FINITARY 2-CATEGORIES ASSOCIATED WITH DUAL PROJECTION FUNCTORS

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ABSTRACT. We study finitary 2-categories associated to dual projection functors for finite dimensional associative algebras. In the case of path algebras of admissible tree quivers (which includes all Dynkin quivers of type A) we show that the monoid generated by dual projection functors is the Hecke-Kiselman monoid of the underlying quiver and also obtain a presentation for the monoid of indecomposable subbimodules of the identity bimodule.

1. Introduction

Study of 2-categories of additive functors operating on a module category of a finite dimensional associative algebra is motivated by recent advances and applications of categorification philosophy, see [CR, Ro, KL, Ma] and references therein. Such 2-categories appear as natural 2-analogues of finite dimensional algebras axiomatized via the notion of finitary 2-categories as introduced in [MM1]. The series [MM1, MM2, MM3, MM5, MM6] of papers develops basics of the structure theory and the 2-representation theory for the so-called fiat 2-categories, that is finitary 2-categories having a weak involution and adjunction morphisms. Natural examples of such fiat 2-categories are 2-categories generated by projective functors, that is functors given by tensoring with projective bimodules, see [MM1, Subsection 7.3]. Fiat 2-categories also naturally appear as quotients of 2-Kac-Moody algebras from [KL, Ro, We], see [MM2, Subsection 7.1] and [MM5, Subsection 7.2] for detailed explanations. There are also many natural constructions which produce new fiat 2-categories from known ones, see e.g. [MM6, Section 6].

Despite of some progress made in understanding fiat 2-categories in the papers mentioned above, the general case of finitary 2-categories remains very mysterious with the only general result being the abstract 2-analogue of the Morita theory developed in [MM4]. One of the major difficulties is that so far there are not that many natural examples of finitary 2-categories which would be "easy enough" for any kind of sensible understanding. In [GrMa], inspired by the study of the so-called projection functors in [Gr, Pa], we defined a finitary 2-category which is a natural 2-analogue of the semigroup algebra of the so-called Catalan monoid of all order-decreasing and order-preserving transformations of a finite chain. This 2-category is associated to the path algebra of a type A Dynkin quiver with a fixed uniform orientation (meaning that all edges are oriented in the same direction).

The main aim of the present paper is to make the next step and consider a similarly defined 2-category for an arbitrary orientation of a type A Dynkin quiver and, more generally, for any admissible orientation of an arbitrary tree quiver. There is one important difference, which we will now explain, between this general case and the case of a uniform orientation in type A. Basic structural properties of a finitary 2-category are encoded in the so-called *multisemigroup* of this 2-category as defined in [MM2, Subsection 3.3]. Elements of this multisemigroup are isomorphism classes

of *indecomposable* 1-morphisms in our 2-category. It turns out that for a uniform orientation of a type A Dynkin quiver any composition of projection functors is either indecomposable or zero. This fails in all other cases in which the orientation is not uniform as well as for all admissible tree quivers outside type A. This is the principal added difficulty of the present paper compared to [GrMa].

For technical reasons it turns out that it is more convenient to work with a dual version of projection functors, which we simply call dual projection functors. Roughly speaking these are the right exact functors given by maximal subfunctors of the identity functor. The first part of the paper is devoted to some basic structure theory for such functors. This is developed in Section 3 after various preliminaries collected in Section 2. In particular, in Proposition 10 we make the connection between projection and dual projection functors very explicit. This, in particular, allows us to transfer, for free, many results of [Gr, Pa] to our situation.

Section 4 contains basic preliminaries on 2-categories. In Section 5 we define finitary 2-categories given by dual projection functors and also finitary 2-categories given by non-exact ancestors of dual projection functors which we call *idealization functors*. Section 6 is the main part of the paper and contains several results. This includes a classification of indecomposable dual projection functors in Theorem 43 and also the statement that composition of indecomposable dual projection functors for any admissible orientation of a tree quiver is indecomposable, see Proposition 45. Our classification is based on a generalization of the Dyck path combinatorics in application to subbimodules of the identity bimodule for admissible tree quivers as described in Subsections 6.5, 6.6, 6.7 and 6.8.

Proposition 45 mentioned above implies that the multisemigroup of the 2-category of dual projection functors associated to any admissible orientation of a tree quiver is, in fact, an ordinary semigroup. This observation automatically makes this semigroup an interesting object of study. In Section 7 we give a presentation for this semigroup in Theorem 56 and also for the semigroup of all idealization functors in Theorem 55. Our proof of Theorem 55 is rather elegant, it exploits the idea of decategorification: the canonical action of our 2-category on the underlying module category gives rise to a linear representation of a certain Hecke-Kiselman monoid from [GM]. Proof of Theorem 55 basically reduces to verification that this representation is effective (in the sense that different elements of the monoid are represented by different linear transformations). This effectiveness was conjectured in [GM] and proved in [Fo]. Theorem 56 requires more technical work as the monoid of indecomposable dual projection functors is not a Hecke-Kiselman monoid on the nose, but after some preparation it also reduces to a similar argument.

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2. Preliminaries

2.1. Notation and setup. In this paper we work over a fixed field k which for simplicity is assumed to be algebraically closed. All categories and functors considered in this paper are supposed to be k-linear, that is enriched over k-Mod. If not explicitly stated otherwise, by a module we always mean a *left* module.

For a finite dimensional associative k-algebra A we denote by A-mod the (abelian) category of all finitely generated A-modules. By A-Mod we denote the (abelian) category of all A-modules. We also denote by A-proj the (additive) category of all finitely generated projective A-modules and by A-inj the (additive) category of all finitely generated injective A-modules.

We denote by $\operatorname{mod-}A$ the category of all finitely generated right A-modules and define $\operatorname{proj-}A$ and $\operatorname{Mod-}A$ respectively.

We denote by A-mod-A the category of all finitely generated A-A-bimodules. Denote by \mathcal{AF}_A the category of all additive \mathbb{k} -linear endofunctors of A-mod. This is an abelian category since A-mod is abelian.

Abusing notation, we write * for both the k-duality functors

$$\operatorname{Hom}_{-\mathbb{k}}(-,\mathbb{k}): A\operatorname{-mod} \to \operatorname{mod-}A \quad \text{ and } \quad \operatorname{Hom}_{\mathbb{k}^{-}}(-,\mathbb{k}): \operatorname{mod-}A \to A\operatorname{-mod}.$$

Let L_1, L_2, \ldots, L_n be a complete and irredundant list of representatives of isomorphism classes of simple A-modules. Then $L_1^*, L_2^*, \ldots, L_n^*$ is a complete and irredundant list of representatives of isomorphism classes of simple right A-modules. For $i, j = 1, 2, \ldots, n$, set $L_{ij} := L_i \otimes_{\mathbb{R}} L_j^*$. This gives a complete and irredundant list of representatives of isomorphism classes of simple A-A-bimodules. For $i = 1, 2, \ldots, n$ we denote by P_i and I_i the indecomposable projective cover and injective envelope of L_i , respectively.

When working with the opposite algebra, we will add the superscript $^{\mathrm{op}}$ to all notation.

We refer the reader to [ARS, Ba, DK, GR] for further generalities and details on representation theory of finite dimensional algebras.

- 2.2. **Trace functors.** With each $N \in A$ -mod one associates the corresponding trace functor $\operatorname{Tr}_N : A$ -mod $\to A$ -mod defined in the following way:
 - For every $M \in A$ -mod, the module $\mathrm{Tr}_N(M) \in A$ -mod is defined as the submodule $\sum_{f:N \to M} \mathrm{Im}(f)$ of M.
 - For every $M, M' \in A$ -mod and every $f: M \to M'$, the corresponding morphism $\operatorname{Tr}_N(f): \operatorname{Tr}_N(M) \to \operatorname{Tr}_N(M')$ is defined as the restriction of f to $\operatorname{Tr}_N(M)$.

Directly from the definition it follows that Tr_N is a subfunctor of the identity functor for every N. We denote by $\iota_N:\operatorname{Tr}_N\hookrightarrow\operatorname{Id}_{A\operatorname{-mod}}$ the corresponding injective natural transformation.

Lemma 1. Let $N \in A$ -mod.

- (i) The functor Tr_N preserves monomorphisms.
- (ii) If N is projective, then Tr_N preserves epimorphisms.
- (iii) We have $\operatorname{Tr}_N \circ \operatorname{Tr}_N \cong \operatorname{Tr}_N$.

Proof. Let $f: M \to M'$ be a monomorphism. In the commutative diagram

$$M^{\leftarrow} \xrightarrow{f} M'$$

$$\downarrow^{\iota_{M}} \downarrow^{\iota_{M'}}$$

$$\operatorname{Tr}_{N}(M) \xrightarrow{\operatorname{Tr}_{N}(f)} \operatorname{Tr}_{N}(M')$$

we have $f \circ \iota_M$ is a monomorphism. Hence $\operatorname{Tr}_N(f)$ is a monomorphism as well. This proves claim (i).

Let $f: M \to M'$ be an epimorphism and $g: N \to M'$ any map. If N is projective, then there is $h: N \to M$ such that $g = f \circ h$. Hence $\operatorname{Im}(g) = f(\operatorname{Im}(h))$ showing that $\operatorname{Tr}_N(M)$ surjects onto $\operatorname{Tr}_N(M')$. This proves claim (ii).

Claim (iii) follows directly from the definition of Tr_N . This completes the proof of the lemma. \Box

Example 2. In general, Tr_N is neither left nor right exact (even if N is projective). Indeed, let A be the path algebra of the quiver $1 \longrightarrow 2$, P_1 be the indecomposable projective A-module $\mathbb{k} \xrightarrow{\operatorname{Id}} \mathbb{k}$, L_1 be the simple A-module $\mathbb{k} \longrightarrow 0$ and L_2 be the simple A-module $0 \longrightarrow \mathbb{k}$. For $N = P_1$, applying Tr_N to the short exact sequence

$$0 \rightarrow L_2 \rightarrow P_1 \rightarrow L_1 \rightarrow 0$$
,

gives the sequence

$$0 \to 0 \to P_1 \to L_1 \to 0$$

which has homology in the middle position.

- 2.3. **Projection functors.** For $N \in A$ -mod we define the corresponding *projection functor* $\Pr_N : A$ -mod $\to A$ -mod as the cokernel of the natural transformation ι_N . Let $\pi_N : \operatorname{Id}_{A-\operatorname{mod}} \to \Pr_N$ denote the corresponding surjective natural transformation. The following properties of projection functors appear in $[\operatorname{Pa}, \operatorname{Gr}]$:
 - For any N, the functor Pr_N preserves epimorphisms.
 - If N is simple, then the functor \Pr_N preserves monomorphisms.
 - If N is simple and $\operatorname{Ext}_A^1(N,N) = 0$, then $\operatorname{Pr}_N \circ \operatorname{Pr}_N \cong \operatorname{Pr}_N$.
 - If N and K are simple and $\operatorname{Ext}_{A}^{1}(K, N) = 0$, then

$$\Pr_N \circ \Pr_K \circ \Pr_N \cong \Pr_K \circ \Pr_N \circ \Pr_K \cong \Pr_N \circ \Pr_K$$
.

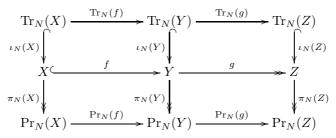
• If N and K are simple and $\operatorname{Ext}_{A}^{1}(N,K) = \operatorname{Ext}_{A}^{1}(K,N) = 0$, then

$$\Pr_N \circ \Pr_K \cong \Pr_K \circ \Pr_N$$
.

For the record, we also point out the following connection between the functors \Pr_N and Tr_N .

Lemma 3. For any fixed N, the functor Tr_N is exact if and only if the functor Pr_N is exact.

Proof. For an exact sequence $X \stackrel{f}{\hookrightarrow} Y \stackrel{g}{\twoheadrightarrow} Z$ in A-mod consider the commutative diagram



Here all columns are exact by construction and the middle row is exact by assumption. Therefore the Nine Lemma (a.k.a. the 3×3 -Lemma) says that the first row is exact if and only if the third row is exact.

3. Dual projection functors

- 3.1. **Idealization functors.** The algebra A is an A-A-bimodule, as usual. Tensoring with this bimodule (over A) is isomorphic to the identity endofunctor of A-mod. We identify subbimodules of ${}_{A}A_{A}$ and two-sided ideals of A. For each two-sided ideal $I \subset A$ denote by Su_{I} the endofunctor of A-mod defined in the following way:
 - For every $M \in A$ -mod, the module $Su_I(M)$ is defined as IM.
 - For every $M, M' \in A$ -mod and $f: M \to M'$, the morphism $Su_I(f)$ is defined as the restriction of f to IM.

We will call Su_I the *idealization functor* associated to I, where the notation Su stands for "Sub".

Let $\gamma: \operatorname{Su}_I \hookrightarrow \operatorname{Id}_{A\operatorname{-mod}}$ denote the injective natural transformation given by the canonical inclusion $IM \hookrightarrow M$. Directly from the definition we obtain that for any two two-sided ideals I and J in A we have

$$\mathrm{Su}_{I}\circ\mathrm{Su}_{J}=\mathrm{Su}_{IJ}.$$

Furthermore, if $I \subset J$, then we have the canonical inclusion $Su_I \hookrightarrow Su_J$.

3.2. **Exactness of idealization.** Here we prove the following property of idealization functors.

Lemma 4. Let I be a two-sided ideal in A.

- (i) The functor Su_I preserves monomorphisms.
- (ii) The functor Su_I preserves epimorphisms.

Proof. Claim (i) follows from the definition of Su_I and the fact that the restriction of a monomorphism is a monomorphism. To prove claim (ii), consider an epimorphism $f: M \to M'$, $v \in M'$ and $a \in I$. Then there is $w \in M$ such that f(w) = v and hence af(w) = f(aw) = av. As $aw \in Su_I(M)$, we obtain that av belongs to the image of $Su_I(f)$, completing the proof.

Example 5. The functor Su_I is neither left nor right exact in general. Indeed, consider the algebra $A = \mathbb{k}[x]/(x^2)$, let L be the (unique up to isomorphism) simple A-module and set $I := \operatorname{Rad}(A)$. Applying Su_I to the short exact sequence

$$0 \to L \to {}_AA \to L \to 0$$
,

we obtain the sequence

$$0 \rightarrow 0 \rightarrow L \rightarrow 0 \rightarrow 0$$

which has homology in the middle position.

3.3. Idealization functors versus trace functors. Let I be a two-sided ideal of A and N an A-module. Since both Su_I and Tr_N are subfunctors of the identity functor, it is natural to ask when they are isomorphic. In this subsection we would like to present some examples showing that, in general, these two families of functors are really different.

Lemma 6. If A is not semi-simple, then $Su_{Rad(A)}$ is not isomorphic to any trace functor.

Proof. We have $\operatorname{Su}_{\operatorname{Rad}(A)}(A) = \operatorname{Rad}(A) \neq 0$ as A is not semi-simple. Hence $\operatorname{Su}_{\operatorname{Rad}(A)}$ is not the zero functor, in particular, it is not isomorphic to Tr_0 . At the same time, let $L := A/\operatorname{Rad}(A)$. Then $\operatorname{Su}_{\operatorname{Rad}(A)}(L) = 0$. On the other hand, for any non-zero $N \in A$ -mod the module N surjects onto some simple A-module. As every simple A-module is a summand of L, we have $\operatorname{Tr}_N(L) \neq 0$. The claim follows.

Lemma 7. If N is simple and not projective, then Tr_N is not isomorphic to any idealization functor.

Proof. Let f: P woheadrightarrow N be a projective cover of N. Then P is indecomposable and has simple top. As N is simple, $\operatorname{Tr}_N(P)$ belongs to the socle of P. As N is not projective, $P \not\cong N$. Consequently, the socle of P belongs to the radical of P. Therefore f annihilates $\operatorname{Tr}_N(P)$ and it follows that $\operatorname{Tr}_N(f)$ is the zero map. We also have the obvious isomorphism $\operatorname{Tr}_N(N) \cong N$. At the same time, each idealization functor preserves epimorphisms by Lemma 4(ii). The obtained contradiction proves the statement.

3.4. **Definition of dual projection functors.** Recall that, for any additive functor F: A-proj $\to A$ -mod, there is a unique, up to isomorphism, right exact functor G: A-mod $\to A$ -mod such that the restriction of G to A-proj is isomorphic to G. As G is an additive generator of G-proj, the condition that the restriction of G to G-proj is isomorphic to G-bimodules G-proj is isomorphic. The functor G-bimodules G-proj is isomorphic. The functor G-proj is isomorphic to G-proj is iso

For an ideal I in A define a dual projection functor corresponding to I as a functor isomorphic to the functor

$$\mathrm{Dp}_I := \mathrm{Su}_I(A) \otimes_{A-} : A\operatorname{-mod} \to A\operatorname{-mod}.$$

Directly from the definition we have that Dp_I is right exact.

Lemma 8. If A is hereditary, then the functor Dp_I is exact for any I.

Proof. As $Su_I(A) \subset A$ and A is hereditary, the right A-module $Su_I(A)$ is projective. This means that Dp_I is exact.

Corollary 9. If A is hereditary, then $\mathrm{Dp}_I \circ \mathrm{Dp}_J \cong \mathrm{Dp}_{IJ}$ for any two two-sided ideals I, J in A.

Proof. Note that for hereditary A the functor Dp_I preserves A-proj. Because of exactness, established in Lemma 8, it is thus enough to prove the isomorphism when restricted to A-proj where it reduces to formula (1).

3.5. **Special dual projection functors.** The radical of A (see e.g. [DK, Section 3]) coincides with the radical of the A-A-bimodule ${}_{A}A_{A}$ and we have a short exact sequence

$$0 \to \operatorname{Rad}(A) \to {}_{A}A_{A} \to \bigoplus_{i=1}^{n} L_{ii} \to 0.$$

For every $i=1,2,\ldots,n$, this gives, using canonical projection onto a component of a direct sum, an epimorphism ${}_AA_A \to L_{ii}$. Let J_i denote the kernel of the latter epimorphism. We will use the shortcut F_i for the corresponding dual projection functor Dp_{J_i} . Setting $n_i := \dim(L_i)$ for $i=1,2,\ldots,n$, we have an isomorphism of left A-modules as follows:

(2)
$$J_i \cong \operatorname{Rad}(P_i)^{\oplus n_i} \oplus \bigoplus_{j \neq i} P_j^{\oplus n_j}.$$

3.6. **Dual projection functors versus projection functors.** In this subsection we explain the name *dual projection functors*.

For i = 1, 2, ..., n denote by G_i the unique, up to isomorphism, left exact endofunctor of A-mod satisfying the condition that

$$G_i|_{A-\text{inj}} \cong \Pr_{L_i}|_{A-\text{inj}}$$
.

For example, we can take

$$G_i = \operatorname{Hom}_A((\operatorname{Pr}_{L_i}(A^*))^*, _{-}),$$

where $(Pr_{L_i}(A^*))^*$ is viewed as an A-A-bimodule in the obvious way, see [GrMa, Subsection 2.3] for details. In other words, G_i is the unique left exact extension of the projection functor corresponding to the simple module L_i .

Proposition 10. There is an isomorphism of functors as follows: $F_i \cong *\circ G_i^{op} \circ *$.

Proof. Both F_i and $* \circ G_i^{op} \circ *$ are right exact functors and hence it is sufficient to prove that they are isomorphic on A-proj. For the additive generator A of the latter category we have

$$(\mathbf{G}_i^{\mathrm{op}}(A^*))^* \cong \mathrm{Hom}_{\text{-}A} \big((\mathrm{Pr}_{L_i^*}^{\mathrm{op}}(A^*))^*, A^* \big)^* \cong \mathrm{Hom}_{\text{-}\Bbbk} \big((\mathrm{Pr}_{L_i^*}^{\mathrm{op}}(A^*))^*, \Bbbk \big)^* \cong (\mathrm{Pr}_{L_i^*}^{\mathrm{op}}(A^*))^*,$$

where the second isomorphism is given by adjunction, and thus the claim of our proposition amounts to finding a natural isomorphism between $(F_i(A))^* \cong J_i^*$ and $\Pr_{L_i^*}^{op}(A^*)$.

Applying * to the exact sequence $J_i \hookrightarrow A \twoheadrightarrow L_{ii}$ results in the exact sequence $L_{ii}^* \hookrightarrow A^* \twoheadrightarrow J_i^*$. As $L_{ii}^* \cong L_{ii}$ and all other simple subbimodules of A^* are of the form L_{jj} for some $j \neq i$, the submodule $\operatorname{Tr}_{L_i^*}^{\operatorname{op}}(A^*)$ coincides with L_{ii}^* . This implies that there is a bimodule isomorphism $J_i^* \cong \operatorname{Pr}_{L_i^*}^{\operatorname{op}}(A^*)$ which completes the proof.

Proposition 10 allows us to freely transfer results for projection functors to dual projection functors and vice versa. For technical reasons in this paper we will mostly work with dual projection functors.

3.7. Dual projection functors and coapproximation functors. In some cases dual projective functors can be interpreted as partial coapproximation functors in the terminology of [KhMa, Subsection 2.4]. For i = 1, 2, ..., n, set

$$Q_i := P_1 \oplus P_2 \oplus \cdots \oplus P_{i-2} \oplus P_{i-1} \oplus P_{i+1} \oplus P_{i+2} \oplus \cdots \oplus P_{n-1} \oplus P_n.$$

The functor C_i of partial coapproximation with respect to Q_i is defined as follows: Given $M \in A$ -mod, consider a short exact sequence $K \hookrightarrow P \twoheadrightarrow M$ with projective P. Then

$$C_i(M) := Tr_{Q_i}(P/Tr_{Q_i}(K))$$

and the action on morphisms is defined by first lifting them using projectivity and then restriction. From [KhMa, Lemma 9] it follows that C_i is right exact. The functor C_i comes together with a natural transformation $\kappa: C_i \to \operatorname{Id}_{A\operatorname{-mod}}$ which is injective on projective modules (note that, if M is projective in the above construction, then we may choose K=0 and $C_i(M)=\operatorname{Tr}_{Q_i}(M)$). In particular, if $\operatorname{Ext}^1_A(L_i,L_i)=0$, then we have

$$\operatorname{Tr}_{Q_i}(P_j) \cong \begin{cases} P_j, & \text{if } i \neq j; \\ \operatorname{Rad}(P_i), & \text{otherwise.} \end{cases}$$

Lemma 11. If $\operatorname{Ext}_{A}^{1}(L_{i}, L_{i}) = 0$, then $C_{i} \cong F_{i}$.

Proof. As both functors are right exact, it is enough to check the bimodule isomorphism $C_i(A) \cong F_i(A)$. Since AA is projective, we have $C_i(A) = \operatorname{Tr}_{Q_i}(A)$. At the same time, if $\operatorname{Ext}_A^1(L_i, L_i) = 0$, then $\operatorname{Tr}_{Q_i}(A) = J_i$. As the action of C_i on morphisms is defined via restriction, it follows that $C_i(A) \cong J_i$ as a bimodule. This completes the proof.

4. Some preliminaries on 2-categories

4.1. Finite and finitary 2-categories. We refer the reader to [Le, McL, Ma] for generalities on 2-categories. Denote by \mathbf{Cat} the category of all small categories. A 2-category is a category enriched over \mathbf{Cat} . A 2-category $\mathscr C$ is called *finite* if it has finitely many objects, finitely many 1-morphisms and finitely many 2-morphisms.

Recall from [MM1] that a 2-category $\mathscr C$ is called *finitary* over k provided that

- \mathscr{C} has finitely many objects;
- each $\mathscr{C}(i,j)$ is an idempotent split additive k-linear category with finitely many isomorphism classes of indecomposable objects and finite dimensional spaces of morphisms;
- all compositions are biadditive and also k-bilinear whenever the latter makes sense;
- all identity 1-morphisms are indecomposable.

For an object i of a 2-category we denote by $\mathbb{1}_i$ the corresponding identity 1-morphism.

4.2. The multisemigroup of a finitary 2-category. For a finitary 2-category \mathscr{C} denote by $\mathscr{S}_{\mathscr{C}}$ the set of isomorphism classes of indecomposable 1-morphisms in \mathscr{C} with an added external zero element 0. By [MM2, Subsection 3.3], the finite set $\mathscr{S}_{\mathscr{C}}$ has the natural structure of a multisemigroup given for $[F], [G] \in \mathscr{S}_{\mathscr{C}}$ by defining

$$[F]\star[G]:=\begin{cases} \{[H]: \ H \ \text{is isomorphic to a direct summand of} \ F\circ G\}\,, & F\circ G\neq 0;\\ 0, & \text{otherwise}. \end{cases}$$

We refer the reader to [KuMa] for more details on multisemigroups.

4.3. &-linearization of finite categories. For a set X denote by &[X] the &-vector space of all formal linear combinations of elements in X with coefficients in &. Then X is naturally identified with a basis in &[X]. Note that $\&[X] = \{0\}$ if $X = \varnothing$.

Let \mathcal{C} be a finite category, that is a category with finitely many objects and morphisms. The \mathbb{k} -linearization of \mathcal{C} is the category $\mathcal{C}_{\mathbb{k}}$ defined as follows:

- $\mathcal{C}_{\mathbb{k}}$ and \mathcal{C} have the same objects;
- $C_{\mathbb{k}}(\mathbf{i},\mathbf{j}) := \mathbb{k}[C(\mathbf{i},\mathbf{j})];$
- composition in \mathcal{C}_{\Bbbk} is induced from composition in \mathcal{C} by \Bbbk -bilinearity.

The additive k-linearization \mathcal{C}_k^{\oplus} of \mathcal{C} is then the "additive closure" of \mathcal{C}_k in the following sense:

- objects in $\mathcal{C}_{\Bbbk}^{\oplus}$ are all expressions of the form $\mathbf{i}_1 \oplus \mathbf{i}_2 \oplus \cdots \oplus \mathbf{i}_k$, where $k \in \{0, 1, 2, \dots\}$ and all \mathbf{i}_i are objects in \mathcal{C}_{\Bbbk} ;
- the set $C_{\mathbb{k}}^{\oplus}(\mathbf{i}_1 \oplus \mathbf{i}_2 \oplus \cdots \oplus \mathbf{i}_k, \mathbf{j}_1 \oplus \mathbf{j}_2 \oplus \cdots \oplus \mathbf{j}_m)$ consists of all matrices of the form

$$\begin{pmatrix} f_{11} & f_{12} & \dots & f_{1k} \\ f_{21} & f_{22} & \dots & f_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ f_{m1} & f_{m2} & \dots & f_{mk} \end{pmatrix}$$

where $f_{st} \in \mathcal{C}_{\mathbb{k}}(\mathbf{i}_t, \mathbf{j}_s)$;

- composition in $\mathcal{C}_{\Bbbk}^{\oplus}$ is given by matrix multiplication.
- 4.4. Finitarization of finite 2-categories. Let $\mathscr C$ be a finite 2-category. Then the *finitarization* of $\mathscr C$ over \Bbbk is the 2-category $\mathscr C_{\Bbbk}$ defined as follows:
 - $\mathscr{C}_{\mathbb{k}}$ has the same objects as \mathscr{C} ;
 - $\bullet \ \mathscr{C}_{\Bbbk}(\mathtt{i},\mathtt{j}) := \mathscr{C}(\mathtt{i},\mathtt{j})^{\oplus}_{\Bbbk};$
 - composition in $\mathscr{C}_{\mathbb{k}}$ is induced from composition in \mathscr{C} using biadditivity and \mathbb{k} -bilinearity.

Directly from the definition it follows that $\mathscr{C}_{\mathbb{k}}$ is finitary if and only if for each 1-morphism f in \mathscr{C} the endomorphism algebra $\operatorname{End}_{\mathscr{C}_{\mathbb{k}}}(f) \cong \mathbb{k}[\operatorname{End}_{\mathscr{C}}(f)]$ is local.

- 4.5. Two 2-categories associated with an ordered monoid. Let (S, e, \cdot) be a finite monoid with a fixed admissible reflexive partial pre-order \leq . Admissibility means that $s \leq t$ implies both $sr \leq tr$ and $rs \leq rt$ for all $s, t, r \in S$. In this situation we may define a finite 2-category \mathscr{C}^S as follows:
 - \mathscr{C}^S has one object \clubsuit ;
 - 1-morphisms in $\mathscr{C}^S(\clubsuit,\clubsuit)$ are elements in S and the horizontal composition of 1-morphisms is given by multiplication in S;
 - for two 1-morphisms s and t, the set of 2-morphisms from s to t is empty if $s \not\preceq t$ and contains one element, denoted (s,t), otherwise (note that in this case all compositions of 2-morphisms are automatically uniquely defined).

The finitarization \mathscr{C}^S_{\Bbbk} of \mathscr{C}^S is then a finitary 2-category as the endomorphism algebra of each 1-morphism is just \Bbbk .

- 5. 2-categories of idealization functors and dual projection functors
- 5.1. Monoid of two-sided ideals. The set \mathcal{I} of all two-sided ideals in A has the natural structure of a monoid given by multiplication of ideals $(I, J) \mapsto IJ$. The identity element of \mathcal{I} is A and the zero element is the zero ideal. We note the following:

Lemma 12. If $\dim_{\mathbb{R}} \operatorname{Hom}_A(P_i, P_j) \leq 1$ for all $i, j \in \{1, 2, ..., n\}$, then $|\mathcal{I}| < \infty$.

Proof. If $a, b \in A$ are idempotents, then, by adjunction, we have

$$\operatorname{Hom}_{A-A}(Aa \otimes_{\mathbb{k}} bA, A) \cong \operatorname{Hom}_{A-}(Aa, Ab) \cong \operatorname{Hom}_{\mathbb{k}}(\mathbb{k}, aAb) = aAb.$$

For $i, j \in \{1, 2, ..., n\}$, the projective cover of the simple bimodule L_{ij} in A-mod-A is isomorphic to $P_i \otimes_{\mathbb{k}} I_j^*$ and hence from our assumptions it follows that the composition multiplicity of L_{ij} in ${}_A A_A$ is at most 1. This means that each subbimodule of ${}_A A_A$ is uniquely determined by its composition subquotients (and equals the sum of images of unique up to scalar nonzero homomorphisms from the projective covers of these simple subquotients). Therefore $|\mathcal{I}| \leq 2^{\dim(A)}$.

Corollary 13. If A is the path algebra of a tree quiver or the incidence algebra of a finite poset, then $|\mathcal{I}| < \infty$.

Proof. Both for the path algebra of a tree quiver and for the incidence algebra of a finite poset, the condition $\dim_{\mathbb{R}} \operatorname{Hom}_A(P_i, P_j) \leq 1$ for all $i, j \in \{1, 2, ..., n\}$ is straightforward and thus the statement follows from Lemma 12.

The monoid \mathcal{I} is naturally ordered by inclusions, moreover, this order is obviously admissible.

- 5.2. A 2-action of $\mathscr{C}^{\mathcal{I}}$ on A-mod by idealization functors. We define a 2-action of the 2-category $\mathscr{C}^{\mathcal{I}}$ associated to the ordered monoid $(\mathcal{I}, A, \cdot, \subset)$ on A-mod as follows:
 - the element $I \in \mathcal{I}$ acts as the functor Su_I ;
 - for $I \subset J$, the 2-morphism (I,J) acts as the canonical inclusion $Su_I \hookrightarrow Su_J$.

This is a strict 2-action because of (1).

This 2-action extends to a 2-action of $\mathscr{C}^{\mathcal{I}}_{\Bbbk}$ on A-mod in the obvious way. Note that this 2-action is clearly faithful both on the level of 1-morphisms and on the level of 2-morphisms. However, this 2-action is not full on the level of 2-morphisms in general. Indeed, in case the algebra A has a non-trivial center, the 1-dimensional endomorphism algebra of the identity 1-morphism in $\mathscr{C}^{\mathcal{I}}_{\Bbbk}$ cannot surject onto the non-trivial endomorphism algebra of the identity functor of A-mod.

5.3. A 2-action of $\mathscr{C}^{\mathcal{I}}$ on A-mod by dual projection functors. The main disadvantage of the 2-action defined in Subsection 5.2 is the fact that the functors Su_J are not exact from any side in general. In particular, they do not induce any reasonable maps on the Grothendieck group of A-mod. To overcome this problem one needs to define another action and dual projection functors are reasonable candidates. However, there is a price to pay. Firstly, in order to avoid weak 2-actions (where equalities of functors are changed to isomorphisms with some coherency conditions, see e.g. [Le]), we will have to change A-mod to an equivalent category. Secondly, we will have to restrict to hereditary algebras.

Denote by \overline{A} -proj the category whose objects are diagrams $P \stackrel{f}{\longrightarrow} Q$ over A-proj and whose morphisms are equivalence classes of solid commutative diagrams



modulo the equivalence relation defined as follows: the solid diagram is equivalent to zero provided that there exists a dashed map h as indicated on the diagram such that $g_1 = f'h$. The category \overline{A} -proj is abelian and, moreover, equivalent to A-mod, see [Fr]. This construction is called *abelianization* in [MM1, MM2].

If A is hereditary, then each Su_I preserves A-proj and hence the 2-actions of both $\mathscr{C}^{\mathcal{I}}$ and $\mathscr{C}^{\mathcal{I}}_{\mathbb{k}}$ defined in Subsection 5.2 extends component-wise to 2-actions of both these categories on \overline{A} -proj. By construction, this is not an action on A-mod but on a category which is only equivalent to A-mod. Moreover, the action is designed so that the ideal I acts by a right exact functor which is isomorphic to Su_I when restricted to A-proj. This means that this is a 2-action by dual projection functors.

- 5.4. The 2-category of idealization functors. The 2-action defined in Subsection 5.2 suggest the following definition. Fix a small category \mathcal{C} equivalent to A-mod. Define the 2-category $\mathcal{Q} = \mathcal{Q}(A,\mathcal{C})$ in the following way:
 - \mathcal{Q} has one object \clubsuit (which we identify with \mathcal{C});
 - 1-morphisms in \mathcal{Q} are endofunctors of \mathcal{C} which belong to the additive closure generated by the identity functor and all idealization functors;
 - 2-morphisms in \mathcal{Q} are all natural transformations of functors;
 - composition in \mathcal{Q} is induced from \mathbf{Cat} .

Our main observation here is the following:

Proposition 14. If A is connected and $|\mathcal{I}| < \infty$, then \mathcal{Q} is a finitary 2-category.

Proof. Connectedness of A ensures that the identity 1-morphism $\mathbb{1}_{\clubsuit}$ is indecomposable. Clearly, \mathscr{Q} has finitely many objects. As $|\mathcal{I}| < \infty$, the 2-category \mathscr{Q} has finitely many isomorphism classes of indecomposable 1-morphisms. It remains to check that all spaces of 2-morphisms are finite dimensional.

Let I and J be two ideals in A. Let $\eta: \operatorname{Su}_I \to \operatorname{Su}_J$ be a natural transformation. We claim that values of η on indecomposable projective A-modules determine η uniquely. Indeed, by additivity these values determine all values of η on all projective A-modules. For $M \in A$ -mod, choose some projective cover $f: P \twoheadrightarrow M$. Then, by Lemma 4(ii), we have the commutative diagram:

$$Su_{I}(P) \xrightarrow{Su_{I}(f)} Su_{I}(M)$$

$$\eta_{P} \downarrow \qquad \qquad \qquad \downarrow \eta_{M}$$

$$Su_{J}(P) \xrightarrow{Su_{J}(f)} Su_{J}(M)$$

From this diagram we see that η_M is uniquely determined by η_P . Consequently, all spaces of 2-morphisms in \mathcal{Q} are finite dimensional.

The fact that Su_I is not right exact implies that, potentially, there might exist a natural transformation $\eta|_{A\text{-proj}} : Su_I|_{A\text{-proj}} \to Su_J|_{A\text{-proj}}$ which cannot be extended to a natural transformation $\eta : Su_I \to Su_J$. Note also that in the case when A has finite representation type the space of natural transformations between any two additive endofunctors on A-mod is finite dimensional (since an additive endofunctor is uniquely determined, up to isomorphism, by its action on indecomposable objects and morphisms between them and in the case when A has finite representation type there are only finitely many indecomposable A-modules).

- 5.5. The 2-category of dual projection functors. The 2-action defined in Subsection 5.3 suggest the following definition. Assume that A is hereditary. Fix a small category \mathcal{C} equivalent to A-mod. Define the 2-category $\mathscr{P} = \mathscr{P}(A, \mathcal{C})$ in the following way:
 - \mathscr{P} has one object \clubsuit (which we identify with \mathscr{C});
 - 1-morphisms in \mathscr{P} are endofunctors of \mathscr{C} which belong to the additive closure generated by the identity functor and all dual projection functors;
 - 2-morphisms in \mathscr{P} are all natural transformations of functors;
 - composition in \mathcal{P} is induced from Cat.

Our main observation here is the following:

Proposition 15. If A is hereditary, connected and $|\mathcal{I}| < \infty$, then \mathscr{P} is a finitary 2-category.

Proof. Similarly to the proof of Proposition 14, the 2-category \mathscr{P} has one object, finitely many isomorphism classes of indecomposable 1-morphisms thanks to the assumption $|\mathcal{I}| < \infty$, and indecomposable identity 1-morphism \mathbb{I}_{\clubsuit} thanks to the assumption that A is connected. Spaces of 2-morphisms are finite dimensional as projection functors are right exact and hence are given by tensoring with finite dimensional bimodules which yields that spaces of 2-morphisms are just bimodule homomorphisms between these finite dimensional bimodules.

5.6. **Decategorification and categorification.** Let \mathscr{C} be a finitary 2-category. Then the *decategorification* of \mathscr{C} is the (1-)category $[\mathscr{C}]$ defined as follows.

- $[\mathscr{C}]$ has same objects as \mathscr{C} ;
- for all $i, j \in \mathscr{C}$ the morphism set $[\mathscr{C}](i, j)$ is defined to be the split Grothendieck group $[\mathscr{C}(i, j)]_{\oplus}$ of the additive category $\mathscr{C}(i, j)$;
- composition in $[\mathscr{C}]$ is induced from composition in \mathscr{C} .

Given a 2-functor Φ from \mathscr{C} to the 2-category of additive categories, taking the split Grothendieck group for each $\Phi(\mathbf{i})$ induces a functor $[\Phi]$ from $[\mathscr{C}]$ to \mathbf{Cat} which is called the *decategorification* of Φ .

Given a 2-functor Φ from $\mathscr C$ to the 2-category of abelian categories and exact functors, taking the usual Grothendieck group for each $\Phi(i)$ induces a functor $[\Phi]$ from $[\mathscr C]$ to \mathbf{Cat} which is also called the *decategorification* of Φ .

Conversely, the 2-category $\mathscr C$ is called a *categorification* of the category $[\mathscr C]$ and the 2-functor Φ is called a *categorification* of the functor Φ . We refer to [Ma, Section 1] for more details and examples.

6. Indecomposable summands of dual projection functors for path algebras of admissible trees

In this section we study both the monoid \mathcal{I} and the multisemigroup $\mathcal{S}_{\mathscr{P}}$ in case A is the path algebra of the quiver Q given by an admissible orientation of a tree.

6.1. Categorification of the Catalan monoid. To start with, we briefly recall the main results from [GrMa]. Let A be the path algebra of the following quiver

$$(3) 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \dots \longrightarrow m.$$

The main result of [GrMa] asserts that the ring $[\mathscr{P}](\clubsuit,\clubsuit)$ in the corresponding decategorification is isomorphic to the integral monoid algebra of the monoid C_{m+1} of all order preserving and decreasing transformations of $\{1,2,\ldots,m,m+1\}$, which is also known as the *Catalan monoid*. Moreover, the monoid C_{m+1} is an ordered monoid and the 2-category \mathscr{P} is biequivalent to the corresponding 2-category $\mathscr{C}^{c_{m+1}}$. We can also observe that in this case the multisemigroup $S_{\mathscr{P}}$ is a usual monoid (i.e. the operation in $S_{\mathscr{P}}$ is single-valued rather than multi-valued) and is, in fact, isomorphic to C_{m+1} .

A very special feature of this example is the fact that the bimodule ${}_{A}A_{A}$ has simple socle. Consequently, all ideals of A are indecomposable as A-A-bimodules. One observation in addition to the results from [GrMa] is the following.

Proposition 16. If A is the path algebra of the quiver (3), then the 2-categories \mathscr{Q} and \mathscr{P} are biequivalent.

Proof. Note that in this situation A is hereditary and connected. From Corollary 13 it follows that $|\mathcal{I}| < \infty$. In particular, both \mathcal{Q} and \mathcal{P} are well-defined and finitary, see Propositions 14 and 15. For both of these 2-categories consider the restriction 2-functor to the 2-category of additive endofunctors on $\mathcal{C}_{\text{proj}}$, where the latter stands for the category of projective objects in \mathcal{C} . This is well defined as the action of Su_I preserves $\mathcal{C}_{\text{proj}}$ for each I as A is hereditary. The restriction 2-functor is clearly faithful both on the level of 1-morphisms and on the level of 2-morphisms.

Now, for any non-zero I and J, the space $\operatorname{Hom}_{A\text{-}A}(I,J)$ is zero if $I \not\subset J$ and is one-dimensional otherwise since both I and J have simple socle (as A-A-bimodules) and the corresponding simple bimodule appears with multiplicity one in both of them. This means that $\operatorname{Hom}_{\mathscr{P}}(\operatorname{Dp}_I,\operatorname{Dp}_J)$ is zero if $I \not\subset J$ and is one-dimensional otherwise.

As the restrictions of Su_I and Dp_I to C_{proj} are isomorphic (by construction), from the previous paragraph and the proof of Proposition 14 it follows that $Hom_{\mathscr{Q}}(Su_I, Su_J)$ is zero if $I \not\subset J$ and is at most one-dimensional otherwise. However, the inclusion $I \subset J$ does give rise to a non-zero natural transformation in $Hom_{\mathscr{Q}}(Su_I, Su_J)$ in the obvious way. Therefore $Hom_{\mathscr{Q}}(Su_I, Su_J)$ is one-dimensional if $I \subset J$. This implies that both restriction 2-functors are full and faithful. As already noted above, by construction of dual projection functors, the values of both these restrictions hit exactly the same isomorphism classes of endofunctors of C_{proj} . The claim follows.

6.2. Setup and some combinatorics. For a vertex i of an oriented graph Γ we denote by $\deg_{\Gamma}(i)$ the degree of i, by $\deg_{\Gamma}^{\mathrm{in}}(i)$ the in-degree of i and by $\deg_{\Gamma}^{\mathrm{out}}(i)$ the out-degree of i. Clearly, $\deg_{\Gamma}(i) = \deg_{\Gamma}^{\mathrm{in}}(i) + \deg_{\Gamma}^{\mathrm{out}}(i)$.

In the rest of the paper we consider an oriented (connected) tree Q with vertex set $Q_0 = \{1, 2, ..., n\}$, where n > 1. Set

$$\mathbf{K}(Q) = \{i \in Q_0 \, ; \, \deg_Q^{\mathrm{in}}(i) \deg_Q^{\mathrm{out}}(i) = 0\}, \qquad \mathbf{K}'(Q) = \{i \in \mathbf{K}(Q) \, ; \, \deg_Q(i) \geq 2\}.$$

In other words, $\mathbf{K}(Q)$ is the set of all sinks and sources in Q and $\mathbf{K}'(Q)$ is the set of all elements $i \in \mathbf{K}(Q)$ which are not leaves. In what follows identify subsets in Q_0 with the corresponding full subgraphs in Q. A function $\alpha: Q_0 \to X$, for any X, will be written $\alpha = (\alpha(1), \alpha(2), \ldots, \alpha(n))$.

Following [Gr], we say that Q is *admissible* provided that all vertices of Q of degree at least 3 belong to $\mathbf{K}(Q)$.

Example 17. The orientation of a D_4 diagram on the left hand side of the following picture is admissible while the one on the right hand side is not.

$$(4) \qquad \qquad \downarrow \qquad \qquad \downarrow$$

For $i \in Q_0$, we denote by \overline{i} the set of all elements in Q_0 to which there is an oriented path (possibly empty) from i in the quiver Q. Elements in \overline{i} will be called *successors* of i. We have $j \in \overline{i}$ if and only if the pair (j,i) belongs to the transitive closure of the binary relation given by elements in Q_1 (our convention is that the arrow from s to t corresponds to the pair (t,s)). For $X \subset Q_0$, we define

$$\overline{X} := \bigcup_{x \in X} \overline{x}.$$

Note that $\overline{\varnothing} = \varnothing$.

Example 18. For the left quiver in (4) we have $\overline{1} = \{1,2\}$ while for the right quiver in (4) we have $\overline{1} = \{1,2,3\}$ and $\overline{\{1,4\}} = Q_0$.

A function $\alpha: Q_0 \to Q_0 \cup \{0\}$ is called a *path function* provided that $\alpha(i) \in \overline{i} \cup \{0\}$ for all i. A path function is called *monotone* provided that, for all i, j such that

 $i \in \overline{j}$ and $\alpha(i) \neq 0$, we have $\alpha(j) \neq 0$ and $\alpha(i) \in \overline{\alpha(j)}$. In particular, a monotone function maps to zero all successors of any preimage of zero.

Example 19. The identity function on Q_0 is a monotone path functions. The function which maps all elements of Q_0 to 0 is a monotone path function. At the same time, for the left quiver in (4), the function which maps 1 to 0 and i to i for all i = 2, 3, 4 is a path function but it is not monotone.

A chain in Q is a subtree isomorphic to (3) for some m. The set of all chains in Q is partially ordered by inclusions. A maximal chain is a chain which is maximal with respect to this partial order. We denote by \mathfrak{M}_Q the set of all maximal chains in Q.

Example 20. For the left quiver in (4), we have $\mathfrak{M}_Q = \{\{1,2\},\{2,3\},\{2,4\}\}.$

For a function $\alpha: Q_0 \to Q_0 \cup \{0\}$, define the $support \operatorname{supp}(\alpha)$ of α as the (not necessarily full) subgraph $Q^{(\alpha)}$ of Q given by the union of all $X \in \mathfrak{M}_Q$ which contain some $i \in Q_0$ such that $\alpha(i) \in X$. If α is a path function, then $X \in \mathfrak{M}_Q$ belongs to $\operatorname{supp}(\alpha)$ if and only if the image of α intersects X.

Example 21. Consider the quiver $1 \longrightarrow 2 \longleftarrow 3 \longrightarrow 4$. For the monotone path function $\alpha = (1, 2, 2, 0)$, the graph $\operatorname{supp}(\alpha)$ is

$$1 \longrightarrow 2 \longleftarrow 3$$

in particular, it is connected. For the monotone path function $\beta = (1, 0, 4, 4)$, the graph supp (β) is

$$1 \longrightarrow 2$$
 $3 \longrightarrow 4$,

in particular, it is disconnected.

A monotone path function $\alpha: Q_0 \to Q_0 \cup \{0\}$ will be called a *special function* provided that its support is connected and, additionally, the equality $\alpha(i) = 0$ for $i \in \mathbf{K}(Q) \cap \operatorname{supp}(\alpha)$ implies that $\deg_{\operatorname{supp}(\alpha)}(i) = 1$.

Lemma 22. Let Q be admissible and α a special function. Then

- (i) for any $i \in \text{supp}(\alpha)$ we have $\deg_{\text{supp}(\alpha)}(i) \in \{\deg_Q(i), 1\}$;
- (ii) $\alpha(i) \neq i$ for any $i \in \mathbf{K}(Q) \cap \operatorname{supp}(\alpha)$ with $\deg_{\operatorname{supp}(\alpha)}(i) = 1$;
- (iii) $\alpha(i) = i$ for any $i \in \mathbf{K}(Q) \cap \operatorname{supp}(\alpha)$ with $\deg_{\operatorname{supp}(\alpha)}(i) = \deg_{Q}(i)$.

Proof. Let $i \in \operatorname{supp}(\alpha)$. First consider the case $i \in Q_0 \setminus \mathbf{K}(Q)$. In this case $\deg_Q(i) = 2$ since Q is admissible. Therefore there is a unique $X \in \mathfrak{M}_Q$ containing i, moreover, i is not a leaf in X. From the definitions we thus have $X \subset \operatorname{supp}(\alpha)$ and hence $\deg_{\operatorname{supp}(\alpha)}(i) = \deg_Q(i) = 2$.

Assume now that $i \in \mathbf{K}(Q)$ is a sink. If $\alpha(i) \neq 0$, then $\alpha(i) = i$ since α is a path function. From $\alpha(i) = i$ and the definitions it follows that each maximal chain $X \in \mathfrak{M}_Q$ containing i belongs to $\operatorname{supp}(\alpha)$. Therefore $\deg_{\operatorname{supp}(\alpha)}(i) = \deg_Q(i)$. If $\alpha(i) = 0$, then $\deg_{\operatorname{supp}(\alpha)}(i) = 1$ as α is special.

Assume, finally, that $i \in \mathbf{K}(Q)$ is a source. If $\alpha(i) = i$, then any $X \in \mathfrak{M}_Q$ containing i belongs to $\operatorname{supp}(\alpha)$ by construction and hence $\deg_{\operatorname{supp}(\alpha)}(i) = \deg_Q(i)$. If $\alpha(i) = 0$, then $\alpha(j) = 0$ for all $j \in \overline{i}$ since α is monotone. Therefore none of the maximal chains starting at i belongs to $\operatorname{supp}(\alpha)$ and thus $i \notin \operatorname{supp}(\alpha)$, which contradicts our assumptions.

It is left to consider the case $\alpha(i) \notin \{0,i\}$. Let $\alpha(i) = j$ and $X \in \mathfrak{M}_Q$ be the unique maximal chain containing i and j. Let $Y \in \mathfrak{M}_Q$ be any other maximal chain starting in i and $s \in Y \setminus \{i\}$. Assume that $\alpha(s) \neq 0$. Then $\alpha(s) \in \overline{s}$ since α is a path function. Further, $\overline{s} \cap X = \emptyset$ since Q is a tree. In particular, we have $\overline{s} \cap \overline{\alpha(i)} = \emptyset$, which contradicts the assumption that α is monotone. Therefore $\alpha(s) = 0$ and hence all non-leaves in Y and all edges in Y are not in $\operatorname{supp}(\alpha)$ by construction. Since $\operatorname{supp}(\alpha)$ is connected by assumptions and Q is a tree, it follows that $Y \cap \operatorname{supp}(\alpha) = i$. Therefore $\deg_{\operatorname{supp}(\alpha)}(i) = 1$ in this case. The claim of the lemma follows.

Example 23. If Q is the quiver given by the left hand side of (4), then possible supports for special functions for Q are: Q_0 , $\{1,2\}$, $\{2,3\}$, $\{2,4\}$ and \varnothing . The following is a complete list of special functions for Q with support Q_0 :

$$(1,2,3,4), (2,2,3,4), (1,2,2,4), (1,2,3,2), (2,2,2,4), (1,2,2,2), (2,2,3,2), (2,2,2,2).$$

For the same Q, here is a complete list of special functions for Q with support $\{1,2\}$:

Finally, (0,0,0,0) is the only special function for Q with support \varnothing . The total number of special functions in this case is 15.

Example 24. If Q is the quiver $1 \longrightarrow 2 \longleftarrow 3$, then possible supports for special functions for Q are: Q_0 , $\{1,2\}$, $\{2,3\}$ and \varnothing . The following is a complete list of special functions for Q with support Q_0 :

For the same Q, here is a complete list of special functions for Q with support $\{1,2\}$:

Finally, (0,0,0) is the only special function for Q with support \varnothing . The total number of special functions in this case is 9.

Example 25. If Q is the quiver $1 \leftarrow 2 \longrightarrow 3$, then possible supports for special functions for Q are: Q_0 , $\{1,2\}$, $\{2,3\}$ and \varnothing . The following is a complete list of special functions for Q with support Q_0 :

For the same Q, here is a complete list of special functions for Q with support $\{1,2\}$:

Finally, (0,0,0) is the only special function for Q with support \varnothing . The total number of special functions in this case is 9.

We denote by **C** the set of all special functions for Q. A subtree Γ of Q (possibly empty) is called a *special subtree* if $\Gamma = \text{supp}(\alpha)$ for some special function α . We denote by **W** the set of all special subtrees of Q. We write

$$\mathbf{C} = \bigcup_{\Gamma \in \mathbf{W}} \mathbf{C}(\Gamma)$$

where $\mathbf{C}(\Gamma)$ stands for the set of all special functions with support Γ .

6.3. **Type** A **enumeration.** Here we enumerate special functions for type A quivers. In this subsection we let Q be the oriented quiver obtained by choosing some orientation of the following Dynkin diagram of type A_n :

$$(5) 1 - 2 - \dots - n$$

As mentioned above, we assume n > 1. We write $\mathbf{K}(Q) = \{l_1, l_2, \dots, l_k\}$, where $1 = l_1 < l_2 < \dots < l_k = n$.

Lemma 26. Let α be a non-zero special function for Q. Then the set $supp(\alpha)$ has the form $\{l_i, l_i + 1, ..., l_j\}$ for some $i, j \in \{1, 2, ..., k\}$ with i < j.

Proof. By definition, $\operatorname{supp}(\alpha)$ is connected, has more than one vertex and both leaves of $\operatorname{supp}(\alpha)$ belong to $\mathbf{K}(Q)$. The claim follows.

After Lemma 26, for $i, j \in \{1, 2, ..., k\}$ with i < j we denote by $\mathbf{C}(i, j)$ the set of all special functions with support $\{l_i, l_i + 1, ..., l_j\}$. For m = 0, 1, 2, ..., we denote by $\operatorname{cat}(m)$ the m-th Catalan number $\frac{1}{m+1}\binom{2m}{m}$ and $\operatorname{set} \underbrace{\operatorname{cat}(m) := \operatorname{cat}(m) - 1}$. For m = 1, 2, 3, ..., we set

$$cat_1(m) := cat(m) - cat(m-1)$$
 and $cat_1(m) := cat_1(m) - 1$.

For m = 2, 3, 4, ..., we set

$$cat_2(m) := cat(m) - 2cat(m-1) + cat(m-2)$$
 and $cat_2(m) := cat_2(m) - 1$.

Proposition 27.

- (i) If k = 2, then $|\mathbf{C}(1,2)| = \underline{\cot}(n+1)$.
- (ii) If k > 3 and $i \in \{2, 3, ..., k 2\}$, then $|\mathbf{C}(i, i + 1)| = \frac{\cot_2(l_{i+1} l_i + 2)}{2}$.
- (iii) If k > 4 and $i, j \in \{2, 3, ..., k-1\}$ with j > i+1, then

$$|\mathbf{C}(i,j)| = \cot_1(l_{i+1} - l_i + 1)\cot_1(l_j - l_{j-1} + 1) \prod_{s=i+1}^{j-2} \cot(l_{s+1} - l_s).$$

(iv) If k > 2, then

$$|\mathbf{C}(1,k)| = \cot(l_2)\cot(l_k - l_{k-1} + 1) \prod_{s=2}^{k-2} \cot(l_{s+1} - l_s).$$

(v) If k > 3 and $j \in \{3, 4, ..., k - 1\}$, then

$$|\mathbf{C}(1,j)| = \cot(l_2)\cot_1(l_j - l_{j-1} + 1) \prod_{s=2}^{j-2} \cot(l_{s+1} - l_s).$$

(vi) If k > 3 and $i \in \{2, 3, ..., k - 2\}$, then

$$|\mathbf{C}(i,k)| = \cot(l_k - l_{k-1} + 1)\cot_1(l_{i+1} - l_i + 1) \prod_{s=i+1}^{k-2} \cot(l_{s+1} - l_s).$$

(vii) If k > 2, then

$$|\mathbf{C}(1,2)| = \underline{\cot}_1(l_2 - l_1 + 2)$$
 and $|\mathbf{C}(k-1,k)| = \underline{\cot}_1(l_k - l_{k-1} + 2)$.

To prove this we will need the combinatorial lemmata below. For $q \in \{1, 2, \dots\}$, we set $q := \{1, 2, \dots, q\}$.

Lemma 28. For $q \in \{1, 2, ...\}$, let X_q denote the set of all transformations f of q satisfying the conditions

- (i) $f(i) \le i$ for all $i \in q$ (i.e. f is order-decreasing);
- (ii) $f(i) \le f(j)$ for all $i, j \in \underline{q}$ such that $i \le j$ (i.e. f is order-preserving).

Then $|X_q| = \operatorname{cat}(q)$.

Proof. See, for example, [Hi].

Lemma 29. For $q \in \{2, 3, ...\}$, let Y_q denote the set of all $f \in X_q$ such that $f(q) \neq q$. Then $|Y_q| = \text{cat}_1(q)$.

Proof. Let Y'_q denote the set of all $f \in X_q$ such that f(q) = q. Then restriction to q-1 defines a bijection from Y'_q to X_{q-1} . Therefore the claim of our lemma follows from Lemma 28.

Lemma 30. For $q \in \{3, 4, ...\}$, let Z_q denote the set of all $f \in Y_q$ such that f(2) = 1. Then $|Z_q| = \text{cat}_2(q)$.

Proof. Let Z'_q denote the set of all $f \in Y_q$ such that f(2) = 2. Then restriction to $\{2, 3, \ldots, q\}$ followed by the identification of the latter set with q-1 given by $x \mapsto x-1$ gives rise to a bijection from Z'_q to Y_{q-1} . Therefore the claim of our lemma follows from Lemma 29.

Proof of Proposition 27. To prove claim (i), let k=2 and assume that Q is given by (3). For convenience, we write n+1 for 0. For a function $\alpha:Q_0\to Q_0\cup\{n+1\}$, let α' denote the extension of α to a transformation of $Q_0\cup\{n+1\}$ via $\alpha'(n+1)=n+1$. Then the fact that α is a path function is equivalent to the requirement $\alpha'(i)\geq i$ for all i (i.e. α is order increasing). Furthermore, the fact that α is monotone is equivalent to the requirement that $i\leq j$ implies $\alpha'(i)\leq \alpha'(j)$ for all i,j (i.e. α is order preserving). Thus the correspondence $\alpha\mapsto\alpha'$ defines a bijection between $\mathbf{C}(1,2)$ and the set of all order increasing and order preserving transformations of $Q_0\cup\{n+1\}$ which are different from the constant transformation with image n+1 (the latter is the unique constant order increasing and order preserving transformation). Hence $|\mathbf{C}(1,2)| = \cot(n+1) - 1$, using Lemma 28 with reversed order.

To prove claim (ii), assume that k>3 and $i\in\{2,3,\ldots,k-2\}$. Consider a special function α supported on $\{l_i,l_i+1,\ldots,l_{i+1}\}$, in particular, α is zero outside this interval. Without loss of generality we may assume that the arrows in this interval are of the form $x\to x-1$ (the other case is similar). Then, using Lemma 22, we have $\alpha(l_i)=0$ and $\alpha(l_{i+1})< l_{i+1}$. Similarly to the previous paragraph, the requirement that α is a path function is equivalent to the fact that it decreases the order and the requirement that α is monotone is equivalent to the fact that it preserves the order. Define a bijection from $\{0,l_i,l_i+1,\ldots,l_{i+1}\}$ to $\{1,2,\ldots,q\}$, where $q=l_{i+1}-l_i+2$, by mapping 0 to 1 and l_i+j to 2+j, for $j=0,1,\ldots$ Under this bijection, special functions supported on $\{l_i,l_i+1,\ldots,l_{i+1}\}$ are mapped exactly to those functions in Z_q which are different from the constant function $\underline{q}\to 1$. Therefore claim (ii) follows from Lemma 30.

To prove claim (vii), assume k > 2. We prove the first equality, the second one follows because of symmetry by swapping the order. Consider a special function α supported on $\{1, 2, \ldots, l_2\}$, in particular, α is zero outside this interval. Without loss of generality we may assume that the arrows in this interval are of the form

 $x \to x-1$ (the other case is similar). Then $\alpha(l_2) < l_2$. Similarly to the above, there is a bijection between such functions and order decreasing and order preserving transformations of $\underline{l_2}$ different from the unique constant transformation and satisfying $\alpha(l_2) < l_2$. Therefore claim (vii) follows from Lemma 29.

To prove claim (iii), assume k>2 and $i,j\in\{2,3,\ldots,k-2\}$ are chosen such that i< j. Consider a special function α supported on $\{l_i,l_i+1,\ldots,l_j\}$, in particular, α is zero outside this interval. Then $\alpha(l_i)\neq l_i$ and $\alpha(l_j)\neq l_j$. At the same time, $\alpha(l_s)=l_s$ for all s such that i< s< j. The value of α can be chosen independently on the intervals of the form $\{l_s,l_s+1,\ldots,l_{s+1}\}$, where s is such that $i\leq s< j$. If $s\neq i,j-1$, then on this interval the values of α correspond precisely to order preserving and order decreasing (or increasing, depending on the orientation of arrows on this interval) transformations of this interval, taking into account the conditions $\alpha(l_s)=l_s$ and $\alpha(l_{s+1})=l_{s+1}$. Hence, from Lemma 28 it follows that we have $\cot(l_{s+1}-l_s)$ choices for the values of α on this interval. On the interval $\{l_i,l_i+1,\ldots,l_{i+1}\}$ we additionally have to take into account the condition $\alpha(l_i)\neq l_i$ to get exactly $\cot(l_{i+1}-l_i+1)$ choices by Lemma 29. By symmetry, we have $\cot(l_j-l_{j-1}+1)$ choices for the last interval. Now claim (iii) follows by applying the product rule.

Claims (iv)—(vi) are proved similarly to claim (iii).

Example 31. For the quiver $1 \longrightarrow 2 \longleftarrow 3 \longleftarrow 4 \longrightarrow 5 \longleftarrow 6$, we have k = 5, $l_2 = 2$, $l_3 = 4$ and $l_4 = 5$. Proposition 27(iii) says that there are exactly $\operatorname{cat}_1(3)\operatorname{cat}_1(2) = 3$ special functions supported on $\{2, 3, 4, 5\}$. These functions are

$$(0,0,3,4,0,0), (0,0,2,4,0,0), (0,0,0,4,0,0).$$

6.4. Some notation for the path algebra. Let us go back to an admissible tree quiver Q as defined in Subsection 6.2. Let A be the path algebra of Q over k. For $i \in Q_0$ denote by e_i the trivial path in vertex i. Then $P_i = Ae_i$ and $L_i = P_i/\text{Rad}(P_i)$. For each $i, j \in \{1, 2, ..., n\}$ such that $j \in \overline{i}$, denote by a_{ji} the unique path from i to j. Then $\{a_{ji}\}$ is a basis in the one-dimensional vector space e_jAe_i .

From now on we assume that $\mathbf{K}'(Q) \neq \emptyset$. This is equivalent to the requirement that Q is not isomorphic to the quiver given by (3).

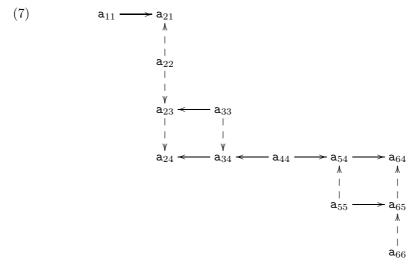
6.5. Graph of the identity bimodule. To study subbimodules of the identity bimodule, it is convenient to use a graphical presentation of the latter. For this we consider ${}_{A}A_{A}$ as an $A\otimes A^{\mathrm{op}}$ -module, cf. [Ba, Chapter II], or, equivalently, as a representation of the quiver $Q\times Q^{\mathrm{op}}$ where we impose all possible commutativity relations. We refer the reader to [Sk] for details concerning the isomorphism between $A\otimes A^{\mathrm{op}}$ and the quiver algebra, see also [ASS, Ri] for details on representations of quivers in general.

Viewing ${}_{A}A_{A}$ as a representation of $Q \times Q^{\text{op}}$ (with relations) can be arranged into a graph, whose vertices are paths in Q, with left multiplication by arrows in Q being depicted using solid arrow and right multiplication by arrows in Q being depicted by dashed arrows.

Example 32. If Q is the quiver

$$(6) 1 \longrightarrow 2 \longleftarrow 3 \longleftarrow 4 \longrightarrow 5 \longrightarrow 6,$$

we obtain the following graphical presentation of ${}_{A}A_{A}$:



Note that the rows of the above are in bijection with indecomposable projective left A-modules.

Example 33. If Q is the quiver given by the left hand side of (4), then we have the following graphical presentation of ${}_{A}A_{A}$:

6.6. **Diagram of a subbimodule in** ${}_{A}A_{A}$. Viewing ${}_{A}A_{A}$ as a representation of $Q \times Q^{\mathrm{op}}$ with all commutativity relations, as described in Subsection 6.5, subbimodules in ${}_{A}A_{A}$ are exactly subrepresentations. As all composition multiplicities in A are at most one (since there is at most one path between any pair of vertices), it follows that subbimodules in A are in bijection with those subsets of \mathbf{a}_{ij} 's which are closed under successors (i.e. under the action of both dashed and solid arrows). In particular, the smallest subbimodules (i.e. the simple ones) are in bijection with the sinks in the diagram of A. Furthermore, we have the following easy observations:

Lemma 34. Let B be a subbimodule of ${}_{A}A_{A}$. Then the set of all a_{ij} contained in B is a basis of B.

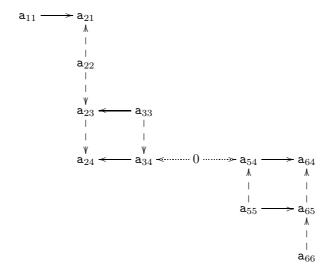
Proof. This follows from the proof of Lemma 12. \Box

Lemma 35. There is a bijection between \mathfrak{M}_Q and simple subbimodules in the socle of ${}_AA_A$.

Proof. Each maximal chain with source s and sink t contributes the simple subbimodule in the socle of ${}_{A}A_{A}$ with basis \mathbf{a}_{ts} . Conversely, if \mathbf{a}_{ts} belongs to the socle, then it cannot be multiplied by any arrow from the left and, similarly, by any arrow from the right. Therefore \mathbf{a}_{ts} is a maximal path, that is it corresponds to a maximal chain.

Lemma 34 yields a graphic presentation of B as a subgraph of the graphic presentation of ${}_{A}A_{A}$ discussed in Subsection 6.5.

Example 36. For the quiver (6) and the graph (7) of the corresponding identity bimodule, the graph of the subbimodule J_4 , see (2), is given by:



Here 0 stands on the place of \mathbf{a}_{44} which is missing in J_4 from the identity bimodule and the dotted arrows depict the corresponding zero multiplication. This clearly shows that J_4 is a decomposable bimodule. In particular, we obtain that in this case the monoid \mathcal{I} should be rather different from the multisemigroup $\mathcal{S}_{\mathscr{P}}$. Note that, for example, the linear span of \mathbf{a}_{44} and \mathbf{a}_{54} is not a subrepresentation as it is not closed with respect to the action of the arrow $\mathbf{a}_{54} \to \mathbf{a}_{64}$. Therefore this linear span is not a subbimodule.

6.7. Special function of an indecomposable subbimodule. Let B be a subbimodule of ${}_{A}A_{A}$. Then

$$B = \bigoplus_{i,j=1}^{n} e_j B e_i$$

with each $e_j B e_i$ being of dimension at most one. Moreover, $e_j B e_i \neq 0$ implies that $j \in \overline{i}$. For i = 1, 2, ..., n, set

$$B_i := \bigoplus_{j=1}^n e_j B e_i.$$

Note that B_i is, by construction, a submodule of AB and also a submodule of $B \cap Ae_i$, where $Ae_i \cong P_i$.

Lemma 37. Let B be an indecomposable subbimodule of ${}_{A}A_{A}$. Then each B_{i} is indecomposable or zero.

Proof. If $\deg_Q^{\text{out}}(i) = 1$, then P_i is uniserial since Q is admissible. Therefore any submodule of P_i is either indecomposable or zero. If $\deg_Q^{\text{out}}(i) > 1$, then i is a source since Q is admissible. If $B_i \cong P_i$, then B_i is indecomposable.

It remains to consider the case $\deg_Q^{\mathrm{out}}(i) > 1$ and $B_i \not\cong P_i$. Consider the full subgraph $Q^{(i)}$ of Q with vertices $Q \setminus \{i\}$. Let $\Gamma^{(j)}$, where $j = 1, 2, \ldots, m$ for $m \geq 2$,

be the list of all connected components of $Q^{(i)}$. For j = 1, 2, ..., m, set

$$B_i^{(j)} := \bigoplus_{t \in \Gamma^{(j)}} e_t B e_i$$
 and we have $B_i = \bigoplus_{j=1}^m B_i^{(j)}$.

For each j = 1, 2, ..., m, the space

$$B_i^{(j)} \oplus \bigoplus_{t \in \Gamma^{(j)}} B_t$$

is a direct summand of B as an A-A-bimodule. Since B is assumed to be indecomposable, we either have $B_i = 0$ or $B_i = B_i^{(j)}$ for some j. Since Q is admissible, $B_i^{(j)}$ is uniserial and hence indecomposable or zero.

Lemma 37 justifies the following definition. For an indecomposable B define the function $\mathbf{x}_B: Q_0 \to Q_0 \cup \{0\}$, in the following way:

- if $B_i = 0$ for $i \in Q_0$, then set $\mathbf{x}_B(i) = 0$;
- if $B_i \neq 0$ for $i \in Q_0$, then B_i is an indecomposable module by Lemma 37 and is projective since A is hereditary, so we can define $\mathbf{x}_B(i)$ as the unique element in Q_0 such that $B_i \cong P_{\mathbf{x}_B(i)}$.

We also define the $support \operatorname{supp}(B)$ of B as the union of all maximal chains in Q which contribute to the bimodule socle of B, confer Lemma 35.

Example 38. The bimodule given in Example 36 decomposes into a direct sum of two indecomposable summands. The first summand corresponds to the part on the left from 0. This summand has the function (1, 2, 3, 3, 0, 0), support $\{1, 2, 3, 4\}$ and a two-dimensional socle with basis a_{21} and a_{24} . The second summand corresponds to the part on the right from 0. This summand has the function (0, 0, 0, 5, 5, 6), support $\{4, 5, 6\}$ and simple socle a_{64} .

Lemma 39. The support of an indecomposable subbimodule B in ${}_{A}A_{A}$ is connected.

Proof. Assume $\operatorname{supp}(B)$ is the disjoint union of two non-empty sets Γ_1 and Γ_2 . Each of these is a union of maximal chains in Q. Let $i \in Q_0$ be such that $B_i \neq 0$. Then AB_iA intersects the socle of B and hence i belongs to some maximal chain $X \subset \operatorname{supp}(B)$, in particular, $i \in \Gamma_1$ or $i \in \Gamma_2$.

For s = 1, 2, let $B^{(s)}$ be the k-span of all B_j , where $j \in \Gamma_s$. Then $B = B^{(1)} \oplus B^{(2)}$. By construction, both $B^{(1)}$ and $B^{(2)}$ are left A-submodules of B. Since each Γ_s is a union of maximal chains, each $B^{(s)}$ is even a right A-submodule of B, in particular, a right A-A-subbimodule. Therefore B is decomposable.

Proposition 40. Let B be an indecomposable subbimodule of ${}_{A}A_{A}$. Then \mathbf{x}_{B} is a special function and $\operatorname{supp}(\mathbf{x}_{B}) = \operatorname{supp}(B)$.

Proof. If P_j is a submodule of P_i , then there is an oriented path from i to j in Q. Therefore $\mathbf{x}_B(i) \in \overline{i}$ and thus \mathbf{x}_B is a path function.

Assume that there is an oriented path α from i to j in Q and that $\mathbf{x}_B(j) \neq 0$. Then right multiplication with α defines an injective homomorphism from P_j to P_i inside A. Restricting this homomorphism to B gives an injective homomorphism from $B_j \cong P_{\mathbf{x}_B(j)}$ to $B_i \cong P_{\mathbf{x}_B(i)}$. This means that $\mathbf{x}_B(i) \neq 0$ and $\mathbf{x}_B(j) \in \overline{\mathbf{x}_B(i)}$. Therefore \mathbf{x}_B is a monotone function.

Let $X \in \mathfrak{M}_Q$ with source i and sink j. If $\mathbf{x}_B(i) \notin X$, then $\mathbf{x}_B(s) = 0$ for all $s \in X \setminus \{i\}$ since \mathbf{x}_B is a monotone path function. In particular, X does not

contribute to the support of \mathbf{x}_B (note that either i or j might still belong to the support via some other maximal chains). In this case we also have $\mathbf{a}_{ii} \notin B$.

Let $X \in \mathfrak{M}_Q$ with source i and sink j. If $\mathbf{x}_B(i) \in X$, then $X \subset \operatorname{supp}(\mathbf{x}_B)$ and \mathbf{a}_{ji} spans a simple subbimodule in the socle of B. From this and the previous paragraph it follows that $\operatorname{supp}(\mathbf{x}_B) = \operatorname{supp}(B)$. In particular, from Lemma 39 we obtain that $\operatorname{supp}(\mathbf{x}_B)$ is connected.

It remains to show that the equality $\mathbf{x}_B(i) = 0$ for $i \in \mathbf{K}(Q) \cap \operatorname{supp}(B)$ implies that $\deg_{\operatorname{supp}(B)}(i) = 1$. If i were a source, then $\mathbf{x}_B(i) = 0$ would imply $i \notin \operatorname{supp}(B)$, which is a contradiction. Therefore i is a sink. Consider the full subgraph $Q^{(i)}$ of Q with vertices $Q \setminus \{i\}$. Let $\Gamma^{(j)}$, for $j = 1, 2, \ldots, m$ where $m \geq 1$, be the list of all connected components of $Q^{(i)}$. For $j = 1, 2, \ldots, m$, denote by $B^{(j)}$ the A-A-bimodule direct summand of B spanned, over \mathbbm{k} , by all B_s , where $s \in \Gamma^{(j)}$. Since B is indecomposable, only one of these direct summand is non-zero. Without loss of generality we assume that this non-zero direct summand is $B^{(1)}$. There are m maximal chains with sink i, one for each $\Gamma^{(j)}$. Since only $B^{(1)}$ is non-zero, only the maximal chain from $\Gamma^{(1)}$ contributes to the socle of B. Hence $\deg_{\operatorname{supp}(B)}(i) = 1$. This completes the proof.

6.8. Subbimodules of ${}_AA_A$ associated with special functions. For a special function $\mathbf{x}=(x_1,x_2,\ldots,x_n):Q_0\to Q_0\cup\{0\}$, denote by $B_{\mathbf{x}}$ the subspace in ${}_AA_A$ obtained as the linear span of all \mathbf{a}_{ts} for which $x_s\neq 0$ and $t\in \overline{x_s}$. The fact that \mathbf{x} is a path function ensures that this definition does make sense. Moreover, for $s\in Q_0$ we have

(9)
$$(B_{\mathbf{x}})_s \cong \begin{cases} P_{x_s}, & x_s \neq 0, \\ 0, & \text{otherwise,} \end{cases}$$

by construction.

Example 41. For the quiver

$$1 \longrightarrow 2 \longrightarrow 3 \longleftarrow 4 \longleftarrow 5 \longrightarrow 6 \longrightarrow 7 \longrightarrow 8 \longleftarrow 9$$

and the special function given by (0,0,0,3,5,7,8,8,8), we have the subbimodule of ${}_{A}A_{A}$ given by the bold and solid part of the diagram in Figure 1, where the the rest of ${}_{A}A_{A}$ is shown as connected by dotted arrows.

Proposition 42. For every special function \mathbf{x} , the subspace $B_{\mathbf{x}}$ of ${}_{A}A_{A}$ is an indecomposable subbimodule and $\operatorname{supp}(\mathbf{x}) = \operatorname{supp}(B_{\mathbf{x}})$.

Proof. From (9) it follows that $B_{\mathbf{x}}$ is closed with respect to the left A-action. From the fact that \mathbf{x} is monotone, it follows that $B_{\mathbf{x}}$ is closed with respect to the right A-action. Therefore $B_{\mathbf{x}}$ is a subbimodule of ${}_{A}A_{A}$.

Let $X \in \mathfrak{M}_Q$ be a maximal chain with source i and sink j. If $\mathbf{x}(i) \notin X$, then $X \not\subset \operatorname{supp}(\mathbf{x})$ by definition and $X \not\subset \operatorname{supp}(B_{\mathbf{x}})$ by construction (as $\mathbf{a}_{ji} \notin B_{\mathbf{x}}$). If $\mathbf{x}(i) \in X$, then $X \subset \operatorname{supp}(\mathbf{x})$ by definition and $X \not\subset \operatorname{supp}(B_{\mathbf{x}})$ by construction (as $\mathbf{a}_{ji} \in B_{\mathbf{x}}$). Hence $\operatorname{supp}(\mathbf{x}) = \operatorname{supp}(B_{\mathbf{x}})$.

It remains to show that $B_{\mathbf{x}}$ is indecomposable. Assume that this is not the case and write $B_{\mathbf{x}} \cong B^{(1)} \oplus B^{(2)}$, where both direct summands are subbimodules. Since $\operatorname{supp}(B_{\mathbf{x}}) = \operatorname{supp}(\mathbf{x})$ is connected, $\operatorname{supp}(B^{(1)})$ and $\operatorname{supp}(B^{(2)})$ must have at least one common element, say s. This common element can be chosen such that it is an end element of a maximal chain, hence it is either a source of a sink. Assume first that s is a source. Then $\mathbf{x}(s) = s$ implies that \mathbf{a}_{ss} must be in both $B^{(1)}$

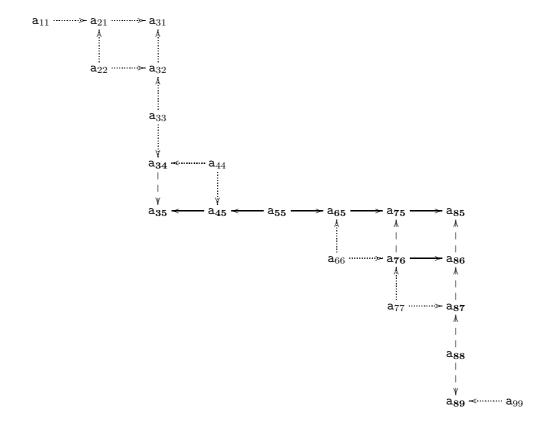


FIGURE 1. Diagram used in Example 41.

and $B^{(2)}$ since $(B_{\mathbf{x}})_s \cong P_s$ is indecomposable, which contradicts the fact that the intersection of these two subbimodules is trivial. Further, $\mathbf{x}(s) \neq s$ implies that only one maximal chain starting with s, namely the one containing $\mathbf{x}(s)$, can contribute to the socle of $B_{\mathbf{x}}$. That socle element must be either in $B^{(1)}$ or $B^{(2)}$, but it cannot be in both of them, which contradicts the fact that s is a common element in $\sup(B^{(1)})$ and $\sup(B^{(2)})$.

Therefore s is a sink. If $\mathbf{x}(s) = s$, then for any $X \in \operatorname{supp}(B_{\mathbf{x}})$ with sink s one can use right multiplication with elements in A to move \mathbf{a}_{ss} to the socle element of $B_{\mathbf{x}}$ corresponding to X. Therefore \mathbf{a}_{ss} must be in both $B^{(1)}$ and $B^{(2)}$, which gives a similar contradiction to the above. If $\mathbf{x}(s) \neq s$, then $\mathbf{x}(s) = 0$ since s is a sink. By definition of a special function, we thus have that s has degree 1 in $\operatorname{supp}(\mathbf{x})$. Therefore only one maximal chain ending at s can contribute to the socle of $B_{\mathbf{x}}$. That socle element must be either in $B^{(1)}$ or $B^{(2)}$, but it cannot be in both of them, which is again a similar contradiction to the above. This completes the proof. \square

6.9. Classification of indecomposable subbimodules of ${}_{A}A_{A}$. We can now collect the above facts into the following statement.

Theorem 43. The maps $B \mapsto \mathbf{x}_B$ and $\mathbf{x} \mapsto B_{\mathbf{x}}$ are mutually inverse bijections between the set of all indecomposable subbimodules of ${}_AA_A$ and the set of all special functions.

Proof. Let \mathbf{x} be a special function. Then $B_{\mathbf{x}}$ is an indecomposable subbimodule of ${}_{A}A_{A}$ by Proposition 42. Furthermore, $\mathbf{x}_{B_{\mathbf{x}}}(i) = \mathbf{x}(i)$ for all $i \in Q_{0}$, and thus also $\mathbf{x}_{B_{\mathbf{x}}} = \mathbf{x}$ follow from the definitions.

Conversely, let B is an indecomposable subbimodule of ${}_AA_A$. Then \mathbf{x}_B is a special function by Proposition 40. Furthermore, $(B_{\mathbf{x}_B})_i = B_i$ for all $i \in Q_0$, and thus also $B_{\mathbf{x}_B} = B$ follow from the definitions.

For a non-empty $\Gamma \in \mathbf{W}$, we denote by $\mathbf{B}(\Gamma)$ the set of all indecomposable subbimodules $B \subset {}_{A}A_{A}$ for which $\operatorname{supp}(B) = \Gamma$. We set $\mathbf{B}(\emptyset) = \{0\}$.

6.10. **Partial order.** We identify $\mathcal{S}_{\mathscr{D}}$ with the subset $\mathcal{I}^{\operatorname{ind}}$ of \mathcal{I} consisting of all indecomposable subbimodules to which we attach an external element 0 (which corresponds to the zero bimodule). To this end, we do not know whether $\mathcal{I}^{\operatorname{ind}}$ is a submonoid of \mathcal{I} , that is, whether $IJ \in \mathcal{I}^{\operatorname{ind}}$ for any $I, J \in \mathcal{I}^{\operatorname{ind}}$. The original multivalued operation in $\mathcal{I}^{\operatorname{ind}}$ sends (I, J) to the set of all indecomposable direct summands of IJ up to isomorphism.

The set \mathcal{I}^{ind} inherits from \mathcal{I} the partial order given by inclusions. Clearly, ${}_{A}A_{A}$ is the maximum element with respect to this order both in \mathcal{I}^{ind} and in \mathcal{I} (note that ${}_{A}A_{A} \in \mathcal{I}^{\text{ind}}$ as we assume Q to be connected).

Consider the set $\mathbf{Q} := \{J_s : s = 1, 2, ..., n\}$ and note that this is the set of maximal elements in $\mathcal{I} \setminus \{AA_A\}$. The bimodule J_s is indecomposable if and only if we have $s \notin \mathbf{K}'(Q)$. For $s \in \mathbf{K}'(Q)$, let $t_1, t_2, ..., t_{m_s}$ be the list of all $t \in Q_0$ for which there is an arrow $t \to s$ or an arrow $s \to t$. Let Γ be the full subgraph of Q with vertex set $Q_0 \setminus \{s\}$. Then

$$\Gamma = \Gamma^{(1)} \cup \Gamma^{(2)} \cup \dots \cup \Gamma^{(m_s)}$$

where $\Gamma^{(q)}$ is the connected component containing t_q for $q = 1, 2, ..., m_s$. We have the decomposition

$$(10) J_s \cong \bigoplus_{q=1}^{m_s} J_s^{(q)}$$

where $J_s^{(q)}$ is the subbimodule of J_s defined as the direct sum of all $e_j(J_s)e_i$ with $i, j \in \Gamma^{(q)} \cup \{s\}$. Clearly, each $J_s^{(q)}$ is indecomposable since $\Gamma^{(q)}$ is connected.

Lemma 44.

- (i) The set $\{J_s: s \notin \mathbf{K}'(Q)\}$ is the set of maximal elements in $\mathbf{B}(Q) \setminus \{AA_A\}$.
- (ii) For $\Omega \in \mathbf{W} \setminus \{Q\}$, there is a unique maximal element, denoted B_{Ω} , in the set $\mathbf{B}(\Omega)$. Moreover, $B_{\varnothing} = 0$ and, for $\Omega \neq \varnothing$, we have

(11)
$$B_{\Omega} = \prod_{t} J_{t}^{(p_{t})} \prod_{s} J_{s}^{(q_{s})}$$

where s runs through the set of sinks $i \in \mathbf{K}'(Q) \cap \Omega$ for which $\deg_{\Omega}(i) = 1$ and q_s is such that the corresponding $\Gamma^{(q_s)}$ has a common vertex with Ω , while t runs through the set of sources $i \in \mathbf{K}'(Q) \cap \Omega$ for which $\deg_{\Omega}(i) = 1$ and p_t is such that the corresponding $\Gamma^{(p_t)}$ has a common vertex with Ω .

Proof. Clearly each J_s with $s \notin \mathbf{K}'(Q)$ is maximal in $\mathbf{B}(Q) \setminus \{AA_A\}$. Assume that $B \in \mathbf{B}(Q)$ is maximal in $\mathbf{B}(Q) \setminus \{AA_A\}$. Then $B \subset J_s$ for some s. If $s \notin \mathbf{K}'(Q)$,

then $B = J_s$ by maximality of B. If $s \in \mathbf{K}'(Q)$, then

$$B = \bigoplus_{q=1}^{m_s} (B \cap J_s^{(q)})$$

since B has a basis consisting of all \mathbf{a}_{st} contained in it and all $J_s^{(q)}$ also have the same property. By indecomposability, we get $B = B \cap J_s^{(q)}$ for some q, which contradicts $B \in \mathbf{B}(Q)$. Therefore this case does not occur, which proves claim (i).

To prove claim (ii) we denote by B'_{Ω} the right hand side of (11). Note that $B_{\varnothing} = 0$ is clear and that for $\Omega \neq \varnothing$ the fact that $B'_{\Omega} \in \mathbf{B}(\Omega)$ follows by construction. The maximal element B_{Ω} in the set $\mathbf{B}(\Omega)$ is the sum of all subbimodules of ${}_{A}A_{A}$ with support Ω . Therefore to complete the proof of claim (ii) it remains to check that $B'_{\Omega} = B_{\Omega}$ for $\Omega \neq \varnothing$.

If there are s and t in (11) which are connected by an edge, then Ω must be the full subgraph of Q with vertices $\{s,t\}$ by connectedness. In this case the only subbimodule of ${}_{A}A_{A}$ with support $\{s,t\}$ is the one with basis \mathbf{a}_{st} . Indeed, since both s and t have degrees higher than 1, appearance of either \mathbf{a}_{ss} or \mathbf{a}_{tt} in a subbimodule would lead to an extra maximal chain in its support. Therefore B_{Ω} has basis \mathbf{a}_{st} . At the same time, \mathbf{a}_{st} appears in B'_{Ω} as the product $\mathbf{a}_{st}\mathbf{a}_{tt}$, where $\mathbf{a}_{st} \in J^{(p_t)}_t$ and $\mathbf{a}_{tt} \in J^{(q_s)}_s$. This means that $B'_{\Omega} = B_{\Omega}$.

In the remaining case (no s and t in (11) are connected by an edge), all factors of (11) commute. Note that $B_{\Omega} \subset J_s$ and $B_{\Omega} \subset J_t$ for any s and t occurring in (11). From indecomposability, it follows that $B_{\Omega} \subset J_s^{(q_s)}$ and $B_{\Omega} \subset J_t^{(p_t)}$ for all s and t occurring in (11). From $J_s^2 = J_s$ and $J_t^2 = J_t$ it follows that $(J_s^{(q_s)})^2 = J_s^{(q_s)}$ and $(J_t^{(p_t)})^2 = J_t^{(p_t)}$. This implies $B_{\Omega}J_s^{(q_s)} = B_{\Omega}$ and $B_{\Omega}J_t^{(p_t)} = B_{\Omega}$ which yields $B_{\Omega}B_{\Omega}' = B_{\Omega} \subset B_{\Omega}'$. From the maximality of B_{Ω} we finally obtain that $B_{\Omega} = B_{\Omega}'$.

6.11. Composition of indecomposable subbimodules. The following is a crucial observation.

Proposition 45. Let B and D be two indecomposable subbimodules in ${}_{A}A_{A}$. Then $B \otimes_{A} D \cong BD$ and the latter is either zero or an indecomposable subbimodule of ${}_{A}A_{A}$.

Proof. As A is hereditary, $B \otimes_{A}$ is exact, in particular, it preserves inclusions. Hence, applying it to $D \hookrightarrow A$ gives $B \otimes_{A} D \hookrightarrow B \otimes_{A} A \cong B$ where the last isomorphism is given by the multiplication map. Therefore the multiplication map $B \otimes_{A} D \to BD$ is an isomorphism.

It remains to prove indecomposability of BD in case the latter subbimodule is nonzero. Let $\Omega := \operatorname{supp}(B) \cap \operatorname{supp}(D)$ which is connected as both $\operatorname{supp}(B)$ and $\operatorname{supp}(D)$ are. As $BD \subset B \cap D$, we have $\operatorname{supp}(BD) \subset \Omega$. If Ω consists of only one maximal chain, it follows that $\operatorname{supp}(BD) = \Omega$, that is BD has simple socle and thus is indecomposable.

Let X and Y be two different maximal chains in Ω with a common vertex i. Then i is either a sink or a source of degree at least two. Note that both X and Y belong to both $\sup(B)$ and $\sup(D)$. As the degree of i is at least two and B is indecomposable, we have $\mathbf{a}_{ii} \in B$ (for otherwise we may decompose B using the decomposition of J_i). Similarly, $\mathbf{a}_{ii} \in D$. Hence $\mathbf{a}_{ii} \in BD$ as well. Using left multiplication, in case i is a source, or right multiplication, in case i is a sink, it

follows that $\operatorname{supp}(BD)$ contains both X and Y. Consequently, $\operatorname{supp}(BD) = \Omega$ as the latter one is connected.

Assume that we can write $BD = B^{(1)} \oplus B^{(2)}$, where $B^{(1)}$ and $B^{(2)}$ are non-zero subbimodules. As Ω is connected, there must be some vertex, say i, in the intersection of the supports of these two subbimodules. From the previous paragraph we have $\mathbf{a}_{ii} \in BD$. Now, using arguments similar to the one used in the proof of Proposition 42, we get a contradiction. This completes the proof.

An immediate consequence of Proposition 45 is the following:

Corollary 46. The multisemigroup $S_{\mathscr{P}}$ is a monoid.

Another consequence from the proof of Proposition 45 is the following:

Corollary 47. Let B and D be two indecomposable subbimodules in ${}_AA_A$ such that $BD \neq 0$. Then $\operatorname{supp}(BD) = \operatorname{supp}(B) \cap \operatorname{supp}(D)$.

We denote by \mathcal{I}^{ind} the submonoid of \mathcal{I} consisting of indecomposable subbimodules in ${}_{A}A_{A}$ and the zero bimodule. By the above, the monoids $\mathcal{S}_{\mathscr{P}}$ and \mathcal{I}^{ind} are isomorphic.

Problem 48. It would be interesting to know for which finite dimensional algebras the product of two indecomposable subbimodules of the identity bimodule is always indecomposable or zero.

7. Presentation for \mathcal{I} and \mathcal{I}^{ind}

The main aim of this section is to obtain presentations for both the monoid \mathcal{I} and the monoid \mathcal{I}^{ind} .

7.1. Minimal generating systems. Set

$$\mathbf{B} := \{ J_s : s \notin \mathbf{K}'(Q) \} \cup \bigcup_{s \in \mathbf{K}'(Q)} \{ J_s^{(q)} : q = 1, 2, \dots, m_s \}.$$

We will need the following technical observation.

Lemma 49. For all $i, j \in Q_0$, we have

(12)
$$J_i P_j = \begin{cases} P_j, & i \neq j; \\ \operatorname{Rad}(P_i), & i = j. \end{cases}$$

Moreover, we also have $J_i \operatorname{Rad}(P_i) = \operatorname{Rad}(P_i)$.

Proof. If $i \neq j$, then J_i contains a_{jj} by definition and thus $J_iP_j = P_j$. If i = j, then J_i does not contain a_{ii} by definition and thus $J_iP_i \neq P_i$. However, since A is hereditary, $Rad(P_i)$ is a direct sum of some P_k , where $k \neq i$. Therefore from the first sentence of the proof we have both $J_iP_i = Rad(P_i)$ and $J_iRad(P_i) = Rad(P_i)$. \square

Proposition 50.

- (i) The set \mathbf{Q} is the unique minimal generating system for the monoid \mathcal{I} .
- (ii) The set **B** is the unique minimal generating system for the monoid \mathcal{I}^{ind} .

Proof of claim (i). As \mathbf{Q} is the set of all maximal elements in $\mathcal{I} \setminus \{AA_A\}$, it must belong to any generating system. Therefore, to prove claim (i) it is enough to show that \mathbf{Q} generates \mathcal{I} . Let S be the submonoid of \mathcal{I} generated by \mathbf{Q} . Assume that $\mathcal{I} \setminus S \neq \emptyset$ and let B be a maximal element in $\mathcal{I} \setminus S$ with respect to inclusions. Certainly, $B \neq 0$, $B \neq AA_A$ and $B \neq J_s$ for $s \in Q_0$.

We split the proof of claim (i) into two cases.

Case 1. Assume first that $B_i \in \{0, P_i\}$ for all i. As $B \neq {}_A A_A$, there is at least one i such that $B_i = 0$. As $B \neq 0$, there is at least one j such that $B_j = P_j$. As B is a bimodule, $B_i = 0$ implies $B_s = 0$ for all $s \in \overline{i}$. Therefore we may choose i such that $B_i = 0$ and for any arrow $j \to i$, where $j \in Q_0$, we have $B_j = P_j$. We have to consider two subcases.

Subcase 1.A. Assume that i is not a source. Then there is some j with an arrow $j \to i$. In particular, P_i has simple socle, say L_s . Then s is a sink and $B_s = 0$.

Lemma 51. With the notation above, the space $B' := B \oplus \mathbb{k}\{a_{si}\}$ is a subbimodule of ${}_AA_A$.

Proof. By construction, $\mathbb{k}\{a_{si}\}$ is the socle of P_i . Therefore B' is a left module. For any $j' \in Q_0$ such that there is an arrow $\alpha : j' \to i$, we have $B_{j'} = P_{j'}$ by our choice of i. Therefore, right multiplication with α maps a_{si} to $a_{sj'} \in B$. The claim follows.

Using (12) and $B_s = 0$, we get $B = J_s B'$. As $B \subsetneq B'$, we have $B' \in S$ by maximality of B. Therefore $B \in S$, a contradiction.

Subcase 1.B. Assume that i is a source and let X be a maximal chain starting at i and ending at some $s \in Q_0$. Then $s \neq i$ is a sink and $B_s = 0$.

Lemma 52. With the notation above, the space $B' = B \oplus \mathbb{k}\{a_{si}\}$ is a subbimodule of ${}_AA_A$.

Proof. By construction, $\mathbb{K}\{\mathbf{a}_{si}\}$ is a socle element of P_i . Therefore B' is a left module. As i is a source, right multiplication with \mathbf{a}_{ab} either preserves \mathbf{a}_{si} , if a=b=i, or annihilates it in all other cases. The claim follows.

Using (12) and $B_s = 0$, we get $B = J_s B'$. As $B \subsetneq B'$, we have $B' \in S$ by maximality of B. Therefore $B \in S$, a contradiction. This completes the proof of Case 1.

Case 2. Now we may assume that there is an i such that $B_i \notin \{0, P_i\}$. In this case we may choose $i \in \{1, 2, ..., n\}$ such that $B_i \notin \{0, P_i\}$ and, additionally, for any j for which there is an arrow $j \to i$ we have $B_j = P_j$ (note that such B_j is automatically non-zero as $B_i \neq 0$ and B is a subbimodule). We again consider two subcases.

Subcase 2.A. Assume that we may choose such i with the additional property that it is not a source. Then there is a unique $s \in \overline{i}$ such that $\operatorname{Rad}(P_s) = B_i$. Then s is not a sink since $B_i \neq 0$. If $t \in \overline{i}$, then B_t cannot have P_s as a direct summand as $B_i \subset \operatorname{Rad}(P_s)$. If $t \notin \overline{i}$, then B_t cannot have P_s as a direct summand as $B_j = P_j$ for all j which have an arrow to i. Therefore, similarly to the above, $B' = B \oplus \mathbb{k}\{a_{si}\}$ is a strictly larger subbimodule than B and $B = J_sB'$, leading again to a contradiction.

Subcase 2.B. Finally, consider the subcase when any i as above in Case 2 is a source. In this case we have $J_iM=M$ for any left submodule $M\subset J_i$ by (12). If $\deg_Q(i)=1$, then P_i is uniserial and there is a unique $s\in \overline{i}$ such that the radical of P_s is isomorphic to B_i . Then s is not a sink (since $B_i\neq 0$) and hence, similarly to the above, $B':=B\oplus \Bbbk\{\mathtt{a}_{si}\}$ is a subbimodule of ${}_AA_A$ and $B=J_sB'$ by (12), a contradiction.

It is left to assume $\deg_Q(i) > 1$. Then J_i decomposes according to (10). For $q = 1, 2, \ldots, m_i$, set

$$B^{(q)} := B \cap J_i^{(q)}$$

and denote by $t_q \in \Gamma^{(q)}$ the unique element such that there is an arrow $\alpha_q: i \to t_q$. If $B_i^{(q)} = P_{t_q}$ for all q, we have $B_i = \operatorname{Rad}(P_i)$. Then, similarly to the above, $B' := B \oplus \mathbb{K}\{\mathbf{a}_{ii}\}$ is a subbimodule of ${}_AA_A$ and $B = J_iB'$ by (12), a contradiction.

If $B_i^{(q)} \neq P_{t_q}$ for some q, we note that P_{t_q} is uniserial since Q is admissible. There is a unique s such that $\operatorname{Rad}(P_s) = B_i^{(q)}$. Then, similarly to the above, $B' := B \oplus \mathbb{k}\{\mathbf{a}_{si}\}$ is a subbimodule of ${}_AA_A$. If s is not a sink, we also have $B = J_sB'$ by (12), a contradiction

If s is a sink, we have $B_i^{(q)}=0$ and thus also $B_s^{(q)}=0$ and even $B_t^{(q)}=0$ for all t in the maximal chain X with source i and sink s. In this case, $B^{(q)}$ is, naturally, a proper subbimodule of the identity bimodule for the path algebra A_q of $\Gamma^{(q)}$. Note that $\Gamma^{(q)}$ has less than n vertices. Using induction by n and the above arguments, we get that there are $x, y \in \Gamma^{(q)}$ such that $(B^{(q)})' := B^{(q)} \oplus \mathbb{k}\{\mathbf{a}_{xy}\}$ is an A_q - A_q -bimodule and $J_x^{(q)}(B^{(q)})' = B^{(q)}$, where $J_x^{(q)}$ denotes the kernel of the bimodule epimorphism $A_q \to L_{xx}$. If $y \notin X$, then $B' := B \oplus \mathbb{k}\{\mathbf{a}_{xy}\}$ is a subbimodule of AA_A and $B = J_xB'$ by the above, a contradiction. If $y \in X$, then x = s and $B' := B \oplus \mathbb{k}\{\mathbf{a}_{si}\}$ is a subbimodule of AA_A and $B = J_xB'$ by the above, a contradiction. Claim (i) follows.

Proof of claim (ii). The fact that **B** generates \mathcal{I}^{ind} follows from claim (i) and Proposition 45 since **B** is exactly the set of indecomposable summands of elements in **Q**.

To prove minimality of \mathbf{B} , assume that we can write some $J_s^{(q)}$ as a product of other elements in \mathbf{B} . By Corollary 47, we thus get that $\operatorname{supp}(J_s^{(q)})$ is the intersection of the supports of the factors in this product. For each factor D of this product, the degree of s in $\operatorname{supp}(D)$ is either 1 or equal to $\deg_Q(s)$. If none of the factors has s with degree 1, then all factors have it with degree $\deg_Q(s)$ and thus the degree of s in the support of the product must be $\deg_Q(s)$ as well, a contradiction. Therefore the product contains at least one factor D for which $\deg_{\operatorname{supp}(D)}(s) = 1$. This means that $D = J_s^{(q')}$ for some q'. As $\operatorname{supp}(D) \supset \operatorname{supp}(J_s^{(q)})$, it follows that q = q', a contradiction. Minimality of \mathbf{B} follows.

The argument from the previous paragraph even shows that, if we write $J_s^{(q)}$ as a product of arbitrary elements in \mathcal{I}^{ind} , then one of the factors must be contained in $J_s^{(q)}$. If the factor is properly contained in $J_s^{(q)}$, then the whole product is properly contained in $J_s^{(q)}$ as well. Therefore one of the factors equals $J_s^{(q)}$, which implies uniqueness of **B**. This completes the proof of claim (ii).

7.2. Relations.

Proposition 53. The ideals J_i , i = 1, 2, ..., n, satisfy the following relations:

- (a) $J_i^2 = J_i$ for all i.
- (b) $J_iJ_j = J_jJ_i$ if there is no arrow between i and j.
- (c) $J_i J_i J_j = J_i J_i J_i = J_j J_i$ if there is an arrow $i \to j$.

Proof. To prove claim (a), note that P_i is not a direct summand of J_i since A is hereditary and finite dimensional. Therefore $J_i^2 = J_i$ for all i follows directly from Lemma 49.

To prove claim (b), note that $(J_iJ_j)_s = (J_jJ_i)_s$ for all $s \in Q_0 \setminus \{i,j\}$. To compare $(J_iJ_j)_s$ with $(J_jJ_i)_s$ for $s \in \{i,j\}$, we observe that, in case there are no arrows between i and j, P_i is not isomorphic to a summand of $(J_j)_j$ and P_j is not isomorphic to a summand of $(J_i)_i$. Therefore, Lemma 49 implies

$$(J_iJ_j)_i = (J_jJ_i)_i = \operatorname{Rad}(P_i)$$
 and $(J_iJ_j)_j = (J_jJ_i)_j = \operatorname{Rad}(P_j)$.

This proves claim (b).

Similarly, to prove claim (c), it is enough to compare $(J_jJ_iJ_j)_s$ with $(J_iJ_jJ_i)_s$ for $s \in \{i, j\}$. From Lemma 49, it follows that

$$(J_j J_i J_j)_j = (J_i J_j J_i)_j = (J_j J_i)_j = \operatorname{Rad}(P_j)$$

since there are no arrows from j to i or from j to j. For s=i, we can write $\operatorname{Rad}(P_i)=P_j\oplus N$, where N contains no direct summand isomorphic to P_i or P_j . Therefore Lemma 49 implies that

$$(J_iJ_iJ_i)_i = (J_iJ_iJ_i)_i = (J_iJ_i)_i = \operatorname{Rad}(P_i) \oplus N.$$

This completes the proof.

Proposition 54. For $i, j \notin \mathbf{K}'(Q)$, $s, t \in \mathbf{K}'(Q)$, $q \in \{1, 2, ..., m_s\}$ and $p \in \{1, 2, ..., m_t\}$, the elements of **B** satisfy the following:

- (a) Relations from Proposition 53(a)-(c) for $i, j \notin \mathbf{K}'(Q)$.
- (b) $(J_s^{(q)})^2 = J_s^{(q)}$.
- (c) $J_s^{(q)} J_s^{(q')} = 0$ for any $q' \in \{1, 2, \dots, m_s\}, q' \neq q$.
- (d) $J_s^{(q)}J_t^{(p)}=J_t^{(p)}J_s^{(q)}$ if there is no arrow between s and t.
- (e) $J_s^{(q)}J_i = J_iJ_s^{(q)}$ if there is no arrow between s and i.
- (f) $J_t^{(p)} J_s^{(q)} J_t^{(p)} = J_s^{(q)} J_t^{(p)} J_s^{(q)} = J_t^{(p)} J_s^{(q)}$ if there is an arrow $s \to t$.
- (g) $J_t^{(p)} J_i J_t^{(p)} = J_i J_t^{(p)} J_i = J_t^{(p)} J_i$ if there is an arrow $i \to t$.
- (h) $J_t^{(p)} J_i J_t^{(p)} = J_i J_t^{(p)} J_i = J_i J_t^{(p)}$ if there is an arrow $t \to i$.
- (i) $J_t^{(p)}J_s^{(q)}J_t^{(p')}=0$ for any $p'\in\{1,2,\ldots,m_t\}, p'\neq p$, if there is an arrow between
- (j) $J_t^{(p)}J_iJ_t^{(p')}=0$ for any $p'\in\{1,2,\ldots,m_t\},\ p'\neq p$, if there is an arrow between t and i.
- (k) $J_s^{(q)} J_t^{(p)} = J_t^{(p)}$ in case $\text{supp}(J_t^{(p)}) \subset \text{supp}(J_s^{(q)})$.
- (1) $J_s^{(q)}J_t^{(p)}=0$ in case $\operatorname{supp}(J_t^{(p)})\cap\operatorname{supp}(J_s^{(q)})=\varnothing$.
- (m) $J_s^{(q)}J_i = J_iJ_s^{(q)} = J_s^{(q)} \text{ if } i \notin \text{supp}(J_s^{(q)}).$

Proof. Relations (a) are clear. From Proposition 53(a), for $s \in \mathbf{K}'(Q)$, we have

$$\big(\bigoplus_{q=1}^{m_s}J_s^{(q)}\big)^2\cong\bigoplus_{q=1}^{m_s}(J_s^{(q)})^2\oplus\bigoplus_{q\neq q'}(J_s^{(q)}J_s^{(q')})\cong\bigoplus_{q=1}^{m_s}J_s^{(q)}.$$

Note that $(J_s^{(q)})^2$ is the only direct summand in the middle with support in $\Gamma^{(q)} \cup \{s\}$ as $(J_s^{(q)})^2 \neq 0$ by Corollary 47. Therefore we get relations (b) and (c).

By Proposition 53(b), for $s, t \in \mathbf{K}'(Q)$ in case there is no arrow between s and t, we have

$$\big(\bigoplus_{q=1}^{m_s}J_s^{(q)}\big)\big(\bigoplus_{p=1}^{m_t}J_t^{(p)}\big)=\big(\bigoplus_{p=1}^{m_t}J_t^{(p)}\big)\big(\bigoplus_{q=1}^{m_s}J_s^{(q)}\big).$$

Opening brackets and matching summands with the same support on the left hand side and on the right hand side, we get relations (d). Relations (e) are obtained similarly from

$$\left(\bigoplus_{q=1}^{m_s} J_s^{(q)}\right) J_i = J_i \left(\bigoplus_{q=1}^{m_s} J_s^{(q)}\right)$$

which is again given by Proposition 53(b).

Relations (f)-(j) are obtained similarly from Proposition 53(c).

To prove relation (k), we compare $J_s^{(q)}J_t^{(p)}P_r$ with $J_t^{(p)}P_r$, for $r \in \{1, 2, ..., n\}$, using (12) and a similar formula for $J_t^{(p)}P_r$, namely

(13)
$$J_{t}^{(p)}P_{r} = \begin{cases} 0, & \text{if } r \notin \Gamma^{(q)} \cup \{t\}; \\ P_{r}, & \text{if } r \in \Gamma^{(q)}; \\ P_{t_{p}}, & \text{if } r = t. \end{cases}$$

Here in the last line we identify P_{t_p} with the corresponding submodule in $\operatorname{Rad}(P_t)$. Note that the only P_i which appear (up to isomorphism) as direct summands of $J_t^{(p)}P_r$ are those for which we have $i\in\operatorname{supp}(J_t^{(p)})\setminus\{t\}$. If $\operatorname{supp}(J_t^{(p)})\subset\operatorname{supp}(J_s^{(q)})$, then $s\not\in\operatorname{supp}(J_t^{(p)})\setminus\{t\}$ and hence $J_s^{(q)}P_i=P_i$ for such i by (13). This implies relation (k). Similarly one checks relations (l) and (m). This completes the proof.

7.3. The main results.

Theorem 55. Any relation between elements in the generating set \mathbf{Q} of the monoid \mathcal{I} is a consequence of the relations given in Proposition 53.

Proof. Let S be the abstract monoid given by generators \mathbf{Q} and relations from Proposition 53. Note that S is a Hecke-Kiselman monoid in the sense of [GM]. Then we have the canonical surjection $\psi: S \twoheadrightarrow \mathcal{I}$.

For any $I \in \mathcal{I}$ the additive functor Su_I acting on A-proj defines an endomorphism of the split Grothendieck group [A-proj] $_{\oplus}$. This gives a homomorphism φ from \mathcal{I} to the monoid of all endomorphisms of [A-proj] $_{\oplus}$. Let T denote the image of φ . Combined together we have surjective composition $\varphi \circ \psi$ as follows: $S \twoheadrightarrow \mathcal{I} \twoheadrightarrow T$.

Consider the standard basis $\{[P_i]: i=1,2,\ldots,n\}$ in $[A\text{-proj}]_{\oplus}$. From isomorphisms in (12), for $i,j=1,2,\ldots,m$ we obtain that T is spanned by $\varphi(J_i)[P_j]$, where

$$\varphi(J_i)[P_j] = \begin{cases} [P_j], & i \neq j; \\ \sum_{i \to s} [P_s], & i = j. \end{cases}$$

This is exactly the image of the linear representation of S considered in [Fo, Theorem 4.5] where it was proved that the corresponding representation map is injective, that is $S \cong T$. Consequently, because of the sandwich position of \mathcal{I} between S and T, we obtain $S \cong \mathcal{I}$ and the proof is complete.

We note that in [Gr] (see also [Fo]) one finds a recursive description of a normal form for elements of the Hecke-Kiselman monoid S from the above proof of Theorem 55. The same thus holds also for the monoid \mathcal{I} . A certain description of a normal form for elements in the monoid \mathcal{I}^{ind} follows from the proof of the following theorem which also describes relations in \mathcal{I}^{ind} .

Theorem 56. Any relation between elements in the generating set **B** of the monoid \mathcal{I}^{ind} is a consequence of the relations given in Proposition 54.

Proof. Let S be the abstract monoid given by generators $\mathbf{X} := \mathbf{B} \cup \{0\}$ and relations from Proposition 54 together with the obvious relations defining 0 as the zero element. As usual, we denote by \mathbf{X}^+ the set of all non-empty words in the alphabet \mathbf{X} . Let $\psi: S \to \mathcal{I}^{\mathrm{ind}}$ be the obvious canonical surjection. Our aim is to prove injectivity of ψ . For this we will describe certain representatives in the fibers of the canonical map $\tau: \mathbf{X}^+ \to S$.

For simplicity, we call all elements in **X** of the form $J_s^{(q)}$, for $s \in \mathbf{K}'(Q)$ and $q \in \{1, 2, ..., m_s\}$, the *split symbols*. For $w \in \mathbf{X}^+$, let $J_{s_1}^{(q_1)}, J_{s_2}^{(q_2)}, ..., J_{s_k}^{(q_k)}$ be the list of all split symbols which appear in w. If w has no split symbols, we set $\Omega = Q$. If w contains 0, we set $\Omega = \emptyset$. Otherwise, set

$$\Omega := \bigcap_{i=1}^k \operatorname{supp}(J_{s_i}^{(q_i)}).$$

Note that Ω is not an invariant of a fiber of τ in general.

If $\Omega=\varnothing$, then either w contains 0 or the fact that Q is a tree implies existence of $i,j\in\{1,2,\ldots,k\}$ such that $\operatorname{supp}(J_{s_i}^{(q_i)})\cap\operatorname{supp}(J_{s_j}^{(q_j)})=\varnothing$. We claim that in the latter case w=0 in S. Without loss of generality we may assume that the indices i and j and the word w (in its equivalence class) are chosen such that $w=xJ_{s_i}^{(q_i)}yJ_{s_j}^{(q_j)}z$ with y shortest possible. Assume that $y\neq 0$ and that there is a J_r with $r\in\operatorname{supp}(J_{s_j}^{(q_j)})$ in y. Take the leftmost occurrence of such element in y. Now we may use relations in Proposition 53(b) to move it past all J_q with $q\not\in\operatorname{supp}(J_{s_j}^{(q_j)})$. Note that, by the minimality of y and relations in Proposition 54(d), (e) and (k), there is no $J_{s_a}^{(q_a)}$ between $J_{s_i}^{(q_i)}$ and J_r , such that r and s_a are connected by an arrow. So J_r commutes with any split symbol between $J_{s_i}^{(q_i)}$ and J_r and so we can move it past $J_{s_i}^{(q_i)}$ making y shorter. Therefore y cannot contain any J_r with $r\in\operatorname{supp}(J_{s_j}^{(q_j)})$.

Similarly, y does not contain any J_r with $r \in \text{supp}(J_{s_i}^{(q_i)})$. Analogously (using also Proposition 54(c)) one shows that y does not contain any split symbol $J_r^{(f)}$ for $r \in \text{supp}(J_{s_i}^{(q_i)}) \cup \text{supp}(J_{s_j}^{(q_j)})$. Using similar arguments, it follows that y may contain only elements J_r where r belong to the unique (unoriented) path between s_i and s_j . Moreover, to avoid application of a similar argument, all vertices from this path must occur. But then one can use, if necessary, relations in Proposition 54(m) to make y shorter. Hence y is empty and we may use relations in Proposition 54(l) to conclude that w = 0.

The case when Ω has only one vertex is dealt with similarly using relations in Proposition 54(c), (i) and (j) and also results in w = 0.

If Ω has at least two vertices, note that, for any split symbol $J_{s_i}^{(q_i)}$ in w, we have $\deg_{\Omega}(s_i) \leq \deg_{\sup(J_{s_i}^{(q_i)})}(s_i) = 1$. Then a similar commutation and deleting procedure as above combined with the relations in Proposition 54(k) shows that w can be changed to an equivalent word u with the property that the only split symbols in u are those $J_{s_i}^{(q_i)}$ for which $s_i \in \mathbf{K}'(Q)$ and $\deg_{\Omega}(s_i) = 1$, moreover, each of them occurs exactly once. Furthermore, one can use relations in Proposition 54(m) and 54(e) to ensure that u contains only J_t for $t \in \Omega$.

Let Ω' be the full subgraph of Ω with vertex set $\Omega \setminus \mathbf{K}'(Q)$. If Ω' is empty, then the above implies that u is a product of split symbols $J_{s_i}^{(q_i)}$ for which $s_i \in \mathbf{K}'(Q)$ and $\deg_{\Omega}(s_i) = 1$. If Ω has two vertices, they are necessarily connected and we can use relation Proposition 54(f) to see that there are exactly two possibilities for u, namely $J_{s_1}^{(q_1)}J_{s_2}^{(q_2)}$ and $J_{s_2}^{(q_2)}J_{s_1}^{(q_1)}$. These two elements are different in $\mathcal{I}^{\mathrm{ind}}$ since from (12) it follows that their actions on $P_{s_1} \oplus P_{s_2}$ are different. Using these actions, we also see that these two elements differ from 0 in S. If Ω has more than two vertices, then all factors in u commute and thus define u uniquely.

It remains to consider the case when Ω' is non-empty. Since Q is admissible, Ω' is a disjoint union of graphs of the form (3). Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$ be the connected components of Ω' . Using relations given by Proposition 54(d) and (e), we can write $u = u_1 u_2 \ldots u_m$, where each element u_r , for $r = 1, 2, \ldots, m$, is a product of J_i or $J_s^{(q)}$ with $i, s \in \Gamma_r$ such that $\deg_{\Gamma_r}(i) = 2$ and $\deg_{\Omega}(s) = 1$. We also have that all factors u_r commute with each other. It remains to show that, if $u' = u'_1 u'_2 \ldots u'_m$ is another word with similar properties which defines the same element in \mathcal{I}^{ind} , then, up to permutation of factors, we have that u_r is equivalent to u'_r in S for each r.

For a fixed r, we are in the situation of the quiver (3). The relations from Proposition 54(a), (b), (e), (g) and (h) guarantee that the J_i 's and $J_s^{(q)}$'s, where $i, s \in \Gamma_r$, satisfy all relations for the corresponding Hecke-Kiselman monoid of type A as defined in [GM]. Let S_r be the submonoid of S generated by J_i and $J_s^{(q)}$, where $i, s \in \Gamma_r$. Now we can proceed similarly to the proof of Theorem 55.

Consider the standard basis $\{[P_i]: i=1,2,\ldots,n\}$ in $[A\text{-proj}]_{\oplus}$. The action of $\mathcal{I}^{\operatorname{ind}}$ on A-proj induces a homomorphism from $\mathcal{I}^{\operatorname{ind}}$ to the monoid $\operatorname{End}([A\text{-proj}]_{\oplus})$. This induces a representation of S_r in $\operatorname{End}([A\text{-proj}]_{\oplus})$. From [GM, Subsection 3.2] it follows that the latter representation map is injective. This means that, in the situation above, u_r is equivalent to u'_r in S and hence the statement of the theorem follows.

We do not know whether the systems of relations described in Theorems 55 and 56 are minimal or not.

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