

A new way to Dirichlet problems for minimal surface systems in arbitrary dimensions and codimensions

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Abstract

In this paper, by considering a special case of the spacelike mean curvature flow investigated by Li and Salavessa [6], we get a condition for the existence of smooth solutions of the Dirichlet problem for the minimal surface equation in arbitrary codimension. We also show that our condition is sharper than Wang's in [13, Theorem 1.1] provided the hyperbolic angle θ of the initial spacelike submanifold M_0 satisfies $\max_{M_0} \cosh \theta > \sqrt{2}$.

1 Introduction

Let Ω be a bounded C^2 domain in the Euclidean n -space \mathbb{R}^n and $\phi : \partial\Omega \rightarrow \mathbb{R}^m$ be a continuous map from the boundary of Ω to \mathbb{R}^m . The Dirichlet problem for the minimal surface system asks whether there exists a Lipschitz map $f : \Omega \rightarrow \mathbb{R}^m$ such that the graph of f is a minimal submanifold in \mathbb{R}^{n+m} and $f|_{\partial\Omega} = \phi$. For $m = 1$ and any mean convex domain Ω , Jenkins and Serrin [4] proved the existence of the solutions for this Dirichlet problem and the smoothness of all of the solutions. The Dirichlet problem is well understood owing to the pioneering works of Jenkins and Serrin [4], De Giorgi [3], and Moser [9]. However, they treated the Dirichlet problem for just hypersurfaces. For surfaces with higher codimension, very little is known. *Is the high codimensional Dirichlet problem solvable, or under what kind of assumptions could one obtain the existence of solutions of this problem?* Lawson and Osserman [5] gave some nice examples to show how important the boundary data is for the solvability of high codimensional Dirichlet problems. If the minimal submanifold is additionally required to be Lagrangian, the minimal surface system becomes a fully nonlinear scalar equation

$$\operatorname{Im} \left(\det(I + \sqrt{-1}D^2f) \right) = 0,$$

where I is the identity matrix and $D^2f = \left(\frac{\partial^2 f}{\partial x^i \partial x^j} \right)$ is the Hessian matrix of f . Caffarelli, Nirenberg, and Spruck [2] solved this Dirichlet problem with the prescribed boundary value of f . For $n = 2$ and any convex planar domain, the existence of solutions of this problem was proved by Radó in

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[11] (see also [5]). For $C^{2,\alpha}$ small Dirichlet boundary data of finite codimension, Smale [12] used the method of linearization to prove the solvability successfully.

Recently, under some assumption about the boundary data, Wang [13] proved the existence of smooth solutions of the Dirichlet problem for minimal surface systems in arbitrary dimensions and codimensions by using his results in [14, 15] on high codimensional mean curvature flow of submanifolds. Surprisingly, by considering a special case of the spacelike mean curvature flow (MCF for short) considered in [6], we can prove the following result.

Theorem 1.1. *Let Ω be a bounded and closed C^2 convex domain in \mathbb{R}^n ($n \geq 2$) with diameter δ . If $\psi : \Omega \rightarrow \mathbb{R}^m$ satisfies*

$$4n\eta_0^2\delta \sup_{\Omega} |D^2\psi| + \sqrt{2} \sup_{\partial\Omega} |D\psi| < 1, \quad (1.1)$$

then the Dirichlet problem for the minimal surface system is solvable for $\psi|_{\partial\Omega}$ in smooth maps. Here η_0 is a constant defined by (3.2), depending only on the spacelike graph of Ω , and for $x \in \overline{\Omega}$,

$$|D\psi|(x) := \sup_{|v|=1} |D\psi(x)(v)|$$

and

$$|D^2\psi|(x) := \sup_{|v|=1} |D^2\psi(x)(v, v)|$$

are the norm and the squared norm of the differential $D\psi(x) : \mathbb{R}^n \rightarrow \mathbb{R}^m$.

The paper is organized as follows. We recall some useful facts about spacelike MCF in [6], and establish the relation between the parametric and the non-parametric forms of the flow in Section 2. At the end of Section 2, as in [13], a boundary gradient estimate is derived by using the initial map as a barrier surface. Theorem 1.1 will be proved in the last section.

2 Some useful facts

Assume the Riemannian manifold (Σ_1, g_1) to be closed and of dimension $n \geq 2$, and the Riemannian manifold (Σ_2, g_2) to be complete, of dimension $m \geq 1$. Let $f : \Sigma_1 \rightarrow \Sigma_2$ be a smooth map from (Σ_1, g_1) to (Σ_2, g_2) . Let $\overline{M} = \Sigma_1 \times \Sigma_2$ be a pseudo-Riemannian manifold with the metric $\bar{g} = g_1 - g_2$. Let M be a spacelike graph defined by

$$M = \Gamma_f = \{(p, f(p)) | p \in \Sigma_1\},$$

and denote by g the induced metric on M . Clearly, if f is a constant map, M is a slice. If we denote this spacelike immersion by $F = id \times_g f$, then we say that the spacelike graph M evolves along the MCF if

$$\begin{cases} \frac{d}{dt}F(x, t) = H(x, t), & \forall x \in M, \forall t > 0, \\ F(\cdot, 0) = F, \end{cases} \quad (2.1)$$

where H is the mean curvature vector of $M_t = (M, F_t^* \bar{g}) = F_t(M)$, and id is the identity map. The hyperbolic angle θ can be defined by (this definition can also be seen in [1, 7])

$$\cosh \theta = \frac{1}{\sqrt{\det(g_1 - f^* g_2)}}, \quad (2.2)$$

which is used to measure the deviation from a spacelike submanifold to a slice. Assume, in addition, that the Ricci curvature of Σ_1 satisfies $Ricci_1(p) \geq 0$, and the sectional curvatures of Σ_1 and Σ_2 satisfy $K_1(p) \geq K_2(q)$, for any $p \in \Sigma_1, q \in \Sigma_2$. Besides, the curvature tensor R_2 of Σ_2 and all its covariant derivatives are bounded. By Theorem 1.1, Propositions 5.1, 5.2 and 5.3 in [6], we have the following.

Theorem 2.1. *Let f be a smooth map from Σ_1 to Σ_2 such that $F_0 : M \rightarrow \overline{M}$ is a compact spacelike graph of f . Then*

- (1) *A unique smooth solution of (2.1) with initial condition F_0 a spacelike graphic submanifold exists in a maximal time interval $[0, T)$ for some $T > 0$.*
- (2) *$\cosh \theta$ defined by (2.2) has a finite upper bound, and the evolving submanifold M_t remains a spacelike graph of a map $f_t : \Sigma_1 \rightarrow \Sigma_2$ whenever the flow (2.1) exists.*
- (3) *$\|B\|, \|H\|, \|\nabla^k B\|$, and $\|\nabla^k H\|$, for all k , are uniformly bounded.*
- (4) *The spacelike MCF (2.1) exists for all the time.*

Now, we would like to explain the connection between the spacelike MCF (2.1) and the high dimensional Dirichlet problem. Let $\Omega \subset \mathbb{R}^n$ be a closed and bounded domain, and $\psi : \Omega \rightarrow \mathbb{R}^m$ be a vector-valued function. Then the graph of ψ can be seen as the spacelike embedding $id \times \psi : \Omega \rightarrow \mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}$ with the pseudo-Riemannian metric $\bar{g} = g_1 - g_2 = ds_1^2 - ds_2^2$, where ds_1^2 and ds_2^2 are the standard Euclidean metrics of \mathbb{R}^n and \mathbb{R}^m , respectively. For the spacelike MCF (2.1), choosing $\Sigma_1 = \Omega \subset \mathbb{R}^n$ and $\Sigma_2 = \mathbb{R}^m$, if we require $F|_{\partial\Omega} = (id \times \psi)|_{\partial\Omega}$, then the immersed mapping F_t , with $F_0 = F$, should be a smooth parametric solution to the Dirichlet problem of the spacelike MCF, that is,

$$\begin{cases} \frac{dF}{dt} = H, \\ F|_{\partial\Omega} = id \times \psi|_{\partial\Omega}. \end{cases}$$

In a local coordinate system $\{x^1, \dots, x^n\}$ on $\Sigma_1 = \Omega$, the spacelike MCF is the solution

$$F = F^A(x^1, \dots, x^n, t), \quad A = 1, 2, \dots, n+m,$$

to the following system of parabolic equations

$$\frac{\partial F}{\partial t} = \left(\sum_{i,j=1}^n g^{ij} \frac{\partial^2 F}{\partial x^i \partial x^j} \right)^\perp,$$

where $g^{ij} = (g_{ij})^{-1}$ is the inverse of the induced metric $g_{ij} = g \left(\frac{\partial F}{\partial x^i}, \frac{\partial F}{\partial x^j} \right)$, and $(\cdot)^\top$ and $(\cdot)^\perp$ denote the tangent and the normal parts of a vector in \mathbb{R}^{n+m} , respectively. *The Einstein summation convention that repeated indices are summed over is adopted in the rest of the paper.* As in the proof of [13, Lemma 2.1], we can easily prove the following lemma.

Lemma 2.2. *We have*

$$\Delta F = \frac{1}{\sqrt{G}} \frac{\partial}{\partial x^i} \left(\sqrt{G} g^{ij} \frac{\partial F}{\partial x^j} \right) = \left(g^{ij} \frac{\partial^2 F}{\partial x^i \partial x^j} \right)^\perp,$$

where $G = \det(g_{ij})$.

Lemma 2.2 tells us that ΔF is always in the normal direction, which implies that $g \left(\frac{\partial F}{\partial x^i}, \Delta F \right) = 0$ for $1 \leq i \leq n$.

Similar to [13, Proposition 2.2], we can also derive a relation between parametric and non-parametric solutions to the spacelike MCF equation.

Proposition 2.3. *Suppose that F is a solution to the Dirichlet problem for spacelike MCF (2.1) and that each $F(\Omega, t)$ can be written as a graph over $\Omega \subset \mathbb{R}^n$. Then there exists a family of diffeomorphisms r_t of Ω such that $\tilde{F}_t = F_t \circ r_t$ is of the form*

$$\tilde{F}(x^1, \dots, x^n) = (x^1, \dots, x^n, f^1, \dots, f^m)$$

and $f = (f^1, \dots, f^m) : \Omega \times [0, T] \rightarrow \mathbb{R}^m$ satisfies

$$\begin{cases} \frac{\partial f^\alpha}{\partial t} = g^{ij} \frac{\partial^2 f^\alpha}{\partial x^i \partial x^j}, & \alpha = 1, \dots, m, \\ f|_{\partial\Omega} = \psi|_{\partial\Omega}, \end{cases} \quad (2.3)$$

where

$$g^{ij} = (g_{ij})^{-1} \quad \text{and} \quad g_{ij} = \delta_{ij} - \sum_{\beta=1}^m \frac{\partial f^\beta}{\partial x^i} \cdot \frac{\partial f^\beta}{\partial x^j}. \quad (2.4)$$

Conversely, if $f = (f^1, \dots, f^m) : \Omega \times [0, T] \rightarrow \mathbb{R}^m$ satisfies (2.3), then $\tilde{F} = I \times f$ is a solution to

$$\left(\frac{\partial}{\partial t} \tilde{F}(x, t) \right)^\perp = \tilde{H}(x, t).$$

By applying the maximum principle for scalar parabolic equations (see, for instance, [10]) to the second-order parabolic equation (2.3), we have the following.

Proposition 2.4. *Let $f = (f^1, \dots, f^m) : \Omega \times [0, T] \rightarrow \mathbb{R}^m$ be a solution to equation (2.3). If $\sup_{\Omega \times [0, T]} |Df|$ is bounded, then*

$$\sup_{\Omega \times [0, T]} f^\alpha \leq \sup_{\Omega} \psi^\alpha,$$

with $\psi = (\psi^1, \dots, \psi^m)$ the initial map given in equation (2.3).

By using the initial data $\psi : \Omega \rightarrow \mathbb{R}^m$ as a barrier surface, we can obtain the boundary gradient estimate as follows.

Theorem 2.5. *Let Ω be a bounded C^2 convex domain in \mathbb{R}^n with diameter δ . Suppose that the flow (2.3) exists smoothly on $\Omega \times [0, T)$. Then we have*

$$|Df| < \frac{4n\delta}{1-\xi} \sup_{\Omega} |D^2\psi| + \sqrt{2} \sup_{\partial\Omega} |D\psi|, \quad \text{on } \partial\Omega \times [0, T),$$

where $\xi = \sup_{\Omega \times [0, T)} |Df|^2$.

Proof. We use a method similar to that of the proof of [13, Theorem 3.1]. Denote by P the supporting $(n-1)$ -dimensional hyperplane at a boundary point p , and d_p the distance function to P . Let $f = (f^1, \dots, f^m)$ be a solution of equation (2.3). Consider the function defined by

$$S(x^1, \dots, x^n, t) = v \log(1 + kd_p) - (f^\alpha - \psi^\alpha)$$

on \mathbb{R}^n for each $\alpha = 1, 2, \dots, m$, where $k, v > 0$ are to be determined. The Laplace operator on the graph $(\Gamma_f, g = F_t^* \bar{g})$ is given by $\Delta = g^{ij} \frac{\partial^2}{\partial x^i \partial x^j}$, with g_{ij} satisfying (2.4). Clearly,

$$g^{ij} = (\delta_{ij} - f_i f_j)^{-1} = \delta_{ij} + \frac{f_i f_j}{1 - |Df|^2},$$

where $f_i f_j = \sum_{\beta=1}^m \frac{\partial f^\beta}{\partial x^i} \frac{\partial f^\beta}{\partial x^j}$. Therefore, the eigenvalues of g^{ij} are between 1 and $1/(1-\xi)$. By direct computation, we know that S satisfies the following evolution equation

$$\left(\frac{d}{dt} - \Delta \right) S = \frac{vk}{1+kd_p} (-\Delta d_p) + \frac{vk^2}{(1+kd_p)^2} g^{ij} \frac{\partial d_p}{\partial x^i} \frac{\partial d_p}{\partial x^j} - \Delta \psi^\alpha. \quad (2.5)$$

Since d_p is a linear function, $\Delta d_p = 0$, then (2.5) is reduced to

$$\left(\frac{d}{dt} - \Delta \right) S = \frac{vk^2}{(1+kd_p)^2} g^{ij} \frac{\partial d_p}{\partial x^i} \frac{\partial d_p}{\partial x^j} - \Delta \psi^\alpha. \quad (2.6)$$

Since $|Dd_p| = 1$, $d_p(y) \leq |p-y| \leq \delta$ for any $y \in \Omega$, and the fact that the eigenvalues of g^{ij} are between 1 and $1/(1-\xi)$, we have

$$\frac{vk^2}{(1+kd_p)^2} g^{ij} \frac{\partial d_p}{\partial x^i} \frac{\partial d_p}{\partial x^j} \geq \frac{vk^2}{(1+k\delta)^2} \geq \frac{vk^2}{(1+k\delta)^2},$$

and

$$\Delta \psi^\alpha = \left| g^{ij} \frac{\partial^2 \psi^\alpha}{\partial x^i \partial x^j} \right| \leq \frac{n}{1-\xi} |D^2 \psi|.$$

Hence, if

$$\frac{vk^2}{(1+k\delta)^2} \geq \frac{n}{1-\xi} \sup_{\Omega} |D^2 \psi|,$$

then, together with (2.6), we have $(\frac{d}{dt} - \Delta)S \geq 0$ on $[0, T)$. On the one hand, by convexity, we have $S > 0$ on the boundary $\partial\Omega$ of Ω except for $S = 0$ at p . On the other hand, $S \geq 0$ on Ω at $t = 0$. By the strong maximum principle for second-order parabolic partial differential equations, we have

$$S > 0, \quad \text{on } \Omega \times (0, T).$$

The same conclusion can also be obtained for a new function $S' := v \log(1 + kd_p) + (f^\alpha - \psi^\alpha)$. Hence, we have that the normal derivatives satisfy

$$\begin{aligned} \left| \frac{\partial (f^\alpha - \psi^\alpha)}{\partial n} \right| (p) &\leq \lim_{d_p(x) \rightarrow 0} \frac{|f^\alpha - \psi^\alpha|}{d_p(x)} \\ &< \lim_{d_p(x) \rightarrow 0} \frac{v \log[1 + kd_p(x)]}{d_p(x)} = vk. \end{aligned}$$

So, by changing coordinates of \mathbb{R}^m , we may assume $\partial f^\alpha / \partial n = 0$ for all α except for $\alpha = 1$ such that the inequality

$$\left| \frac{\partial f}{\partial n} \right| < vk + \left| \frac{\partial \psi}{\partial n} \right| \quad (2.7)$$

holds.

The Dirichlet boundary condition implies

$$\left| D^{\partial\Omega} f \right| = \left| D^{\partial\Omega} \psi \right|, \quad \text{on } \partial\Omega, \quad (2.8)$$

where $|D^{\partial\Omega} f|$ is defined by $|D^{\partial\Omega} f| := \sup_w |Df(x)w|$ for $x \in \partial\Omega$ and w being taken over all unit vectors tangent to $\partial\Omega$. Combining (2.7) and (2.8), we have

$$|Df| < \sqrt{\left(vk + \left| \frac{\partial \psi}{\partial n} \right| \right)^2 + |D^{\partial\Omega} \psi|^2} \leq \sqrt{2} |D\psi| + vk, \quad \text{on } \partial\Omega.$$

Now, it is not difficult to find out that if we want to prove our assertion here, we actually need to minimize vk under the constraint $\frac{vk^2}{(1+k\delta)^2} \geq \frac{n}{1-\xi} \sup_{\Omega} |D^2 \psi|$. In fact, the minimum $(vk)_{\min}$ of the function vk is obtained when $k = \delta^{-1}$, and $(vk)_{\min} = 4n\delta(1-\xi)^{-1} \sup_{\Omega} |D^2 \psi|$. The theorem follows. \square

3 Proof of the main theorem

Now, by applying the conclusions we recalled and derived in Section 2, we can prove Theorem 1.1 as follows.

Proof of Theorem 1.1. We divide the proof of Theorem 1.1 into five steps.

Step 1. By the Schauder fixed-point theorem (see, for instance, Theorem 8.1 on p. 199 of [8] for a detailed description and the proof of *the Schauder fixed-point theorem*), the solvability of the parabolic system (2.3) can be reduced to the estimates of the solution (f^α) of the uniformly parabolic system

$$\begin{cases} \frac{df^\alpha}{dt} = \tilde{g}^{ij} \frac{\partial^2 f^\alpha}{\partial x^i \partial x^j}, & \alpha = 1, \dots, m, \\ f|_{\partial\Omega} = \psi|_{\partial\Omega}, \end{cases} \quad (3.1)$$

with the coefficients

$$\tilde{g}^{ij} = \left(\delta_{ij} - \sum_{\beta=1}^m \frac{\partial u^\beta}{\partial x^i} \frac{\partial u^\beta}{\partial x^j} \right)^{-1} = \delta_{ij} + \frac{\sum_{\beta=1}^m \frac{\partial u^\beta}{\partial x^i} \frac{\partial u^\beta}{\partial x^j}}{1 - |Du|^2}$$

for any $u = (u^1, \dots, u^m)$ with uniform $C^{1,\gamma}$ bound. The property of being uniformly parabolic of (3.1) is equivalent to $|Du| < 1$ for all time $t \in [0, T)$. Fortunately, $|Du| < 1$ for $0 \leq t < T$ essentially corresponds to the fact that the evolving submanifold M_t is spacelike for all the time $t \in [0, T)$, which can be obtained directly from Theorem 2.1 (2). Now, (3.1) is a decoupled system of linear parabolic equations, which is uniformly parabolic and whose required estimate follows from linear theory for scalar equations. Therefore, we know that the system (2.3) has the solution on a finite time interval.

In fact, there is another way to show the short-time existence of the solution of (2.3). More precisely, by Theorem 2.1 (1), we can also get the short-time existence, since in our case, as explained before, we choose Σ_1 to be a closed domain in \mathbb{R}^n and $\Sigma_2 = \mathbb{R}^m$, which implies that the system (2.3) is just a special case of the spacelike MCF (2.1) provided we additionally require $F|_{\partial\Omega} = (id \times \psi)|_{\partial\Omega}$, i.e., $f|_{\partial\Omega} = \psi|_{\partial\Omega}$.

Step 2. Denote the graph of f_t by M_t . We show that $|Df_t| < 1$ holds under the assumption of Theorem 1.1. Similar to (3.4) and (3.5) in [6], for each point $p \in \Sigma_1$, we can choose an orthonormal basis for the tangent space $T_p M$ and for the normal space $N_p M$ given as follows

$$e_i = \frac{1}{\sqrt{1 - \sum_{\beta} \lambda_{i\beta}^2}} \left(a_i + \sum_{\beta} \lambda_{i\beta} a_\beta \right), \quad i = 1, \dots, n,$$

$$e_\alpha = \frac{1}{\sqrt{1 - \sum_j \lambda_{j\alpha}^2}} \left(a_\alpha + \sum_j \lambda_{j\alpha} a_j \right), \quad \alpha = n+1, \dots, n+m,$$

where $\{a_i\}_{i=1, \dots, n}$ is a g_1 -orthonormal basis of $T_p \Sigma_1$ of eigenvectors of $f^* g_2$, $\{a_\alpha\}_{\alpha=n+1, \dots, n+m}$ is a g_2 -orthonormal basis of $T_{f(p)} \Sigma_2$, and $df = -\lambda_{i\alpha}$ with $\lambda_{i\alpha} = \delta_{\alpha, n+i} \lambda_i$. Here λ_i , $1 \leq i \leq n$, are the eigenvalues of $f^* g_2$. So, the spacelike condition on M implies $\lambda_i^2 < 1$ for each $1 \leq i \leq n$. We list them non-increasingly as $\lambda_1^2 \geq \lambda_2^2 \geq \dots \geq \lambda_n^2 \geq 0$. By the classical Weyl's perturbation theorem, the ordering eigenvalues $\lambda_i^2 : \Sigma_1 \rightarrow [0, 1)$ is a continuous and locally Lipschitz function. For each $p \in \Sigma_1$, denote by $s = s(p) = \{1, 2, \dots, n\}$ the rank of f at the point p , which implies $\lambda_s^2 > 0$ and $\lambda_{s+1} = \lambda_{s+2} = \dots = \lambda_n = 0$. Therefore, we have $s \leq \min\{m, n\}$. In fact, after this setting, we have $\lambda_{i\alpha} = \delta_{\alpha, n+i} = 0$ if $i > s$, or $\alpha > n+s$. Under the orthonormal basis $\{e_1, \dots, e_n, \dots, e_{n+m}\}$, by (2.2) the hyperbolic angle θ satisfies

$$\cosh \theta = \frac{1}{\sqrt{\det(g_1 - f^* g_2)}} = \frac{1}{\sqrt{\prod_{i=1}^n (1 - \lambda_i^2)}}.$$

Let

$$\eta_t := \max_{M_t} \cosh \theta. \quad (3.2)$$

By [6, Proposition 4.3], we know that (*in our case*, $\Sigma_1 = \Omega \subset \mathbb{R}^n$, $\Sigma_2 = \mathbb{R}^m$) the evolution equation of $\cosh\theta$ here should be

$$\frac{d}{dt} \ln(\cosh\theta) = \Delta \ln(\cosh\theta) - \left\{ \|B\|^2 - \sum_{k,i=1}^n \lambda_i^2 (h_{ik}^{m+i})^2 - 2 \sum_{k,i < j} \lambda_i \lambda_j h_{ik}^{m+j} h_{jk}^{m+i} \right\},$$

where $\|B\|^2 = \sum_{i,j=1}^n \sum_{\alpha=n+1}^{n+m} (h_{ij}^\alpha)^2$ is the squared norm of the second fundamental form. Hence, there exists some nonnegative constant ℓ such that

$$\frac{d}{dt} \ln(\cosh\theta) \leq \Delta \ln(\cosh\theta) - \ell \|B\|^2 \leq \Delta \ln(\cosh\theta).$$

Then by the maximum principle for parabolic equations, we can obtain

$$\eta_t \leq \eta_0 = \max_{M_0} \cosh\theta$$

for $0 < t \leq T \leq \infty$, which implies $\lambda_i^2(t) < 1$ and

$$1 - \lambda_i^2(t) \geq \prod_{i=1}^n (1 - \lambda_i^2(t)) \geq \frac{1}{\eta_t^2} \geq \frac{1}{\eta_0^2}$$

for any $0 < t \leq T \leq \infty$. On the other hand, if we assume

$$4n\eta_0^2 \delta \sup_{\Omega} |D^2\psi| + \sqrt{2} \sup_{\partial\Omega} |D\psi| < 1,$$

then by integrating along a path in Ω , we have $\sup_{\Omega} |Df_0| = \sup_{\partial\Omega} |D\psi| < 1$ initially. Hence, by the above arguments, we have $|Df_t| < 1$ for any $0 \leq t \leq T \leq \infty$.

Step 3. By Theorem 2.1 (3) and (4), we know that the norms of the second fundamental form and all of its derivatives are bounded, and the flow (2.1) exists for all time. Hence, the solution to the spacelike MCF (2.3) exists smoothly in $[0, \infty)$.

Step 4. By [6, Corollary 6.1], we know that there exists a time sequence $t_n \rightarrow \infty$ such that $\sup_{\Sigma_1} \|H_{t_n}\| \rightarrow 0$ when $t_n \rightarrow \infty$. Since we also have a gradient bound (see Theorem 2.1 (3)), we can extract a subsequence t_i such that the graph M_{t_i} converges to a Lipschitz graph with $\sup \|H\| = 0$ and $|Df| < 1$, which implies that the limit submanifold is a minimal spacelike submanifold.

Step 5. Interior regularity of the limit follows from [13, Theorem 4.1], since the singular values λ_i of Df satisfy $|\lambda_i \lambda_j| \leq 1 - 1/\eta_0^2$ almost everywhere for any $i \neq j$. Boundary regularity follows from [5, Theorem 2.3]. Our proof is finished. \square

Remark 3.1. Clearly, our condition (1.1) is better than that in [13, Theorem 1.1] provided $\eta_0 = \max_{M_0} \cosh\theta > \sqrt{2}$.

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