conformal field theory*, *Nucl.Phys.* **B548** (1999) 139–166, [[hep-th/9806104](#)].
- [44] M. Gunaydin, G. Sierra, and P. Townsend, *The Unitary Supermultiplets of  $d = 3$  Anti-de Sitter and  $d = 2$  Conformal Superalgebras*, *Nucl.Phys.* **B274** (1986) 429.
- [45] S. Deger, A. Kaya, E. Sezgin, and P. Sundell, *Spectrum of  $D = 6$ ,  $N=4b$  supergravity on  $AdS$  in three-dimensions  $\times S^{**3}$* , *Nucl.Phys.* **B536** (1998) 110–140, [[hep-th/9804166](#)].
- [46] H. Nicolai and H. Samtleben, *Kaluza-Klein supergravity on  $AdS(3) \times S^{**3}$* , *JHEP* **0309** (2003) 036, [[hep-th/0306202](#)].
- [47] J. Strathdee, *Extended Poincare Supersymmetry*, *Int.J.Mod.Phys.* **A2** (1987) 273.
- [48] C. Vafa, *Puzzles at large  $N$* , [[hep-th/9804172](#)].
- [49] W. Nahm, *Supersymmetries and their Representations*, *Nucl.Phys.* **B135** (1978) 149.
- [50] P. S. Howe, G. Sierra, and P. Townsend, *Supersymmetry in Six-Dimensions*, *Nucl.Phys.* **B221** (1983) 331.
- [51] J. L. Cardy, *Operator content and modular properties of higher dimensional conformal field theories*, *Nucl.Phys.* **B366** (1991) 403–419.
- [52] D. Kutasov and F. Larsen, *Partition sums and entropy bounds in weakly coupled CFT*, *JHEP* **0101** (2001) 001, [[hep-th/0009244](#)].
- [53] M. Beccaria, X. Bekaert, and A. A. Tseytlin, *Partition function of free conformal higher SPIN theory*, *JHEP* **1408** (2014) 113, [[arXiv:1406.3542](#)].
- [54] C. P. Herzog and K.-W. Huang, *Stress Tensors from Trace Anomalies in Conformal Field Theories*, *Phys.Rev.* **D87** (2013) 081901, [[arXiv:1301.5002](#)].
- [55] D. E. Diaz and H. Dorn, *Partition functions and double-trace deformations in  $AdS/CFT$* , *JHEP* **0705** (2007) 046, [[hep-th/0702163](#)].
- [56] R. Camporesi and A. Higuchi, *Spectral functions and zeta functions in hyperbolic spaces*, *J.Math.Phys.* **35** (1994) 4217–4246.
- [57] R. R. Metsaev, *Lowest eigenvalues of the energy operator for totally (anti)symmetric massless fields of the  $n$ -dimensional anti-de Sitter group*, *Class.Quant.Grav.* **11** (1994) L141–L145.
- [58] R. R. Metsaev, *Massive totally symmetric fields in  $AdS_d$* , *Phys.Lett.* **B590** (2004) 95–104, [[hep-th/0312297](#)].
- [59] R. Gopakumar, R. K. Gupta, and S. Lal, *The Heat Kernel on  $AdS$* , *JHEP* **1111** (2011) 010, [[arXiv:1103.3627](#)].
- [60] M. Vasiliev, *Higher spin superalgebras in any dimension and their representations*, *JHEP* **0412** (2004) 046, [[hep-th/0404124](#)].