Local distributions for eigenfunctions and for perfect colorings of q-ary hypercube [1]

Anastasia Vasil'eva vasilan@math.nsc.ru
Sobolev Institute of Mathematics,
Novosibirsk State University, Novosibirsk, RUSSIA

Abstract. Under study are the eigenfunctions and perfect colorings of the graph of n-dimensional q-ary Hamming space. We obtain the interdependence of local distributions of an eigenfunction in two orthogonal faces. We prove also an analogous result for perfect colorings.

1 Introduction

We study the eigenfunctions and perfect colorings of n-dimensional q-ary hypercube. The particular case of perfect colorings, which is extensively investigated now, corresponds to the completely regular codes. The aim of the paper is to provide a connection between the local distributions in two orthogonal faces. Earlier this question was considered in [2, 4–6] for the 1-error correcting perfect codes and perfect colorings in binary case ($q = 2$). In case $q > 2$ the question is investigated in [1] for the 1-error-correcting codes. In [3] a more general case of the direct product of graphs is studied; however, the formula is not extended for the classes of graphs.

The paper is organised as follows: In Section 2 we give some necessary notations and propositions. In Section 3 we establish a formula for local weight enumerators of an eigenfunction in a pair of orthogonal faces. Using this, we obtain in Section 4 the formula for local weight enumerators of a perfect coloring in a pair of orthogonal faces. Both derived formulas are symmetric under choice of the face from the pair.

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2 Preliminaries

Consider the set $\mathbf{F}_q = \{0, 1, \ldots, q-1\}$ as the group modulo q and $\mathbf{F}_q^n$ as the abelian group $\mathbf{F}_q \times \ldots \times \mathbf{F}_q$. We investigate functions and the colorings on the graph of $\mathbf{F}_q^n$ of *q-ary n-dimensional hypercube*; in this graph two vertices are adjacent if they differ in exactly one position.

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Let $\alpha \in \mathbf{F}_q^n$ be an arbitrary vertex. Here and elsewhere I denotes a subset of $\{1,\ldots,n\}$ and $\overline{I} = \{1,\ldots,n\}\backslash I$. We denote the *support* of the vertex α by $s(\alpha)$ (i.e., the set of its nonzero positions); the cardinality of the support is the *Hamming weight* of α and is denoted by $wt(\alpha)$; the *Hamming distance* between two vertices α and β that equals the Hamming weight of $\alpha - \beta$ is denoted by $\rho(\alpha,\beta)$. We write $W_i(\alpha)$ for the *sphere* of radius i centered at the vertex α (i.e., the set of all vertices with distance i from α) and we write $B_i(\alpha)$ for the *ball* of radius i centered at the vertex α (i.e., the set of all vertices with distance at most i from α). By definition, put

$$\Gamma_I(\alpha) = \{\beta \in \mathbf{F}_q^n \ : \ \beta_i = \alpha_i \ \forall \ i \notin I\},$$

then $\Gamma_I(\alpha)$ is an $|I|$-*dimensional face*, it has the structure of $\mathbf{F}_q^{|I|}$. We write simply W_i and Γ_I instead of $W_i(\alpha)$ and $\Gamma_I(\alpha)$ in the case of the all-zero vertex α. Two faces $\Gamma_I(\alpha)$ and $\Gamma_J(\beta)$ are *orthogonal* if $J = \overline{I}$. Obviously, two orthogonal faces have exactly one common vertex. Given $\alpha, \beta \in \mathbf{F}_q^n$, we denote $\langle \alpha, \beta \rangle = \alpha_1\beta_1 + \ldots + \alpha_n\beta_n \mod q$.

Let us consider the set of all functions $f : \mathbf{F}_q^n \longrightarrow \mathbb{C}$ as q^n-dimensional vector space V over the complex field $\mathbb{C}$. Let $\xi = e^{2\pi\sqrt{-1}/q}$. For $\beta \in \mathbf{F}_q^n$, the function $\varphi^\beta \in V$, where

$$\varphi^\beta(\alpha) = \xi^{\langle \alpha, \beta \rangle}, \qquad \alpha \in \mathbf{F}_q^n,$$

is called the *character*. All characters φ^β, $\beta \in \mathbf{F}_q^n$, form the orthogonal basis of the vector space V with respect to the *inner product* $\langle\ ,\ \rangle$ defined as follows:

$$\langle f, g \rangle = \sum_{\beta \in \mathbf{F}_q^n} f(\beta)\overline{g(\beta)}.$$

The *Fourier transform* $\widehat{f}$ of the function f is defined as the inner product with the characters:

$$\widehat{f}(\alpha) = \langle f, \varphi^\alpha \rangle = \sum_{\beta \in \mathbf{F}_q^n} f(\beta)\overline{\xi^{\langle \alpha, \beta \rangle}}, \quad \alpha \in \mathbf{F}_q^n. \tag{1}$$

The initial function f can be presented in the basis of the characters:

$$f(\alpha) = q^{-n} \sum_{\beta \in \mathbf{F}_q^n} \widehat{f}(\beta)\xi^{\langle \alpha, \beta \rangle}, \quad \alpha \in \mathbf{F}_q^n. \tag{2}$$

Lemma 1. *Let $\beta \in \mathbf{F}_q^n$ and $I \subseteq \{1,\ldots,n\}$. Then*

$$\sum_{\alpha \in \Gamma_I} \varphi^\beta(\alpha) x^{|I|-|s(\alpha)|} y^{|s(\alpha)|} = (x-y)^{|I \bigcap s(\beta)|}(x+(q-1)y)^{|I|-|I \bigcap s(\beta)|}.$$

Proof. Let $|I| = k$. Without loss of generality assume that $I = \{1, \dots, n\}$. By definition of the characters,

$$\sum_{\alpha \in \Gamma_I} \varphi^{\beta}(\alpha) x^{|I|-|s(\alpha)|} y^{|s(\alpha)|} = \sum_{\alpha_1=0}^{q-1} \cdots \sum_{\alpha_k=0}^{q-1} \prod_{i=1}^{k} \xi^{\alpha_i \beta_i} x^{1-|s(\alpha_i)|} y^{|s(\alpha_i)|}.$$

(For $a \in \mathbf{F}_q$ it holds $|s(a)| = 0$ if $a = 0$ and $|s(a)| = 1$ if $a \neq 0$.) Then we change the order of summations and multiplication:

$$\prod_{i=1}^{k} \sum_{\alpha_i=0}^{q-1} \xi^{\alpha_i \beta_i} x^{1-|s(\alpha_i)|} y^{|s(\alpha_i)|}. \tag{3}$$

Owing to the properties of the primitive root of unity, we have

$$\sum_{a=0}^{q-1} \xi^{ab} = \begin{cases} 0, & b \neq 0, \\ q, & b = 0 \end{cases},$$

and therefore

$$\sum_{a=0}^{q-1} \xi^{ab} x^{1-|s(a)|} y^{|s(a)|} = \begin{cases} x - y, & b \neq 0, \\ x + (q-1)y, & b = 0. \end{cases}$$

Applying this to (3), we finally obtain

$$(1-t)^{|I \bigcap s(\beta)|} (1 + (q-1)t)^{|I| - |I \bigcap s(\beta)|}.$$

□

Now we introduce the concept of a local distribution. By definition, put

$$v_j^{I,f}(\alpha) = \sum_{\beta \in \Gamma_I(\alpha) \bigcap W_j(\alpha)} f(\beta),$$

the vector $v^{I,f}(\alpha) = (v_0^{I,f}(\alpha), \dots, v_{|I|}^{I,f}(\alpha))$ is called the *local distribution of the function f in the face $\Gamma_I(\alpha)$ with respect to the vertex α* or shortly the (I, α)-*local distribution* of f. We say that the polynomial

$$g_f^{I,\alpha}(x, y) = \sum_{j=0}^{|I|} v_j^{I,f}(\alpha) y^j x^{|I|-j} = \sum_{\beta \in \Gamma_I(\alpha)} f(\beta) y^{|s(\beta-\alpha)|} x^{|I|-|s(\beta-\alpha)|}$$

is a *local weight enumerator of the function f in the face $\Gamma_I(\alpha)$ with respect to the vertex α* or shortly the (I, α)-*local weight enumerator* of f. We omit α (in all notations) if $\alpha = (0, \dots, 0)$.

Let us describe the local weight enumerator of an arbitrary function in terms of its Fourier coefficients:

Lemma 2. *Let f be an arbitrary function. Then*

$$g_f^I(x,y) = q^{-n} \sum_{\beta\in\mathbf{F}_q^n} \widehat{f}(\beta)(x+(q-1)y)^{|I|-|I\bigcap s(\beta)|}(x-y)^{|I\bigcap s(\beta)|}. \qquad (4)$$

Proof. By Lemma 1,

$$g_f^I(x,y) = \sum_{\beta\in\Gamma_I} f(\beta)y^{|s(\beta)|}x^{|I|-|s(\beta)|}$$

$$= q^{-n} \sum_{\delta\in\mathbf{F}_q^n} \widehat{f}(\delta) \sum_{\beta\in\Gamma_I} \xi^{\langle\beta,\delta\rangle} x^{|I|-|s(\beta)|}y^{|s(\beta)|}.$$

Then we can apply Lemma 1 and obtain (4). □

3 Eigenfunctions

The first object of our consideration is the set of all eigenfunctions of the n-dimensional q-ary hypercube $\mathbf{F}_q^n$. As usual, we refer to as the *eigenvalue of a graph* the eigenvalue of its adjacency matrix. It is known that the eigenvalues λ of the graph of n-dimensional q-ary hypercube are equal to

$$\lambda_h = (q-1)n - qh, \quad h = 0,1,\ldots,n,$$

here h is called the *number of the eigenvalue* λ_h. Obviously, an eigenvalue λ has the number $h = h(\lambda) = \frac{(q-1)n-\lambda}{q}$. The corresponding eigenfunctions (we call them λ-*functions*) satisfy the equations

$$\sum_{\beta\in W_1(\alpha)} f(\beta) = \lambda_h f(\alpha), \quad \alpha\in\mathbf{F}_q^n, \qquad (5)$$

or in the matrix form:

$$Df = \lambda_h f,$$

where D is the adjacency matrix of $\mathbf{F}_q^n$ and f is a vector of the function f values. It is easy to see that the Fourier coefficients $\widehat{f}(\alpha)$ of a λ-function f equal zero apart from the case, where the Hamming weight of α is equal to the number of λ.

We are going to derive the interdependence between the local weight enumerators for an eigenfunction in two orthogonal faces.

Theorem 1. *Let λ be an eigenvalue of $\mathbf{F}_q^n$ with the number $h = \frac{(q-1)n-\lambda}{q}$, let f be an arbitrary λ-function, and let $\alpha\in\mathbf{F}_q^n$. Then*

$$(x+(q-1)y)^{h-|\overline{I}|} g_f^{\overline{I},\alpha}(x,y) = (x'+(q-1)y')^{h-|I|} g_f^{I,\alpha}(x',y'),$$

where $x' = x+(q-2)y,\ y' = -y$.

Proof. The faces $\Gamma_I(\alpha)$ and $\Gamma_{\overline{I}}(\alpha)$ are orthogonal. Without loss of generality assume that α is the all-zero vertex. Using Lemma 2, we can express the $(\overline{I}, \mathbf{0})$-local weight enumerator of the λ-function f in terms of the Fourier coefficients:

$$g_f^{\overline{I}}(x,y) = q^{-n} \sum_{\beta \in \mathbf{F}_q^n} \widehat{f}(\beta)(x+(q-1)y)^{n-|I|-|s(\beta)|+|I \bigcap s(\beta)|}(x-y)^{|s(\beta)|-|I \bigcap s(\beta)|}.$$

Since $\widehat{f}(\beta) = 0$ for every $\beta \notin W_h$, the summation can be taken over all vertices of weight h instead of all vertices of $\mathbf{F}_q^n$. This implies

$$g_f^{\overline{I}}(x,y) = q^{-n}(x+(q-1)y)^{n-|I|-h}(x-y)^{h-|I|}$$
$$\times \sum_{\beta \in W_h} \widehat{f}(\beta)(x+(q-1)y)^{|I \bigcap s(\beta)|}(x-y)^{|I|-|I \bigcap s(\beta)|}.$$

We choose new variables x' and y' such that

$$\begin{cases} x' + (q-1)y' &= x - y, \\ x' - y' &= x + (q-1)y, \end{cases} \quad \text{or} \quad \begin{cases} x' &= x + (q-2)y, \\ y' &= -y. \end{cases}$$

Hence,

$$g_f^{\overline{I}}(x,y) = q^{-n}(x+(q-1)y)^{n-|I|-h}(x-y)^{h-|I|}$$
$$\times \sum_{\beta \in W_h} \widehat{f}(\beta)(x'-y')^{|I \bigcap s(\beta)|}(x'+(q-1)y')^{|I|-|I \bigcap s(\beta)|}.$$

Comparing with Lemma 2, we finally have

$$g_f^{\overline{I}}(x,y) = (x+(q-1)y)^{n-|I|-h}(x'+(q-1)y')^{h-|I|} g_f^I(x',y').$$

□

4 Perfect colorings

In this section we prove an analog of Theorem 1 for perfect colorings.

The partition $C = (C_1, \ldots, C_r)$ of $\mathbf{F}_q^n$ is called a *perfect r-coloring* (or an *equitable partition*, or a *partition design*) with the *parameter matrix* $S = (s_{ij})_{i,j=1,\ldots,r}$ if for every $i, j \in \{1, \ldots, r\}$ and each vertex $\alpha \in C_i$ the number of vertices $\beta \in C_j$ at distance 1 from α is equal to s_{ij}. Present a perfect r-coloring by $(0,1)$-matrix C of size $q^n \times r$ with the rows corresponding to the vertices of $\mathbf{F}_q^n$ and the columns corresponding to the colors $\{1, \ldots, r\}$. The matrix C is defined as follows: each row has only one nonzero position that marks the color of the corresponding vertex. In these terms the coloring is perfect if

$$DC = CS, \tag{6}$$

where D is the adjacency matrix of the hypercube $\mathbf{F}_q^n$.

We define a local distribution of a coloring as a local distribution of characteristic functions of the colors. More precisely, a *local distribution of the coloring C in the face $\Gamma_I(\alpha)$ with respect to the vertex α* (or (I,α)-*local distribution*) is the $r \times (|I|+1)$-matrix

$$v^{I,C}(\alpha) = \begin{pmatrix} v_0^{I,C_1}(\alpha) & \dots & v_{|I|}^{I,C_1}(\alpha) \\ \vdots & & \vdots \\ v_0^{I,C_r}(\alpha) & \dots & v_{|I|}^{I,C_r}(\alpha) \end{pmatrix},$$

where $v_j^{I,C_i}(\alpha) = |C_i \bigcap W_j(\alpha) \bigcap \Gamma_I(\alpha)|, \quad i = 1,\dots,r,$ and $j = 0,\dots,|I|$. Let $g_{C_i}^{I,\alpha}(x,y),\ i = 1\dots,r$, be the (I,α)-local weight enumerator of the ith color C_i; i.e.,

$$g_{C_i}^{I,\alpha}(x,y) = \sum_{j=0}^{|I|} v_j^{I,C_i}(\alpha) y^j x^{|I|-j}.$$

The vector-function

$$g_C^{I,\alpha}(x,y) = (g_{C_1}^{I,\alpha}(x,y),\dots,g_{C_r}^{I,\alpha}(x,y))$$

is called the *local weight enumerator of the coloring C in the face $\Gamma_I(\alpha)$ with respect to the vertex α* (or the (I,α)-*local weight enumerator*).

The next theorem is an analog of Theorem 1 for perfect colorings.

Theorem 2. *Let $C = (C_1,\dots,C_r)$ be an arbitrary perfect coloring of $\mathbf{F}_q^n$ with parameter matrix S and $\alpha \in \mathbf{F}_q^n$. Put $h(S) = \frac{(q-1)nE-S}{q}$, where E is an identity matrix. Then*

$$g_C^{\overline{I},\alpha}(x,y)(x+(q-1)y)^{h(S)-|\overline{I}|E} = g_C^{I,\alpha}(x',y')(x'+(q-1)y')^{h(S)-|I|E}. \tag{7}$$

Proof. Without loss of generality assume that $\alpha = (0,\dots,0)$.

Perfect colorings are closely related with eigenfunctions of the hypercube. Indeed, let $\mu_1,\dots,\mu_r$ be the all eigenvalues (not necessarily distinct) of the parameter matrix S and let $T^1,\dots,T^r$ be the linearly independent eigenvectors of S that corresponds to the eigenvalues; i.e.,

$$ST^i = \mu_i T^i, \quad i = 1,\dots,r.$$

Thus, for the matrices $T = [T^1, \dots, T^r]$ and $M = diag\{\mu_1, \dots, \mu_r\}$ it holds

$$ST = TM.$$

Multiplying both sides of (6) by T and applying the last equation, we have for the matrix

$$F = CT \tag{8}$$

that

$$DF = DCT = CST = CTM = FM.$$

It means that the columns $F^1, \dots, F^r$ of F are the eigenfunctions of D or λ-functions; i.e.,

$$DF^i = \mu_i F^i, \quad i = 1 \dots, r.$$

Applying Theorem 1 to these λ-functions, we have

$$(x+(q-1)y)^{h_i-|\overline{I}|} g_{F^i}^{\overline{I}}(x,y) = (x'+(q-1)y')^{h_i-|I|} g_{F^i}^{I}(x',y'), \quad i = 1, \dots, r, \tag{9}$$

where for $i = 1 \dots, r$ the value h_i is equal to the number of the eigenvalue μ_i of the hypercube $\mathbf{F}_q^n$; i.e., $h_i = \frac{(q-1)n-\mu_i}{q}$. Put $g_F = (g_{F^1}, \dots, g_{F^r})$ and

$$M_I(x,y) = diag\left\{(x+(q-1)y)^{h_1-|I|}, \dots, (x+(q-1)y)^{h_r-|I|}\right\}.$$

So we can rewrite the equations (9) in terms of these matrices:

$$g_F^{\overline{I}}(x,y) M_{\overline{I}}(x,y) = g_F^{I}(x',y') M_I(x',y').$$

It follows from (8) that

$$g_F = (g_{F^1}, \dots, g_{F^r}) = (g_{C^1}, \dots, g_{C^r})T = g_C T.$$

Therefore, we obtain

$$g_C^{\overline{I}}(x,y) T M_{\overline{I}}(x,y) = g_C^{I}(x',y') T M_I(x',y'). \tag{10}$$

Then we multiply both sides of (10) by T^{-1} and recall the definition of a matrix function:

$$(x+(q-1)y)^{\frac{(q-1)nE-S}{q}-|I|E} = TM_I(x,y)T^{-1},$$

which gives (7) and concludes the proof. □

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