Sigma-model limit of Yang-Mills instantons in higher dimensions

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Abstract

We consider the Hermitian Yang-Mills (instanton) equations for connections on vector bundles over a 2n-dimensional Kähler manifold X which is a product $Y \times Z$ of p- and q-dimensional Riemannian manifold Y and Z with p+q=2n. We show that in the adiabatic limit, when the metric in the Z direction is scaled down, the gauge instanton equations on $Y \times Z$ become sigma-model instanton equations for maps from Y to the moduli space \mathcal{M} (target space) of gauge instantons on Z if $q \geq 4$. For q < 4 we get maps from Y to the moduli space \mathcal{M} of flat connections on Z. Thus, the Yang-Mills instantons on $Y \times Z$ converge to sigma-model instantons on Y while Z shrinks to a point. Put differently, for small volume of Z, sigma-model instantons on Y with target space \mathcal{M} approximate Yang-Mills instantons on $Y \times Z$.

1 Introduction and summary

The Yang-Mills equations in two, three and four dimensions were intensively studied both in physics and mathematics. In mathematics, this study (e.g. projectively flat unitary connections and stable bundles in d=2 [1], the Chern-Simons model and knot theory in d=3, instantons and Donaldson invariants [2] in d=4 dimensions) has yielded a lot of new results in differential and algebraic geometry. There are also various interrelations between gauge theories in two, three and four dimensions. In particular, Chern-Simons theory in d=3 dimensions reduces to the theory of flat connections in d=2 (see e.g. [3, 4]). On the other hand, the gradient flow equations for Chern-Simons theory on a d=3 manifold Y are the first-order anti-self-duality equations on $Y \times \mathbb{R}$, which play a crucial role in d=4 gauge theory.

The program of extending familiar constructions in gauge theory, associated to problems in low-dimensional topology, to higher dimensions was proposed by Donaldson and Thomas in the seminal paper [5] (see also [6]) and developed in [7]-[14] among others. An important role in this investigation is played by first-order gauge-field equations which are a generalization of the anti-self-duality equations in d=4 to higher-dimensional manifolds with special holonomy (or, more generally, with G-structure [15, 16]). Such equations were first introduced in [17] and further considered in [18]-[22] (see also references therein).

Instanton equations on a d-dimensional Riemannian manifold X can be introduced as follows [17, 5, 10]. Suppose there exist a 4-form Q on X. Then there exists a (d-4)-form $\Sigma := *Q$, where * is the Hodge operator on X. Let A be a connection on a bundle E over X with curvature $\mathcal{F} = \mathrm{d}\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$. The generalized anti-self-duality (instanton) equation on the gauge field then is [10]

$$*\mathcal{F} + \Sigma \wedge \mathcal{F} = 0. \tag{1.1}$$

For d > 4 these equations can be defined on manifolds X with special holonomy, i.e. such that the holonomy group G of the Levi-Civita connection on the tangent bundle TX is a subgroup in SO(d). Solutions of (1.1) satisfy the Yang-Mills equation

$$d * \mathcal{F} + \mathcal{A} \wedge *\mathcal{F} - (-1)^d * \mathcal{F} \wedge \mathcal{A} = 0.$$
 (1.2)

The instanton equation (1.1) is also well defined on manifolds X with non-integrable G-structures, i.e. when $d\Sigma \neq 0$. In this case (1.1) implies the Yang-Mills equation with (3-form) torsion $T := *d\Sigma$, as is discussed e.g. in [23]-[27].

Manifolds X with a (d-4)-form Σ which admits the instanton equation (1.1) are usually calibrated manifolds with calibrated submanifolds. Recall that a calibrated manifold is a Riemannian manifold (X,g) equipped with a closed p-form φ such that for any oriented p-dimensional subspace ζ of T_xX , $\varphi \mid_{\zeta} \leq vol_{\zeta}$ for any $x \in X$, where vol_{ζ} is the volume of ζ with respect to the metric g [28]. A p-dimensional submanifold Y of X is said to be a calibrated submanifold with respect to φ (φ -calibrated) if $\varphi \mid_{Y} = vol_{Y}$ [28]. In particular, suitably normalized powers of the Kähler form on a Kähler manifold are calibrations, and the calibrated submanifolds are complex submanifolds. On a G_2 -manifold one has a 3-form which defines a calibration, and on a Spin(7)-manifold the defining 4-form (the Cayley form) is a calibration as well [5, 6].

It is not easy to construct solutions of (1.1) for d > 4 and to describe their moduli space.¹ It was shown by Donaldson, Thomas, Tian [5, 10] and others that the *adiabatic limit* method provides

Some explicit solutions for particular manifolds X were constructed e.g. in [21, 23, 25, 14, 27].

a useful and powerful tool. The adiabatic limit refers to the geometric process of shrinking a metric in some directions while leaving it fixed in the others.² It is assumed that on X there is a family Σ_{ε} of (d-4)-forms with a real parameter ε such that $\Sigma_0 = \lim_{\varepsilon \to 0} \Sigma_{\varepsilon}$ defines a calibrated submanifold Y of X. Then one can define a normal bundle N(Y) of Y with a projection

$$\pi: N(Y) \to Y. \tag{1.3}$$

The metric on X induces on N(Y) a Riemannian metric

$$g_{\varepsilon} = \pi^* g_V + \varepsilon^2 g_Z \,, \tag{1.4}$$

where $Z \cong \mathbb{R}^4$ is a typical fibre. In fact, the fibres are calibrated by a 4-form Q_{ε} dual to Σ_{ε} . The metric (1.4) extends to a tubular neighborhood of Y in X, and (1.1) may be considered on this subset of X. Anyway, it was shown [5, 10, 6] that solutions of the instanton equation (1.1) defined by the form Σ_{ε} on (X, g_{ε}) in the adiabatic limit $\varepsilon \to 0$ converge to sigma-model instantons describing a map from the (d-4)-dimensional submanifold Y into the hyper-Kähler moduli space of framed Yang-Mills instantons on fibres \mathbb{R}^4 of the normal bundle N(Y).

The submanifold $Y \hookrightarrow X$ is calibrated by the (d-4)-form Σ defining the instanton equation (1.1). However, on X there may exist other p-forms φ and associated φ -calibrated submanifolds Y of dimension $p \neq d-4$. In such a case one can define a different normal bundle (1.3) with fibres \mathbb{R}^{d-p} and deform the metric as in (1.4). However, this task is quite difficult technically and will be postponed for a future work. As a more simple task, one may take a direct product manifold $X = Y \times Z$ with $\dim_{\mathbb{R}} Y = p$ and $\dim_{\mathbb{R}} Z = q = d-p$ with a p-form $\varphi = vol_Y$, or consider non-flat manifolds Z and a (d-4)-form Σ defining (1.1). In string theory $\dim_{\mathbb{R}} X = 10$, and calibrated submanifolds Y are identified with worldvolumes of p-branes where p varies from zero to ten.

In this short paper we explore the direct product case $X=Y\times Z$ with $\dim_{\mathbb{R}}Y=p\neq d-4$ for Kähler manifolds X and the adiabatic limit of the Hermitian Yang-Mills equations on bundles over X. We will show that for even p (and hence even q) the adiabatic limit of (1.1) yields sigma-model instanton equations describing holomorphic maps from Y into the moduli space of Hermitian Yang-Mills instantons on Z. For odd p and q the consideration is more involved, and we describe only the case p=q=3 in which we obtain maps from Y into the moduli space of flat connections on Z. For the purpose of this paper, this special case sufficiently illustrates the main features of the odd-dimensional cases.

2 Moduli space of instantons in $d \ge 4$

Bundles. Let X be an oriented smooth manifold of dimension d, G a semisimple compact Lie group, \mathfrak{g} its Lie algebra, P a principal G-bundle over X, \mathcal{A} a connection 1-form on P and $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ its curvature. We consider also the bundle of groups $\operatorname{Int} P = P \times_G G$ (G acts on itself by internal automorphisms: $h \mapsto ghg^{-1}$, $h, g \in G$) associated with P, the bundle of Lie algebras $\operatorname{Ad} P = P \times_G \mathfrak{g}$ and a complex vector bundle $E = P \times_G V$, where V is the space of some irreducible representation of G. All these associated bundles inherit their connection \mathcal{A} from P.

²In lower dimensions, the adiabatic limit was successfully used for a description of solutions to the d=2+1 Ginzburg-Landau equations and to the d=4 Seiberg-Witten monopole equations (see e.g. reviews [29, 30] and references therein).

Gauge transformations. We denote by \mathbb{A}' the space of connections on P and by \mathcal{G}' the infinite-dimensional group of gauge transformations (automorphisms of P which induce the identity transformation of X),

$$\mathcal{A} \mapsto \mathcal{A}^g = g^{-1}\mathcal{A}g + g^{-1}\mathrm{d}g , \qquad (2.1)$$

which can be identified with the space of global sections of the bundle IntP. Correspondingly, the infinitesimal action of \mathcal{G}' is defined by global sections χ of the bundle AdP,

$$\mathcal{A} \mapsto \delta_{\chi} \mathcal{A} = d\chi + [\mathcal{A}, \chi] =: D_{\mathcal{A}} \chi \tag{2.2}$$

with $\chi \in \text{Lie}\mathcal{G}' = \Gamma(X, \text{Ad}P)$.

Moduli space of connections. We restrict ourselves to the subspace $\mathbb{A} \subset \mathbb{A}'$ of irreducible connections and to the subgroup $\mathcal{G} = \mathcal{G}'/Z(\mathcal{G}')$ of \mathcal{G}' which acts freely on \mathbb{A} . Then the *moduli space* of irreducible connections on P (and on E) is defined as the quotient \mathbb{A}/\mathcal{G} . We do not distinguish connections related by a gauge transformation. Classes of gauge equivalent connections are points $[\mathcal{A}]$ in \mathbb{A}/\mathcal{G} .

Metric on \mathbb{A}/\mathcal{G} . Since \mathbb{A} is an affine space, for each $\mathcal{A} \in \mathbb{A}$ we have a canonical identification between the tangent space $T_{\mathcal{A}}\mathbb{A}$ and the space $\Lambda^1(X, \operatorname{Ad}P)$ of 1-forms on X with values in the vector bundle $\operatorname{Ad}P$. We consider \mathfrak{g} as a matrix Lie algebra, with the metric defined by the trace. The metrics on X and on the Lie algebra \mathfrak{g} induce an inner product on $\Lambda^1(X, \operatorname{Ad}P)$,

$$\langle \xi_1, \xi_2 \rangle = \int_X \operatorname{tr} (\xi_1 \wedge *\xi_2) \quad \text{for} \quad \xi_1, \xi_2 \in \Lambda^1(X, \operatorname{Ad}P) .$$
 (2.3)

This inner product is transferred to $T_{\mathcal{A}}\mathbb{A}$ by the canonical identification. It is invariant under the \mathcal{G} -action on \mathbb{A} , whence we get a metric (2.3) on the moduli space \mathbb{A}/\mathcal{G} .

Instantons. Suppose there exists a (d-4)-form Σ on X which allows us to introduce the instanton equation

$$*\mathcal{F} + \Sigma \wedge \mathcal{F} = 0 \tag{2.4}$$

discussed in Section 1. We denote by $\mathcal{N} \subset \mathbb{A}$ the space of irreducible connections subject to (2.4) on the bundle $E \to X$. This space \mathcal{N} of instanton solutions on X is a subspace of the affine space \mathbb{A} , and we define the moduli space \mathcal{M} of instantons as the quotient space

$$\mathcal{M} = \mathcal{N}/\mathcal{G} \tag{2.5}$$

together with a projection

$$\pi: \mathcal{N} \xrightarrow{\mathcal{G}} \mathcal{M}$$
 (2.6)

According to the bundle structure (2.6), at any point $A \in \mathcal{N}$, the tangent bundle $T_A \mathcal{N} \to \mathcal{N}$ splits into the direct sum

$$T_{\mathcal{A}}\mathcal{N} = \pi^* T_{[\mathcal{A}]} \mathcal{M} \oplus T_{\mathcal{A}} \mathcal{G} .$$
 (2.7)

In other words,

$$T_{\mathcal{A}}\mathcal{N} \ni \tilde{\xi} = \xi + D_{\mathcal{A}}\chi \quad \text{with} \quad \xi \in \pi^* T_{[\mathcal{A}]}\mathcal{M} \quad \text{and} \quad D_{\mathcal{A}}\chi \in T_{\mathcal{A}}\mathcal{G} ,$$
 (2.8)

where $\tilde{\xi}, \xi \in \Lambda^1(X, AdP)$ and $\chi \in \Lambda^0(X, AdP) = \Gamma(X, AdP)$. The choice of ξ corresponds to a local fixing of a gauge.

Metric on \mathcal{M} . Denote by ξ_{α} a local basis of vector fields on \mathcal{M} (sections of the tangent bundle $T\mathcal{M}$) with $\alpha = 1, \ldots, \dim_{\mathbb{R}} \mathcal{M}$. Restricting the metric (2.3) on \mathbb{A}/\mathcal{G} to the subspace \mathcal{M} provides a metric $\mathbb{G} = (G_{\alpha\beta})$ on the instanton moduli space,

$$G_{\alpha\beta} = \int_X \operatorname{tr} \left(\xi_\alpha \wedge * \xi_\beta \right) .$$
 (2.9)

Kähler forms on \mathcal{M} . If X is Kähler with a complex structure J and a Kähler form $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$, then the Kähler 2-form $\Omega = (\Omega_{\alpha\beta})$ on \mathcal{M} is given by

$$\Omega_{\alpha\beta} = -\int_X \operatorname{tr} \left(J\xi_\alpha \wedge *\xi_\beta \right) . \tag{2.10}$$

It is well known that the moduli space of framed instantons³ on a hyper-Kähler 4-manifold X (with three integrable almost complex structures J^i) is hyper-Kähler, with three Kähler forms

$$\Omega^{i}_{\alpha\beta} = -\int_{X} \operatorname{tr} \left(J^{i} \xi_{\alpha} \wedge * \xi_{\beta} \right) . \tag{2.11}$$

3 Hermitian Yang-Mills equations

Instanton equations. On any Kähler manifold X of dimension d=2n there exists an integrable almost complex structure $J \in \text{End}(TX)$, $J^2 = -\text{Id}$, and a Kähler (1,1)-form $\omega(\cdot, \cdot) = g(J \cdot, \cdot)$ compatible with J. The natural 4-form

$$Q = \frac{1}{2}\omega \wedge \omega \tag{3.1}$$

and its dual $\Sigma = *Q$ allow one to formulate the instanton equation (2.4) for a connection \mathcal{A} on a complex vector bundle E over X associated to the principal bundle P(X,G). The fibres \mathbb{C}^N of E support an irreducible G-representation. For simplicity, we have in mind the fundamental representation of SU(N). One can endow the bundle E with a Hermitian metric and choose \mathcal{A} to be compatible with the Hermitian structure on E.

The instanton equations in the form (2.4) with $\Sigma = \frac{1}{2} * (\omega \wedge \omega)$ may then be rewritten as the following pair of equations,

$$\mathcal{F}^{0,2} = -(\mathcal{F}^{2,0})^{\dagger} = 0 \tag{3.2}$$

and

$$\omega^{n-1} \wedge \mathcal{F} = 0 \qquad \Leftrightarrow \qquad \omega \, \lrcorner \, \mathcal{F} = \omega^{\hat{\mu}\hat{\nu}} \mathcal{F}_{\hat{\mu}\hat{\nu}} = 0 , \qquad (3.3)$$

where $\hat{\mu}, \hat{\nu}, \ldots = 1, \ldots, 2n$, and the notation ω exploits the underlying Riemannian metric of X for raising indices of ω . The equations (3.2)-(3.3) were introduced by Donaldson, Uhlenbeck and Yau [19] and are called the Hermitian Yang-Mills (HYM) equations.⁴ The HYM equations

³Framed instantons are instantons modulo gauge transformations which approach the identity at a fixed point.

⁴Instead of (3.3) one sometimes finds $\omega \, \lrcorner \, \mathcal{F} = i \, \lambda \, \mathrm{Id}_E$ with $\lambda \in \mathbb{R}$. We take $\lambda = 0$, i.e. assume $c_1(E) = 0$, since one may always pass from a rank-N bundle of non-zero degree to one of zero degree by considering $\tilde{\mathcal{F}} = \mathcal{F} - \frac{1}{N} (\mathrm{tr} \mathcal{F}) \mathbf{1}_N$.

have the following algebro-geometric interpretation. Equation (3.2) implies that the curvature $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ is of type (1,1) with respect to J, whence the connection \mathcal{A} defines a holomorphic structure on E. Equation (3.3) means that $E \to X$ is a polystable vector bundle. The moduli space \mathcal{M}_X of HYM connections on E, the metric $\mathbb{G} = (G_{\alpha\beta})$ and the Kähler form $\Omega = (\Omega_{\alpha\beta})$ on \mathcal{M}_X are introduced as described in Section 2 after specializing X to be Kähler.

Direct product of Kähler manifolds. The subject of this paper is the adiabatic limit of the HYM equations (3.2)-(3.3) on a direct product

$$X = Y \times Z \tag{3.4}$$

of Kähler manifolds Y and Z. The dimensions p and q of Y and Z are even, and p+q=2n. Let $\{e^a\}$ with $a=1,\ldots,p$ and $\{e^\mu\}$ with $\mu=p+1,\ldots,2n$ be local frames for the cotangent bundles T^*Y and T^*Z , respectively. Then $\{e^{\hat{\mu}}\}=\{e^a,e^\mu\}$ with $\hat{\mu}=1,\ldots,2n$ will be a local frame for the cotangent bundle $T^*X=T^*Y\oplus T^*Z$. We introduce on $Y\times Z$ the metric

$$g = g_Y + g_Z = \delta_{ab} e^a \otimes e^b + \delta_{\mu\nu} e^{\mu} \otimes e^{\nu} = \delta_{\hat{\mu}\hat{\nu}} e^{\hat{\mu}} \otimes e^{\hat{\nu}}$$

$$(3.5)$$

and an integrable almost complex structure

$$J = J_Y \oplus J_Z \in \operatorname{End}(TY) \oplus \operatorname{End}(TZ)$$
, $J_Y^2 = -\operatorname{Id}_Y$ and $J_Z^2 = -\operatorname{Id}_Z$, (3.6)

whose components are defined by $J_Y e^a = J_b^a e^b$ and $J_Z e^\mu = J_\nu^\mu e^\nu$. Likewise, the Kähler form $\omega(\cdot,\cdot) = g(J\cdot,\cdot)$ on $Y\times Z$ decomposes as

$$\omega = \omega_Y + \omega_Z \tag{3.7}$$

with components $\omega_Y = (\omega_{ab})$ and $\omega_Z = (\omega_{\mu\nu})$.

Splitting of the HYM equations. We introduce on $X = Y \times Z$ local coordinates $\{y^a\}$ and $\{z^{\mu}\}$ and choose $e^a = dy^a$, $e^{\mu} = dz^{\mu}$. Any connection on the bundle $E \to X$ is decomposed as

$$A = A_Y + A_Z = A_a dy^a + A_\mu dz^\mu , \qquad (3.8)$$

where the components \mathcal{A}_a and \mathcal{A}_μ depend on $(y, z) \in Y \times Z$. The curvature \mathcal{F} of \mathcal{A} has components \mathcal{F}_{ab} along Y, $\mathcal{F}_{\mu\nu}$ along Z, and $\mathcal{F}_{a\mu}$ which we call "mixed".

Note that the holomorphicity conditions (3.2) may be expressed through the projector

$$\bar{P} = \frac{1}{2} (\text{Id} + iJ) , \qquad \bar{P}^2 = \bar{P}$$
 (3.9)

onto the (0,1)-part of the complexification of the cotangent bundle $T^*X = T^*Y \oplus T^*Z$ as

$$\bar{P}\bar{P}\mathcal{F} = 0 , \qquad (3.10)$$

which in components reads

$$\left(\delta_{\hat{\mu}}^{\hat{\sigma}} + iJ_{\hat{\mu}}^{\hat{\sigma}}\right) \left(\delta_{\hat{\nu}}^{\hat{\lambda}} + iJ_{\hat{\nu}}^{\hat{\lambda}}\right) \mathcal{F}_{\hat{\sigma}\hat{\lambda}} = 0.$$

$$(3.11)$$

From (3.6) it follows that these equations split into three parts:

$$\left(\delta_a^c + iJ_a^c\right)\left(\delta_b^d + iJ_b^d\right)\mathcal{F}_{cd} = 0 \qquad \Leftrightarrow \qquad \mathcal{F}_V^{0,2} = 0 , \qquad (3.12)$$

$$\left(\delta^{\sigma}_{\mu} + iJ^{\sigma}_{\mu}\right)\left(\delta^{\lambda}_{\nu} + iJ^{\lambda}_{\nu}\right)\mathcal{F}_{\sigma\lambda} = 0 \qquad \Leftrightarrow \qquad \mathcal{F}^{0,2}_{Z} = 0 , \qquad (3.13)$$

and

$$\mathcal{F}_{a\nu}J^{\nu}_{\mu} + J^{c}_{a}\mathcal{F}_{c\mu} = 0 \qquad \Leftrightarrow \qquad \mathcal{F}_{a\mu} - J^{c}_{a}J^{\nu}_{\mu}\mathcal{F}_{c\nu} = 0. \tag{3.14}$$

Finally, with the help of (3.7) the stability equation (3.3) takes the form

$$\omega_Y \, \lrcorner \, \mathcal{F}_Y + \omega_Z \, \lrcorner \, \mathcal{F}_Z = \omega^{ab} \mathcal{F}_{ab} + \omega^{\mu\nu} \mathcal{F}_{\mu\nu} = 0 . \tag{3.15}$$

4 Adiabatic limit of the HYM equations for even p and q

Moduli space \mathcal{M}_Z . In order to investigate the adiabatic limit of (3.12)-(3.15), we introduce on $X = Y \times Z$ the deformed metric and Kähler form

$$g_{\varepsilon} = g_Y + \varepsilon^2 g_Z$$
 and $\omega_{\varepsilon} = \omega_Y + \varepsilon^2 \omega_Z$, (4.1)

while the complex structure $J = J_Y \oplus J_Z$ does not depend on ε according to (3.6). Since J_Y and J_Z are untouched, (3.12)-(3.14) keep their form in the adiabatic limit $\varepsilon \to 0$. In particular, (3.12) implies that $\mathcal{F}_Y^{0,2} = 0$, i.e. the bundle $E \to Y \times Z$ is holomorphic along Y for any $z \in Z$.⁵ On the other hand, (3.15) for $\varepsilon \to 0$ becomes

$$\omega_Z \, \lrcorner \, \mathcal{F}_Z = \omega^{\mu\nu} \mathcal{F}_{\mu\nu} = 0 , \qquad (4.2)$$

which together with (3.13) means that A_Z is a HYM connection (framed instanton) on Z for any given $y \in Y$. We denote the moduli space of such connections by

$$\mathcal{M}_Z = \mathcal{N}_Z/\mathcal{G}_Z \,, \tag{4.3}$$

where \mathcal{N}_Z is the space of all instanton solutions on Z for a fixed $y \in Y$, and \mathcal{G}_Z consists of the elements of \mathcal{G} with the same fixed value of y. We here suppress the y dependence in our notation. The moduli space \mathcal{M}_Z is a Kähler manifold on which we introduce the metric \mathbb{G} and Kähler form Ω with components

$$G_{\alpha\beta} = \int_{Z} \operatorname{tr} \left(\xi_{\alpha} \wedge *_{Z} \xi_{\beta} \right) \quad \text{and} \quad \Omega_{\alpha\beta} = -\int_{Z} \operatorname{tr} \left(J_{Z} \xi_{\alpha} \wedge *_{Z} \xi_{\beta} \right)$$
 (4.4)

similar to (2.9) and (2.10) but now with $\xi_{\alpha} \in \Lambda^{1}(Z, \operatorname{Ad}P)$ and the Hodge operator $*_{Z}$ defined on Z. Note that for $\dim_{\mathbb{R}} Z = 2$ the HYM equations (3.13) and (4.2) enforce $\mathcal{F}_{Z} = 0$, i.e. \mathcal{M}_{Z} becomes the moduli space of flat connections on bundles E(y) over a two-dimensional Riemannian manifold Z.

A map into \mathcal{M}_Z . The bundle E(y) is a HYM vector bundle over Z for any $y \in Y$. Letting the point y vary, the connection $\mathcal{A}_Z = \mathcal{A}_{\mu}(y,z) dz^{\mu}$ on E(y) defines a map

$$\phi: Y \to \mathcal{M}_Z \quad \text{with} \quad \phi(y) = \{\phi^{\alpha}(y)\},$$
 (4.5)

where ϕ^{α} with $\alpha = 1, ..., \dim_{\mathbb{R}} \mathcal{M}_Z$ are local coordinates on \mathcal{M}_Z . This map is constrained by our remaining set of equations, namely (3.14) for the mixed field-strength components

$$\mathcal{F}_{a\mu} = \partial_a \mathcal{A}_{\mu} - \partial_{\mu} \mathcal{A}_a + [\mathcal{A}_a, \mathcal{A}_{\mu}] = \partial_a \mathcal{A}_{\mu} - D_{\mu} \mathcal{A}_a . \tag{4.6}$$

Similarly to (2.7) and (2.8), $\partial_a A_\mu$ decomposes into two parts,

$$T_{\mathcal{A}_Z}\mathcal{N}_Z = \pi^*T_{[\mathcal{A}_Z]}\mathcal{M}_Z \oplus T_{\mathcal{A}_Z}\mathcal{G}_Z \qquad \Leftrightarrow \qquad \partial_a \mathcal{A}_\mu = (\partial_a \phi^\alpha)\xi_{\alpha\mu} + D_\mu \epsilon_a , \qquad (4.7)$$

where $\{\xi_{\alpha} = \xi_{\alpha\mu} dz^{\mu}\}$ is a local basis of vector fields on \mathcal{M}_Z . Here, ϵ_a are \mathfrak{g} -valued gauge parameters which are determined by the gauge-fixing equations

$$(\partial_a \phi^\alpha) g^{\mu\nu} D_\mu \xi_{\alpha\nu} = 0 \qquad \Rightarrow \qquad g^{\mu\nu} D_\mu D_\nu \epsilon_a = g^{\mu\nu} D_\mu \partial_a \mathcal{A}_\nu . \tag{4.8}$$

⁵We can always choose a gauge such that $\mathcal{A}_Y^{0,1} = 0$ and locally $\mathcal{A}_Y^{1,0} = h^{-1}\partial_Y h$ for a G-valued function h(y,z).

Substituting (4.7) into (4.6), the mixed field-strength components simplify to

$$\mathcal{F}_{a\mu} = (\partial_a \phi^\alpha) \, \xi_{\alpha\mu} - D_\mu (\mathcal{A}_a - \epsilon_a) \ . \tag{4.9}$$

Inserting this expression into our remaining equations (3.14), we obtain

$$(\partial_a \phi^\alpha) \, \xi_{\alpha\mu} - J_a^c J_\mu^\sigma (\partial_c \phi^\alpha) \, \xi_{\alpha\sigma} = D_\mu (\mathcal{A}_a - \epsilon_a) - J_a^c J_\mu^\sigma D_\sigma (\mathcal{A}_c - \epsilon_c) \tag{4.10}$$

as a condition on the map ϕ .

Sigma-model instantons. In order to better interpret the above equations, we multiply both sides with $dz^{\mu} \wedge *_Z \xi_{\beta}$, take the trace over \mathfrak{g} , integrate over Z and recognize the integrals in (4.4). The integral of the right-hand side of (4.10) vanishes due to (4.7)-(4.8) (orthogonality of $\xi_{\alpha} \in T\mathcal{M}_Z$ and $D\chi \in T\mathcal{G}_Z$), and we end up with

$$(\partial_a \phi^\alpha) G_{\alpha\beta} + J_a^c (\partial_c \phi^\alpha) \Omega_{\alpha\beta} = 0. (4.11)$$

Inverting the moduli-space metric G and introducing the almost complex structure \mathcal{J} on \mathcal{M}_Z via its components

$$\mathcal{J}^{\alpha}_{\beta} := \Omega_{\beta\gamma} G^{\gamma\alpha} \,, \tag{4.12}$$

we rewrite (4.11) as

$$\partial_a \phi^{\alpha} = -J_a^c (\partial_c \phi^{\beta}) \mathcal{J}_{\beta}^{\alpha} \qquad \Leftrightarrow \qquad \mathrm{d}\phi = -\mathcal{J} \circ \mathrm{d}\phi \circ J \ . \tag{4.13}$$

Using $J_c^a J_b^c = -\delta_b^a$ and $\mathcal{J}_{\gamma}^{\alpha} \mathcal{J}_{\beta}^{\gamma} = -\delta_{\beta}^{\alpha}$, alternative versions are

$$(\partial_a \phi^{\beta}) \mathcal{J}^{\alpha}_{\beta} - J^b_a (\partial_b \phi^{\alpha}) = 0 \qquad \Leftrightarrow \qquad \mathcal{J} \circ d\phi = d\phi \circ J \tag{4.14}$$

and

$$(\delta_a^b + iJ_a^b)(\partial_b \phi^\beta)(\delta_\beta^\alpha - iJ_\beta^\alpha) = 0 \qquad \Leftrightarrow \qquad \mathcal{P} \circ d\phi \circ \bar{P} = 0 , \qquad (4.15)$$

with the obvious definition for \mathcal{P} .

These equations mean that $\phi^1 + i\phi^2$, $\phi^3 + i\phi^4$, ... are holomorphic functions of complex coordinates on Y, i.e. ϕ is a holomorphic map. It is clear that our equations (4.15) are BPS-type (instanton) first-order equations for the sigma model on Y with target space \mathcal{M}_Z , whose field equations define harmonic maps from Y into \mathcal{M}_Z . For $\dim_{\mathbb{R}} Y = \dim_{\mathbb{R}} Z = 2$ these equations have appeared in [31] as the adiabatic limit of the HYM equations on the product of two Riemann surfaces.⁶ Our (4.15) generalize [31] to the case $\dim_{\mathbb{R}} Y > 2$ and $\dim_{\mathbb{R}} Z \geq 2$. From the implicit function theorem it follows that near every solution ϕ of (4.15) there exists a solution $\mathcal{A}_{\varepsilon}$ of the HYM equations (3.2)-(3.3) for ε sufficiently small. In other words, solutions of (4.15) approximate solutions of the HYM equations on X.

⁶See also [32] where this limit was discussed in the framework of topological Yang-Mills theories.

5 Adiabatic limit of gauge instantons for p = q = 3

If the Kähler manifold X is a direct product of two odd-dimensional manifolds Y and Z, i.e. if $p = \dim_{\mathbb{R}} Y$ and $q = \dim_{\mathbb{R}} Z$ are both odd, then we may need to impose conditions on the geometry of Y and Z for $X = Y \times Z$ to be Kähler. However, we are not aware of these demands outside of special cases, such as products of tori. Therefore, we restrict ourselves to tori Y and Z with p = q = 3 since already this case illustrates essential differences from the case of even p and q. More general situations demand more effort and will be considered elsewhere.

Deformed structures. We consider the Calabi-Yau space

$$X = Y \times Z = T^3 \times T_r^3 , \qquad (5.1)$$

where T^3 is a 3-torus and T_r^3 is another 3-torus, with r marked points (punctures). We endow X with the deformed metric

$$g_{\varepsilon} = g_{T^3} + \varepsilon^2 g_{T_{\tau}^3} = e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3 + \varepsilon^2 (e^4 \otimes e^4 + e^5 \otimes e^5 + e^6 \otimes e^6)$$
 (5.2)

and choose the basis of (1,0)-forms as

$$\theta^1 = e^1 + i\varepsilon e^4$$
, $\theta^2 = e^2 + i\varepsilon e^5$ and $\theta^3 = e^3 + i\varepsilon e^6$ (5.3)

with a real deformation parameter ε .

The combined torus $T^3 \times T_r^3$ supports an integrable almost complex structure J satisfying $J\theta^j = i\theta^j$ for j = 1, 2, 3, which determines its components,

$$Je^{\hat{\mu}} = J_{\hat{\nu}}^{\hat{\mu}} e^{\hat{\nu}} : \quad J_4^1 = J_5^2 = J_6^3 = -\varepsilon \quad \text{and} \quad J_1^4 = J_2^5 = J_3^6 = \varepsilon^{-1} .$$
 (5.4)

For the Kähler form $\omega(\cdot,\cdot)=g(J\cdot,\cdot)$ the components are

$$\omega_{14} = \omega_{25} = \omega_{36} = \varepsilon$$
 and $\omega_{41} = \omega_{52} = \omega_{63} = -\varepsilon$. (5.5)

Adiabatic limit for instantons. The HYM equations (3.2) and (3.3) on $T^3 \times T_r^3$ with J and ω given by (5.4) and (5.5) read

$$\mathcal{F}_{ab} + i\mathcal{F}_{a\mu}J_{b}^{\mu} + iJ_{a}^{\mu}\mathcal{F}_{\mu b} - J_{a}^{\mu}J_{b}^{\nu}\mathcal{F}_{\mu \nu} = 0 ,$$

$$\mathcal{F}_{\mu\nu} + i\mathcal{F}_{\mu b}J_{\nu}^{b} + iJ_{\mu}^{b}\mathcal{F}_{b\nu} - J_{\mu}^{a}J_{\nu}^{b}\mathcal{F}_{ab} = 0 ,$$

$$\mathcal{F}_{a\mu} + i\mathcal{F}_{ab}J_{\mu}^{b} + iJ_{a}^{\nu}\mathcal{F}_{\nu\mu} - J_{a}^{\nu}J_{\mu}^{b}\mathcal{F}_{\nu b} = 0 ,$$
(5.6)

with a, b = 1, 2, 3 and $\mu, \nu = 4, 5, 6$, as well as

$$\mathcal{F}_{14} + \mathcal{F}_{25} + \mathcal{F}_{36} = 0. ag{5.7}$$

In the adiabatic limit $\varepsilon \to 0$ the first two lines of (5.6) reduce to

$$\mathcal{F}_{45} = \mathcal{F}_{46} = \mathcal{F}_{56} = 0 \tag{5.8}$$

while the mixed-component part of (5.6) together with (5.7) produces

$$\mathcal{F}_{16} - \mathcal{F}_{34} = 0$$
, $\mathcal{F}_{35} - \mathcal{F}_{26} = 0$, $\mathcal{F}_{24} - \mathcal{F}_{15} = 0$ and $\mathcal{F}_{14} + \mathcal{F}_{25} + \mathcal{F}_{36} = 0$. (5.9)

Recall that

$$\mathcal{A} = \mathcal{A}_Y + \mathcal{A}_Z = \mathcal{A}_a(y, z) dy^a + \mathcal{A}_\mu(y, z) dz^\mu$$
 (5.10)

is a connection on a vector bundle E over $X = T^3 \times T_r^3$. From (5.8) we learn that \mathcal{A}_Z is a flat connection on $Z = T_r^3$ for any $y \in Y = T^3$. We denote by \mathcal{N}_Z the space of solutions to (5.8) and by \mathcal{M}_Z the moduli space of all such connections. From (5.9) we see that in the adiabatic limit there are no restrictions on \mathcal{A}_Y , since the components \mathcal{A}_a and \mathcal{F}_{ab} no longer appear.

Sigma-model equations. For the mixed components $\mathcal{F}_{a\mu}$ of the field strength we have

$$\mathcal{F}_{a\mu} = \partial_a \mathcal{A}_{\mu} - D_{\mu} \mathcal{A}_a = (\partial_a \phi^{\alpha}) \xi_{\alpha\mu} - D_{\mu} (\mathcal{A}_a - \epsilon_a)$$
 (5.11)

where, as in Section 4, we used for $\partial_a \mathcal{A}_{\mu}$ the decomposition formula (4.7) and introduced the map

$$\phi: T^3 \to \mathcal{M}_{T_x^3} \quad \text{with} \quad \phi(y) = \left\{\phi^{\alpha}(y)\right\},$$
(5.12)

where ϕ^{α} with $\alpha = 1, ..., \dim_{\mathbb{R}} \mathcal{M}_{T_{\sigma}^{3}}$ are local coordinates on $\mathcal{M}_{T_{\sigma}^{3}}$.

Substituting (5.11) into (5.9), we obtain the equations

$$(\partial_{1}\phi^{\alpha})\xi_{\alpha6} - (\partial_{3}\phi^{\alpha})\xi_{\alpha4} = D_{6}(\mathcal{A}_{1} - \epsilon_{1}) - D_{4}(\mathcal{A}_{3} - \epsilon_{3}),$$

$$(\partial_{3}\phi^{\alpha})\xi_{\alpha5} - (\partial_{2}\phi^{\alpha})\xi_{\alpha6} = D_{5}(\mathcal{A}_{3} - \epsilon_{3}) - D_{6}(\mathcal{A}_{2} - \epsilon_{2}),$$

$$(\partial_{2}\phi^{\alpha})\xi_{\alpha4} - (\partial_{1}\phi^{\alpha})\xi_{\alpha5} = D_{4}(\mathcal{A}_{2} - \epsilon_{2}) - D_{5}(\mathcal{A}_{1} - \epsilon_{1})$$

$$(5.13)$$

and

$$(\partial_1 \phi^{\alpha}) \xi_{\alpha 4} + (\partial_2 \phi^{\alpha}) \xi_{\alpha 5} + (\partial_3 \phi^{\alpha}) \xi_{\alpha 6} = D_4(\mathcal{A}_1 - \epsilon_1) + D_5(\mathcal{A}_2 - \epsilon_2) + D_6(\mathcal{A}_3 - \epsilon_3) . \tag{5.14}$$

Multiplying both sides with $\xi_{\beta\mu}$ for $\mu = 4, 5, 6$ and integrating $\operatorname{tr}(\xi_{\alpha\mu}\xi_{\beta\nu})$ over T_r^3 , the above four equations yield the $3\dim_{\mathbb{R}}\mathcal{M}_{T_r^3}$ relations

$$\partial_a \phi^{\alpha} + \pi_a {}_c^b (\partial_b \phi^{\beta}) \Pi^c {}_{\beta}^{\alpha} = 0 , \qquad (5.15)$$

where

$$\pi_{a\ c}^{\ b} := \varepsilon_{ac}^{b} \quad \text{and} \quad \Pi^{a\ \alpha}_{\ \beta} := \Pi^{a}_{\beta\gamma}G^{\gamma\alpha}$$
 (5.16)

with

$$G_{\alpha\beta} = \int_{T_r^3} d^3z \, \delta^{\mu\nu} \operatorname{tr} \left(\xi_{\alpha\mu} \xi_{\beta\nu} \right) \quad \text{and} \quad \Pi_{\alpha\beta}^a = \int_{T_r^3} d^3z \, \varepsilon^{a+3\,\mu\nu} \operatorname{tr} \left(\xi_{\alpha\mu} \xi_{\beta\nu} \right) \,. \tag{5.17}$$

The integrals of the right-hand sides of (5.13) and (5.14) vanish due to the orthogonality of $\xi_{\alpha} \in T\mathcal{M}_{T_r^3}$ and $D_{\mu}\chi \in T\mathcal{G}_{T_r^3}$.

The (1,1) tensors $\pi_a = (\varepsilon_{ac}^b)$, a = 1, 2, 3, on T^3 and the (1,1) tensors $\Pi_a = (\delta_{ab}\Pi^b{}_{\beta}^{\alpha})$ on $\mathcal{M}_{T_r^3}$ satisfy the identities

$$\pi_a^3 + \pi_a = 0$$
 and $\Pi_a^3 + \Pi_a = 0$, (5.18)

i.e. they define three so-called f-structures [33] correspondingly on T^3 and on $\mathcal{M}_{T_r^3}$. To clarify their meaning we observe that (5.18) defines orthogonal projectors

$$P_a := -\pi_a^2 \quad \text{and} \quad P_a^{\perp} := \mathbb{1}_3 + \pi_a^2$$
 (5.19)

of rank two and rank one on T^3 and similarly orthogonal projectors

$$\mathcal{P}_a := -\Pi_a^2 \quad \text{and} \quad \mathcal{P}_a^{\perp} := \operatorname{Id} + \Pi_a^2$$
 (5.20)

on $\mathcal{M}_{T_r^3}$, where Id is the identity tensor. The tangent bundle $T(T^3)$ splits into eigenspaces of P_a ,

$$T(T^3) = T(T_a^2 \times S_a^1) = T(T_a^2) \oplus T(S_a^1) = L_a \oplus N_a \quad \text{for} \quad a = 1, 2, 3,$$
 (5.21)

which defines on T^3 two distributions L_a and N_a of rank two and one, respectively, and decomposes the 3-torus in three different ways. Analogously, the projector \mathcal{P}_a yields a splitting

$$T(\mathcal{M}_{T_a^3}) = \mathcal{L}_a \oplus \mathcal{N}_a \tag{5.22}$$

which is in fact induced by the factorization of T_r^3 into a two-dimensional torus and a circle.

Our equations (5.15) look similar to the adiabatic form of the G_2 -instanton equations (for a definition see e.g. [5, 6, 12, 14]) on the 7-manifold

$$X = Y \times Z = T^3 \times Z$$
 with $Z = T^4$, $K3$ or \mathbb{R}^4 . (5.23)

In the adiabatic limit of $\varepsilon \to 0$ with the deformed metric $g_{\varepsilon} = g_Y + \varepsilon^2 g_Z$ the G_2 -instanton equations become

$$\partial_a \phi^{\alpha} + \varepsilon_{ac}^b \left(\partial_b \phi^{\beta} \right) \mathcal{J}^{c \alpha}_{\beta} = 0 . \tag{5.24}$$

This looks similar to (5.15) and features three complex structures $\mathcal{J}^c = (\mathcal{J}^c{}^\alpha{}_\beta)$ (instead of f-structures Π^c) on the hyper-Kähler moduli space \mathcal{M}_Z of framed Yang-Mills instantons on the hyper-Kähler 4-manifold Z. These equations were discussed e.g. in [6, 13] in the form of Fueter equations. In the above case (5.23) they define maps $\phi: T^3 \to \mathcal{M}_Z$ which are sigma-model instantons minimizing the standard sigma-model energy functional.

The moduli space of Yang-Mills instantons in (5.23)-(5.24) has dimensionality divisible by four (a hyper-Kähler manifold), and it allows for three complex structures \mathcal{J}^a . In distinction, the dimension of the moduli space of flat connections on 3-tori T^3 is a multiple of three [34, 35, 36] (r punctures add 3r real parameters to the above moduli), and the three f-structures Π^a in (5.16) play the role of degenerate complex structures on the moduli space $\mathcal{M}_{T_r^3}$. Solutions of (5.15) approximate solutions of the HYM equation on $X = T^3 \times T_r^3$.

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