

EXOTIC CLUSTER STRUCTURES ON SL_n WITH BELAVIN–DRINFELD DATA OF MINIMAL SIZE, I. THE STRUCTURE

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ABSTRACT. Abstract. Using the notion of compatibility between Poisson brackets and cluster structures in the coordinate rings of simple Lie groups, Gekhtman Shapiro and Vainshtein conjectured the existence of a cluster structure for each Belavin–Drinfeld solution of the classical Yang–Baxter equation compatible with the corresponding Poisson–Lie bracket on the simple Lie group. Poisson Lie groups are classified by the Belavin–Drinfeld classification of solutions to the classical Yang–Baxter equation. For any non trivial Belavin–Drinfeld data of minimal size for SL_n , we give an algorithm for constructing an initial seed Σ in $\mathcal{O}(SL_n)$. The cluster structure $\mathcal{C} = \mathcal{C}(\Sigma)$ is then proved to be compatible with the Poisson bracket associated with that Belavin–Drinfeld data, and the seed Σ is locally regular.

This is the first of two papers, and the second one proves the rest of the conjecture: the upper cluster algebra $\overline{\mathcal{A}}_{\mathcal{C}}(\mathcal{C})$ is naturally isomorphic to $\mathcal{O}(SL_n)$, and the correspondence between Belavin–Drinfeld classes and cluster structures is one to one.

1. INTRODUCTION

Upon the introduction of cluster algebras in [7], a natural question has arisen: do (multiple) cluster structures in the coordinate rings of a given algebraic variety V exist? Partial answers were given for Grassmannians $V = Gr_k(n)$ [15] and double Bruhat cells [2]. If $V = \mathcal{G}$ is a simple Lie group, one can extend the cluster structure found in the double Bruhat cell to one in $\mathcal{O}(\mathcal{G})$. The compatibility of cluster structures and Poisson brackets, as characterized in [9] suggested a connection between the two: given a Poisson bracket, does a compatible cluster structure exist? Is there a way to find it?

In the case that $V = \mathcal{G}$ is a simple complex Lie group, R-matrix Poisson brackets on \mathcal{G} are classified by the Belavin–Drinfeld classification of solutions to the classical Yang–Baxter equation [1]. Given a solution of that kind, a Poisson bracket can be defined on \mathcal{G} , making it a Poisson–Lie group.

The Belavin–Drinfeld (BD) classification is based on pairs of isometric subsets of simple roots of the Lie algebra \mathfrak{g} of \mathcal{G} . The trivial case when the subsets are empty corresponds to the standard Poisson bracket on \mathcal{G} . It has been shown in [11] that extending the cluster structure introduced in [2] from the double Bruhat cell to the whole Lie group V yields a cluster structure that is compatible with the standard Poisson bracket. This led to naming this cluster structure “standard”, and trying to find other cluster structures, compatible with brackets associated with non trivial

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BD subsets. The term “exotic” was suggested for these non standard structures [12].

Gekhtman, Shapiro and Vainshtein conjectured the existence of a corresponding cluster structure for every BD class for a given simple Lie group [11, 13]. According to the conjecture, for a given BD class for \mathcal{G} , there exists a cluster structure on \mathcal{G} , with rank determined by the BD data. This cluster structure is compatible with the associated Poisson bracket. The conjecture also states that the structure is regular, and that the upper cluster algebra coincides with the ring of regular functions on \mathcal{G} . The conjecture was proved for the standard case and for $\mathcal{G} = SL_n$ with $n < 5$ in [11]. The Cremmer – Gervais case, which in some sense is the “furthest” from the standard one, was proved in [13]. It was also found to be true for all possible BD classes for SL_5 [5].

This paper proves parts of the conjecture for SL_n when the BD data is of minimal size, i.e., the two subsets contain only one simple root. Starting with two such subsets $\{\alpha\}$ and $\{\beta\}$, Section 3.1 describes an algorithm for construction of a set $\mathcal{B}_{\alpha\beta}$ of functions that will serve as the initial cluster. It is then proved that this set is *log canonical* with respect to the associated Poisson bracket $\{\cdot, \cdot\}_{\alpha\beta}$. Adding a quiver $Q_{\alpha\beta}$ (or an exchange matrix $\tilde{B}_{\alpha\beta}$) defines a cluster structure on SL_n . It is shown in Section 4 that this structure is indeed compatible with the Poisson bracket. Then Section 5 proves that the initial seed is locally regular.

This proves that for minimal size BD data for SL_n there exists a cluster structure which is compatible with the associated Poisson bracket. The companion paper [6] will complete the proof of the conjecture: the bijection between cluster structures and BD classes of this type, the fact that the upper cluster algebra is naturally isomorphic to the ring of regular functions on SL_n , and the description of a global toric action.

2. BACKGROUND AND MAIN RESULTS

2.1. Cluster structures. Let $\{z_1, \dots, z_m\}$ be a set of independent variables, and let S denote the ring of Laurent polynomials generated by z_1, \dots, z_m -

$$S = \mathbb{Z} [z_1^{\pm 1}, \dots, z_m^{\pm 1}].$$

(Here and in what follows $z^{\pm 1}$ stands for z, z^{-1}). The *ambient field* \mathcal{F} is the field of rational functions in n independent variables (distinct from z_1, \dots, z_m), with coefficients in the field of fractions of S .

A *seed* (of geometric type) is a pair (\mathbf{x}, \tilde{B}) , where $\mathbf{x} = (x_1, \dots, x_n)$ is a transcendence basis of \mathcal{F} over the field of fractions of S , and \tilde{B} is an $n \times (n + m)$ integer matrix whose principal part B (that is, the $n \times n$ matrix formed by columns $1 \dots n$) is skew-symmetric. The set \mathbf{x} is called a *cluster*, and its elements (x_1, \dots, x_n) are called *cluster variables*. Set $x_{n+i} = z_i$ for $i \in [1, m]$ (where we use the notation $[a, b]$ for the set of integers $\{a, a + 1, \dots, b\}$). Sometimes we write just $[m]$ for the set $[1, m]$. The elements x_{n+1}, \dots, x_{n+m} are called *stable variables* (or *frozen variables*). The set $\tilde{\mathbf{x}} = (x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m})$ is called an *extended cluster*. The square matrix B is called the *exchange matrix*, and \tilde{B} is called the *extended exchange matrix*. We sometimes denote the entries of \tilde{B} by b_{ij} , or say that \tilde{B} is skew-symmetric when the matrix B has this property.

Let $\Sigma = (\tilde{\mathbf{x}}, \tilde{B})$ be a seed. The adjacent cluster in direction $k \in [n]$ is $\tilde{\mathbf{x}}_k = (\tilde{\mathbf{x}} \setminus x_k \cup \{x'_k\})$, where x'_k is defined by the *exchange relation*

$$(2.1) \quad x_k \cdot x'_k = \prod_{b_{kj} > 0} x_j^{b_{kj}} + \prod_{b_{kj} < 0} x_j^{-b_{kj}}$$

A *matrix mutation* $\mu_k(\tilde{B})$ of \tilde{B} in direction k is defined by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + \frac{1}{2} (|b_{ik}b_{kj} + b_{ik}b_{kj}|) & \text{otherwise.} \end{cases}$$

Seed mutation in direction k is then defined $\mu_k(\Sigma) = (\tilde{\mathbf{x}}_k, \mu_k(\tilde{B}))$.

Two seeds are said to be mutation equivalent if they can be connected by a sequence of seed mutations.

Given a seed $\Sigma = (\mathbf{x}, \tilde{B})$, the *cluster structure* $\mathcal{C}(\Sigma)$ (sometimes denoted $\mathcal{C}(\tilde{B})$, if \mathbf{x} is understood from the context) is the set of all seeds that are mutation equivalent to Σ . The number n of rows in the matrix \tilde{B} is called the *rank* of \mathcal{C} .

Let Σ be a seed as above, and $\mathbb{A} = \mathbb{Z}[x_{n+1}, \dots, x_{n+m}]$. The *cluster algebra* $\mathcal{A} = \mathcal{A}(\mathcal{C}) = \mathcal{A}(\tilde{B})$ associated with the seed Σ is the \mathbb{A} -subalgebra of \mathcal{F} generated by all cluster variables in all seeds in $\mathcal{C}(\tilde{B})$. The *upper cluster algebra* $\overline{\mathcal{A}} = \overline{\mathcal{A}}(\mathcal{C}) = \overline{\mathcal{A}}(\tilde{B})$ is the intersection of the rings of Laurent polynomials over \mathbb{A} in cluster variables taken over all seeds in $\mathcal{C}(\tilde{B})$. The famous *Laurent phenomenon* [8] claims the inclusion $\mathcal{A}(\mathcal{C}) \subseteq \overline{\mathcal{A}}(\mathcal{C})$.

It is sometimes convenient to describe a cluster structure $\mathcal{C}(\tilde{B})$ in terms of the *quiver* $Q(\tilde{B})$: it is a directed graph with $n+m$ nodes labeled x_1, \dots, x_{n+m} (or just $1, \dots, n+m$), and an arrow pointing from x_i to x_j with weight b_{ij} if $b_{ij} > 0$.

Let V be a quasi-affine variety over \mathbb{C} , $\mathbb{C}(V)$ be the field of rational functions on V , and $\mathcal{O}(V)$ be the ring of regular functions on V . Let \mathcal{C} be a cluster structure in \mathcal{F} as above, and assume that $\{f_1, \dots, f_{n+m}\}$ is a transcendence basis of $\mathbb{C}(V)$. Then the map $\varphi : x_i \rightarrow f_i$, $1 \leq i \leq n+m$, can be extended to a field isomorphism $\varphi : \mathcal{F}_{\mathbb{C}} \rightarrow \mathbb{C}(V)$, with $\mathcal{F}_{\mathbb{C}} = \mathcal{F} \otimes \mathbb{C}$ obtained from \mathcal{F} by extension of scalars. The pair (\mathcal{C}, φ) is then called a cluster structure in $\mathbb{C}(V)$ (or just a cluster structure on V), and the set $\{f_1, \dots, f_{n+m}\}$ is called an extended cluster in (\mathcal{C}, φ) . Sometimes we omit direct indication of φ and just say that \mathcal{C} is a cluster structure on V . A cluster structure (\mathcal{C}, φ) is called *regular* if $\varphi(x)$ is a regular function for any cluster variable x , and a seed Σ is called *locally regular* if all the cluster variables in Σ and in all the adjacent seeds are regular functions. The two algebras defined above have their counterparts in $\mathcal{F}_{\mathbb{C}}$ obtained by extension of scalars; they are denoted $\mathcal{A}_{\mathbb{C}}$ and $\overline{\mathcal{A}}_{\mathbb{C}}$. If, moreover, the field isomorphism φ can be restricted to an isomorphism of $\mathcal{A}_{\mathbb{C}}$ (or $\overline{\mathcal{A}}_{\mathbb{C}}$) and $\mathcal{O}(V)$, we say that $\mathcal{A}_{\mathbb{C}}$ (or $\overline{\mathcal{A}}_{\mathbb{C}}$) is *naturally isomorphic* to $\mathcal{O}(V)$.

Let $\{\cdot, \cdot\}$ be a Poisson bracket on the ambient field \mathcal{F} . Two elements $f_1, f_2 \in \mathcal{F}$ are *log canonical* if there exists a rational number ω_{f_1, f_2} such that $\{f_1, f_2\} = \omega_{f_1, f_2} f_1 f_2$. A set $F \subseteq \mathcal{F}$ is called a log canonical set if every pair $f_1, f_2 \in F$ is log canonical.

A cluster structure \mathcal{C} in \mathcal{F} is said to be compatible with the Poisson bracket $\{\cdot, \cdot\}$ if every cluster is a log canonical set with respect to $\{\cdot, \cdot\}$. In other words, for every cluster \mathbf{x} and every two cluster variables $x_i, x_j \in \tilde{\mathbf{x}}$ there exists ω_{ij} s.t.

$$(2.2) \quad \{x_i, x_j\} = \omega_{ij} x_i x_j$$

The skew symmetric matrix $\Omega^{\tilde{\mathbf{x}}} = (\omega_{ij})$ is called the *coefficient matrix* of $\{\cdot, \cdot\}$ (in the basis $\tilde{\mathbf{x}}$).

If $\mathcal{C}(\tilde{B})$ is a cluster structure of maximal rank (i.e., $\text{rank } \tilde{B} = n$), one can give a complete characterization of all Poisson brackets compatible $\mathcal{C}(\tilde{B})$ (see [9], and also [10, Ch. 4]). In particular, an immediate corollary of Theorem 1.4 in [9] is:

Proposition 2.1. *If $\text{rank } \tilde{B} = n$ then a Poisson bracket is compatible with $\mathcal{C}(\tilde{B})$ if and only if its coefficient matrix $\Omega^{\tilde{\mathbf{x}}}$ satisfies $\tilde{B}\Omega^{\tilde{\mathbf{x}}} = [D \ 0]$, where D is a diagonal matrix.*

2.2. Poisson–Lie groups. A Lie group \mathcal{G} with a Poisson bracket $\{\cdot, \cdot\}$ is called a *Poisson–Lie group* if the multiplication map $\mu : \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$, $\mu : (x, y) \mapsto xy$ is Poisson. That is, \mathcal{G} with a Poisson bracket $\{\cdot, \cdot\}$ is a Poisson–Lie group if

$$\{f_1, f_2\}(xy) = \{\rho_y f_1, \rho_y f_2\}(x) + \{\lambda_x f_1, \lambda_x f_2\}(y),$$

where ρ_y and λ_x are, respectively, right and left translation operators on \mathcal{G} .

Given a Lie group \mathcal{G} with a Lie algebra \mathfrak{g} , let $(\ , \)$ be a nondegenerate bilinear form on \mathfrak{g} , and $\mathfrak{t} \in \mathfrak{g} \otimes \mathfrak{g}$ be the corresponding Casimir element. For an element $r = \sum_i a_i \otimes b_i \in \mathfrak{g} \otimes \mathfrak{g}$ denote

$$[[r, r]] = \sum_{i,j} [a_i, a_j] \otimes b_i \otimes b_j + \sum_{i,j} a_i \otimes [b_i, a_j] \otimes b_j + \sum_{i,j} a_i \otimes a_j \otimes [b_i, b_j]$$

and $r^{21} = \sum_i b_i \otimes a_i$.

The *Classical Yang–Baxter equation (CYBE)* is

$$(2.3) \quad [[r, r]] = 0,$$

an element $r \in \mathfrak{g} \otimes \mathfrak{g}$ that satisfies (2.3) together with the condition

$$(2.4) \quad r + r^{21} = \mathfrak{t}$$

is called a classical R-matrix.

A classical R-matrix r induces a Poisson–Lie structure on \mathcal{G} : choose a basis $\{I_\alpha\}$ in \mathfrak{g} , and denote by ∂_α (resp., ∂'_α) the left (resp., right) invariant vector field whose value at the unit element is I_α . Let $r = \sum_{\alpha, \beta} r_{\alpha, \beta} I_\alpha \otimes I_\beta$, then

$$(2.5) \quad \{f_1, f_2\}_r = \sum_{\alpha, \beta} r_{\alpha, \beta} (\partial_\alpha f_1 \partial_\beta f_2 - \partial'_\alpha f_1 \partial'_\beta f_2)$$

defines a Poisson bracket on \mathcal{G} . This is called the *Sklyanin bracket* corresponding to r .

In [1] Belavin and Drinfeld give a classification of classical R-matrices for simple complex Lie groups: let \mathfrak{g} be a simple complex Lie algebra with a fixed nondegenerate invariant symmetric bilinear form $(\ , \)$. Fix a Cartan subalgebra \mathfrak{h} , a root system Φ of \mathfrak{g} , and a set of positive roots Φ^+ . Let $\Delta \subseteq \Phi^+$ be a set of positive simple roots.

A Belavin–Drinfeld (BD) triple is two subsets $\Gamma_1, \Gamma_2 \subset \Delta$ and an isometry $\gamma : \Gamma_1 \rightarrow \Gamma_2$ with the following property: for every $\alpha \in \Gamma_1$ there exists $m \in \mathbb{N}$ such that $\gamma^j(\alpha) \in \Gamma_1$ for $j = 0, \dots, m-1$, but $\gamma^m(\alpha) \notin \Gamma_1$. The isometry γ extends in a natural way to a map between root systems generated by Γ_1, Γ_2 . This allows one to define a partial ordering on the root system: $\alpha \prec \beta$ if $\beta = \gamma^j(\alpha)$ for some $j \in \mathbb{N}$.

Select now root vectors $E_\alpha \in \mathfrak{g}$ that satisfy $(E_\alpha, E_{-\alpha}) = 1$. According to the Belavin–Drinfeld classification, the following is true (see, e.g., [4, Ch. 3]).

Proposition 2.2. (i) Every classical R-matrix is equivalent (up to an action of $\sigma \otimes \sigma$ where σ is an automorphism of \mathfrak{g}) to

$$(2.6) \quad r = r_0 + \sum_{\alpha \in \Phi^+} E_{-\alpha} \otimes E_{\alpha} + \sum_{\substack{\alpha < \beta \\ \alpha, \beta \in \Phi^+}} E_{-\alpha} \wedge E_{\beta}$$

(ii) $r_0 \in \mathfrak{h} \otimes \mathfrak{h}$ in (2.6) satisfies

$$(2.7) \quad (\gamma(\alpha) \otimes \mathbb{1}) r_0 + (\mathbb{1} \otimes \alpha) r_0 = 0$$

for any $\alpha \in \Gamma_1$, and

$$(2.8) \quad r_0 + r_0^{21} = \mathfrak{t}_0,$$

where \mathfrak{t}_0 is the $\mathfrak{h} \otimes \mathfrak{h}$ component of \mathfrak{t} .

(iii) Solutions r_0 to (2.7),(2.8) form a linear space of dimension $k_T = |\Delta \setminus \Gamma_1|$.

Two classical R-matrices of the form (2.6) that are associated with the same BD triple are said to belong to the same Belavin–Drinfeld class. The corresponding bracket defined in (2.5) by an R-matrix r associated with a triple T will be denoted by $\{ , \}_T$.

Given a BD triple T for \mathcal{G} , write

$$\mathfrak{h}_T = \{h \in \mathfrak{h} : \alpha(h) = \beta(h) \text{ if } \alpha < \beta\},$$

and define the torus $\mathcal{H}_T = \exp \mathfrak{h}_T \subset \mathcal{G}$.

2.3. Main results and outline. The following conjecture was given by Gekhtman, Shapiro and Vainshtein in [11]:

Conjecture 2.3. Let \mathcal{G} be a simple complex Lie group. For any Belavin–Drinfeld triple $T = (\Gamma_1, \Gamma_2, \gamma)$ there exists a cluster structure \mathcal{C}_T on \mathcal{G} such that

- (1) the number of stable variables is $2k_T$, and the corresponding extended exchange matrix has a full rank.
- (2) \mathcal{C}_T is regular.
- (3) the corresponding upper cluster algebra $\overline{\mathcal{A}}_{\mathbb{C}}(\mathcal{C}_T)$ is naturally isomorphic to $\mathcal{O}(\mathcal{G})$;
- (4) the global toric action of $(\mathbb{C}^*)^{2k_T}$ on $\mathbb{C}(\mathcal{G})$ is generated by the action of $\mathcal{H}_T \otimes \mathcal{H}_T$ on \mathcal{G} given by $(H_1, H_2)(X) = H_1 X H_2$;
- (5) for any solution of CYBE that belongs to the Belavin–Drinfeld class specified by T , the corresponding Sklyanin bracket is compatible with \mathcal{C}_T ;
- (6) a Poisson–Lie bracket on \mathcal{G} is compatible with \mathcal{C}_T only if it is a scalar multiple of the Sklyanin bracket associated with a solution of CYBE that belongs to the Belavin–Drinfeld class specified by T .

The main result of this paper is the following theorem:

Theorem 2.4. For any Belavin–Drinfeld triple $T = (\{\alpha\}, \{\beta\}, \gamma : \alpha \mapsto \beta)$, there exists a cluster structure on SL_n with a locally regular initial seed and with $2k_T$ stable variables, that is compatible with the Sklyanin bracket associated with T .

In other words, Theorem 2.4 states that part 1 of Conjecture 2.3 is true for SL_n for BD triple with $|\Gamma_1| = 1$.

For a given n and a BD triple $T_{\alpha\beta}$, a set $\mathcal{B}_{\alpha\beta}$ of functions in $\mathcal{O}(SL_n)$ is constructed in Section 3.1. The rest of Section 3 is dedicated to proving that this set is

log canonical with respect to the Sklyanin bracket $\{\cdot, \cdot\}_{\alpha\beta}$ associated with $T_{\alpha\beta}$. After declaring some of these functions as frozen variables and introducing the quiver $Q_{\alpha\beta}^n$ in Section 4.2, the initial seed $(\mathcal{B}_{\alpha\beta}, Q_{\alpha\beta}^n)$ determines a cluster structure $\mathcal{C}_{\alpha\beta}$. Theorem 4.2 states that $\mathcal{C}_{\alpha\beta}$ is compatible with the bracket $\{\cdot, \cdot\}_{\alpha\beta}$, and Section 5 proves that the initial seed is locally regular. Last, Section 6 has some technical computations and results that were used through the paper.

Parts and 2 – 6 of the conjecture will be proved in the companion paper [6].

A Poisson–Lie bracket on SL_n can be extended to one on GL_n , with the determinant being a Casimir function. From here on we discuss GL_n , and any statement can be restricted to SL_n by removing the determinant function.

3. A LOG CANONICAL BASIS

This section describes a log canonical set of function, that will serve as an initial cluster for the structure $\mathcal{C}_{\alpha\beta}$. After constructing this set in Section 3.1, we show it is log canonical with respect to the bracket $\{\cdot, \cdot\}_{\alpha\beta}$ in Section 3.2, using results from Section 6.

Before moving on, note the following two isomorphisms of the BD data for SL_n : the first reverses the direction of γ and transposes Γ_1 and Γ_2 , while the second one takes each root α_j to $\alpha_{\omega_0(j)}$, where ω_0 is the longest element in the Weyl group (which in SL_n is naturally identified with the symmetric group S_{n-1}). These two isomorphisms correspond to the automorphisms of SL_n given by $X \mapsto -X^t$ and $X \mapsto \omega_0 X \omega_0$, respectively. Since R-matrices are considered up to an action of $\sigma \otimes \sigma$, from here on we do not distinguish between BD triples obtained one from the other via these isomorphisms. We will also assume that in the map $\gamma : \alpha_i \mapsto \alpha_j$ we always have $i < j$.

Slightly abusing the notation, we sometime refer to a root $\alpha_i \in \Delta$ just as i , and write $\gamma : i \mapsto j$ instead of $\gamma : \alpha_i \mapsto \alpha_j$. For shorter notation, denote the BD triple $(\{\alpha\}, \{\beta\}, \gamma : \alpha \mapsto \beta)$ by $T_{\alpha\beta}$, and naturally the corresponding Sklyanin bracket will be $\{\cdot, \cdot\}_{\alpha\beta}$.

3.1. Constructing a log canonical basis . For a triple $T_{\alpha\beta}$ we will construct a set of matrices \mathcal{M} such that the set of all their trailing principal minors is log canonical with respect to $\{\cdot, \cdot\}_{\alpha\beta}$. A trailing principal minor of an $n \times n$ matrix M is a minor of M of the form $\det M_{[i,n]}^{[j,n]}$.

Following [14], recall the construction of *the Drinfeld double* of a Lie algebra \mathfrak{g} with the Killing form $\langle \cdot, \cdot \rangle$: define $D(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}$, with an invariant nondegenerate bilinear form

$$\langle \langle (\xi, \eta), (\xi', \eta') \rangle \rangle = \langle \xi, \xi' \rangle - \langle \eta, \eta' \rangle.$$

Define subalgebras \mathfrak{d}_{\pm} of $D(\mathfrak{g})$ by

$$(3.1) \quad \mathfrak{d}_+ = \{(\xi, \xi) : \xi \in \mathfrak{g}\}, \quad \mathfrak{d}_- = \{(R_+(\xi), R_-(\xi)) : \xi \in \mathfrak{g}\},$$

where $R_{\pm} \in \text{End } \mathfrak{g}$ are defined for any R-matrix r by

$$(3.2) \quad \langle R_+(\eta), \zeta \rangle = -\langle R_-(\zeta), \eta \rangle = \langle r, \eta \otimes \zeta \rangle_{\otimes},$$

and $\langle \cdot, \cdot \rangle_{\otimes}$ is the corresponding Killing form on the tensor square of \mathfrak{g} .

For a matrix X let $M_{ij}(X)$ be the maximal contiguous submatrix of X with x_{ij} at the upper left hand corner. That is,

$$M_{ij}(X) = \begin{cases} \begin{bmatrix} x_{ij} & \cdots & x_{in} \\ \vdots & & \vdots \\ x_{n-j+i,j} & \cdots & x_{n-j+i,n} \end{bmatrix} & \text{if } j > i \\ \begin{bmatrix} x_{ij} & \cdots & x_{i,n-i+j} \\ \vdots & & \vdots \\ x_{nj} & \cdots & x_{n,n-i+j} \end{bmatrix} & \text{otherwise.} \end{cases}$$

Slightly abusing the notation, define $M_{ij}(X, Y)$ on the double $D(\mathfrak{gl}_n)$ by

$$M_{ij}(X, Y) = \begin{cases} M_{ij}(X) & \text{if } i \geq j \\ M_{ij}(Y) & \text{otherwise,} \end{cases}$$

and we can then write $M_{ij}(X) = M_{ij}(X, X)$. Let X_R^C denote the submatrix of X with rows in the set R and columns in C (with $R, C \subseteq [n]$). Then define two special families of matrices: for $1 \leq j \leq \alpha$ and $i = n + j - \alpha$ set

$$\tilde{M}_{ij}(X, Y) = \begin{bmatrix} X_{[i,n]}^{[j,\alpha+1]} & 0_{(n-i+1) \times \mu} \\ 0_{\mu \times (n-i+1)} & Y_{[1,\mu]}^{[\beta,n]} \end{bmatrix}$$

with $\mu = n - \beta$, and for $1 \leq i \leq \beta$ and $j = n + i - \beta$, set

$$\tilde{M}_{ij}(X, Y) = \begin{bmatrix} Y_{[i,\beta+1]}^{[j,n]} & 0_{(n-j+1) \times \mu} \\ 0_{\mu \times (n-j+1)} & X_{[\alpha,n]}^{[1,\mu]} \end{bmatrix},$$

and here $\mu = n - \alpha$. Note that these matrices are not block diagonal: in the first case the number of columns in each of the two blocks is greater than the number of rows by one, while in the second case the number of rows in each block is greater than the number of columns by one. As above, we set $\tilde{M}_{ij}(X) = \tilde{M}_{ij}(X, X)$.

When n is even there are two special cases - $\alpha = \frac{n}{2}$ or $\beta = \frac{n}{2}$. We discuss here the case $\beta = \frac{n}{2}$, as the case of $\alpha = \frac{n}{2}$ is symmetric (and isomorphic under $\alpha \longleftrightarrow \beta$): for $i = j + \alpha$ the matrix $\tilde{M}_{ij}(X, Y)$ now involves three blocks (submatrices of X and Y), and it has the form

$$\tilde{M}_{ij}(X, Y) = \begin{bmatrix} x_{ij} & \cdots & \cdots & x_{i,\alpha+1} & 0 & \cdots & \cdots & 0 \\ & \ddots & & \vdots & \vdots & & & \vdots \\ x_{n1} & \cdots & x_{n\alpha} & x_{n,\alpha+1} & 0 & \cdots & & \vdots \\ 0 & \cdots & y_{1\beta} & y_{1,\beta+1} & \cdots & y_{1n} & 0 & \cdots & 0 \\ & & \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ & & \vdots & y_{\beta j} & \cdots & y_{\beta n} & x_{\alpha 1} & \cdots & x_{\alpha \mu} \\ & & y_{\beta+1} & y_{\beta+1,j} & \cdots & y_{\beta+1,n} & x_{\alpha+1,1} & & \vdots \\ & & & 0 & \cdots & 0 & \vdots & \ddots & \vdots \\ & & & \vdots & & \vdots & x_{n1} & \cdots & x_{n\mu} \end{bmatrix}.$$

Now define

$$(3.3) \quad f_{ij}(X) = \det M_{ij}(X).$$

The set $\mathcal{B}_{std} = \{f_{ij}(X) | i, j \in [n]\}$ of determinants of all matrices $M_{ij}(X)$ forms a log canonical set with respect to the standard bracket [10, Ch. 4.3]. For the $\alpha \mapsto \beta$ case, take this set and for all $i = n + j - \alpha$ and $j = n + i - \beta$ replace $M_{ij}(X)$ with $\tilde{M}_{ij}(X)$. This assures that for a fixed pair $(i, j) \in [n] \times [n]$ there is still a unique matrix in the set. Denote this matrix (either $M_{ij}(X)$ or $\tilde{M}_{ij}(X)$) by \overline{M}_{ij} , and set

$$\varphi_{ij} = \det \overline{M}_{ij}.$$

Our set of log canonical functions (with respect to the bracket $\{\cdot, \cdot\}_{\alpha\beta}$) – that will later serve as an initial cluster – is the set $\mathcal{B}_{\alpha\beta} = \{\varphi_{ij} | i, j \in [n]\}$. Further on we will also need the set \mathcal{B}^D of functions on $D(\mathfrak{gl}_n)$, defined by $\mathcal{B}^D = \{\varphi_{ij}^D(X, Y) = \det M_{ij}(X, Y) | i, j \in [n]\}$.

Some matrices in the above construction contain others: for example, $M_{1j}(X)$ contains all matrices M_{ik} with $k = j + i - 1$. Therefore, we can see the set \mathcal{B} as the set of all trailing principal minors of matrices $M_{1j}(X)$ and $M_{i1}(X)$, excluding $M_{\alpha+1,1}(X)$ and $M_{1,\beta+1}(X)$. So the set \mathcal{B}_{std} can be viewed as all trailing principal minors of the matrices M_{1j} and M_{i1} with $i, j \in [n]$. We will denote this set of matrices by \mathcal{M}_{std} . Equivalently, define the set

$$(3.4) \quad \mathcal{M}_{\alpha\beta} = \{\overline{M}_{1j}, \overline{M}_{i1} | i, j \in [n]\} \setminus \{\overline{M}_{1,\beta+1}, \overline{M}_{\alpha+1,1}\},$$

and it is not hard to see that $\mathcal{B}_{\alpha\beta} = \{\varphi_{ij} | i, j \in [n]\}$ is the set of trailing principal minors of all matrices in $\mathcal{M}_{\alpha\beta}$.

Clearly, $|\mathcal{B}_{\alpha\beta}| = n^2$, since the map $(i, j) \mapsto \varphi_{ij}$ is a bijection between $[n] \times [n]$ and $\mathcal{B}_{\alpha\beta}$. In Section 3.2 we show that $\mathcal{B}_{\alpha\beta}$ is log canonical with respect to the bracket $\{\cdot, \cdot\}_{\alpha\beta}$.

Remark 3.1. Further details about the construction of a log canonical set from determinants of matrices as above can be found in [5]. The special case of $n = 5$ is addressed there, with any BD data, but it can be easily generalized to any n (with the restriction $|\Gamma_1| = |\Gamma_2| = 1$).

3.2. The log canonical set $\mathcal{B}_{\alpha\beta}$. Comparing the bracket $\{\cdot, \cdot\}_{\alpha\beta}$ with the standard one will allow us to compute $\{f, g\}_{\alpha\beta}$ for every pair of functions $f, g \in \mathcal{B}_{\alpha\beta}$. We will use results from Section 6.

Since some of the proofs involve the standard Poisson bracket and cluster structure on SL_n , we start with a reminder: there are multiple Poisson brackets on SL_n that correspond to the trivial BD data $\Gamma_1 = \Gamma_2 = \emptyset$, since r_0 is not uniquely determined. For a pair α, β we will use r_0 as defined in (6.7), and call the associated Poisson bracket *the* standard Poisson bracket on SL_n . The corresponding cluster structure on SL_n that will be called the standard one and denoted \mathcal{C}_{std} is described in [2] and [11]. Note that this cluster structure is independent on the choice of r_0 and the Poisson bracket. The initial seed of this cluster structure is the set $\{f_{ij}\}_{i,j=1}^n$ defined by

$$(3.5) \quad f_{ij} = \begin{cases} \det X_{\begin{smallmatrix} [j,n] \\ [i, n+i-j] \end{smallmatrix}} & \text{if } j \geq i, \\ \det X_{\begin{smallmatrix} [j, n-i+j] \\ [i, n] \end{smallmatrix}} & \text{otherwise.} \end{cases}$$

looks as follows: set $\mu(i, j) = \min(n, n + i - j)$ and write

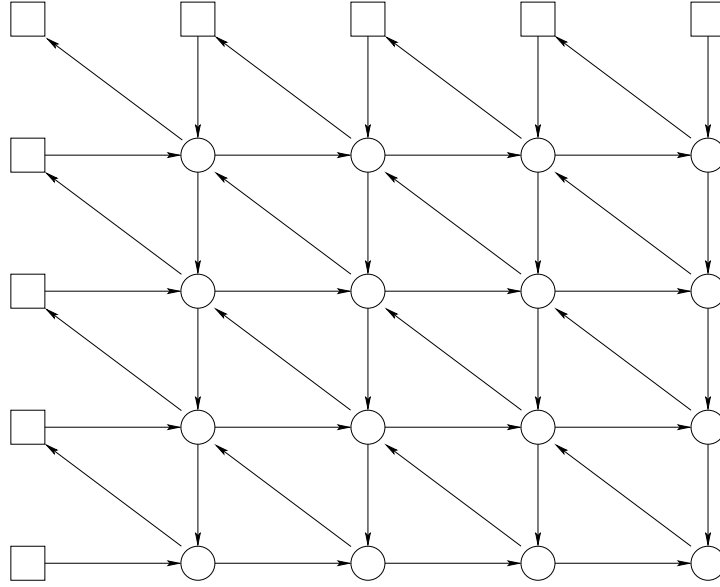


FIGURE 3.1. The standard quiver for GL_5

This definition coincides with (3.3), and for all $\varphi_{ij} \in \mathcal{B}_{\alpha\beta} \cap \mathcal{B}_{std}$ we have $\varphi_{ij} = f_{ij}$. The function $f_{11} = \det X$ is constant on SL_n . Take the set

$$\{f_{ij}\}_{i,j=1}^n \setminus \{f_{11}\}$$

as the set of cluster variables. Set the variables f_{i1} and f_{1j} to be frozen, so there are $n^2 - 1$ cluster variables with $2(n - 1)$ of them frozen. Let Q_{std}^n be the quiver of \mathcal{C}_{std}^n (see [2, 11]). The vertices of Q_{std}^n are placed on an $n \times n$ grid with rows numbered from top to bottom and columns numbered from left to right. The cluster variable f_{ij} corresponds to the node (i, j) (that is, the node on the i -th row and the j -th column). There are arrows from each node (i, j) to $(i, j + 1)$ (as long as $j \neq n$), from (i, j) to $(i + 1, j)$ (when $i \neq n$) and from $(i + 1, j + 1)$ to (i, j) . Arrows connecting two frozen variables can be ignored. As explained at the end of Section 2.3, we can extend from SL_n to GL_n by adding the function $f_{11} = \det X$. Figure 3.1 shows the initial quiver of the standard cluster structure on GL_5 (remove the upper left node with the arrow incident to it to get the initial standard quiver for SL_5). Mutable variables are represented by circles, while frozen ones are represented by squares. Then $\Sigma_{std} = (\mathcal{B}_{std}, Q_{std}^n)$ is an initial seed for the standard cluster structure on SL_n [2, 11].

We will use the following notations:

$$(3.6) \quad f^{i \leftarrow j} = (\nabla f \cdot X)_{ij} = \sum_{k=1}^n \frac{\partial f}{\partial x_{ki}} x_{kj}$$

$$(3.7) \quad f_{j \leftarrow i} = (X \cdot \nabla f)_{ij} = \sum_{k=1}^n \frac{\partial f}{\partial x_{jk}} x_{ik}.$$

Note that if f is a determinant of a submatrix S of a matrix X , then $f^{i \leftarrow j}$ (or $f_{i \leftarrow j}$) is the same determinant, with column (or row) i replaced by column (row) j . If S

does not contain column (row) i , then $f^{i\leftarrow j} = 0$ ($f_{i\leftarrow j} = 0$). If $f = f_{ij} = \det S$, where $S = X_{[i,k]}^{[j,\ell]}$ is a dense submatrix of X , we write

$$\begin{aligned} f^{\rightarrow} &= f^{\ell\leftarrow\ell+1} \\ f^{\leftarrow} &= f^{j\leftarrow j-1} \\ f^{\uparrow} &= f_{i\leftarrow i-1} \\ f^{\downarrow} &= f_{k\leftarrow k+1}. \end{aligned}$$

For a pair (f, g) of log canonical functions, denote by $\omega_{f,g}$ the Poisson coefficient

$$(3.8) \quad \omega_{f,g} = \frac{\{f, g\}_{std}}{fg}.$$

Our first result states that the functions we defined in 3.1 are indeed log canonical:

Theorem 3.2. *The set $\mathcal{B}_{\alpha\beta}$ is log canonical with respect to the bracket $\{\cdot, \cdot\}_{\alpha\beta}$.*

Proof. Compute the bracket $\{f, g\}_{\alpha\beta}$ for all $f, g \in \mathcal{B}_{\alpha\beta}$: first, by Corollary 6.3, if $f, g \in \mathcal{B}_{\alpha\beta} \cap \mathcal{B}_{std}$ then $\{f, g\}_{\alpha\beta} = \{f, g\}_{std}$, and therefore f, g are log canonical with respect to $\{\cdot, \cdot\}_{\alpha\beta}$. Now turn to $\{f, g\}_{\alpha\beta}$ where f or g are non standard basis functions. These are functions of the form φ_{ij} with $i = n + j - \alpha$ or $j = n + i - \beta$, so for $k \in [\alpha]$ and $m \in [\beta]$ define

$$\begin{aligned} \theta_k &= \varphi_{n+k-\alpha, k} = f_{n+k-\alpha, k} \cdot f_{1, \beta+1} - f_{n+k-\alpha, k}^{\rightarrow} \cdot f_{1, \beta+1}^{\leftarrow} \\ \psi_m &= \varphi_{m, n+m-\beta} = f_{m, n+m-\beta} \cdot f_{\alpha+1, 1} - f_{m, n+m-\beta}^{\downarrow} \cdot f_{\alpha+1, 1}^{\uparrow}, \end{aligned}$$

and so f or g (or both) are either θ_k or ψ_m .

Take the bracket $\{\theta_k, g\}_{\alpha\beta}$ with $g \in \mathcal{B}_{\alpha\beta} \cap \mathcal{B}_{std}$ and look at three cases:

1. If $g \neq f_{m, \beta+1}$ for some $m \in [n]$, and $g \neq f_{n+m-\alpha, m}$ for some $m > k$, we can write

$$(3.9) \quad \begin{aligned} \{\theta_k, g\}_{\alpha\beta} &= \{f_{n+k-\alpha, k} \cdot f_{1, \beta+1}, g\}_{\alpha\beta} - \{f_{n+k-\alpha, k}^{\rightarrow} \cdot f_{1, \beta+1}^{\leftarrow}, g\}_{\alpha\beta} \\ &= \{f_{n+k-\alpha, k} \cdot f_{1, \beta+1}, g\}_{std} - \{f_{n+k-\alpha, k}^{\rightarrow} \cdot f_{1, \beta+1}^{\leftarrow}, g\}_{std}. \end{aligned}$$

The last equality holds since by Lemma 6.2 the difference between the $\alpha\beta$ bracket and the standard bracket is

$$\begin{aligned} \{f, g\}_{\alpha\beta} - \{f, g\}_{std} &= f^{\alpha\leftarrow\alpha+1} g^{\beta+1\leftarrow\beta} - f^{\beta+1\leftarrow\beta} g^{\alpha\leftarrow\alpha+1} \\ &\quad + f_{\beta\leftarrow\beta+1} g_{\alpha+1\leftarrow\alpha} - f_{\alpha+1\leftarrow\alpha} g_{\beta\leftarrow\beta+1}. \end{aligned}$$

Since $g \in \mathcal{B}_{\alpha\beta} \cap \mathcal{B}_{std}$ and $g \neq f_{m, \beta+1}$ we get $g^{\beta+1\leftarrow\beta} = 0$, and also $g^{\alpha\leftarrow\alpha+1} = 0$, because $g \neq f_{n+m-\alpha, m}$. Similarly, $f_{\beta\leftarrow\beta+1} = 0$ for the functions $f = f_{n+k-\alpha, k}$ and $f = f_{n+k-\alpha, k}^{\rightarrow}$, and of course $g_{\beta\leftarrow\beta+1} = 0$ (unless g is of the form $g = \det X_{[1, \beta]}^{[j, \mu]}$ but this function can not in $\mathcal{B}_{\alpha\beta}$). Therefore, the $\alpha\beta$ bracket and the standard bracket are equal in this case.

According to Lemma 6.4, the functions $f_{n+k-\alpha, k}^{\rightarrow}$ and $f_{1, \beta+1}^{\leftarrow}$ are both log canonical with g (w.r.t. the standard bracket) with Poisson coefficients

$$\begin{aligned} \omega_{f_{n+k-\alpha, k}^{\rightarrow}, g} &= \omega_{f_{n+k-\alpha, k}, g} + \omega_{x_{n, \alpha+1}, g} - \omega_{x_{n, \alpha}, g}, \\ \omega_{f_{1, \beta+1}^{\leftarrow}, g} &= \omega_{f_{1, \beta+1}, g} + \omega_{x_{n, \beta}, g} - \omega_{x_{n, \beta+1}, g}, \end{aligned}$$

so (3.9) turns to

$$\begin{aligned} \{\theta_k, g\}_{\alpha\beta} &= \omega_1 f_{n+k-\alpha, k} \cdot f_{1, \beta+1} \cdot g \\ &\quad - (\omega_1 - s\omega_{\alpha\beta}(g)) f_{n+k-\alpha, k}^{\leftarrow} \cdot f_{1, \beta+1}^{\rightarrow} \cdot g, \end{aligned}$$

with

$$\omega_1 = \omega_{f_{n+k-\alpha, k}, g} + \omega_{f_{1, \beta+1}, g}$$

and

$$s\omega_{\alpha\beta}(g) = \omega_{f_{n\alpha}, g} - \omega_{f_{n, \alpha+1}, g} - \omega_{f_{n\beta}, g} + \omega_{f_{n, \beta+1}, g},$$

as defined in (6.31).

Now, using Lemma 6.10 we get

$$\{\theta_k, g\}_{\alpha\beta} = (\omega_{f_{n+k-\alpha, k}, g} + \omega_{f_{1, \beta+1}, g}) \theta_k \cdot g.$$

2. If $g = f_{m, \beta+1}$ for some $m \in [2, n]$, we write $q_k = f_{n+k-\alpha, k}$, and then

$$\begin{aligned} \{\theta_k, g\}_{\alpha\beta} &= \{q_k \cdot f_{1, \beta+1}, g\}_{\alpha\beta} - \{q_k^{\rightarrow} \cdot f_{1, \beta+1}^{\leftarrow}, g\}_{\alpha\beta} \\ &= q_k \{f_{1, \beta+1}, g\}_{\alpha\beta} + f_{1, \beta+1} \{q_k, g\}_{\alpha\beta} \\ &\quad - q_k^{\rightarrow} \{f_{1, \beta+1}^{\leftarrow}, g\}_{\alpha\beta} - f_{1, \beta+1}^{\leftarrow} \{q_k^{\rightarrow}, g\}_{\alpha\beta} \\ &= q_k \{f_{1, \beta+1}, g\}_{std} + f_{1, \beta+1} \{q_k, g\}_{std} \\ &\quad + f_{1, \beta+1} q_k^{\rightarrow} g^{\leftarrow} - q_k^{\rightarrow} \{f_{1, \beta+1}^{\leftarrow}, g\}_{std} \\ &\quad - f_{1, \beta+1}^{\leftarrow} \{q_k^{\rightarrow}, g\}_{std}. \end{aligned}$$

These are brackets of log canonical functions, except for $\{f_{1, \beta+1}^{\leftarrow}, g\}_{std}$ which is given in Lemma 6.6, so it is

$$\begin{aligned} \{\theta_k, g\}_{\alpha\beta} &= (\omega_{f_{1, \beta+1}, g} + \omega_{q_k, g}) q_k \cdot f_{1, \beta+1} \cdot g \\ &\quad + f_{1, \beta+1} q_k^{\rightarrow} g^{\leftarrow} \\ &\quad - (\omega_{f_{1, \beta+1}, g} + \omega_{x_{n\beta}, g} - \omega_{x_{n, \beta+1}, g}) q_k^{\rightarrow} \cdot f_{1, \beta+1}^{\leftarrow} \cdot g \\ &\quad - q_k^{\rightarrow} f_{1, \beta+1} g^{\leftarrow} \\ &\quad - (\omega_{q_k, g} - \omega_{x_{n\alpha}, g} + \omega_{x_{n, \alpha+1}, g}) f_{1, \beta+1}^{\leftarrow} \cdot q_k^{\rightarrow} \cdot g \\ &= (\omega_{f_{1, \beta+1}, g} + \omega_{q_k, g}) q_k \cdot f_{1, \beta+1} \cdot g - \\ &\quad (\omega_{f_{1, \beta+1}, g} + \omega_{q_k, g} - s\omega_{\alpha\beta}(g)) f_{1, \beta+1}^{\leftarrow} \cdot q_k^{\rightarrow} \cdot g, \end{aligned}$$

and with Lemma 6.10 this comes down to

$$(3.10) \quad \{\theta_k, g\}_{\alpha\beta} = (\omega_{f_{1, \beta+1}, g} + \omega_{q_k, g}) \theta_k g.$$

3. Now look at $g = f_{n+m-\alpha, m}$ for some $m > k$: with Lemma 6.6 we can compute

$$\begin{aligned} \{\theta_k, g\}_{\alpha\beta} &= \{q_k \cdot f_{1, \beta+1}, g\}_{\alpha\beta} - \{q_k^{\rightarrow} \cdot f_{1, \beta+1}^{\leftarrow}, g\}_{\alpha\beta} \\ &= q_k \{f_{1, \beta+1}, g\}_{\alpha\beta} + f_{1, \beta+1} \{q_k, g\}_{\alpha\beta} \\ &\quad - q_k^{\rightarrow} \{f_{1, \beta+1}^{\leftarrow}, g\}_{\alpha\beta} - f_{1, \beta+1}^{\leftarrow} \{q_k^{\rightarrow}, g\}_{\alpha\beta} \\ &= q_k \{f_{1, \beta+1}, g\}_{std} - q_k f_{1, \beta+1}^{\leftarrow} g^{\rightarrow} + f_{1, \beta+1} \{q_k, g\}_{std} \\ &\quad - q_k^{\rightarrow} \{f_{1, \beta+1}^{\leftarrow}, g\}_{std} - f_{1, \beta+1}^{\leftarrow} \{q_k^{\rightarrow}, g\}_{std} \\ &= (\omega_{f_{1, \beta+1}, g} + \omega_{q_k, g}) q_k \cdot f_{1, \beta+1} \cdot g \\ &\quad - (\omega_{q_k, g} + \omega_{f_{1, \beta+1}, g} - s\omega_{\alpha\beta}(g)) q_k^{\rightarrow} \cdot f_{1, \beta+1}^{\leftarrow} \cdot g, \end{aligned}$$

and with Lemma 6.10 this is

$$(3.11) \quad \{\theta_k, g\}_{\alpha\beta} = (\omega_{f_{1,\beta+1},g} + \omega_{q_k,g}) \theta_k \cdot g.$$

We now turn to look at $\{\theta_k, \theta_m\}_{\alpha\beta}$: w.l.o.g. assume $m > k$:

$$(3.12) \quad \begin{aligned} \{\theta_k, \theta_m\}_{\alpha\beta} &= \{q_k \cdot f_{1,\beta+1} - q_k^\rightarrow \cdot f_{1,\beta+1}^\leftarrow, q_m \cdot f_{1,\beta+1} - q_m^\rightarrow \cdot f_{1,\beta+1}^\leftarrow\}_{\alpha\beta} \\ &= \{q_k \cdot f_{1,\beta+1}, q_m \cdot f_{1,\beta+1}\}_{\alpha\beta} - \{q_k^\rightarrow \cdot f_{1,\beta+1}^\leftarrow, q_m \cdot f_{1,\beta+1}\}_{\alpha\beta} \\ &\quad - \{q_k \cdot f_{1,\beta+1}, q_m^\rightarrow \cdot f_{1,\beta+1}^\leftarrow\}_{\alpha\beta} \\ &\quad + \{q_k^\rightarrow \cdot f_{1,\beta+1}^\leftarrow, q_m^\rightarrow \cdot f_{1,\beta+1}^\leftarrow\}_{\alpha\beta}. \end{aligned}$$

The Poisson bracket satisfy the Leibniz rule:

$$\{f_1 \cdot f_2, f_3\} = f_1 \cdot \{f_2, f_3\} + \{f_1, f_3\} \cdot f_2,$$

so each of the four brackets above can break into four terms of the form $f_1 \cdot f_2 \cdot \{f_3, f_4\}$. We have already seen that

$$(3.13) \quad \begin{aligned} \{q_k, q_m^\rightarrow\}_{\alpha\beta} &= \begin{cases} (\omega_{q_k, q_m} - \omega_{q_k, x_n, \alpha} + \omega_{q_k, x_n, \alpha+1}) q_k q_m^\rightarrow & \text{if } m > k \\ (\omega_{q_k, q_m} - \omega_{q_k, x_n, \alpha} + \omega_{q_k, x_n, \alpha+1}) q_k q_m^\rightarrow + q_k^\rightarrow q_m & \text{if } m < k \end{cases} \\ \{q_k, f_{1,\beta+1}^\leftarrow\}_{\alpha\beta} &= (\omega_{q_k, f_{1,\beta+1}} + \omega_{q_k, x_n, \beta} - \omega_{q_k, x_n, \beta+1}) q_k f_{1,\beta+1}^\leftarrow, \\ \{f_{1,\beta+1}, f_{1,\beta+1}^\leftarrow\}_{\alpha\beta} &= (\omega_{f_{1,\beta+1}, x_n, \beta} - \omega_{f_{1,\beta+1}, x_n, \beta+1}) f_{1,\beta+1} f_{1,\beta+1}^\leftarrow, \\ \{q_k^\rightarrow, f_{1,\beta+1}^\leftarrow\}_{\alpha\beta} &= (\omega_{q_k, f_{1,\beta+1}^\leftarrow} - \omega_{x_n, \alpha, f_{1,\beta+1}^\leftarrow} + \omega_{x_n, \alpha+1, f_{1,\beta+1}^\leftarrow}) q_k^\rightarrow f_{1,\beta+1}^\leftarrow \\ &= [(\omega_{q_k, f_{1,\beta+1}} + \omega_{q_k, x_n, \beta} - \omega_{q_k, x_n, \beta+1}) \\ &\quad - (\omega_{x_n, \alpha, f_{1,\beta+1}} + \omega_{x_n, \alpha, x_n, \beta} - \omega_{x_n, \alpha, x_n, \beta+1}) \\ &\quad + (\omega_{x_n, \alpha+1, f_{1,\beta+1}} + \omega_{x_n, \alpha+1, x_n, \beta} - \omega_{x_n, \alpha+1, x_n, \beta+1})] q_k^\rightarrow f_{1,\beta+1}^\leftarrow. \end{aligned}$$

We will look at the four brackets of (3.12) one at a time. The first one is

$$(3.14) \quad \begin{aligned} \{q_k \cdot f_{1,\beta+1}, q_m \cdot f_{1,\beta+1}\}_{\alpha\beta} &= (f_{1,\beta+1})^2 \{q_k, q_m\}_{\alpha\beta} + q_k f_{1,\beta+1} \{f_{1,\beta+1}, q_m\}_{\alpha\beta} \\ &\quad + q_m f_{1,\beta+1} \{q_k, f_{1,\beta+1}\}_{\alpha\beta} \\ &= (f_{1,\beta+1})^2 \{q_k, q_m\}_{std} + q_k f_{1,\beta+1} \{f_{1,\beta+1}, q_m\}_{std} \\ &\quad - q_k f_{1,\beta+1} f_{1,\beta+1}^\leftarrow q_m^\rightarrow + q_m f_{1,\beta+1} \{q_k, f_{1,\beta+1}\}_{std} \\ &\quad + q_m f_{1,\beta+1} q_k^\rightarrow f_{1,\beta+1}^\leftarrow \\ &= (\omega_{q_k, q_m} + \omega_{q_k, f_{1,\beta+1}} + \omega_{f_{1,\beta+1}, q_m}) (f_{1,\beta+1})^2 q_k q_m \\ &\quad + q_k^\rightarrow f_{1,\beta+1}^\leftarrow q_m f_{1,\beta+1} - q_k f_{1,\beta+1} f_{1,\beta+1}^\leftarrow q_m^\rightarrow. \end{aligned}$$

The second bracket:

$$\begin{aligned} &\{q_k^\rightarrow \cdot f_{1,\beta+1}^\leftarrow, q_m \cdot f_{1,\beta+1}\}_{\alpha\beta} \\ &= f_{1,\beta+1}^\leftarrow q_m \{q_k^\rightarrow, f_{1,\beta+1}\}_{\alpha\beta} + f_{1,\beta+1}^\leftarrow f_{1,\beta+1} \{q_k^\rightarrow, q_m\}_{\alpha\beta} \\ &\quad + q_k^\rightarrow q_m \{f_{1,\beta+1}^\leftarrow, f_{1,\beta+1}\}_{\alpha\beta} + q_k^\rightarrow f_{1,\beta+1} \{f_{1,\beta+1}^\leftarrow, q_m\}_{\alpha\beta} \\ &= f_{1,\beta+1}^\leftarrow q_m \{q_k^\rightarrow, f_{1,\beta+1}\}_{std} + f_{1,\beta+1}^\leftarrow f_{1,\beta+1} \{q_k^\rightarrow, q_m\}_{std} \\ &\quad + q_k^\rightarrow q_m \{f_{1,\beta+1}^\leftarrow, f_{1,\beta+1}\}_{std} + q_k^\rightarrow f_{1,\beta+1} \{f_{1,\beta+1}^\leftarrow, q_m\}_{std}, \end{aligned}$$

and with (3.13) and Lemma 6.10 this is

$$(3.15) \quad \{q_k^{\rightarrow} \cdot f_{1,\beta+1}^{\leftarrow}, q_m \cdot f_{1,\beta+1}\}_{\alpha\beta} = \omega_2 q_k^{\rightarrow} f_{1,\beta+1}^{\leftarrow} q_m f_{1,\beta+1} - q_k f_{1,\beta+1} f_{1,\beta+1}^{\leftarrow} q_m^{\rightarrow},$$

where

$$\omega_2 = \omega_{q_k, q_m} + \omega_{q_k, f_{1,\beta+1}} + \omega_{f_{1,\beta+1}, q_m} + 1.$$

The third one is

$$\begin{aligned} & \{q_k \cdot f_{1,\beta+1}, q_m^{\rightarrow} \cdot f_{1,\beta+1}^{\leftarrow}\}_{\alpha\beta} \\ &= q_k q_m^{\rightarrow} \{f_{1,\beta+1}, f_{1,\beta+1}^{\leftarrow}\}_{\alpha\beta} + q_k f_{1,\beta+1}^{\leftarrow} \{f_{1,\beta+1}, q_m^{\rightarrow}\}_{\alpha\beta} \\ & \quad + f_{1,\beta+1} q_m^{\rightarrow} \{q_k, f_{1,\beta+1}^{\leftarrow}\}_{\alpha\beta} + f_{1,\beta+1} f_{1,\beta+1}^{\leftarrow} \{q_k, q_m^{\rightarrow}\}_{\alpha\beta} \\ &= q_k q_m^{\rightarrow} \{f_{1,\beta+1}, f_{1,\beta+1}^{\leftarrow}\}_{std} + q_k f_{1,\beta+1}^{\leftarrow} \{f_{1,\beta+1}, q_m^{\rightarrow}\}_{std} \\ & \quad + f_{1,\beta+1} q_m^{\rightarrow} \{q_k, f_{1,\beta+1}^{\leftarrow}\}_{std} + f_{1,\beta+1} f_{1,\beta+1}^{\leftarrow} \{q_k, q_m^{\rightarrow}\}_{std} \end{aligned}$$

and with Lemma 6.10 it makes

$$(3.16) \quad \{q_k \cdot f_{1,\beta+1}, q_m^{\rightarrow} \cdot f_{1,\beta+1}^{\leftarrow}\}_{\alpha\beta} = \omega_3 q_k f_{1,\beta+1} q_m^{\rightarrow} f_{1,\beta+1}^{\leftarrow},$$

with

$$\omega_3 = \omega_{q_k, q_m} + \omega_{q_k, f_{1,\beta+1}} + \omega_{f_{1,\beta+1}, q_m}.$$

The last bracket is

$$\begin{aligned} & \{q_k^{\rightarrow} \cdot f_{1,\beta+1}^{\leftarrow}, f_j^{\rightarrow} \cdot f_{1,\beta+1}^{\leftarrow}\}_{\alpha\beta} \\ &= q_k^{\rightarrow} f_{1,\beta+1}^{\leftarrow} \{f_{1,\beta+1}^{\leftarrow}, f_j^{\rightarrow}\}_{\alpha\beta} + f_{1,\beta+1}^{\leftarrow} f_j^{\rightarrow} \{q_k^{\rightarrow}, f_{1,\beta+1}^{\leftarrow}\}_{\alpha\beta} \\ & \quad + f_{1,\beta+1}^{\leftarrow} f_{1,\beta+1}^{\leftarrow} \{q_k^{\rightarrow}, f_j^{\rightarrow}\}_{\alpha\beta} \\ &= q_k^{\rightarrow} f_{1,\beta+1}^{\leftarrow} \{f_{1,\beta+1}^{\leftarrow}, f_j^{\rightarrow}\}_{std} + f_{1,\beta+1}^{\leftarrow} f_j^{\rightarrow} \{q_k^{\rightarrow}, f_{1,\beta+1}^{\leftarrow}\}_{std} \\ & \quad + f_{1,\beta+1}^{\leftarrow} f_{1,\beta+1}^{\leftarrow} \{q_k^{\rightarrow}, f_j^{\rightarrow}\}_{std} \end{aligned}$$

and again, Lemma 6.10 turns it to

$$(3.17) \quad \{q_k^{\rightarrow} \cdot f_{1,\beta+1}^{\leftarrow}, f_j^{\rightarrow} \cdot f_{1,\beta+1}^{\leftarrow}\}_{\alpha\beta} = \omega_4 q_k^{\rightarrow} \cdot f_{1,\beta+1}^{\leftarrow} \cdot f_j^{\rightarrow} \cdot f_{1,\beta+1}^{\leftarrow},$$

with

$$\omega_4 = \omega_{q_k, q_m} + \omega_{q_k, f_{1,\beta+1}} + \omega_{f_{1,\beta+1}, q_m}.$$

Summing (3.14)–(3.17) proves that

$$\{\theta_k, \theta_m\}_{\alpha\beta} = (\omega_{q_k, q_m} + \omega_{q_k, f_{1,\beta+1}} + \omega_{f_{1,\beta+1}, q_m}) \theta_k \theta_m.$$

Last, we check that every pair θ_k, ψ_m is log canonical w.r.t. $\{\cdot, \cdot\}_{\alpha\beta}$. The process is pretty much like the one for θ_k and θ_m : break the two functions into their components,

$$\theta_k = f_{n+k-\alpha, k} f_{1,\beta+1} - f_{n+k-\alpha, k}^{\rightarrow} f_{1,\beta+1}^{\leftarrow}$$

and

$$\psi_m = f_{m, n+m-\beta} f_{\alpha+1, 1} - f_{m, n+m-\beta}^{\downarrow} f_{\alpha+1, 1}^{\uparrow}.$$

Then compute all brackets of these components. Most of these brackets can be computed as above, but here Lemma 6.5 will be needed as well. Setting

$$\begin{aligned} \omega_{\theta_k, \psi_m} &= \omega_{f_{n+k-\alpha, k}, f_{m, n+m-\beta}} + \omega_{f_{n+k-\alpha, k}, f_{\alpha+1, 1}} \\ & \quad + \omega_{f_{1,\beta+1}, f_{m, n+m-\beta}} + \omega_{f_{1,\beta+1}, f_{\alpha+1, 1}}, \end{aligned}$$

the result is

$$\{\theta_k, \psi_m\}_{\alpha\beta} = \omega_{\theta_k, \psi_m} \theta_k \psi_m.$$

The other possible combinations are symmetric (e.g., $\{\psi_k, \psi_m\}$ is symmetric to $\{\theta_k, \theta_m\}$). Lemmas 6.7 and 6.9 can be used instead of 6.4 and 6.6, respectively. \square

4. THE CLUSTER STRUCTURE $\mathcal{C}_{\alpha\beta}$

4.1. Stable variables. Recall the definition of $\mathcal{B}^D = \varphi_{ij}^D(X, Y) | i, j \in [n]$ from Section 3.1. Look at the set $S = \{\varphi_{i1}^D, \varphi_{1j}^D | i \neq \alpha + 1, j \neq \beta + 1\}$ and let \tilde{S} be the restrictions of these functions to the diagonal subgroup.

Though the following proposition is not required for the proof of the main theorem, it does give further information about the cluster structure: the set \tilde{S} will be the set of stable variables. As indicated in [13], in all known cluster structures on Poisson varieties, the frozen variables have two important properties: they behave well under certain natural group actions, and they are log canonical with certain globally defined coordinate functions. Proposition (4.1) states that these two properties hold in our case, and therefore supports the choice of \tilde{S} as the set of stable variables.

Proposition 4.1. *1. The elements of S are semi-invariants of the left and right action of D_- in $D(GL_n)$.*

2. The elements of \tilde{S} are log canonical with all matrix entries x_{ij} .

Proof. 1. The subgroup D_- of $D(GL_n)$ that corresponds to the subalgebra \mathfrak{g}_- of \mathfrak{g} is given by

$$D_- = (U, L)$$

with

$$U = \begin{bmatrix} a_1 & \star & \star & \star & \star & \star \\ 0 & \ddots & \star & \star & \star & \star \\ 0 & 0 & a_{\alpha-1} & \star & \star & \star \\ 0 & 0 & 0 & A & \star & \star \\ 0 & 0 & 0 & 0 & a_{\alpha+2} & \star \\ 0 & 0 & 0 & 0 & 0 & \ddots \end{bmatrix}$$

and

$$L = \begin{bmatrix} a_{n+\alpha-\beta+1} & 0 & 0 & 0 & 0 & 0 & 0 \\ \star & \ddots & 0 & 0 & 0 & 0 & 0 \\ \star & \star & a_n & 0 & 0 & 0 & 0 \\ \star & \star & \star & \ddots & 0 & 0 & 0 \\ \star & \star & \star & \star & a_{\alpha-1} & 0 & 0 \\ \star & \star & \star & \star & \star & A & 0 \\ \star & \star & \star & \star & \star & \star & \ddots \end{bmatrix}.$$

where $A \in GL_2$ and the indices of the diagonal entries a_i are taken modulo n . The \star 's will not play any role in further computations. The left and right action of D_- can be parametrized by

$$(X, Y) \mapsto (A_1 X A_1', A_2 Y A_2')$$

with matrices

$$A_1 = \begin{bmatrix} a_1 & \star & \star & \star & \star & \star \\ 0 & \ddots & \star & \star & \star & \star \\ 0 & 0 & a_{\alpha-1} & \star & \star & \star \\ 0 & 0 & 0 & A & \star & \star \\ 0 & 0 & 0 & 0 & a_{\alpha+2} & \star \\ 0 & 0 & 0 & 0 & 0 & \ddots \end{bmatrix},$$

$$A'_1 = \begin{bmatrix} a'_1 & \star & \star & \star & \star & \star \\ 0 & \ddots & \star & \star & \star & \star \\ 0 & 0 & a'_{\alpha-1} & \star & \star & \star \\ 0 & 0 & 0 & A' & \star & \star \\ 0 & 0 & 0 & 0 & a'_{\alpha+2} & \star \\ 0 & 0 & 0 & 0 & 0 & \ddots \end{bmatrix},$$

$$A_2 = \begin{bmatrix} a_{n+\alpha-\beta+1} & 0 & 0 & 0 & 0 & 0 & 0 \\ \star & \ddots & 0 & 0 & 0 & 0 & 0 \\ \star & \star & a_n & 0 & 0 & 0 & 0 \\ \star & \star & \star & \ddots & 0 & 0 & 0 \\ \star & \star & \star & \star & a_{\alpha-1} & 0 & 0 \\ \star & \star & \star & \star & \star & A & 0 \\ \star & \star & \star & \star & \star & \star & \ddots \end{bmatrix},$$

and

$$A'_2 = \begin{bmatrix} a'_{n+\alpha-\beta+1} & 0 & 0 & 0 & 0 & 0 & 0 \\ \star & \ddots & 0 & 0 & 0 & 0 & 0 \\ \star & \star & a'_n & 0 & 0 & 0 & 0 \\ \star & \star & \star & \ddots & 0 & 0 & 0 \\ \star & \star & \star & \star & a'_{\alpha-1} & 0 & 0 \\ \star & \star & \star & \star & \star & A' & 0 \\ \star & \star & \star & \star & \star & \star & \ddots \end{bmatrix}.$$

There are three kinds of functions in S : minors of X , minors of Y and “mixed” functions. A function $f_X \in S$ that is a minor of X is a semi-invariant of this action: f_X is the determinant of a submatrix $X_{[i,n]}^{[1,\mu]}$ with $\mu = n - i + 1$. One has $i \notin \{\alpha + 1, n - \alpha + 1\}$ (see the construction in Section 3.1). The action of D_- multiplies each row $k \in [i, n]$ of X by the corresponding entry a_k and each column $\ell \in [\mu]$ of X by $a'_{\ell+n+\alpha-\beta-1}$, with two exceptions: rows α and $\alpha + 1$ are multiplied together by A , and columns α and $\alpha + 1$ are multiplied by A' . So as long as one of these rows (or columns) does not occur in the submatrix $X_{[i,n]}^{[1,\mu]}$ without the other, f_X is still a semi-invariant of the action. If $\alpha \in [i, n]$ then clearly $\alpha + 1 \in [i, n]$. On the other hand, the only case with $\alpha + 1 \in [i, n]$ and $\alpha \notin [i, n]$ is when $i = \alpha + 1$. But such a minor can not be in S according to the construction (Section 3.1). Looking at columns, it is easy to see that if $\alpha + 1 \in [1, \mu]$ (that is, the column $\alpha + 1$ occurs in the submatrix $X_{[i,n]}^{[1,\mu]}$), then $\alpha \in [1, \mu]$. The only way to have $\alpha \in [1, \mu]$ and

$\alpha + 1 \notin [1, \mu]$ is $\mu = \alpha$. But this implies $i = n - \alpha + 1$, and this minor is also not in the set S .

For $f_Y \in S$ which is a minor of Y similar arguments hold.

Look now at the function

$$\theta = \det \begin{bmatrix} x_{i1} & \cdots & x_{i\alpha} & x_{i,\alpha+1} & 0 & \cdots \\ \vdots & \ddots & \vdots & \vdots & \vdots & \\ x_{n1} & \cdots & x_{n\alpha} & x_{n,\alpha+1} & 0 & \cdots \\ 0 & \cdots & y_{1\beta} & y_{1\beta+1} & \cdots & y_{1n} \\ \vdots & & \vdots & & \ddots & \vdots \\ 0 & \cdots & y_{\mu\beta} & \cdots & \cdots & y_{\mu n} \end{bmatrix},$$

with $i = n + 1 - \alpha$. It is not hard to see that θ is a semi-invariant of the action of D_- : the block of x_{ij} 's is subject to the same arguments as above, except for when $\alpha = \frac{n}{2}$, which will be treated later. The same holds for the block of y_{ij} 's, unless $\beta = \frac{n}{2}$. Therefore θ is a semi-invariant of this action.

Symmetric arguments show that

$$\psi = \det \begin{bmatrix} y_{1,n+1-\beta} & \cdots & y_{1n} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ y_{\beta j} & \cdots & y_{\beta n} & x_{\alpha 1} & \cdots & x_{\alpha \mu} \\ y_{\beta+1,j} & \cdots & y_{\beta+1,n} & x_{\alpha+1,1} & & \vdots \\ 0 & \cdots & 0 & \vdots & \ddots & \vdots \\ \vdots & & \vdots & x_{n1} & \cdots & x_{n\mu} \end{bmatrix}$$

is also semi-invariant.

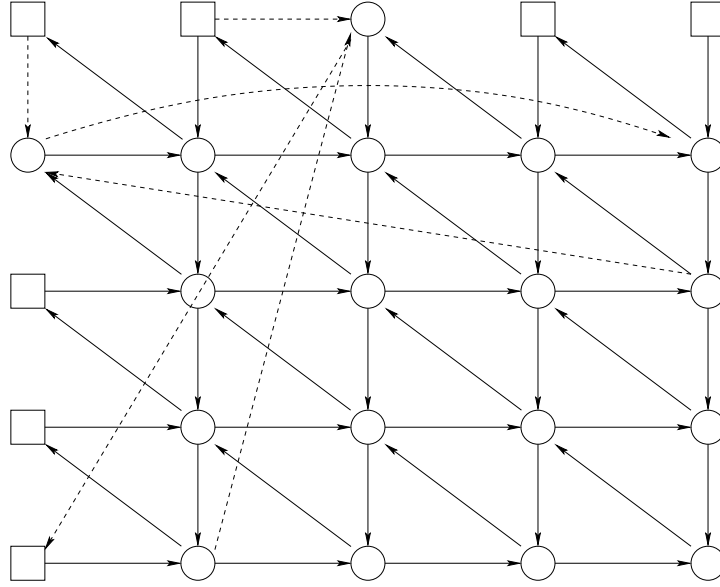
Last, we look at the special case $\alpha = \frac{n}{2}$: here there is only one matrix in \mathcal{M} with elements of both X and Y . The ‘‘building blocks’’ of this matrix are submatrices of X and Y that satisfy the restrictions above, so the determinant of this matrix is also a semi-invariant of the action. The case $\beta = \frac{n}{2}$ is symmetric.

2. First, look at a function $\varphi \in \tilde{S} \cap \tilde{\mathcal{B}}_{std}$. In this case it is not hard to see that $\{\varphi, x_{ij}\}_{\alpha\beta} = \{\varphi, x_{ij}\}_{std}$ (according to Lemma 6.2) and therefore φ is log canonical with all x_{ij} . There are only two other functions in \tilde{S} : $\theta = \varphi_{n-\alpha+1,1}$ and $\psi = \varphi_{1,n-\beta+1}$. Start with $\theta = f_{n-\alpha+1,1}f_{1,\beta+1} - f_{n-\alpha+1,1}^{\rightarrow}f_{1,\beta+1}^{\leftarrow}$. Following the line of the proof of Lemma 6.10, it can be shown that $f_{n-\alpha+1,1}^{\rightarrow}$ is log canonical with every x_{ij} with $j \neq \alpha$, and similarly $f_{1,\beta+1}^{\leftarrow}$ is log canonical with every x_{ij} with $j \neq \beta + 1$, with respect to the standard bracket. The cases $j = \alpha$ and $j = \beta + 1$ are exactly the cases when the standard bracket and the $\alpha\beta$ bracket do not coincide. Adding the difference that was described in Lemma 6.2,

$$\{f_{n-\alpha+1,1}, x_{ij}\}_{\alpha\beta} = \begin{cases} \{f_{n-\alpha+1,1}, x_{ij}\}_{std} & \text{if } j \neq \beta + 1 \\ \{f_{n-\alpha+1,1}, x_{ij}\}_{std} + f_{n-\alpha+1,1}^{\rightarrow}x_{i\beta} & \text{if } j = \beta + 1, \end{cases}$$

and

$$\{f_{1,\beta+1}, x_{ij}\}_{\alpha\beta} = \begin{cases} \{f_{1,\beta+1}, x_{ij}\}_{std} & \text{if } j \neq \alpha \\ \{f_{1,\beta+1}, x_{ij}\}_{std} + f_{1,\beta+1}^{\leftarrow}x_{i,\alpha+1} & \text{if } j = \alpha, \end{cases}$$


 FIGURE 4.1. The $1 \mapsto 2$ quiver for GL_5

shows that with respect to the bracket $\{\cdot, \cdot\}_{\alpha\beta}$, the pairs $f_{n-\alpha+1,1}^{\rightarrow}, x_{i\alpha}$ and $f_{1,\beta+1}^{\leftarrow}, x_{i,\beta+1}$ are log canonical. The coefficients $\omega_{f_{n-\alpha+1,1}^{\rightarrow}, x_{ij}} = \frac{\{f_{n-\alpha+1,1}^{\rightarrow}, x_{ij}\}}{f_{n-\alpha+1,1}^{\rightarrow} \cdot x_{ij}}$ and $\omega_{f_{1,\beta+1}^{\leftarrow}, x_{ij}} = \frac{\{f_{1,\beta+1}^{\leftarrow}, x_{ij}\}}{f_{1,\beta+1}^{\leftarrow} \cdot x_{ij}}$ can be computed like in the proof of Lemma 6.10, showing that θ is log canonical with x_{ij} . Symmetric arguments hold for ψ . \square

Note that the set of stable variables \tilde{S} is the set of determinants of all matrices in the set $\mathcal{M}_{\alpha\beta}$. By its definition in 3.4, $\mathcal{M}_{\alpha\beta}$ has $2(n-1)-2$ matrices, and therefore

$$|\tilde{S}| = 2(n-1) - 2 = 2|\Delta \setminus \Gamma_1|,$$

as in Statement 1 of Conjecture 2.3.

4.2. The quiver $Q_{\alpha\beta}^n$. To describe the quiver $Q_{\alpha\beta}^n$, start with the standard quiver Q_{std}^n as given in Section 3.2. The vertex in the i -th row and j -th column corresponds to the cluster variable $f_{ij} = \det X_{[i,\mu(i,j)]}^{[j,\mu(i,j)]}$. The quiver of the exotic cluster structure with BD data $\alpha \mapsto \beta$ is very close: a vertex (i, j) now represents the cluster variable φ_{ij} , and the quiver takes these changes:

- (1) Vertices $(\alpha+1, 1)$ and $(1, \beta+1)$ are not frozen.
- (2) The arrows $(\alpha, 1) \rightarrow (\alpha+1, 1)$ and $(1, \beta) \rightarrow (1, \beta+1)$ are added.
- (3) The arrows $(n, \alpha+1) \rightarrow (1, \beta+1)$, $(1, \beta+1) \rightarrow (n, \alpha)$ are added.
- (4) The arrows $(\beta+1, n) \rightarrow (\alpha+1, 1)$, $(\alpha+1, 1) \rightarrow (\beta, n)$ are added.

The example of $1 \mapsto 2$ on SL_5 is given in Figure 4.1. The dashed arrows are the arrows that were added to the standard quiver.

Through this section we use B, ω, Ω to denote the exchange matrix, Poisson coefficients and Poisson matrix in the standard case, and $\bar{B}, \bar{\Omega}, \bar{\omega}$ for their counterparts in the $\alpha \mapsto \beta$ case. We now prove that the cluster structure described in the previous section is indeed compatible with the bracket $\{\cdot, \cdot\}_{\alpha\beta}$:

Theorem 4.2. *The cluster structure $\mathcal{C}_{\alpha\beta} = \mathcal{C}(\bar{B}_{\alpha\beta})$ is compatible with the bracket $\{\cdot, \cdot\}_{\alpha\beta}$.*

Proof. According to Proposition 2.1, it is sufficient to prove

$$(4.1) \quad \bar{B}\bar{\Omega} = [I \ 0],$$

which can be rephrased as

$$(\bar{B}\bar{\Omega})_{pq} = \sum_{p \leftarrow k} \bar{\omega}_{kq} - \sum_{p \rightarrow k} \bar{\omega}_{kq} = \begin{cases} 1 & p = q \\ 0 & p \neq q, \end{cases}$$

where the first sum is over vertices k with an arrow pointing from k to p , and the second sum is over vertices k with an arrow pointing from p to k . Recall that the standard case has

$$\sum_{p \leftarrow k} \omega_{kq} - \sum_{p \rightarrow k} \omega_{kq} = \begin{cases} 1 & p = q \\ 0 & p \neq q. \end{cases}$$

Label a row (or column) of the exchange matrices B and \bar{B} by (i, j) if it corresponds to the cluster variable f_{ij} (in B) or φ_{ij} (in \bar{B}). Now compute $(\bar{B}\bar{\Omega})_{pq}$ in the following cases:

1. The p -th row of \bar{B} is equal to the p -th row of B . This is true for almost all rows of \bar{B} , or more precisely when

$$(4.2) \quad p \notin \{(1, \beta + 1), (n, \alpha), (n, \alpha + 1), (\alpha + 1, 1), (\beta, n), (\beta + 1, n)\},$$

and we have these possible situations:

(a) p corresponds to a cluster variable in $\mathcal{B}_{std} \cap \mathcal{B}_{\alpha\beta}$.

i. Assume all cluster variables adjacent to p are in $\mathcal{B}_{std} \cap \mathcal{B}_{\alpha\beta}$. If q is also in $\mathcal{B}_{std} \cap \mathcal{B}_{\alpha\beta}$, this is just the same as the standard case, and $(\bar{B}\bar{\Omega})_{pq} = (B\Omega)_{pq}$. If q is not a standard basis function, then either it corresponds to a cluster variable of the form $\varphi_{n+m-\alpha, m}$, and then

$$\bar{\omega}_{kq} = \omega_{kq} + \omega_{k, f_{n, \alpha+1}} - \omega_{k, f_{n\alpha}}$$

or one of the form $\varphi_{m, n+m-\beta}$, and then

$$\bar{\omega}_{kq} = \omega_{kq} + \omega_{k, f_{\beta+1, n}} - \omega_{k, f_{\beta n}}.$$

So in the first case,

$$\begin{aligned} \sum_{p \leftarrow k} \bar{\omega}_{kq} - \sum_{p \rightarrow k} \bar{\omega}_{kq} &= \sum_{p \leftarrow k} \omega_{kq} - \sum_{p \rightarrow k} \omega_{kq} + \sum_{p \leftarrow k} \omega_{k, f_{n, \alpha+1}} \\ &\quad - \sum_{p \rightarrow k} \omega_{k, f_{n, \alpha+1}} - \sum_{p \leftarrow k} \omega_{k, f_{n\alpha}} + \sum_{p \rightarrow k} \omega_{k, f_{n\alpha}} \end{aligned}$$

and since $p \neq (n, \alpha), (n, \alpha + 1)$ (from (4.2)), the standard case tells us

$$\sum_{p \leftarrow k} \omega_{k, f_{n, \alpha+1}} - \sum_{p \rightarrow k} \omega_{k, f_{n, \alpha+1}} = \sum_{p \leftarrow k} \omega_{k, f_{n\alpha}} - \sum_{p \rightarrow k} \omega_{k, f_{n\alpha}} = 0.$$

The case of $\varphi_{m, n+m-\beta}$ is symmetric.

ii. The cluster variable p has at least one neighbor that is not in $\mathcal{B}_{std} \cap \mathcal{B}_{\alpha\beta}$.

Looking at the quiver one can easily see that the number of such neighbors can be either one or two.

A. p has exactly one such neighbor. The quiver has only two such vertices: $p = (n, \alpha + 1)$ or $p = (\beta + 1, n)$. In both cases it means that the i -th row of \overline{B} is different from that row of B , because the quiver $Q_{\alpha\beta}^n$ has arrows $(n, \alpha + 1) \rightarrow (1, \beta + 1)$ and $(\beta + 1, n) \rightarrow (\alpha + 1, 1)$, which Q_{std}^n does not have. This case will be handled later on.

B. p has two non standard neighbors. In this case these two neighbors are connected to p by arrows in opposite directions (i.e., one of them is pointing at p and the other one from p). These two non standard neighbors must both belong to the same “family” of functions, either $\{\psi_m\}$ or $\{\theta_m\}$ (as defined in Section 3.2). We have seen that the Poisson coefficients of these function differ from their standard counterparts by a constant, e.g., for every function $g \in \mathcal{B}_{std}$,

$$\omega_{\varphi_{n+m-\alpha, m, g}} = \omega_{f_{n+m-\alpha, m, g}} + \omega_{f_{1, \beta+1, g}}.$$

When summing over all neighbors of p , this constant is then added once, for the vertex with an arrow pointing at p , and subtracted once, for the vertex with an arrow pointing from p to it. These cancel each other and the sum remains as it was in the standard case.

(b) p is not in $\mathcal{B}_{std} \cap \mathcal{B}_{\alpha\beta}$, which means $p = (n + m - \alpha, m)$ or $p = (m, n + m - \beta)$. Assume $m < \alpha$ (for the first one) or $m < \beta$ (second), because $p = (n, \alpha)$ and $p = (\beta, n)$ are in (4.2) and will be treated later. If $m = 1$ it is a frozen variable. Again, look at the first case (second is just the same): if $1 < m < \alpha$ then two neighbors of $p = (n + m - \alpha, m)$ are non standard. These are $(n + m + 1 - \alpha, m + 1)$ and $(n + m - 1 - \alpha, m - 1)$ with arrows pointing in opposite directions. Since we know that

$$\overline{\omega}_{f_{n+m-\alpha, m, q}} = \omega_{f_{n+m-\alpha, m, q}} + \omega_{f_{1, \beta+1, q}}$$

and therefore a constant is added to the sum for the vertex $(n + m + 1 - \alpha, m + 1)$ and then subtracted for the vertex $(n + m - 1 - \alpha, m - 1)$. This constant is added to all ω 's in the sum, and they cancel each other.

2. Here the p -th row of \overline{B} is not equal to the p -th row of B .

(a) If $p = (1, \beta + 1)$ then B does not have this row (it was a frozen variable in the standard case). Its neighbors are now $\varphi_{n, \alpha+1}, \varphi_{2, \beta+2}, \varphi_{1\beta}$ with arrows pointing to p , and $\varphi_{2, \beta+1}, \varphi_{n\alpha}$ with arrows from p to them. So we have

$$\begin{aligned} (4.3) \quad \sum_{p \leftarrow k} \overline{\omega}_{kq} - \sum_{p \rightarrow k} \overline{\omega}_{kq} &= +\overline{\omega}_{\varphi_{1\beta, \varphi_q}} + \overline{\omega}_{\varphi_{n, \alpha+1, \varphi_q}} + \overline{\omega}_{\varphi_{2, \beta+2, \varphi_q}} \\ &\quad - \overline{\omega}_{\varphi_{2, \beta+1, \varphi_q}} + \overline{\omega}_{\varphi_{n\alpha, \varphi_q}} \\ &= \omega_{f_{1\beta, f_q}} + \omega_{f_{n, \alpha+1, f_q}} + \omega_{f_{2, \beta+2, f_q}} \\ &\quad - \omega_{f_{2, \beta+1, f_q}} - \omega_{f_{n\alpha, f_q}} - \omega_{f_{1, \beta+1, f_q}} \end{aligned}$$

In the standard case, since exchange relation must hold at $(2, \beta + 1)$ we can write

(4.4)

$$\omega_{f_{1, \beta, f_j}} + \omega_{f_{2, \beta+2, f_j}} + \omega_{f_{3, \beta+1, f_j}} - \omega_{f_{1, \beta+1, f_j}} - \omega_{f_{2, \beta, f_j}} - \omega_{f_{3, \beta+2, f_j}} = \begin{cases} 1 & j = (2, \beta + 1) \\ 0 & j \neq (2, \beta + 1) \end{cases}$$

and we continue, using standard exchange relation at $(i, \beta + 1)$

$$\omega_{f_{i-1,\beta},f_j} + \omega_{f_{i,\beta+2},f_j} + \omega_{f_{i+1,\beta+1},f_j} - \omega_{f_{i-1,\beta+1},f_j} - \omega_{f_{i,\beta},f_j} - \omega_{f_{i+1,\beta+2},f_j} = \begin{cases} 1 & j = (i, \beta + 1) \\ 0 & j \neq (i, \beta + 1) \end{cases}$$

and assuming $j \neq (i, \beta + 1)$

$$(4.5) \quad \omega_{f_{i-1,\beta},f_j} - \omega_{f_{i-1,\beta+1},f_j} + \omega_{f_{i,\beta+2},f_j} = \omega_{f_{i,\beta},f_j} - \omega_{f_{i+1,\beta+1},f_j} + \omega_{f_{i+1,\beta+2},f_j}$$

and eventually for $i = n$

$$\omega_{f_{n,\beta},f_j} + \omega_{f_{n-1,\beta+1},f_j} - \omega_{f_{n-1,\beta},f_j} - \omega_{f_{n,\beta+2},f_j} = \begin{cases} 1 & j = (n, \beta + 1) \\ 0 & j \neq (n, \beta + 1) \end{cases}.$$

The standard exchange relation at $(n, \beta + 1)$ implies

$$\omega_{f_{n,\beta},f_j} + \omega_{f_{n-1,\beta+1},f_j} - \omega_{f_{n-1,\beta},f_j} - \omega_{f_{n,\beta+2},f_j} = 0$$

or

$$\omega_{f_{n,\beta},f_j} = \omega_{f_{n,\beta+2},f_j} - \omega_{f_{n-1,\beta+1},f_j} + \omega_{f_{n-1,\beta},f_j}.$$

So

$$\omega_{f_{n,\beta},f_j} - \omega_{f_{n,\beta+1},f_j} = \omega_{f_{n-2,\beta},f_j} + \omega_{f_{n,-1\beta+2},f_j} - \omega_{f_{n-1,\beta+1},f_j} - \omega_{f_{n-2,\beta+1},f_j},$$

and using (4.5) recursively

$$(4.6) \quad \omega_{f_{n,\beta},f_j} - \omega_{f_{n,\beta+1},f_j} = \omega_{f_{i-2,\beta},f_j} + \omega_{f_{i,-1\beta+2},f_j} - \omega_{f_{i-1,\beta+1},f_j} - \omega_{f_{i-2,\beta+1},f_j}.$$

Now we only need $\omega_{f_{n,\alpha+1},f_j} - \omega_{f_{n,\alpha},f_j} = \omega_{f_{n,\beta+1},f_j} - \omega_{f_{n,\beta},f_j}$. This is true from Lemma 6.10 and the assumption $j \neq (i, \beta + 1)$, so (4.3) turns to

$$\sum_{p \leftarrow k} \bar{\omega}_{kq} - \sum_{p \rightarrow k} \bar{\omega}_{kq} = 0, \quad \forall q \neq (p, \beta + 1).$$

If, on the other hand $q = (1, \beta + 1)$, this still holds, but Lemma 6.10 now says $\omega_{f_{n,\alpha+1},f_j} - \omega_{f_{n,\alpha},f_j} = \omega_{f_{n,\beta+1},f_j} - \omega_{f_{n,\beta},f_j} + 1$, so that

$$\sum_{p \leftarrow k} \bar{\omega}_{kq} - \sum_{p \rightarrow k} \bar{\omega}_{kq} = 1.$$

Last, let $q = (p, \beta + 1)$ with $p > 1$. So in (4.5) we need to add 1 to the right hand side. This 1 is then added to the sum of coefficients over neighbors of $(1, \beta + 1)$, but now Lemma 6.10 says $\omega_{f_{n,\alpha+1},f_j} - \omega_{f_{n,\alpha},f_j} = \omega_{f_{n,\beta+1},f_j} - \omega_{f_{n,\beta},f_j} + 1$, so again

$$\sum_{p \leftarrow k} \bar{\omega}_{kq} - \sum_{p \rightarrow k} \bar{\omega}_{kq} = 0.$$

The special case $\beta = n - 1$ is somewhat different, because here vertices $(i, \beta + 1) = (i, n)$ do not have neighbors on the right. However, the same arguments still hold, and since the exchange relations in the standard quiver are similar, the final conclusion is identical.

(b) Let $p = (n, \alpha)$ then in the standard quiver its neighbors were $f_{n,\alpha+1}, f_{n-1,\alpha-1}$ (with arrows from p to them), and $f_{n,\alpha-1}, f_{n-1,\alpha}$ (with arrows pointing to p). In

$Q_{\alpha\beta}^n$ an arrow is added from $(1, \beta + 1)$ to p . We then have $\varphi_{n-1, \alpha-1} = f_{n-1, \alpha-1} \cdot f_{1, \beta+1} - f_{n-1, \alpha-1}^{\rightarrow} \cdot f_{1, \beta+1}^{\leftarrow}$. So using $\bar{\omega}_{\varphi_{n-1, \alpha-1}, g} = \omega_{f_{n-1, \alpha-1}, g} + \omega_{f_{1, \beta+1}, g}$ we get

$$\begin{aligned} \sum_{p \leftarrow k} \bar{\omega}_{kq} - \sum_{p \rightarrow k} \bar{\omega}_{kq} &= \bar{\omega}_{\varphi_{n, \alpha-1}, \varphi_q} + \bar{\omega}_{\varphi_{n-1, \alpha}, \varphi_q} + \bar{\omega}_{\varphi_{1, \beta+1}, \varphi_q} \\ &\quad - \bar{\omega}_{\varphi_{n, \alpha+1}, \varphi_q} - \bar{\omega}_{\varphi_{n-1, \alpha-1}, \varphi_q} \\ &= \omega_{f_{n, \alpha-1}, f_q} + \omega_{f_{n-1, \alpha}, f_q} + \omega_{f_{1, \beta+1}, f_q} \\ &\quad - \omega_{f_{n, \alpha+1}, f_q} - \omega_{f_{n-1, \alpha-1}, f_q} - \omega_{f_{1, \beta+1}, f_q} \\ &= \sum_{p \leftarrow k} \omega_{k, f_q} - \sum_{p \rightarrow k} \omega_{k, f_q}, \end{aligned}$$

and the last term is the one from the standard case, which equals δ_{ij} .

(c) $p = (n, \alpha + 1)$. In the standard quiver there are arrows from p to (n, α) , $(n-1, \alpha + 1)$ and from $(n, \alpha + 2)$, $(n-1, \alpha)$ to p . In $Q_{\alpha\beta}^n$ there is a new arrow $(n, \alpha + 1) \rightarrow (1, \beta + 1)$. Again,

$$\begin{aligned} \sum_{p \leftarrow k} \bar{\omega}_{kq} - \sum_{p \rightarrow k} \bar{\omega}_{kq} &= \bar{\omega}_{f_{n, \alpha+2}} + \bar{\omega}_{f_{n-1, \alpha}} + \bar{\omega}_{f_{1, \beta+1}} \\ &\quad - \bar{\omega}_{\varphi_{n\alpha}} - \bar{\omega}_{f_{n-1, \alpha+1}} \\ &= \omega_{f_{n, \alpha+2}} + \omega_{f_{n-1, \alpha}} + \omega_{f_{1, \beta+1}} \\ &\quad - \omega_{f_{n\alpha}} - \omega_{f_{1, \beta+1}} - \omega_{f_{n-1, \alpha+1}} \\ &= \sum_{p \leftarrow k} \omega_{k, f_{n, \alpha+1}} - \sum_{p \rightarrow k} \omega_{k, f_{n, \alpha+1}} \end{aligned}$$

which is also equal to the standard. \square

Note that an immediate corollary from Theorem 4.2 is that the exchange matrix \bar{B} is of maximal rank, since $\text{rank}(\bar{B}\bar{\Omega}) \leq \min(\text{rank}\bar{B}, \text{rank}\bar{\Omega})$, and (4.1) implies that $\bar{B}\bar{\Omega}$ has maximal rank.

5. LOCAL REGULARITY

We now show that the initial seed $\mathcal{B}_{\alpha\beta}$ defined in Section 3.1 is locally regular. Clearly, all the elements of $\mathcal{B}_{\alpha\beta}$ are regular functions, so local regularity is equivalent to:

Theorem 5.1. *For every exchangeable variable φ in the initial cluster, the exchanged variable φ' is a regular function.*

Proof. We can use the similarity of the exchange quivers $Q_{\alpha\beta}^n$ and Q_{std}^n . The exchange relation (2.1) involves the cluster variable φ and its neighbors in the exchange quiver.

Consider the following cases:

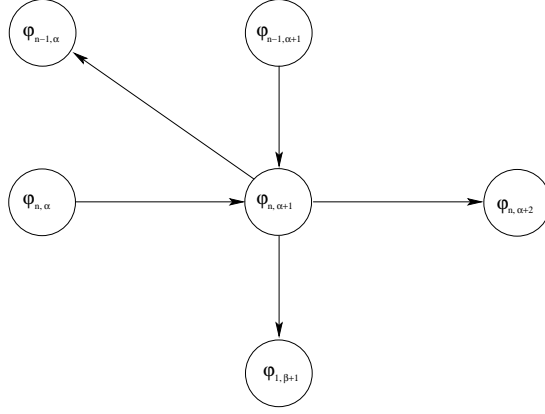
1. φ is in $\mathcal{B}_{\alpha\beta} \cap \mathcal{B}_{std}$ and all its neighbors are also in $\mathcal{B}_{\alpha\beta} \cap \mathcal{B}_{std}$.

This means the exchange rule is the same as in the standard case, and therefore the exchanged cluster variable is equal to the one in the standard case, which is regular.

2. φ is in $\mathcal{B}_{\alpha\beta} \cap \mathcal{B}_{std}$, but at least one of its neighbors is not in $\mathcal{B}_{\alpha\beta} \cap \mathcal{B}_{std}$.

(a) Two neighbors of φ are not in $\mathcal{B}_{\alpha\beta} \cap \mathcal{B}_{std}$.

The exchange rule is now $\varphi \cdot \varphi' = \varphi_{ij} \cdot p_1 + \varphi_{i+1, j+1} \cdot p_2$ where φ_{ij} and $\varphi_{i+1, j+1}$ are the two non standard neighbors and p_1, p_2 some monomials. Now recall that in

FIGURE 5.1. The neighbors of $f_{n,\alpha+1}$

this case $\varphi_{ij} = f_{ij}h - \bar{f}_{ij}\bar{h}$ where \bar{f}_{ij} is either f_{ij}^{\rightarrow} or f_{ij}^{\downarrow} , and \bar{h} is either $f_{1,\beta+1}^{\leftarrow}$ or $f_{\alpha+1,1}^{\uparrow}$, respectively. The exchange rule is then

$$\begin{aligned} \varphi \cdot \varphi' &= (f_{ij}h - \bar{f}_{ij}\bar{h})p_1 + (f_{i+1,j+1}h - \bar{f}_{i+1,j+1}\bar{h})p_2 \\ &= h(f_{ij}p_1 + f_{i+1,j+1}p_2) - \bar{h}(\bar{f}_{ij}p_1 + \bar{f}_{i+1,j+1}p_2) \end{aligned}$$

the first part is just the standard exchange rule multiplied by h . The term in the second parenthesis can be regarded as a Desnanot–Jacobi identity (6.13). It is equal to the standard one with just one change: the last column (row) that was α (β) in the standard case is now replaced by $\alpha+1$ ($\beta+1$). It is not hard to conclude that the result is a product of φ and some regular function, as it was in the standard case.

(b) Only one neighbor of φ is not in $\mathcal{B}_{\alpha\beta} \cap \mathcal{B}_{std}$.

There are only two such vertices: $\varphi_{n,\alpha+1}$ and $\varphi_{\beta+1,n}$. The vertex $\varphi_{n,\alpha+1} = x_{n,\alpha+1}$ has neighbors $\varphi_{n,\alpha}, \varphi_{n-1,\alpha}, \varphi_{n-1,\alpha+1}, \varphi_{n,\alpha+2}$ and $\varphi_{1,\beta+1}$. Figure 5.1 shows the relevant subquiver of $Q_{\alpha\beta}^n$. Recall that $\varphi_{n,\alpha} = x_{n,\alpha}f_{1,\beta+1} - x_{n,\alpha+1}f_{1,\beta+1}^{\leftarrow}$, so

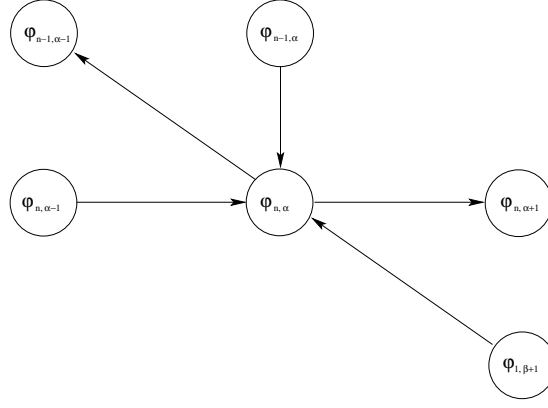
$$\begin{aligned} \varphi_{n,\alpha+1} \cdot \varphi_{n,\alpha}^{\leftarrow} &= \varphi_{n,\alpha}\varphi_{n-1,\alpha+1} + \varphi_{n-1,\alpha}\varphi_{n,\alpha+2}\varphi_{1,\beta+1} \\ &= f_{1,\beta+1}(x_{n,\alpha}f_{n-1,\alpha+1} + f_{n-1,\alpha}f_{n,\alpha+2}) - f_{n,\alpha+1}f_{1,\beta+1}^{\beta+1\leftarrow\beta}f_{n-1,\alpha+1}. \end{aligned}$$

The term in parenthesis is the exchange rule in the standard case, so it is the product of $f_{n,\alpha+1}$ and some other regular function, and the second term is clearly divisible by $f_{n,\alpha+1}$. Therefore the exchanged variable is regular. Similar arguments hold for the vertex $\varphi_{\beta+1,n}$.

3. φ is not in $\mathcal{B}_{\alpha\beta} \cap \mathcal{B}_{std}$.

(a) φ is either $\varphi_{n\alpha}$ or $\varphi_{\beta n}$.

Assume $\varphi = \varphi_{n,\alpha} = x_{n,\alpha}f_{1,\beta+1} - x_{n,\alpha+1}f_{1,\beta+1}^{\leftarrow}$. Assume $\alpha > 1$ because if $\alpha = 1$, the variable φ_{n1} must be frozen. The adjacent vertices correspond to $\varphi_{n,\alpha-1}, \varphi_{n-1,\alpha}, \varphi_{n,\alpha+1}, \varphi_{n-1,\alpha-1}, \varphi_{1,\beta+1}$ where $\varphi_{n-1,\alpha-1} = f_{n-1,\alpha-1}f_{1,\beta+1} - f_{n-1,\alpha-1}^{\rightarrow}f_{1,\beta+1}^{\leftarrow}$

FIGURE 5.2. The neighbors of $\varphi_{n\alpha}$

, as shown in Figure 5.2. The exchange rule is

$$\begin{aligned} \varphi_{n,\alpha} \cdot \varphi'_{n,\alpha} &= \varphi_{n,\alpha+1}\varphi_{n-1,\alpha-1} + \varphi_{n,\alpha-1}\varphi_{n-1,\alpha}\varphi_{1,\beta+1} \\ &= x_{n,\alpha+1} (f_{n-1,\alpha-1}f_{1,\beta+1} - f_{n-1,\alpha-1}^{\rightarrow}f_{1,\beta+1}^{\leftarrow}) + x_{n,\alpha-1}f_{n-1,\alpha}f_{1,\beta+1} \\ &= f_{1,\beta+1} (x_{n,\alpha+1}f_{n-1,\alpha-1} + x_{n,\alpha-1}f_{n-1,\alpha}) - x_{n,\alpha+1}f_{n-1,\alpha-1}^{\rightarrow}f_{1,\beta+1}^{\leftarrow} \end{aligned}$$

and the term in parenthesis is just the standard exchange rule, which is $x_{n\alpha} \cdot f_{n-1,\alpha-1}^{\rightarrow}$. Therefore,

$$\varphi_{n,\alpha} \cdot \varphi'_{n,\alpha} = (x_{n\alpha}f_{1,\beta+1} - x_{n,\alpha+1}f_{1,\beta+1}^{\leftarrow}) f_{n-1,\alpha-1}^{\rightarrow} = \varphi_{n,\alpha} f_{n-1,\alpha-1}^{\rightarrow},$$

and $\varphi'_{n\alpha} = f_{n-1,\alpha-1}^{\rightarrow}$ is a regular function.

Symmetric arguments show that $\varphi'_{\beta n}$ is also regular.

(b) φ has two non standard neighbors.

This happens when $\varphi = \varphi_{ij}$, and the two non standard neighbors are

$$\begin{aligned} \varphi_{i-1,j-1} &= f_{i-1,j-1}f_{1,\beta+1} - \bar{f}_{i-1,j-1}\bar{f}_{1,\beta+1} \\ \varphi_{i+1,j+1} &= f_{i+1,j+1}f_{1,\beta+1} - \bar{f}_{i+1,j+1}\bar{f}_{1,\beta+1}. \end{aligned}$$

The other neighbors are the same neighbors from the standard case. Denote the corresponding standard exchange rule at f_{ij} by $e_{f_{ij}}$. This is a Desnanot-Jacobi identity (6.13) or the modified version of it (6.14). Let $\bar{e}_{f_{ij}}$ be the same identity with column α (or row β) replaced by column $\alpha + 1$ (or row $\beta + 1$, respectively). In other words if $e_{f_{ij}} = f_{ij} \cdot g$ then $\bar{e}_{f_{ij}} = \bar{f}_{ij} \cdot g$, and the exchange rule is

$$\begin{aligned} \varphi_{ij} \cdot \varphi'_{ij} &= f_{1,\beta+1}e_{f_{ij}} - \bar{f}_{1,\beta+1}\bar{e}_{f_{ij}} \\ &= (f_{1,\beta+1}f_{ij} - \bar{f}_{1,\beta+1}\bar{f}_{ij})g = \varphi_{ij} \cdot g \end{aligned}$$

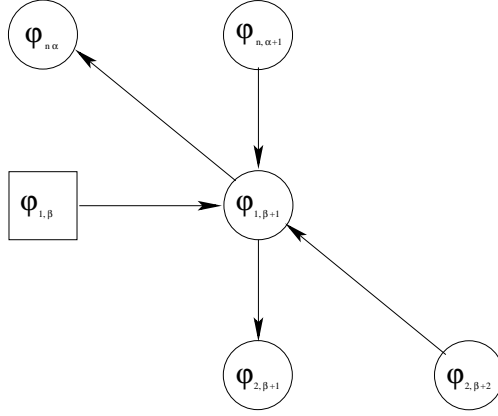
and g is the same regular function as in the standard case.

4. φ is $f_{1,\beta+1}$ or $f_{\alpha+1,1}$ (which were frozen in the standard case).

Assume $\varphi = f_{1,\beta+1}$ with neighbors $\varphi_{n,\alpha}, \varphi_{n,\alpha+1}, \varphi_{1,\beta}, \varphi_{2,\beta+1}, \varphi_{2,\beta+2}$ (Figure 5.3).

The exchange rule is then

$$\varphi \cdot \varphi' = \varphi_{n\alpha}\varphi_{2,\beta+1} + \varphi_{n,\alpha+1}\varphi_{1,\beta}\varphi_{2,\beta+2}.$$

FIGURE 5.3. The neighbors of $\varphi_{1,\beta+1}$

If we put

$$A = \begin{bmatrix} x_{n\alpha} & x_{n,\alpha+1} & 0 & \cdots & 0 \\ x_{1,\beta} & x_{1,\beta+1} & \cdots & \cdots & x_{1n} \\ \vdots & x_{2,\beta+1} & x_{2,\beta+2} & & \vdots \\ \vdots & & & \ddots & \end{bmatrix},$$

then the exchange rule reads

$$\varphi \cdot \varphi' = \det A \cdot \det A_{12}^{\hat{1}\hat{n}} + \det A_2^{\hat{1}} \cdot \det A_1^{\hat{n}}$$

and according to (6.13) $\varphi \cdot \varphi' = \det A_1^{\hat{1}} \det A_2^{\hat{n}}$. Since $\varphi = \det A_1^{\hat{1}}$, we get $\varphi' = \det A_2^{\hat{n}}$, which is regular. The case of $\varphi = \varphi_{\alpha+1,1}$ is symmetric.

This completes the proof, since in all cases the exchanged variable φ' is a regular function. \square

6. TECHNICAL RESULTS AND COMPUTATIONS

We present here the proofs to some technical results that were used in previous sections. The bracket $\{\cdot, \cdot\}$ will be computed through the operator R_+ . Lemma 6.1 explains this operator, while the rest of this section gives more information about the standard bracket and the difference between the $\alpha\beta$ bracket and the standard bracket.

6.1. The operator R_+ . Following Lemma 4.1 in [12], we compute the Sklyanin bracket $\{f, g\}$ associated with an R-matrix r through

$$(6.1) \quad \{f, g\}(X) = \langle R_+(\nabla f(X)X), \nabla g(X)X \rangle - \langle R_+(X\nabla f(X)), X\nabla g(X) \rangle,$$

where $\langle X, Y \rangle = \text{Tr}(XY)$, ∇ is the gradient with respect to the trace-form, and $R_+ \in \text{End } \mathfrak{gl}_n$ as defined in (3.2). For the computations it will be convenient to describe R_+ in a different way: for an element $\eta \in \mathfrak{gl}_n$, let $\eta_{>0}$ and η_0 be the projections of η onto the subalgebra spanned by positive roots, and the Cartan

subalgebra \mathfrak{h} , respectively. Let $h_i = e_{ii} - e_{i+1,i+1}$ be a basis for \mathfrak{h} . The dual basis (defined by $\langle \hat{h}_i, h_j \rangle = \delta_{ij}$) is then

$$\hat{h}_i = \frac{1}{n} \operatorname{diag} \left(\underbrace{(n-i), \dots, (n-i)}_i, \underbrace{-i, \dots, -i}_{(n-i)} \right).$$

Defining

$$(6.2) \quad s_k(j) = \begin{cases} n-j & j \geq k \\ -j & j < k \end{cases}$$

we can write $(\hat{h}_i)_{kk} = \frac{1}{n} s_k(i)$. Next, define the operator R_{diag} on \mathfrak{h} by

$$(6.3) \quad \begin{aligned} R_{\operatorname{diag}}(e_{kk}) &= \sum_{j=1}^{n-1} s_k(j) (\hat{h}_j - \hat{h}_{j-1}) + (s_k(\beta-1) - s_k(\beta)) \hat{h}_\alpha \\ &+ (s_k(\alpha) - s_k(\alpha+1)) \hat{h}_\beta + s_k(\beta) \hat{h}_{\alpha+1} - s_k(\alpha) \hat{h}_{\beta-1}, \end{aligned}$$

Last, for the Belavin–Drinfeld data $\{\alpha\} \mapsto \{\beta\}$ define

$$R_{BD}(\eta) = \eta_{\alpha, \alpha+1} e_{\beta, \beta+1} - \eta_{\beta+1, \beta} e_{\alpha+1, \alpha}.$$

Lemma 6.1. *The operator R_+ acts on $\eta \in \mathfrak{gl}_n$ by*

$$(6.4) \quad R_+(\eta) = \eta_{>0} + R_{\operatorname{diag}}(\eta) + R_{BD}(\eta).$$

Proof. Recall the construction of the R-matrix $r^{\alpha\beta}$ according to (2.6): there is some freedom in choosing the diagonal part r_0 . Following [4, Ch. 3], $r_0 = \sum_{i,j} a_{ij} \hat{h}_{\alpha_i} \otimes \hat{h}_{\alpha_j}$ is determined by the coefficient matrix (a_{ij}) , which is subject to the conditions

$$(6.5) \quad a_{ij} + a_{ji} = (\alpha_i, \alpha_j) \quad \text{if } \alpha_i, \alpha_j \in \Delta$$

$$(6.6) \quad a_{\gamma(i), j} + a_{ji} = 0 \quad \text{if } \alpha_i \in \Gamma_1, \alpha_j \in \Delta.$$

Define two matrices,

$$A_{ij} = \begin{cases} 1 & i = j \\ -1 & i = j + 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$B_{ij} = \begin{cases} 1 & (\alpha_i, \alpha_j) \in \{(\alpha, \beta), (\beta-1, \alpha), (\beta, \alpha+1)\} \\ -1 & (\alpha_i, \alpha_j) \in \{(\beta, \alpha), (\alpha, \beta-1), (\alpha+1, \beta)\} \\ 0 & \text{otherwise.} \end{cases}$$

It is not hard to see that for the BD triple $T_{\alpha\beta} = (\{\alpha\}, \{\alpha+1\}, \gamma: \alpha \mapsto \alpha+1)$ (i.e., $\beta = \alpha+1$) the matrix A satisfies conditions (6.5) and (6.6), and can serve as the coefficient matrix that determines r_0 . When $\beta > \alpha+1$, we take $A+B$ as the coefficient matrix, and conditions (6.5) and (6.6) are satisfied again. So for $\beta = \alpha+1$ take

$$(6.7) \quad r_0^{\alpha\beta} = \sum_{i=1}^{n-1} \hat{h}_i \otimes \hat{h}_i - \sum_{i=1}^{n-2} \hat{h}_{i+1} \otimes \hat{h}_i + \hat{h}_\alpha \wedge \hat{h}_\beta + \hat{h}_{\beta-1} \wedge \hat{h}_\alpha + \hat{h}_\beta \wedge \hat{h}_{\alpha+1},$$

and for $\beta \neq \alpha + 1$

$$(6.8) \quad r_0^{\alpha\beta} = \sum_{i=1}^{n-1} \hat{h}_i \otimes \hat{h}_i - \sum_{i=1}^{n-2} \hat{h}_{i+1} \otimes \hat{h}_i;$$

Note that in the standard case condition (6.6) is empty, so we can use $r_0^{\alpha\beta}$ in the the standard case as well.

To prove the Lemma, it is enough to show that (6.4) holds for all elements of the basis $\{E_\delta\}_{\delta \in \Phi} \cup \{h_k\}_{k=1}^{n-1}$. Recall that $\langle E_{-i}, E_j \rangle = \delta_{ij}$:

1. $\eta = E_\delta$, with $\delta \in \Phi^+$. so

$$\begin{aligned} \langle r, \eta \otimes \zeta \rangle &= \left\langle \left(E_{-\delta} \otimes E_\delta + \sum_{\alpha \prec \beta} E_{-\alpha} \wedge E_\beta \right), E_\delta \otimes \zeta \right\rangle \\ &= \begin{cases} \langle E_\delta, \zeta \rangle & \delta \notin \Gamma_1 \\ \langle E_\delta, \zeta \rangle + \sum_{\delta \prec \beta} \langle E_\beta, \zeta \rangle & \delta \in \Gamma_1, \end{cases} \end{aligned}$$

$$\text{and } R_+(E_\delta) = \begin{cases} E_\delta & \delta \notin \Gamma_1 \\ \left(E_\delta - \sum_{\delta \prec \beta} E_\beta \right) & \delta \in \Gamma_1, \end{cases} \text{ in accordance with (6.4).}$$

2. $\eta = E_{-\delta}$, with $\delta \in \Phi^+$.

$$\begin{aligned} \langle r, \eta \otimes \zeta \rangle &= \left\langle \left(\sum_{\alpha \prec \beta} E_{-\alpha} \wedge E_\beta \right), E_{-\delta} \otimes \zeta \right\rangle \\ &= \begin{cases} 0 & \delta \notin \Gamma_2 \\ -\sum_{\alpha \prec \delta} \langle E_{-\alpha}, \zeta \rangle & \delta \in \Gamma_2, \end{cases} \end{aligned}$$

$$\text{hence } R_+(E_{-\delta}) = \begin{cases} 0 & \delta \notin \Gamma_2 \\ -\sum_{\alpha \prec \delta} \langle E_{-\alpha}, \zeta \rangle & \delta \in \Gamma_2, \end{cases} \text{ which also fits (6.4).}$$

3. $\eta = h_k$.

$$\begin{aligned} \langle r, \eta \otimes \zeta \rangle &= \left\langle \sum_{i=1}^{n-2} \hat{h}_i \wedge \hat{h}_{i+1}, h_k \otimes \zeta \right\rangle \\ &= \langle \hat{h}_{k+1}, \zeta \rangle - \langle \hat{h}_{k-1}, \zeta \rangle \end{aligned}$$

(with $\hat{h}_0 = \hat{h}_n = 0$). Therefore $R_+(h_k) = (\hat{h}_{k+1} - \hat{h}_{k-1})$. Expressing e_{kk} as a linear combination of $\{h_i\}_{i=1}^{n-1} \cup \{\mathbb{1}\}$ implies (6.4).

4. Last, look at $\eta = \mathbb{1}$. Here it is clear that

$$\langle r, \mathbb{1} \otimes \zeta \rangle = 0.$$

This implies $R_+(\mathbb{1}) = 0$, and the proof is complete. \square

6.2. Bracket computations .

Lemma 6.2. *For any two functions f, g on SL_n ,*

$$(6.9) \quad \begin{aligned} \{f, g\}_{\alpha\beta} - \{f, g\}_{std} &= f^{\alpha \leftarrow \alpha + 1} g^{\beta + 1 \leftarrow \beta} - f^{\beta + 1 \leftarrow \beta} g^{\alpha \leftarrow \alpha + 1} \\ &\quad + f_{\beta \leftarrow \beta + 1} g_{\alpha + 1 \leftarrow \alpha} - f_{\alpha + 1 \leftarrow \alpha} g_{\beta \leftarrow \beta + 1}. \end{aligned}$$

Proof. Let $r_{\alpha\beta}$ and r_{std} be the R-matrices associated with BD data $\{\alpha\} \rightarrow \{\beta\}$ and $\emptyset \rightarrow \emptyset$, respectively. Using (6.1), it is easy to see that the difference (6.9) comes from the difference $R_+^{\alpha\beta} - R_+^{std}$. According to Lemma 6.1, this is

$$(6.10) \quad R_+^{\alpha\beta}(\eta) - R_+^{std}(\eta) = \eta_{\alpha,\alpha+1}e_{\beta,\beta+1} - \eta_{\beta+1,\beta}e_{\alpha+1,\alpha}.$$

Write $R_d = R_+^{\alpha\beta} - R_+^{std}$, so now

$$\begin{aligned} & \{f, g\}_{\alpha\beta} - \{f, g\}_{std} \\ &= \langle R_d(\nabla f(X)X), \nabla g(X)X \rangle - \langle R_d(X\nabla f(X)), X\nabla g(X) \rangle \\ &= (\nabla f(X)X)_{\alpha,\alpha+1}(\nabla g(X)X)_{\beta+1,\beta} - (\nabla f(X)X)_{\beta+1,\beta}(\nabla g(X)X)_{\alpha,\alpha+1} \\ &\quad - (X\nabla f(X))_{\alpha,\alpha+1}(X\nabla g(X))_{\beta+1,\beta} - (X\nabla f(X))_{\beta+1,\beta}(X\nabla g(X))_{\alpha,\alpha+1} \\ &= f^{\alpha\leftarrow\alpha+1}g^{\beta+1\leftarrow\beta} - f^{\beta+1\leftarrow\beta}g^{\alpha\leftarrow\alpha+1} - f_{\alpha+1\leftarrow\alpha}g_{\beta\leftarrow\beta+1} + f_{\beta\leftarrow\beta+1}g_{\alpha+1\leftarrow\alpha}. \end{aligned}$$

□

Corollary 6.3. *If $f, g \in \mathcal{B}_{\alpha\beta} \cap \mathcal{B}_{std}$ then $\{f, g\}_{\alpha\beta} = \{f, g\}_{std}$.*

Proof. All functions in \mathcal{B}_{std} are determinants of submatrices of X . Let f_{ij} be such a function as defined in (3.5). The term $f_{ij}^{k\leftarrow m}$ is the determinant of a similar submatrix, with column m replacing column k (i.e., every instance of x_{pk} is replaced by x_{pm}). Therefore, for $f_{ij} \in \mathcal{B}_{std}$, the function $f_{ij}^{\alpha\leftarrow\alpha+1}$ is non zero only if f_{ij} is a determinant of a submatrix that contains column α but not column $\alpha + 1$. The only functions with this property in \mathcal{B}_{std} are determinants of submatrices of the form $X_{[n+j-\alpha, n]}^{[j, \alpha]}$, that is, the functions $f_{n+j-\alpha, j}$ with $j \in [\alpha]$. But these functions are not in $\mathcal{B}_{\alpha\beta}$, because $\alpha \in \Gamma_1$ (see the construction in Section 3.1). Similarly, $f_{m\leftarrow k}$ is the determinant of the matrix obtained by replacing the m -th of X row by the k -th row. So the function $(f_{ij})_{\beta\leftarrow\beta+1}$ is non zero only if f_{ij} is the determinant of a submatrix with row β and without row $\beta + 1$. The only functions in \mathcal{B}_{std} that satisfy this condition are $f_{i, n+i-\beta}$, and these functions are not in $\mathcal{B}_{\alpha\beta}$ because $\beta \in \Gamma_2$ (see Section 3.1 again). □

The next Lemma describes the “building blocks” of the functions in $\mathcal{B}_{\alpha\beta} \setminus \mathcal{B}_{std}$ and the Poisson coefficients of these functions with respect to the standard bracket.

Lemma 6.4. *1. The function $f_{n+k-\alpha, k}^{\rightarrow}$ (with $k \in [\alpha]$) is log canonical with all functions $g \in \mathcal{B}_{\alpha\beta} \cap \mathcal{B}_{std}$, provided $g \neq f_{n+m-\alpha, m}$ for some $m > k$, w.r.t. the standard bracket $\{\cdot, \cdot\}_{std}$. In this case the Poisson coefficient is*

$$(6.11) \quad \omega_{f_{n+k-\alpha, k}^{\rightarrow}, g} = \omega_{f_{n+k-\alpha, k}, g} + \omega_{x_{n, \alpha+1}, g} - \omega_{x_{n, \alpha}, g}.$$

2. The function $f_{1, \beta+1}^{\leftarrow}$ is log canonical with all functions $g \in \mathcal{B}_{\alpha\beta} \cap \mathcal{B}_{std}$, provided $g \neq f_{m, \beta+1}$ for some $m \in [2, n]$ w.r.t. the standard bracket $\{\cdot, \cdot\}_{std}$. In this case the Poisson coefficient is

$$(6.12) \quad \omega_{f_{1, \beta+1}^{\leftarrow}, g} = \omega_{f_{1, \beta+1}, g} + \omega_{x_{n, \beta}, g} - \omega_{x_{n, \beta+1}, g}.$$

Proof. The proof will use the *Desnanot–Jacobi* identity (see [3]): for a square matrix A , denote by “hatted” subscripts and superscripts deleted rows and columns, respectively. Then

$$(6.13) \quad \det A \cdot \det A_{\hat{r}_1, \hat{r}_2}^{\hat{c}_1, \hat{c}_2} = \det A_{\hat{r}_1}^{\hat{c}_1} \cdot \det A_{\hat{r}_2}^{\hat{c}_2} - \det A_{\hat{r}_2}^{\hat{c}_1} \cdot \det A_{\hat{r}_1}^{\hat{c}_2}.$$

By adding an appropriate row, we get a similar result for a non square matrix B with number of rows greater by one than the number of columns:

$$(6.14) \quad \det B_{\hat{r}_1} \det B_{\hat{r}_2 \hat{r}_3}^{\hat{c}_1} = \det B_{\hat{r}_2} \det B_{\hat{r}_1 \hat{r}_3}^{\hat{c}_1} - \det B_{\hat{r}_3} \det B_{\hat{r}_1 \hat{r}_2}^{\hat{c}_1},$$

and naturally, a similar identity can be obtained for a matrix with number of columns greater by one than the number of rows.

Start with statement 1. of the Lemma. We will show that $f_{n+k-\alpha, k}^{\rightarrow}$ is a cluster variable that can be obtained from the initial cluster through the mutation sequence $(f_{n\alpha}, f_{n-1, \alpha-1}, \dots, f_{n+k-1-\alpha, k-1})$. Look at the initial quiver described in Section 3.2, and mutate in direction $f_{n\alpha}$. We can assume $\alpha > 1$ because if $\alpha = 1$ then $f_{n\alpha}$ is frozen. In this case statement 1 holds trivially, with $f_{n\alpha}^{\rightarrow} = f_{n, \alpha+1} \in \mathcal{B}_{std}$. For $\alpha > 1$ the exchange rule is

$$(6.15) \quad \begin{aligned} f_{n\alpha} \cdot f'_{n\alpha} &= f_{n, \alpha-1} f_{n-1, \alpha} + f_{n-1, \alpha-1} f_{n, \alpha+1} \\ &= x_{n, \alpha-1} \begin{vmatrix} x_{n-1, \alpha} & x_{n-1, \alpha+1} \\ x_{n\alpha} & x_{n, \alpha+1} \end{vmatrix} + \begin{vmatrix} x_{n-1, \alpha-1} & x_{n-1, \alpha} \\ x_{n, \alpha-1} & x_{n\alpha} \end{vmatrix} x_{n, \alpha+1} \\ &= x_{n\alpha} \begin{vmatrix} x_{n-1, \alpha-1} & x_{n-1, \alpha+1} \\ x_{n, \alpha-1} & x_{n, \alpha+1} \end{vmatrix} = x_{n\alpha} f_{n-1, \alpha-1}^{\rightarrow}, \end{aligned}$$

which implies $f'_{n\alpha} = f_{n-1, \alpha-1}^{\rightarrow}$. The arrows of the quiver take the following changes: the arrows $(n, \alpha+1) \rightarrow (n-1, \alpha)$, $(n-1, \alpha-1) \rightarrow (n, \alpha-1)$ and $(n-1, \alpha-1) \rightarrow (n-1, \alpha)$ are removed, and an arrow $(n, \alpha-1) \rightarrow (n, \alpha+1)$ is added. All arrows incident to (n, α) are inverted. Therefore the exchange rule at $(n-1, \alpha-1)$ is now

$$f_{n-1, \alpha-1} \cdot f'_{n-1, \alpha-1} = f'_{n\alpha} f_{n-2, \alpha-2} + f_{n-2, \alpha-1} f_{n-1, \alpha-2}.$$

Proceed with the mutation sequence $(f_{n\alpha}, f_{n-1, \alpha-1}, \dots, f_{n+k-1-\alpha, k-1})$. Assume by induction that for $m \in [\alpha]$, mutating at $f_{n+m-\alpha, m}$ yields the exchanged variable

$$(6.16) \quad f'_{n+m-\alpha, m} = f_{n-m-1, \alpha-m-1}^{\rightarrow},$$

and that the exchange rule at $f_{n-m-1, \alpha-m-1}$ is now

$$\begin{aligned} f_{n-m-1, \alpha-m-1} \cdot f'_{n-m-1, \alpha-m-1} &= f'_{n+m-\alpha, m} f_{n-m-2, \alpha-m-2} \\ &\quad + f_{n-m-2, \alpha-m-1} f_{n-m-1, \alpha-m-2}. \end{aligned}$$

Write $A = X_{[n-m-2, n]}^{[\alpha-m-2, \alpha+1]}$ and let ℓ be the last column of A . Using (6.14) we get

$$\begin{aligned} &f_{n-m-1, \alpha-m-1} \cdot f'_{n-m-1, \alpha-m-1} \\ &= \det A_{\hat{1}, \widehat{\ell-1}}^{\hat{1}, \ell-1} \det A^{\hat{\ell}} + \det A_{\hat{1}, \widehat{\ell-1}, \hat{\ell}}^{\widehat{\ell-1}, \hat{\ell}} \det A^{m \hat{-} 2} \\ &= \det A_{\hat{1}, \hat{\ell}}^{\hat{1}, \hat{\ell}} \det A^{\widehat{\ell-1}} \\ &= f_{n-m-1, \alpha-m-1} \cdot f_{n-m-2, \alpha-m-2}^{\rightarrow}, \end{aligned}$$

and therefore

$$f'_{n-m-1, \alpha-m-1} = f_{n+m-2, \alpha-m-2}^{\rightarrow}$$

The quiver mutates as follows: arrows $(n-m-2, \alpha-m-2) \rightarrow (n-m-2, \alpha-m-1)$ and $(n-m-2, \alpha-m-2) \rightarrow (n-m-1, \alpha-m-2)$ are removed, arrows $(n-m-2, \alpha-m-1) \rightarrow (n+m-\alpha, m)$ and $(n-m-1, \alpha-m-2) \rightarrow (n+m-\alpha, m)$ added, and all arrows incident to $n-m-1, \alpha-m-1$ are inverted. Therefore the mutation rule at

the next cluster variable of the sequence will be now

$$\begin{aligned} f_{n-m-2,\alpha-m-2} \cdot f'_{n-m-2,\alpha-m-2} &= f'_{n-m-1,\alpha-m-1} f_{n-m-3,\alpha-m-3} \\ &\quad + f_{n-m-3,\alpha-m-2} f_{n-m-2,\alpha-m-3}. \end{aligned}$$

That proves that after the mutation sequence $(f_{n\alpha}, f_{n-1,\alpha-1}, \dots, f_{n-k+1,\alpha-k+1})$ we get $f'_{n-k+1,\alpha-k+1} = f_{n+k-\alpha,k}^{\rightarrow}$ and therefore it is log canonical with all cluster variables of the initial cluster, except for $(f_{n\alpha}, f_{n-1,\alpha-1}, \dots, f_{n-k+1,\alpha-k+1})$ that were mutated.

Now for $g \neq f_{n+m-\alpha,m}$ with $m > k+1$, the coefficient $\omega_{f_{n+k-\alpha,k},g}$ can be computed: from the Leibniz rule for Poisson brackets, any triple of functions f_1, f_2, g such that $\{f_1, g\} = \omega_1 f_1 g$ and $\{f_2, g\} = \omega_2 f_2 g$, must satisfy

$$\{f_1 f_2, g\} = (\omega_2 + \omega_1) f_1 f_2 g,$$

or, in other words $\omega_{f_1 f_2, g} = \omega_{f_1, g} + \omega_{f_2, g}$. Applying this together with the linearity of the bracket to the exchange rule (6.15) we get

$$\omega_{f_{n\alpha}, g} + \omega_{f'} = \omega_{f_{n-1,\alpha-1}, g} + \omega_{x_{n,\alpha+1}, g},$$

which is

$$(6.17) \quad \omega_{f_{n-1,\alpha-1}, g}^{\rightarrow} = \omega_{f_{n-1,\alpha-1}, g} + \omega_{x_{n,\alpha+1}, g} - \omega_{f_{n\alpha}, g}.$$

Again, we proceed inductively: assume that

$$\omega_{f_{n-k+1,\alpha-k+1}, g}^{\rightarrow} = \omega_{f_{n+k-\alpha,k}, g} + \omega_{x_{n,\alpha+1}, g} - \omega_{x_{n\alpha}, g}$$

and the exchange rule at $f_{n+k-\alpha,k}$ is

$$f_{n+k-\alpha,k} \cdot f_{n-k-1,\alpha-k-1}^{\rightarrow} = f_{n-k+1,\alpha-k+1}^{\rightarrow} f_{n-k-1,\alpha-k-1} + f_{n+k-\alpha,k-1} f_{n-k-1,\alpha-k}.$$

This means that

$$\omega_{f_{n-k-1,\alpha-k-1}, g}^{\rightarrow} = \omega_{f_{n-k+1,\alpha-k+1}, g}^{\rightarrow} + \omega_{f_{n-k-1,\alpha-k-1}, g} - \omega_{f_{n+k-\alpha,k}, g},$$

and recursively this leads to

$$(6.18) \quad \omega_{f_{n-k-1,\alpha-k-1}, g}^{\rightarrow} = \omega_{f_{n-k-1,\alpha-k-1}, g} + \omega_{x_{n,\alpha+1}, g} - \omega_{x_{n\alpha}, g},$$

which complete the proof of statement 1.

Next, look at statement 2. Here also, we will show that $f_{1,\beta+1}^{\leftarrow}$ is a cluster variable that can be obtained through a mutation sequence, which in this case is $(f_{n,\beta+1}, f_{n-1,\beta+1}, \dots, f_{2,\beta+1})$. First, mutate at $f_{n,\beta+1}$. It is easy to see, just like in (6.15) that

$$f' = \begin{vmatrix} x_{n-1,\beta} & x_{n-1,\beta+2} \\ x_{n,\beta} & x_{n,\beta+2} \end{vmatrix} = f_{n-1,\beta+1}^{\leftarrow}.$$

Just like we have already showed above, arrows $(n, \beta+2) \rightarrow (n-1, \beta+1)$, $(n-1, \beta) \rightarrow (n, \beta)$ and $(n-1, \beta) \rightarrow (n-1, \beta+1)$ are removed from the quiver, and an arrow $(n, \beta) \rightarrow (n, \beta+2)$ is added to it. In addition, all the arrows adjacent to $(n, \beta+1)$ are inverted. The exchange rule at $f_{n-1,\beta+1}$ is now

$$f_{n-1,\beta+1} f'_{n-1,\beta+1} = f'_{n,\beta+1} f_{n-2,\beta+1} + f_{n-1,\beta+2} f_{n-2,\beta}.$$

Again we use induction on m with the mutation sequence $(f_{n,\beta+1}, f_{n-1,\beta+1}, \dots, f_{m,\beta+1})$. Assume that after mutating at $f_{m+1,\beta+1}$ we got

$$f' = f_{m,\beta+1}^{\leftarrow}$$

and that the exchange rule at $f_{m,\beta+1}$ is

$$(6.19) \quad f_{m,\beta+1} \cdot f'_{m,\beta+1} = f'_{m+1,\beta+1} f_{m-1,\beta+1} + f_{m,\beta+2} f_{m-1,\beta}.$$

If $m > \beta + 1$ then we can set $\mu = \mu(\beta, m-1)$ and $B = X_{[m-1,n]}^{[\beta,\mu+1]}$. Then the exchange rule is

$$\begin{aligned} f_{m,\beta+1} \cdot f'_{m,\beta+1} &= \det B^{\hat{\beta}} \det B_{m-1}^{\widehat{2\mu+\beta}} + \det B^{\widehat{\mu+1}} \det B_{m-1}^{\widehat{2\beta}} \\ &= \det B^{\hat{2}} \det B_{m-1}^{\widehat{\beta\mu+1}} = f_{m-1,\beta+1}^{\leftarrow} f_{m,\beta+1}. \end{aligned}$$

If, on the other hand, $m \leq \beta + 1$ we set $\mu = \mu(\beta, m-1)$ and $A = X_{[m-1,\mu]}^{[\beta,n]}$ so that the exchange rule becomes

$$\begin{aligned} f_{m,\beta+1} \cdot f'_{m,\beta+1} &= \det A \det A_{m-1}^{\widehat{\beta\beta+1}} + \det A_{\mu}^{\hat{\beta}} \det A_{m-1}^{\widehat{\beta+1}} \\ &= \det A_{m-1}^{\hat{\beta}} \det A_{\mu}^{\widehat{\beta+1}} = f_{m,\beta+1} f_{m-1,\beta+1}^{\leftarrow}, \end{aligned}$$

hence

$$f'_{m,\beta+1} = f_{m-1,\beta+1}^{\leftarrow}.$$

It is easy to see that the mutation of the quiver also agrees with the induction hypothesis, and we can conclude that after the mutation sequence

$$(6.20) \quad f'_{2,\beta+1} = f_{1,\beta+1}^{\leftarrow},$$

and therefore $f_{1,\beta+1}^{\leftarrow}$ is log canonical with all functions $f_{ij} \in \mathcal{B}_{std}$, excluding the functions $f_{m,\beta+1}$ that were mutated on the way.

We can now compute the coefficients $\omega_{f_{m-1,\beta+1}^{\leftarrow},g}$ recursively like we did in the first statement and get for every $f_{m,\beta+1} \neq g \in \mathcal{B}_{std}$,

$$(6.21) \quad \omega_{f_{1,\beta+1}^{\leftarrow},g} = \omega_{f_{1,\beta+1},g} + \omega_{x_{n,\beta},g} - \omega_{x_{n,\beta+1},g}.$$

This completes the proof for statement 2. \square

The functions $f_{1,\beta+1}^{\leftarrow}$ and $f_{\alpha+1,1}^{\uparrow}$ need some special attention:

Lemma 6.5. *For $k \in [\beta]$, the function $f_{k,n+k-\beta}^{\downarrow}$ is log canonical with $f_{1,\beta+1}^{\leftarrow}$ and $f_{\alpha+1,1}^{\uparrow}$. The Poisson coefficients are*

$$\begin{aligned} \omega_{f_{k,n+k-\beta}^{\downarrow}, f_{1,\beta+1}^{\leftarrow}} &= \omega_{f_{k,n+k-\beta}, f_{1,\beta+1}} + \omega_{f_{\beta+1,n}, f_{1,\beta+1}} - \omega_{f_{\beta n}, f_{1,\beta+1}} \\ &\quad + \omega_{f_{k,n+k-\beta}, f_{n\beta}} + \omega_{f_{\beta+1,n}, f_{n\beta}} - \omega_{f_{\beta n}, f_{n\beta}} \\ &\quad - \omega_{f_{k,n+k-\beta}, f_{n,\beta+1}} - \omega_{f_{\beta+1,n}, f_{n,\beta+1}} + \omega_{f_{\beta n}, f_{n,\beta+1}}, \\ \omega_{f_{k,n+k-\beta}^{\downarrow}, f_{\alpha+1,1}^{\uparrow}} &= \omega_{f_{k,n+k-\beta}, f_{\alpha+1,1}} + \omega_{f_{\beta+1,n}, f_{\alpha+1,1}} - \omega_{f_{\beta n}, f_{\alpha+1,1}} \\ &\quad + \omega_{f_{k,n+k-\beta}, f_{\alpha n}} + \omega_{f_{\beta+1,n}, f_{\alpha n}} - \omega_{f_{\beta n}, f_{\alpha n}} \\ &\quad - \omega_{f_{k,n+k-\beta}, f_{\alpha+1,n}} - \omega_{f_{\beta+1,n}, f_{\alpha+1,n}} + \omega_{f_{\beta n}, f_{\alpha+1,n}}. \end{aligned}$$

Proof. Naturally, Lemma 6.4 could be helpful, but it may seem that the proof does not hold: since the path $(f_{\beta n}, f_{\beta-1,n-1}, \dots)$ crosses the paths $(f_{\alpha+1,n}, f_{\alpha+1,n-1}, \dots)$ and $(f_{n,\beta+1}, f_{n-1,\beta+1}, \dots)$, applying the first mutation sequence followed by the second (or the third) one, will not yield the function $f_{\alpha+1,1}^{\uparrow}$ (or $f_{1,\beta+1}^{\leftarrow}$), because one of the cluster variables had been mutated in the first sequence. However, this can be easily settled. First, apply the sequence $(f_{\beta n}, f_{\beta-1,n-1}, \dots)$. Now shift every mutated vertex $(\beta - m, n - m)$ of the new quiver to the place $(\beta - m - 1, n - m - 1)$

i.e., move it one row up and one column to the left. The quiver now looks locally just like the initial one, with two changes at $f_{\beta n}$ and at $f_{1, n+1-\beta}$. Then, set $\ell = 2\beta - n + 1$. Note that if $\ell \leq 1$ the paths do not cross each other, and there is no problem. Now apply the sequence $(f_{n, \beta+1}, f_{n-1, \beta+1}, \dots, f_{\ell+1, \beta+1})$. The quiver then reads the exact same exchange rules as the initial quiver. At $f_{\ell+1, \beta+1}$ the exchange rule is then almost the same as it was in the proof above, with one change: the function $f_{\ell, \beta+1}$ is now replaced by $f_{\ell-1, \beta+1}^\downarrow$. The exchange rule is

$$f_{\ell+1, \beta+1} \cdot f'_{\ell+1, \beta+1} = f_{\ell, \beta} f_{\ell+2, \beta+2} + f_{\ell+1, \beta+1}^{\leftarrow} f_{\ell, \beta+1}^\downarrow.$$

So write $A = X_{[\ell, \beta+1]}^{[\beta, n]}$ and then

$$\begin{aligned} f_{\ell+1, \beta+1} \cdot f'_{\ell+1, \beta+1} &= \det A \det A_{\ell \ell+1}^{\widehat{\beta \beta+1}} + \det A_{\ell}^{\widehat{\beta+1}} \det A_{\beta}^{\widehat{\beta}} \\ &= \det A_{\ell}^{\widehat{\beta}} \det A_{\beta}^{\widehat{\beta+1}} = f_{\ell+1, \beta+1} f_{\ell, \beta+1}^{\leftarrow}. \end{aligned}$$

The picture is slightly different in the special case of $\beta = n - 1$, because now the column $\beta + 1$ is the last one, but it is not hard to see that the result is still

$$(6.22) \quad f'_{\ell+1, \beta+1} = f'_{nn} = x_{n-2, n-1} = f_{\ell, \beta+1}^{\leftarrow}.$$

Moving to the next step of the sequence, we mutate at $(\ell, \beta + 1)$. The corresponding cluster variable is $f'_{\ell, \beta+1} = f_{\ell-1, \beta+1}^{\leftarrow}$ (since it was mutated in the sequence $(f_{\beta n}, f_{\beta-1, n-1}, \dots)$). The exchange rule here reads

$$\begin{aligned} f'_{\ell, \beta+1} \cdot f''_{\ell, \beta+1} &= f'_{\ell, \beta+1} f_{\ell, \beta+2} + f'_{\ell+1, \beta+1} f_{\ell-1, \beta+1} \\ &= f_{\ell-1, \beta+1}^{\leftarrow} f_{\ell, \beta+2} + f_{\ell, \beta}^\downarrow f_{\ell-1, \beta+1} \end{aligned}$$

and (6.13) can be used again, with $A = X_{[\ell-1, \beta-1, \beta+1]}^{[\beta, n]}$. The result is

$$f''_{\ell, \beta+1} = f_{\ell, \beta+1}^{\leftarrow},$$

and again, the same result can be obtained in the case $\beta = n - 1$. So just like in the proof of Lemma 6.4 we still get $f_{1, \beta+1}^{\leftarrow}$ as a cluster variable, and so it is log canonical with all the functions of the form $f_{k, n+k-\beta}^\downarrow$.

The Poisson coefficients can now be computed just like in Lemma 6.4 so

$$\begin{aligned} \omega_{f_{k, n+k-\beta}^\downarrow, f_{1, \beta+1}^{\leftarrow}} &= \omega_{f_{k, n+k-\beta}, f_{1, \beta+1}} + \omega_{f_{\beta+1, n}, f_{1, \beta+1}} - \omega_{f_{\beta n}, f_{1, \beta+1}} \\ &\quad + \omega_{f_{k, n+k-\beta}, f_{n\beta}} + \omega_{f_{\beta+1, n}, f_{n\beta}} - \omega_{f_{\beta n}, f_{n\beta}} \\ &\quad - \omega_{f_{k, n+k-\beta}, f_{n, \beta+1}} - \omega_{f_{\beta+1, n}, f_{n, \beta+1}} + \omega_{f_{\beta n}, f_{n, \beta+1}}. \end{aligned}$$

This can be done in the same way with the sequence $(f_{\alpha+1, n}, \dots, f_{\alpha+1, 2})$ to show that $f_{\alpha+1, 1}^\uparrow$ is also log canonical with all $f_{k, n+k-\beta}^\downarrow$. The Poisson coefficient will be

$$\begin{aligned} \omega_{f_{k, n+k-\beta}^\downarrow, f_{\alpha+1, 1}^\uparrow} &= \omega_{f_{k, n+k-\beta}, f_{\alpha+1, 1}} + \omega_{f_{\beta+1, n}, f_{\alpha+1, 1}} - \omega_{f_{\beta n}, f_{\alpha+1, 1}} \\ &\quad + \omega_{f_{k, n+k-\beta}, f_{\alpha n}} + \omega_{f_{\beta+1, n}, f_{\alpha n}} - \omega_{f_{\beta n}, f_{\alpha n}} \\ &\quad - \omega_{f_{k, n+k-\beta}, f_{\alpha+1, n}} - \omega_{f_{\beta+1, n}, f_{\alpha+1, n}} + \omega_{f_{\beta n}, f_{\alpha+1, n}}. \end{aligned}$$

□

The following Lemma computes the brackets of a function $f \in \mathcal{B}_{\alpha\beta} \cap \mathcal{B}_{std}$ with certain families of functions in \mathcal{B}_{std} .

Lemma 6.6. 1. Let $g = f_{k,\beta+1}$ with $k \in [2, n]$. Then

$$(6.23) \quad \{f_{1,\beta+1}^{\leftarrow}, g\}_{std} = (\omega_{f_{1,\beta+1},g} + \omega_{x_n,\beta,g} - \omega_{x_n,\beta+1,g}) f_{1,\beta+1}^{\leftarrow} g + f_{1,\beta+1} g^{\leftarrow}$$

2. For $k \in [\alpha - 1]$, let $g = f_{n+m-\alpha,m}$ with $m \in [k + 1, \alpha]$. Then

$$(6.24) \quad \{g, f_{n+k-\alpha,k}^{\rightarrow}\}_{std} = (\omega_{g,f_{n+k-\alpha,k}} - \omega_{g,x_n\alpha} + \omega_{g,x_n,\alpha+1}) g f_{n+k-\alpha,k}^{\rightarrow} + f_{n+k-\alpha,k} g^{\rightarrow}$$

Proof. 1. Let $g = f_{k,\beta+1}$. we compute the bracket $\{f_{1,\beta+1}^{\leftarrow}, g\}_{std}$ directly using (6.1). Recall that

$$(\nabla f_{1,\beta+1}^{\leftarrow} \cdot X)_{ij} = \sum_{m=1}^n \frac{\partial f_{1,\beta+1}^{\leftarrow}}{\partial x_{mi}} x_{mj} = (f_{1,\beta+1}^{\leftarrow})^{i \leftarrow j}$$

and since $f_{1,\beta+1}^{\leftarrow} = \det X_{[1,n-\beta]}^{[\beta,\beta+2,\dots,n]}$ we have

$$(f_{1,\beta+1}^{\leftarrow})^{i \leftarrow j} = 0$$

for $i < \beta$ and $i = \beta + 1$. Similarly, the term

$$(\nabla g \cdot X)_{ij} = \sum_{m=1}^n \frac{\partial g}{\partial x_{mi}} x_{mj} = g^{i \leftarrow j}$$

vanishes for $i < \beta + 1$. On the other hand, looking at the second trace form in (6.1),

$$(X \cdot \nabla f_{1,\beta+1}^{\leftarrow})_{ij} = \sum_{m=1}^n \frac{\partial f_{1,\beta+1}^{\leftarrow}}{\partial x_{jm}} x_{im} = (f_{1,\beta+1}^{\leftarrow})_{j \leftarrow i},$$

which vanishes for $j > n - \beta$, and also

$$(X \cdot \nabla g)_{ij} = \sum_{m=1}^n \frac{\partial g}{\partial x_{jm}} x_{im} = g_{j \leftarrow i}$$

is non zero only for $k \leq j \leq n + k - \beta - 1$. Applying R_+ to the matrices $\nabla \tilde{f}_k \cdot X$ and $X \cdot \nabla \tilde{f}_k$ vanishes all entries below the main diagonal. On the main diagonal we have only the original function with some coefficients ξ_i . So we can write (6.1) as:

$$\begin{aligned} \{f_{1,\beta+1}^{\leftarrow}, g\} &= \langle R_+ (\nabla f_{1,\beta+1}^{\leftarrow} \cdot X), \nabla g \cdot X \rangle - \langle R_+ (X \cdot \nabla f_{1,\beta+1}^{\leftarrow}), X \cdot \nabla g \rangle^{\leftarrow} \\ &= \sum_{i < j} (f_{1,\beta+1}^{\leftarrow})^{i \leftarrow j} g^{j \leftarrow i} + \sum_i \xi_i f_{1,\beta+1}^{\leftarrow} g^{i \leftarrow i} \\ &\quad - \sum_{i < j} (f_{1,\beta+1}^{\leftarrow})_{j \leftarrow i} g_{i \leftarrow j} - \sum_i \xi'_i f_{1,\beta+1}^{\leftarrow} g_{i \leftarrow i}, \end{aligned}$$

Look at the term $(f_{1,\beta+1}^{\leftarrow})^{i \leftarrow j}$: whenever $(i, j) \neq (\beta, \beta + 1)$ it vanishes, because $f_{1,\beta+1}^{\leftarrow} = \det X_{[1,n-\beta]}^{[\beta,\beta+2,\dots,n]}$ and so $(f_{1,\beta+1}^{\leftarrow})^{i \leftarrow j}$ is the determinant of a submatrix with two identical columns ($j > i$). The only non zero term here is then $(f_{1,\beta+1}^{\leftarrow})^{\beta \leftarrow \beta+1} = f_{1,\beta+1}$. Similarly, $(f_{1,\beta+1}^{\leftarrow})_{j \leftarrow i}$ must vanish when $i < j$, because it is the determinant of a submatrix with two identical rows. Therefore, the only non zero terms of the trace form are $f_{1,\beta+1} g^{\leftarrow}$ and the diagonal ones. The latter are just the product of the two functions multiplied by the coefficients ξ_i and ξ'_i . Note that $f_{1,\beta+1}^{i \leftarrow i}$ vanishes when $i < \beta + 1$, and $\tilde{f}_k^{i \leftarrow i}$ vanishes for $i < \beta$ and for $i = \beta + 1$.

Comparing these coefficients with the coefficients of the bracket $\{f_{1,\beta+1}, g\}$, we see that the only difference is the contribution of the elements in entries (β, β) and $(\beta + 1, \beta + 1)$:

$$\begin{aligned} (\nabla f_{1,\beta+1} \cdot X)_{\beta,\beta} &= 0 \\ (\nabla f_{1,\beta+1} \cdot X)_{\beta+1,\beta+1} &= f_{1,\beta+1} \\ (\nabla f_{1,\beta+1}^{\leftarrow} \cdot X)_{\beta,\beta} &= f_{1,\beta+1}^{\leftarrow} \\ (\nabla f_{1,\beta+1}^{\leftarrow} \cdot X)_{\beta+1,\beta+1} &= 0 \end{aligned}$$

And this is just the same for $x_{n,\beta}$ and $x_{n,\beta+1}$:

$$\begin{aligned} (\nabla x_{n,\beta+1} \cdot X)_{\beta,\beta} &= 0 \\ (\nabla x_{n,\beta+1} \cdot X)_{\beta+1,\beta+1} &= x_{n,\beta+1} \\ (\nabla x_{n,\beta} \cdot X)_{\beta,\beta} &= x_{1,\beta} \\ (\nabla x_{n,\beta} \cdot X)_{\beta+1,\beta+1} &= 0. \end{aligned}$$

Hence, we can conclude

$$(6.25) \quad \{f_{1,\beta+1}^{\leftarrow}, g\}_{std} = (\omega_{f_{1,\beta+1},g} + \omega_{x_{n,\beta},g} - \omega_{x_{n,\beta+1},g}) f_{1,\beta+1}^{\leftarrow} g + f_{1,\beta+1} g^{\leftarrow}.$$

2. The proof here follows a similar path: from (6.1) we have

$$\begin{aligned} \{g, f_{n+k-\alpha,k}^{\rightarrow}\}_{std} &= \langle R_+ (\nabla g \cdot X), \nabla f_{n+k-\alpha,k}^{\rightarrow} \cdot X \rangle \\ &\quad - \langle R_+ (X \cdot \nabla g), X \cdot \nabla f_{n+k-\alpha,k}^{\rightarrow} \rangle \end{aligned}$$

and since R_+ annihilates all the entries below the main diagonal,

$$\begin{aligned} \{g, f_{n+k-\alpha,k}^{\rightarrow}\}_{std} &= \sum_{i=m}^{\alpha} \sum_{j=i+1}^{\alpha-1} g^{i \leftarrow j} (f_{n+k-\alpha,k}^{\rightarrow})^{j \leftarrow i} \\ &\quad + \sum_{i=m}^{\alpha} g^{i \leftarrow \alpha+1} (f_{n+k-\alpha,k}^{\rightarrow})^{\alpha+1 \leftarrow i} + \sum_{j=1}^n \xi_j g f_{n+k-\alpha,k}^{\rightarrow} \\ &\quad - \sum_{j=n+m-\alpha}^n \sum_{i < j} g^{j \leftarrow i} (f_{n+k-\alpha,k}^{\rightarrow})_{i \leftarrow j} \\ &\quad - \sum_{j=1}^n \xi'_j g f_{n+k-\alpha,k}^{\rightarrow} \end{aligned}$$

where ξ_j and ξ'_j are some coefficients. But $f_{n+k-\alpha,k}^{\rightarrow} = \det X_{[n-k,n]}^{[n-k,\dots,\alpha-1,\alpha+1]}$, and therefore for every $i \in [m, \alpha - 1]$ and $j \in [i + 1, \alpha - 1]$ we get

$$(f_{n+k-\alpha,k}^{\rightarrow})^{j \leftarrow i} = 0,$$

because it is the determinant of a matrix with two identical columns. For the same reason, $(f_{n+k-\alpha,k}^{\rightarrow})^{\alpha+1 \leftarrow i}$ vanishes for every $i \neq \alpha$. Likewise, the term $(f_{n+k-\alpha,k}^{\rightarrow})_{i \leftarrow j}$ is zero for every $j \in [n + m - \alpha, n]$ and $i < j$, because this is also a determinant of a matrix with two identical columns. So we are left with

$$\begin{aligned} \{g, f_{n+k-\alpha,k}^{\rightarrow}\}_{std} &= \xi g f_{n+k-\alpha,k}^{\rightarrow} + g^{\alpha \leftarrow \alpha+1} (f_{n+k-\alpha,k}^{\rightarrow})^{\alpha+1 \leftarrow \alpha} \\ &= \xi g f_{n+k-\alpha,k}^{\rightarrow} + g^{\rightarrow} f_{n+k-\alpha,k}, \end{aligned}$$

for some coefficient ξ . Now, compare the coefficients ξ_j and ξ'_j in the bracket $\{g, f_{n+k-\alpha}^{\rightarrow}\}$ to those of $\{g, f_{n+k-\alpha,k}\}$. The difference is equal to the difference between these coefficients in $\{g, x_{n,\alpha+1}\}$ and $\{g, x_{n\alpha}\}$. to see that, note that these functions are determinants of submatrices of X that are distinguished only by the last column, which is $\alpha + 1$ in the first case and α in the second. The result, like in (6.23) is

$$(6.26) \quad \{g, f_{n+k-\alpha,k}^{\rightarrow}\}_{std} = (\omega_1 - \omega_2 + \omega_3) f_{n+k-\alpha,k}^{\rightarrow} g + f_{n+k-\alpha,k} g^{\rightarrow},$$

with

$$\begin{aligned} \omega_1 &= \omega_{g, f_{n+k-\alpha,k}} \\ \omega_2 &= \omega_{g, x_{n,\alpha}} \\ \omega_3 &= \omega_{g, x_{n,\alpha+1}}. \end{aligned}$$

□

The Lemmas 6.4, 6.5 and 6.6 can be rephrased in a symmetric way: transpose rows and columns of the matrix, so $x_{ij} \longleftrightarrow x_{ji}$ (and therefore $f_{ij} \longleftrightarrow f_{ji}$) and switch α and β . The proofs are identical.

Lemma 6.7. 1. The function $f_{k,n+k-\beta}^{\downarrow}$ (with $k \in [\beta]$) is log canonical with all functions $g \in \mathcal{B}_{\alpha\beta} \cap \mathcal{B}_{std}$, provided $g \neq f_{m,n+m-\beta}$ for some $m < k$, w.r.t. the standard bracket $\{\cdot, \cdot\}_{std}$. In this case the Poisson coefficient is

$$(6.27) \quad \omega_{f_{k,n+k-\beta}^{\downarrow}, g} = \omega_{f_{k,n+k-\beta}, g} + \omega_{x_{\beta+1,n}, g} - \omega_{x_{\beta n}, g}.$$

2. The function $f_{\alpha+1,1}^{\uparrow}$ is log canonical with all functions $g \in \mathcal{B}_{\alpha\beta} \cap \mathcal{B}_{std}$, provided $g \neq f_{\alpha+1,m}$ for some $m \in [2, n]$ w.r.t. the standard bracket $\{\cdot, \cdot\}_{std}$. In this case the Poisson coefficient is

$$(6.28) \quad \omega_{f_{\alpha+1,1}^{\uparrow}, g} = \omega_{f_{\alpha+1,1}, g} + \omega_{x_{\alpha n}, g} - \omega_{x_{n,\alpha+1}, g}.$$

Proof. See proof of Lemma 6.4. □

Lemma 6.8. For $k \in [\alpha]$, the function $f_{n+k-\alpha,k}^{\rightarrow}$ is log canonical with $f_{1,\beta+1}^{\leftarrow}$ and $f_{\alpha+1,1}^{\uparrow}$.

Proof. See Proof of Lemma 6.5. The path $((n, \alpha), (n-1, \alpha-1), \dots)$ can not cross the path $((n, \beta+1), (n-1, \beta+1), \dots)$ since we assume $\alpha < \beta$. Proving that $f_{n+k-\alpha,k}^{\rightarrow}$ and $f_{\alpha+1,1}^{\uparrow}$ are log canonical is symmetric proving Lemma 6.5. □

Lemma 6.9. 1. Let $g = f_{\alpha+1,k}$ with $k \in [2, n]$. Then

$$(6.29) \quad \left\{ f_{\alpha+1,1}^{\uparrow}, g \right\}_{std} = (\omega_{f_{\alpha+1,1}, g} + \omega_{x_{\alpha n}, g} - \omega_{x_{\alpha+1,n}, g}) f_{\alpha+1,1}^{\uparrow} g + f_{1,\beta+1} g^{\uparrow}$$

2. For $k \in [\beta-1]$, let $g = f_{m,n+m-\beta}$ with $m \in [k+1, \beta]$. Then

$$(6.30) \quad \left\{ g, f_{k,n+k-\beta}^{\downarrow} \right\}_{std} = (\omega_{g, f_{k,n+k-\beta}} - \omega_{g, x_{\beta n}} + \omega_{g, x_{\beta+1,n}}) g f_{k,n+k-\beta}^{\downarrow} - f_{k,n+k-\beta} g^{\downarrow}$$

Proof. Same as the proof of Lemma 6.6. □

Lemma 6.10. 1. Let $g \in \mathcal{B}_{std}$ be a function of the initial standard cluster, and let

$$(6.31) \quad s\omega_{\alpha\beta}(g) = \omega_{f_{n\alpha},g} - \omega_{f_{n,\alpha+1},g} - \omega_{f_{n\beta},g} + \omega_{f_{n,\beta+1},g}.$$

Then

$$(6.32) \quad s\omega_{\alpha\beta}(g) = \begin{cases} 1 & \text{if } g = f_{n+k-\alpha,k} \\ -1 & \text{if } g = f_{i,\beta+1} \\ 0 & \text{otherwise.} \end{cases}$$

2. Let $g \in \mathcal{B}_{std}$ be a function of the initial standard cluster and let

$$s'\omega_{\alpha\beta}(g) = \omega_{f_{\alpha n},g} - \omega_{f_{\alpha+1,n},g} - \omega_{f_{\beta n},g} + \omega_{f_{\beta+1,n},g}.$$

Then

$$(6.33) \quad s'\omega_{\alpha\beta}(g) = \begin{cases} 1 & \text{if } g = f_{k,n+k-\beta} \\ -1 & \text{if } g = f_{\alpha+1,j} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. 1. We will compute the coefficients through (6.1). Since $\nabla x_{nk} = e_{kn}$ we have

$$(\nabla x_{nk} X)_{ij} = \begin{cases} x_{nj} & i = k \\ 0 & i \neq k, \end{cases} \quad \text{and} \quad (X \nabla x_{nk})_{ij} = \begin{cases} x_{ik} & j = n \\ 0 & j \neq n. \end{cases}$$

According to Lemma 6.1,

$$R_+(\nabla x_{nk} X)_{ij} = \begin{cases} x_{nj} & i = k, j > k \\ \xi_j x_{nk} & i = j \end{cases}$$

with some coefficients ξ_j , and

$$R_+(X \nabla x_{nk})_{ij} = \begin{cases} x_{ik} & j = n \\ \xi'_j x_{nk} & i = j \end{cases}$$

with other coefficients ξ'_j . Plugging this into (6.1) gives

$$\begin{aligned} \{x_{nk}, g\}_{std} &= \sum_{j=k+1}^n x_{nj} \sum_{i=1}^n \frac{\partial g}{\partial x_{ij}} x_{ik} + \sum_{j=1}^n \xi_j x_{nk} g \\ &\quad - \sum_{i=1}^{n-1} x_{ik} \sum_{j=1}^n x_{nj} \frac{\partial g}{\partial x_{ij}} - \sum_{j=1}^n \xi'_j x_{nk} g. \end{aligned}$$

Split the last term into the ‘‘diagonal’’ part

$$D = \sum_{j=1}^n \xi_j x_{nk} g - \sum_{j=1}^n \xi'_j x_{nk} g$$

and the ‘‘non diagonal’’ part

$$N = \sum_{j=k+1}^n x_{nj} \sum_{i=1}^n \frac{\partial g}{\partial x_{ij}} x_{ik} - \sum_{i=1}^{n-1} x_{ik} \sum_{j=1}^n x_{nj} \frac{\partial g}{\partial x_{ij}}.$$

Start with the diagonal part D . We need the coefficients ξ_j and ξ'_j for $k = \alpha, \beta, \alpha + 1, \beta + 1$. Recall (6.3):

$$\begin{aligned}
R_+(e_{\alpha\alpha}) &= \frac{1}{n} \sum_{j=1}^{n-1} s_\alpha(j) (\hat{h}_j - \hat{h}_{j-1}) + \hat{h}_\alpha \\
&\quad + \hat{h}_\beta - (n - \alpha) \hat{h}_{\beta-1} + (n - \beta) \hat{h}_{\alpha+1} \\
R_+(e_{\beta\beta}) &= \frac{1}{n} \sum_{j=1}^{n-1} s_\beta(j) (\hat{h}_j - \hat{h}_{j-1}) \\
&\quad + (1 - n) \hat{h}_\alpha + \alpha \hat{h}_{\beta-1} + (n - \beta) \hat{h}_{\alpha+1} \\
&\quad + \hat{h}_\beta \begin{cases} 1 & \beta > \alpha + 1 \\ (1 - n) & \beta = \alpha + 1 \end{cases} \\
R_+(e_{\alpha+1, \alpha+1}) &= \frac{1}{n} \sum_{j=1}^{n-1} s_{\alpha+1}(j) (\hat{h}_j - \hat{h}_{j-1}) \\
&\quad + (1 - n) \hat{h}_\beta + \alpha \hat{h}_{\beta-1} + (n - \beta) \hat{h}_{\alpha+1} \\
&\quad + \hat{h}_\alpha \begin{cases} 1 & \beta > \alpha + 1 \\ (1 - n) & \beta = \alpha + 1 \end{cases} \\
R_+(e_{\beta+1, \beta+1}) &= \frac{1}{n} \sum_{j=1}^{n-1} s_{\beta+1}(j) (\hat{h}_j - \hat{h}_{j-1}) + \hat{h}_\alpha + \hat{h}_\beta + \alpha \hat{h}_{\beta-1} - \beta \hat{h}_{\alpha+1} \\
R_+(e_{nn}) &= \frac{1}{n} \sum_{j=1}^{n-1} -j (\hat{h}_j - \hat{h}_{j-1}) + \hat{h}_\alpha + \hat{h}_\beta + \alpha \hat{h}_{\beta-1} - \beta \hat{h}_{\alpha+1}.
\end{aligned}$$

Using $(\hat{h}_j - \hat{h}_{j-1}) = \frac{1}{n} \text{diag}(-1, \dots, -1) + e_{jj}$ and the fact

$$s_\alpha(j) - s_{\alpha+1}(j) = \begin{cases} n & j = \alpha \\ 0 & j \neq \alpha \end{cases}$$

we get

$$\begin{aligned}
&\sum_{j=1}^{n-1} s_\alpha(j) (\hat{h}_j - \hat{h}_{j-1}) - \sum_{j=1}^{n-1} s_{\alpha+1}(j) (\hat{h}_j - \hat{h}_{j-1}) \\
&= \text{diag}(-1, \dots, -1) + ne_{\alpha\alpha},
\end{aligned}$$

and

$$\sum_{j=1}^{n-1} s_{\beta+1}(j) (\hat{h}_j - \hat{h}_{j-1}) - \sum_{j=1}^{n-1} s_\beta(j) (\hat{h}_j - \hat{h}_{j-1}) = \text{diag}(1, \dots, 1) - ne_{\beta\beta}.$$

Putting everything together gives

$$(6.34) \quad R_+(e_{\alpha\alpha} - e_{\beta\beta} - e_{\alpha+1, \alpha+1} + e_{\beta+1, \beta+1})$$

$$(6.35) \quad = n (\hat{h}_\alpha + \hat{h}_\beta - \hat{h}_{\alpha+1} - \hat{h}_{\beta-1}) + ne_{\alpha\alpha} - ne_{\beta\beta},$$

for $\beta > \alpha + 1$, or in the case $\beta = \alpha + 1$:

$$R_+(e_{\alpha\alpha} - e_{\beta\beta} - e_{\alpha+1, \alpha+1} + e_{\beta+1, \beta+1}) = e_{\alpha\alpha} - e_{\alpha+1, \alpha+1},$$

and since $\hat{h}_\alpha - \hat{h}_{\alpha+1} = \frac{1}{n} \text{diag}(1, \dots, 1) - e_{\alpha+1, \alpha+1}$, and $\hat{h}_\beta - \hat{h}_{\beta-1} = \frac{1}{n} \text{diag}(-1, \dots, -1) + e_{\beta\beta}$, (6.34) turns to

$$R_+(e_{\alpha\alpha} - e_{\beta\beta} - e_{\alpha+1, \alpha+1} + e_{\beta+1, \beta+1}) = (e_{\alpha\alpha} - e_{\alpha+1, \alpha+1}).$$

Since D is a trace of two matrices, we are only interested in products of the diagonal elements in $R_+(\nabla x_{nk} \cdot X)$, $R_+(X \cdot \nabla x_{nk})$ with the corresponding diagonal elements in $\nabla g \cdot X$ and $X \cdot \nabla g$. These products vanish for all $g \in \mathcal{B}_{std}$ except $g = f_{i, \alpha+1}$ (which is a the determinant of a submatrix that has column $\alpha + 1$ but not col. α) or $g = f_{n-\alpha+k, k}$ (a determinant of a submatrix that has column α but not column $\alpha + 1$). Write $\omega_{f_{nk}, g}^D = \frac{D}{f_{nk}g}$, So the sum of coefficients of the diagonal part is

$$\omega_{f_{n\alpha}, g}^D - \omega_{f_{n\beta}, g}^D - \omega_{f_{n, \alpha+1}, g}^D + \omega_{f_{n, \beta+1}, g}^D = \begin{cases} 1 & g = f_{n-\alpha+k, k} \\ -1 & g = f_{i, \alpha+1} \\ 0 & \text{otherwise.} \end{cases}$$

We now turn to the non diagonal part N : recall

$$\begin{aligned} (\nabla x_{nk} X)_{ij} &= (x_{nk})^{i \leftarrow j} = \begin{cases} x_{nj} & i = k \\ 0 & i \neq k, \end{cases} \\ (X \nabla x_{nk})_{ij} &= (x_{nk})_{j \leftarrow i} = \begin{cases} x_{ik} & j = n \\ 0 & j \neq n, \end{cases} \end{aligned}$$

and we have

$$R_+(\nabla x_{nk} X) = \begin{bmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & x_{n, k+1} & \cdots & & x_{n, n} \\ & & & \ddots & & \\ & & & & \ddots & \\ & & & & & \ddots \\ & & & & & & \ddots \end{bmatrix},$$

and

$$R_+(X \nabla x_{nk}) = \begin{bmatrix} 0 & & & & x_{1k} \\ & \ddots & & & \vdots \\ & & \ddots & & \vdots \\ & & & \ddots & \vdots \\ & & & & x_{n-1, k} \\ & & & & \ddots \end{bmatrix},$$

so when computing the bracket with (6.1),

$$\begin{aligned}
N &= \sum_{j=k+1}^n x_{nj} \sum_{i=1}^n \frac{\partial g}{\partial x_{ij}} x_{ik} - \sum_{i=1}^{n-1} x_{ik} \sum_{j=1}^n x_{nj} \frac{\partial g}{\partial x_{ij}} \\
&= x_{nk} \sum_{j=k+1}^n x_{nj} \frac{\partial g}{\partial x_{nj}} - \sum_{i=1}^{n-1} x_{ik} \sum_{j=1}^k x_{nj} \frac{\partial g}{\partial x_{ij}} \\
&= x_{nk} \sum_{j=k+1}^n x_{nj} \frac{\partial g}{\partial x_{ij}} - \sum_{i=1}^n x_{ik} \sum_{j=1}^k x_{nj} \frac{\partial g}{\partial x_{ij}} + x_{nk} \sum_{j=1}^k x_{nj} \frac{\partial g}{\partial x_{nj}} \\
&= x_{nk} \sum_{j=1}^n x_{nj} \frac{\partial g}{\partial x_{nj}} - \sum_{i=1}^n x_{ik} \sum_{j=1}^k x_{nj} \frac{\partial g}{\partial x_{ij}} \\
&= x_{nk} g - \sum_{i=1}^n x_{ik} \sum_{j=1}^k x_{nj} \frac{\partial g}{\partial x_{ij}}.
\end{aligned}$$

Now, since g is a determinant of some submatrix A of X , let g^{\max} and g^{\min} denote the maximal (right) and minimal (left) columns of A . Similarly, let g_{\max} be the last row of A . Then

$$\sum_{i=1}^n x_{ik} \sum_{j=1}^k x_{nj} \frac{\partial g}{\partial x_{ij}} = \begin{cases} 0 & g^{\min} > k \\ x_{nk} g & g^{\min} \leq k \leq g^{\max} \\ x_{nk} g & g^{\max} < k \longrightarrow g_{\max} = n \end{cases}$$

so that

$$N = \begin{cases} x_{nk} g & g^{\min} > k \\ 0 & g^{\min} \leq k. \end{cases}$$

Defining $\omega_{f_{nk},g}^N = \frac{N}{f_{nk}g}$, summing over $k = \alpha, \alpha + 1, \beta, \beta + 1$ we get $\sum \omega_{f_{nk},g}^N \neq 0$ only when $g = f_{i,\alpha+1}$ or $g = f_{i,\beta+1}$, or in the ‘‘special’’ case $\beta = \alpha + 1$: we can then write the sum of these coefficients:

$$\sum \omega_{f_{nk},g}^N = \begin{cases} 0 & g^{\min} = \alpha \\ 1 & g^{\min} = \alpha + 1 \\ 0 & g^{\min} = \beta \text{ and } \beta > \alpha + 1 \\ -1 & g^{\min} = \beta + 1. \end{cases}$$

We now add the diagonal part coefficients, for $s\omega_{\alpha\beta}(g) = \sum \omega_{f_{nk},g}^N + \sum \omega_{f_{nk},g}^D$, so

- (1) If $g = f_{i,\alpha+1}$, then the sum of non diagonal coefficients is 1. We have seen that in this case the sum of diagonal coefficients is -1 , and therefore $s\omega_{\alpha\beta}(g) = 0$.
- (2) If $g = g_{i,\beta}$ and $\beta = \alpha + 1$, just like in 1., it is $s\omega_{\alpha\beta}(g) = 0$.
- (3) If $g = f_{i,\beta+1}$ then $s\omega_{\alpha\beta}(g) = -1$.
- (4) If $g = f_{n+k-\alpha,k}$ then $s\omega_{\alpha\beta}(g) = -1$.
- (5) For any other $g \in \mathcal{B}_{std}$, $s\omega_{\alpha\beta}(g) = 0$.

This completes the proof of part 1. of the Lemma. Part 2. is similar, using the symmetries $x_{ij} \longleftrightarrow x_{ji}$ (and therefore $f_{ij} \longleftrightarrow f_{ji}$), and $\alpha \longleftrightarrow \beta$. \square

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