ON THE KÄHLER RICCI FLOW ON PROJECTIVE MANIFOLDS OF GENERAL TYPE

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ABSTRACT. We consider the Kähler Ricci flow on a smooth minimal model of general type, we show that if the Ricci curvature is uniformly bounded below along the Kähler-Ricci flow, then the diameter is uniformly bounded. As a corollary we show that under the Ricci curvature lower bound assumption, the Gromov-Hausdorff limit of the flow is homeomorphic to the canonical model. Moreover, we can give a purely analytic proof of a recent result of Tosatti-Zhang ([29]) that if the canonical line bundle K_X is big and nef, but not ample, then the flow is of Type IIb.

1. INTRODUCTION

The Ricci flow ([10]) has been one of the most powerful tools in geometric analysis with remarkable applications to the study of 3-manifolds. The complex analogue, the Kähler-Ricci flow, has been used by Cao ([2]) to give an alternative proof of the existence of Kähler Einstein metrics on manifolds with negative or vanishing first Chern class ([31, 1]). Tsuji [30] applied the Kähler-Ricci flow to construct a singular Kähler Einstein metrics on smooth minimal manifolds of general type. The analytic Minimal Model program, introduced in [19, 20], aims to find the minimal model of an algebraic variety, by running the Kähler-Ricci flow. It is conjectured ([20]) that the Kähler-Ricci flow will deform a given projective variety to its minimal model and eventually to its canonical model coupled with a canonical metric of Einstein type, in the sense of Gromov-Hausdorff.

Let X be a projective n-dimensional manifold, with the canonical bundle K_X big and nef. We consider the Kähler-Ricci flow

$$(1.1) \qquad \frac{\partial \omega}{\partial t} = -\mathrm{Ric}(\omega) - \omega, \quad \omega(0) = \omega_0,$$

where ω_0 is a Kähler metric on X. It's well-known that the equation (1.1) is equivalent to the following complex Monge-Ampere equation

$$(1.2) \qquad \begin{cases} \dfrac{\partial \varphi}{\partial t} = \log \dfrac{(\chi + e^{-t}(\omega_0 - \chi) + i\partial\bar\partial\varphi)^n}{\Omega} - \varphi \\ \varphi(0) = 0, \end{cases}$$

where Ω is a smooth volume form, $\chi = i\partial\bar\partial \log \Omega \in c_1(K_X) = -c_1(X)$, and $\omega(t) = \chi + e^{-t}(\omega_0 - \chi) + i\partial\bar\partial\varphi$. It's also well-known ([30, 27]) that the equation (1.2) has long time existence, if K_X is nef. We will prove the following result:

Theorem 1.1. *Let X be a projective manifold with K_X big and nef. If along the Kähler Ricci flow (1.1), the Ricci curvature is uniformly bounded below for any $t \geq 0$, i.e.,*

$$\mathrm{Ric}(\omega(t)) \geq -K\omega(t),$$

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for some $K > 0$, then there is a constant $C > 0$ such that the diameter of $(X, \omega(t))$ remain bounded, i.e.,

$$\mathrm{diam}(X, \omega(t)) \leq C.$$

Remark 1.1. *If we use Kawamata's theorem ([11]) that the nef and big canonical line bundle K_X is semi-ample, by [32, 21] the scalar curvature along the Kähler Ricci flow (1.1) is uniformly bounded, hence Ricci curvature lower bound implies that Ricci curvature is uniformly bounded on both sides. Then in the proof of Theorem 1.1, we can use Cheeger-Colding-Tian ([4]) theory to identify the regular sets. Moreover, if K_X is semi-ample and big, then the L^∞ bound of φ in (1.2) will simplify the proof. However, following Song's ([18]) recent analytic proof of base point freeness for nef and big K_X, our proof of Theorem 1.1 does not rely on Kawamata's theorem.*

It is conjectured by Song-Tian in [20] that the Kähler Ricci flow (1.1) will converge to the the canonical model of X coupled with the unique Kähler Einstein current with bounded potential, in the Gromov-Hausdorff sense. Under the assumption that the Ricci curvature is uniformly bounded below, we can partially confirm this conjecture.

Corollary 1.1. *Under the same assumptions as Theorem 1.1, then as $t \to \infty$,*

$$(X, \omega(t)) \xrightarrow{d_{GH}} (X_\infty, d_\infty),$$

the limit space X_∞ is homeomorphic to the canonical model X_{can} of X. Moreover, (X_∞, d_∞) is isometric to the metric completion of (X_{can}°, g_{KE}), where g_{KE} is the unique Kähler-Einstein current with bounded local potentials and X_{can}° is the regular part of X_{can}.

Consider the unnormalized Kähler Ricci flow

(1.3) $$\frac{\partial}{\partial t}\omega = -\mathrm{Ric}(\omega), \quad \omega(0) = \omega_0,$$

with long time existence. The flow (1.3) is called to be of Type III, if

$$\sup_{X \times [0,\infty)} t|Rm|(x,t) < \infty,$$

otherwise it is of Type IIb, here $|Rm|(\omega(t))$ denotes the Riemann curvature of $\omega(t)$. It's well-known that Type III condition is equivalent to the curvature is uniformly bounded along the normalized Kähler Ricci flow (1.1). As a by-product of our proof of Theorem 1.1, we obtain a purely analytic proof of a recent result of Tosatti-Zhang ([29]), namely,

Theorem 1.2. [29] *Let X be a projective manifold with K_X big and nef, if the Kähler Ricci flow (1.1) is of type III, then the canonical line bundle K_X is ample.*

Throughout this paper, the constants C may be different from lines to lines, but they are all uniform. We also use g as the associated Riemannian metric of a Kähler form ω, for example the metric space $(X, \omega(t))$ means the space $(X, g(t))$.

2. Preliminaries

In this section, we will recall some definitions and theorems we will use in this note.

Definition 2.1. *Let $L \to X$ be a holomorphic line bundle over a projective manifold X. L is said to be semi-ample if the linear system $|kL|$ is base point free for some $k \in \mathbb{Z}^+$. L is said to be big if the Iitaka dimension of L is equal to the dimension of X. L is called numerically effective (nef) if $L \cdot C \geq 0$ for any irreducible curve $C \subset X$.*

We can define a semi-group

$$\mathcal{F}(X, L) = \{k \in \mathbb{Z}^+ | kL \text{ is base point free}\}.$$

For any $k \in \mathcal{F}(X, L)$, the linear system $|kL|$ induces a morphism

$$\Phi_k = \Phi_{|kL|} : X \to X_k = \Phi_k(X) \subset \mathbb{P}^{N_k}$$

where $N_k + 1 = \dim H^0(X, kL)$. It's well-known that ([13]) that for large enough $k, l \in \mathcal{F}(X, L)$, $(\Phi_k)_* \mathcal{O}_X = \mathcal{O}_{X_k}$, $\Phi_k = \Phi_l$ and $X_k = X_l$.

We will need the following version of L^2 estimates due to Demailly ([7])

Theorem 2.1. *Suppose X is an n-dimensional projective manifold equipped with a smooth Kähler metric ω. Let L be a holomorphic line bundle over X equipped with a possibly singular hermitian metric h such that $\mathrm{Ric}(h) + \mathrm{Ric}(\omega) \geq \delta\omega$ in the current sense for some $\delta > 0$. Then for any L-valued $(0,1)$-form τ satisfying*

$$\bar{\partial}\tau = 0, \quad \int_X |\tau|^2_{h,\omega} \omega^n < \infty,$$

there exists a smooth section u of L such that $\bar{\partial}u = \tau$ and

$$\int_X |u|^2_{h,\omega} \omega^n \leq \frac{1}{2\pi\delta} \int_X |\tau|^2_{h,\omega} \omega^n.$$

3. Identify the regular sets

Since K_X is big and nef, by Kodaira lemma, there exists an effective divisor $D \subset X$, such that $K_X - \varepsilon D$ is ample, hence there exists a hermitian metric on $[D]$ such that

$$\chi - \varepsilon \mathrm{Ric}(h_D) > 0.$$

Let's recall a few known estimates of the flow, (see [15, 30] or [17, 18] without assuming that K_X is semi-ample)

Lemma 3.1.   (i) *There is a constant $C > 0$ such that for any $t \geq 0$,*

$$\sup_X \varphi(t) \leq C, \quad \sup_X \dot{\varphi}(t) = \sup_X \frac{\partial}{\partial t}\varphi \leq C.$$

(ii) *For any $\delta \in (0, 1)$, there is a constant $C_\delta > 0$ such that*

$$\varphi \geq \delta \log |\sigma_D|^2_{h_D} - C_\delta,$$

where σ_D is a holomorphic section of the line bundle $[D]$ associated to the divisor D.
(iii) *Along the Kähler Ricci flow, there exist constants $C > 0, \lambda > 0$ such that*

$$\mathrm{tr}_{\omega_0}\omega(t) \leq C|\sigma_D|^{-2\lambda}_{h_D}$$

(iv) *For any compact subset $K \subset X \backslash D$, any $\ell \in \mathbb{Z}^+$, there exists a constant $C_{\ell,K} > 0$ such that*

$$\|\varphi\|_{C^\ell(K)} \leq C_{\ell,K}.$$

Hence we can conclude that

$$\omega(t) \xrightarrow{C^\infty_{loc}(X \backslash D)} \omega_\infty,$$

for some smooth Kähler metric ω_∞ on $X \backslash D$. On the other hand, it can be shown that $\dot\varphi(t) \to 0$ on any $K \subset X \backslash D$ as $t \to \infty$, hence ω_∞ satisfies the equation

$$\omega_\infty^n = (\chi + i\partial\bar\partial\varphi_\infty)^n = e^{\varphi_\infty}\Omega, \text{ on } X \backslash D,$$

and ω_∞ is a Kähler-Einstein metric on $X \backslash D$, i.e.,

$$\mathrm{Ric}(\omega_\infty) = -\omega_\infty.$$

Proposition 3.1. [18] *For any holomorphic section $\sigma \in H^0(X, mK_X)$, there is a constant $C = C(\sigma)$ such that for any $t \geq 0$, we have*

$$\sup_X |\sigma|^2_{h_t^m} \leq C, \qquad \sup_X |\nabla_t \sigma|^2_{h_t^m} \leq C$$

where $h_t = \frac{1}{\omega(t)^n}$ is the hermtian metric on K_X induced by the Kähler metric $\omega(t)$ and ∇_t is the covariant derivative with respect to h_t^m.

Letting $t \to \infty$, we have $h_t \to h_\infty = h_{KE} = h_\chi e^{-\varphi_\infty}$ on $X \backslash D$ (here $h_\chi = \frac{1}{\Omega}$), and

(3.1) $$\sup_{X \backslash D} |\sigma|^2_{h_\infty^m} \leq C, \quad \sup_{X \backslash D} |\nabla_\infty \sigma|^2_{h_\infty^m} \leq C.$$

Definition 3.1. *We define a set $\mathcal{R}_X \subset X$ to be the points $p \in X$ such that the μ-jets at p are generated by global sections of mK_X for some $m \in \mathbb{Z}^+$, for any $\mu \in \mathbb{N}^n$ with $|\mu| \leq 2$.*

Proposition 3.2. [18] *$\mathcal{R}_X$ is an open dense set of X and on $\mathcal{R}_X$ we have locally smooth convergence of $\omega(t)$ to ω_∞.*

By the smooth convergence of $\omega(t)$ on $X \backslash D$, we can choose a point $p \in X \backslash D$ and a small $r_0 > 0$ such that (we write the associated Riemannian metric of $\omega(t)$ as $g(t)$)

$$B_{g(t)}(p, r_0) \subset\subset X \backslash D, \qquad Vol_{g(t)}(B_{g(t)}(p, r_0)) \geq v_0, \quad \forall t \geq 0$$

for some $v_0 > 0$. For any sequence $t_i \to \infty$, $(X, g(t_i), p)$ is a sequence of almost Kähler-Einstein manifolds (see the Appendix), in the sense of Tian-Wang ([24]). By the structure theorem in Tian-Wang ([24]), we have

(3.2) $$(X, g(t_i), p) \xrightarrow{d_{GH}} (X_\infty, d_\infty, p_\infty).$$

Moreover, X_∞ has a regular-singular decomposition, $X_\infty = \mathcal{R} \cup \mathcal{S}$; the singular $\mathcal{S}$ is closed and of Hausdroff dimension $\leq 2n - 4$; the regular set $\mathcal{R}$ is an open smooth Kähler manifold, and $d_\infty|_\mathcal{R}$ is induced by some smooth Kähler-Einstein metric g'_∞, i.e. on $\mathcal{R}$, $\mathrm{Ric}(g'_\infty) = -g'_\infty$.

We define a subset $\mathcal{S}_X \subset X_\infty$ to be a set consisting of the points $q \in X_\infty$ such that there exist a sequence of points $q_k \in X \backslash \mathcal{R}_X$ such that $q_k \to q$ along the Gromov-Hausdroff convergence.

By a theorem of Rong-Zhang (see Theorem 4.1 in [16]), there exists a surjective map

$$\overline{(\mathcal{R}_X, g_\infty)} \to (X_\infty, d_\infty),$$

where $\overline{(\mathcal{R}_X, g_\infty)}$ denotes the metric completion of the metric space $(\mathcal{R}_X, g_\infty)$, and a homeo-morphism $(\mathcal{R}_X, g_\infty) \to (X_\infty \backslash \mathcal{S}_X, d_\infty)$ which is a local isometry.

It's not hard to see that $\mathcal{S}_X$ is closed in X_∞ and any tangent cone at $q \notin \mathcal{S}_X$ is $\mathbb{R}^{2n}$, hence $X_\infty \backslash \mathcal{S}_X \subset \mathcal{R}$, i.e., $\mathcal{S} \subset \mathcal{S}_X$.

Proposition 3.3. *We have*
$$\mathcal{S}_X \subset \mathcal{S}$$
hence $\mathcal{S} = \mathcal{S}_X$.

Proof. Suppose not, there exists $q \in \mathcal{S}_X \cap \mathcal{R}$, then there exist $q_k \in (X \backslash \mathcal{R}_X, g(t_k))$ converging to q along the Gromov-Hausdroff convergence (3.2). Since $\mathcal{R}$ is open and tangent cones at points in $\mathcal{R}$ is the Euclidean space $\mathbb{R}^{2n}$, for any small $\delta > 0$, there exists a sufficiently small $r_0 > 0$ such that
$$B_{d_\infty}(q, 3r_0) \subset\subset \mathcal{R}, \qquad Vol_{g'_\infty}(B_{d_\infty}(q, 3r_0)) > (1 - \delta/2)Vol_{g_E}(B(0, 3r_0)),$$
where g_E is the standard Euclidean metric on $\mathbb{R}^{2n}$ and $B(0, 3r_0)$ is the Euclidean ball. Since Ricci curvatures are bounded below, by volume continuity for the Gromov-Hausdorff convergence ([6]) we have for k large enough,
$$Vol_{g(t_k)}(B_{g(t_k)}(q_k, 3r_0)) > (1 - \delta)Vol_{g_E}(B(0, 3r_0)).$$
By assumption that the Ricci curvature is uniformly bounded below along the Kähler Ricci flow, hence Perelman's pseudo-locality ([14, 24]) implies that if δ is small enough, there exists a small but uniform constant $\varepsilon_0 > 0$ such that
$$\sup_{B_{g(t_k)}(q_k, 2r_0)} |Rm(g(t_k + \varepsilon_0))| \leq \frac{2}{\varepsilon_0}.$$
Moreover, by Theorem 4.2 in [24],
$$(3.3) \qquad (B_{g(t_k)}(q_k, 2r_0), g(t_k + \varepsilon_0), q_k) \xrightarrow{d_{GH}} (B_{d_\infty}(q, 2r_0), d_\infty, q).$$
By Shi's derivative estimate, we have
$$\sup_{B_{g(t_k)}(q_k, 3r_0/2)} |\nabla^l Rm(t_k + \varepsilon_0)| \leq C(\varepsilon_0, l),$$
for any $l \in \mathbb{N}$ and some constant $C(\varepsilon_0, l)$. Thus we have smooth convergence of $g(t_k + \varepsilon_0)$ to a Kähler metric $\tilde{g}_\infty$ on $(B_{d_\infty}(q, r_0), J_\infty)$ along the Gromov-Hausdorff convergence (3.3), where J_∞ is the limit complex structure.

Without loss of generality we can assume the injectivity radii of $g(t_k + \varepsilon_0)$ at q_k are bounded below by r_0 ([5]), since the Riemann curvatures and volumes of $B_{g(t_k)}(q_k, r_0)$ are uniformly bounded. For k large enough, there exists (see [25]) a local holomorphic coordinates system $\{z_\alpha^{(k)}\}_{\alpha=1}^n$ on the ball $(B_{g(t_k)}(q_k, r_0), g(t_k + \varepsilon_0))$ such that $|z^{(k)}|^2 = \sum_{\alpha=1}^n |z_\alpha^{(k)}|^2 \leq r_0^2$, $|z^{(k)}|^2(q_k) = 0$ and under these coordinates $g_{\alpha\bar\beta} = g_{t_k+\varepsilon_0}(\nabla z_\alpha^{(k)}, \bar\nabla z_\beta^{(k)})$ satisfies
$$\frac{1}{C}\delta_{\alpha\beta} \leq g_{\alpha\bar\beta} \leq C\delta_{\alpha\beta}, \quad \|g_{\alpha\bar\beta}\|_{C^{1,\gamma}} \leq C, \quad \text{for some } \gamma \in (0, 1).$$
This implies that the Euclidean metric under these coordinates
$$(3.4) \qquad \sum_{\alpha=1}^n \sqrt{-1}dz_\alpha^{(k)} \wedge d\bar{z}_\alpha^{(k)}$$

is uniformly equivalent to $g(t_k + \varepsilon_0)$ on the ball $B_{g(t_k)}(q_k, r_0)$.

Recall that along Kähler-Ricci flow

$$\mathrm{Ric}(\omega(t)) = -\chi - i\partial\bar{\partial}(\varphi + \dot{\varphi}).$$

Take a cut-off function η on $\mathbb{R}$ such that $\eta(x) = 1$ for $x \in (-\infty, 1/2)$ and vanishes for $x \in [1, \infty)$. Choose a function

$$\Phi_k = (|\mu| + 1 + n)\eta\Big(\frac{|z^{(k)}|^2}{r_0^2/2}\Big) \log |z^{(k)}|^2 + \varphi(t_k + \varepsilon_0) + \dot{\varphi}(t_k + \varepsilon_0).$$

Note that Φ_k is a globally defined function on X (with a log-pole at q_k) when k is large enough.

Since the metrics (3.4) and $g(t_k + \varepsilon_0)$ are uniformly equivalent for k large enough on the support of $i\partial\bar{\partial}\Big((n + 1 + |\mu|)\eta\Big(\frac{|z^{(k)}|^2}{r_0^2/2}\Big) \log |z^{(k)}|^2\Big)$, we see that there is a uniform constant Λ independent of k such that

$$i\partial\bar{\partial}\Big((n + 1 + |\mu|)\eta\Big(\frac{|z^{(k)}|^2}{r_0^2/2}\Big) \log |z^{(k)}|^2\Big) \geq -\Lambda\omega(t_k + \varepsilon_0).$$

We will fix an integer $m \geq 10\Lambda$.

Define a (singular) hermitian metric on K_X by

$$h_k = h_\chi e^{-\frac{\varphi(t_k + \varepsilon_0)}{2} - \frac{\epsilon}{m} \log |\sigma_D|^2_{h_D}},$$

for some small $\epsilon > 0$. Then we have for k large enough (we denote below $\omega_k = \omega(t_k + \varepsilon_0)$, and $[D]$ the current of integration over the divisor D.)

$$\begin{aligned}
\mathrm{Ric}(h_k^m) + \mathrm{Ric}(\omega_k) + i\partial\bar{\partial}\Phi_k =& m\chi + \frac{1}{2}i\partial\bar{\partial}\varphi - \epsilon\mathrm{Ric}(h_D) + \epsilon[D] - \chi \\
& + i\partial\bar{\partial}\Big((n + 1 + |\mu|)\eta\Big(\frac{|z^{(k)}|^2}{r_0^2/2}\Big) \log |z^{(k)}|^2\Big) \\
=& \frac{m}{2}\omega_k + \frac{m\chi}{2} - \epsilon\mathrm{Ric}(h_D) - \frac{m}{2}e^{-t_k - \varepsilon_0}(\omega_0 - \chi) \\
& + \epsilon[D] + i\partial\bar{\partial}\Big((n + 1 + |\mu|)\eta\Big(\frac{|z^{(k)}|^2}{r_0^2/2}\Big) \log |z^{(k)}|^2\Big) \\
\geq& \frac{m}{4}\omega_k,
\end{aligned}$$

in the current sense, for k large enough. The above inequality follows since $\frac{m}{2}\chi - \epsilon\mathrm{Ric}(h_D)$ is a fixed Kähler metric, which is greater than $\frac{m}{2}e^{-t_k - \varepsilon_0}(\omega_0 - \chi)$ for k large enough.

Define an mK_X-valued $(0, 1)$ form

$$\eta_{k,\mu} = \bar{\partial}\Big(\eta\Big(\frac{|z^{(k)}|^2}{r_0^2/2}\Big)(z^{(k)})^\mu\Big),$$

where

$$(z^{(k)})^\mu = \prod_{\alpha=1}^{n}(z_\alpha^{(k)})^{\mu_\alpha}, \quad \mu = (\mu_1, \ldots, \mu_n) \in \mathbb{N}^n.$$

It's not hard to see (noting that the pole order along D is $\leq \epsilon$)

$$\int_X |\eta_{k,\mu}|^2_{h_k^m} e^{-\Phi_k}\omega_k^n < \infty.$$

Then we can apply the Hormander's L^2 estimate (see Theorem 2.1 with $L = mK_X$) to solve the following $\bar{\partial}$-equation

$$\bar{\partial} u_{k,\mu} = \eta_{k,\mu},$$

with $u_{k,\mu}$ a smooth section of mK_X satisfying

$$\int_X |u_{k,\mu}|_{h_k^m} e^{-\Phi_k} \omega_k^n \leq \frac{4}{m} \int_X |\eta_{k,\mu}|_{h_k^m}^2 e^{-\Phi_k} \omega_k^n < \infty.$$

By checking the pole order of $e^{-\Phi_k}$ at q_k we can see that $u_{k,\mu}$ vanishes at q_k up to order $|\mu|$, and hence

$$\sigma_{k,\mu} := u_{k,\mu} - \eta\Big(\frac{|z^{(k)}|^2}{r_0^2/2}\Big)(z^{(k)})^\mu$$

is a nontrivial global holomorphic section of mK_X. Hence we see the global sections of mK_X generates the μ-jets at q_k for k large enough. This gives the contradiction. Hence $\mathcal{S}_X \subset \mathcal{S}$. $\square$

Thus we have a local isometry homeomorphism

$$(\mathcal{R}_X, g_\infty) \to (X_\infty \backslash \mathcal{S}_X, d_\infty) = (\mathcal{R}, d_\infty).$$

Hence we can identify $\mathcal{R}_X$ and $\mathcal{R}$, and $d_\infty|_{\mathcal{R}}$ is induced by the Kähler-Einstein metric $g_\infty|_{\mathcal{R}_X}$.

4. ESTIMATES NEAR THE SINGULAR SET

Throughout this section, we fix an effective divisor $D \subset X$ such that

$$K_X - \varepsilon[D] > 0$$

for sufficiently small $\varepsilon > 0$. By the previous section, we see $X \backslash D \subset \mathcal{R}_X$. Choose a log-resolution of (X, D),

$$\pi_1 : Z \to X$$

such that $\pi_1^{-1}(D)$ is a smooth divisor with simple normal crossings. Fix a point O in a smooth component of $\pi_1^{-1}(D)$ and blow up Z at the point O, we get a map

$$\pi_2 : \tilde{X} \to Z,$$

for some smooth projective manifold $\tilde{X}$. Denote $\pi = \pi_1 \circ \pi_2 : \tilde{X} \to X$.

By Adjunction formula, we have

$$K_{\tilde{X}} = \pi^* K_X + (n-1)E + F, \qquad F = \sum_k a_k F_k,$$

where E is the exceptional locus of the blow up π_2, and F_k is a prime divisor in the exceptional locus of π. We also note that $a_k > 0$ for any k.

Since $\tilde{\chi} = \pi^*\chi \in \pi^* K_X$ is big and nef, Kodaira's lemma implies there exists an effective divisor $\tilde{D}$ whose support coincide with the exceptional locus E, F and

$$\tilde{\chi} - \varepsilon[\tilde{D}] \text{ is Kähler},$$

hence there exists a hermitian metric $h_{\tilde{D}}$ on the line bundle associated to D such that

$$\tilde{\chi} - \varepsilon \text{Ric}(h_{\tilde{D}}) > 0.$$

We write $\tilde{D} = \tilde{D}' + \tilde{D}''$, where $\text{supp} D'' = E$, and $E \not\subset \tilde{D}'$. Let $\sigma_E, \sigma_F, \sigma_{\tilde{D}}$ be the defining section of E, F and $\tilde{D}$, respectively. Here these sections are multi-valued holomorphic sections

which become global after taking some power. There also exist hermtian metrics h_E, h_F, and $h_{\tilde{D}}$ such that

$$\pi^*\Omega = |\sigma_E|_{h_E}^{2(n-1)}|\sigma_F|_{h_F}^2 \tilde{\Omega},$$

for some smooth volume form $\tilde{\Omega}$ on $\tilde{X}$.

We fix a Kähler metric $\tilde{\omega}$ on $\tilde{X}$. The Kähler Ricci flow on X is pulled back to $\tilde{X}$ by the map π, and it safeties the equation

$$(4.1)\qquad \frac{\partial}{\partial t}\pi^*\varphi = \log\frac{(\tilde{\chi} + e^{-t}(\pi^*\omega_0 - \tilde{\chi}) + i\partial\bar{\partial}\pi^*\varphi)^n}{|\sigma_E|_{h_E}^{2(n-1)}|\sigma_F|_{h_F}^2 \tilde{\Omega}} - \pi^*\varphi,$$

with the initial $\pi^*\varphi(0) = 0$. By the previous estimates, we see that $\pi^*\varphi$ satisfies the estimates

$$\delta \log|\sigma_{\tilde{D}}|_{h_{\tilde{D}}}^2 - C_\delta \le \pi^*\varphi(t) \le C, \forall t \ge 0 \text{ and } \forall \delta \in (0,1).$$

We will consider a family of perturbed parabolic Monge-Ampere equations for $\epsilon \in (0,1)$

$$(4.2)\qquad \begin{cases} \dfrac{\partial}{\partial t}\tilde{\varphi}_\epsilon = \log\dfrac{(\tilde{\chi} + e^{-t}(\pi^*\omega_0 - \tilde{\chi}) + \epsilon\tilde{\omega} + i\partial\bar{\partial}\pi^*\varphi)^n}{(|\sigma_E|_{h_E}^{2(n-1)} + \epsilon)(|\sigma_F|_{h_F}^2 + \epsilon)\tilde{\Omega}} - \tilde{\varphi}_\epsilon, \\[2ex] \tilde{\varphi}_\epsilon(0) = 0 \end{cases}$$

where $\tilde{\varphi}_\epsilon(t) \in PSH(\tilde{X}, \tilde{\chi} + e^{-t}(\pi^*\omega_0 - \tilde{\chi}) + \epsilon\tilde{\omega})$. The equation (4.2) has long time existence [27], and we will show that solutions to (4.2) converge to that of (4.1) in some sense.

It's easy to check that the Kähler metrics $\tilde{\omega}_\epsilon = \tilde{\chi} + e^{-t}(\pi^*\omega_0 - \tilde{\chi}) + \epsilon\tilde{\omega} + i\partial\bar{\partial}\tilde{\varphi}_\epsilon$ satisfies the following evolution equation

$$(4.3)\qquad \frac{\partial}{\partial t}\tilde{\omega}_\epsilon = -\mathrm{Ric}(\tilde{\omega}_\epsilon) - \tilde{\omega}_\epsilon + \tilde{\chi} + \epsilon\tilde{\omega} - i\partial\bar{\partial}\log\left((|\sigma_E|_{h_E}^{2(n-1)} + \epsilon)(|\sigma_F|_{h_F}^2 + \epsilon)\tilde{\Omega}\right).$$

By direct calculations, for any smooth nonnegative function f, we have

$$\begin{aligned} i\partial\bar{\partial}\log(\epsilon + f) &= \frac{i\partial\bar{\partial}f}{\epsilon + f} - \frac{\partial f \wedge \bar{\partial}f}{(f + \epsilon)^2} \\ &= \frac{f}{f + \epsilon}i\partial\bar{\partial}\log f + \frac{\epsilon}{f(f + \epsilon)^2}\partial f \wedge \bar{\partial}f \\ &\ge \frac{f}{f + \epsilon}i\partial\bar{\partial}\log f, \end{aligned}$$

in the smooth sense on $\tilde{X}\backslash\{f = 0\}$ and globally as currents. So

$$\begin{aligned} &i\partial\bar{\partial}\log\left((|\sigma_E|_{h_E}^{2(n-1)} + \epsilon)(|\sigma_F|_{h_F}^2 + \epsilon)\tilde{\Omega}\right) \\ &\ge -\frac{(n-1)|\sigma_E|_{h_E}^{2(n-1)}}{|\sigma_E|_{h_E}^{2(n-1)} + \epsilon}\mathrm{Ric}(h_E) - \frac{|\sigma_F|_{h_F}^2}{|\sigma_F|_{h_F}^2 + \epsilon}\mathrm{Ric}(h_F) - \mathrm{Ric}(\tilde{\Omega}) \\ &\ge -C\tilde{\omega}, \end{aligned}$$

for some uniform constant C independent of ϵ. Thus away from $\mathrm{supp}\tilde{D} = \mathrm{supp}E \cup \mathrm{supp}F$, we have

$$(4.4)\qquad \frac{\partial}{\partial t}\tilde{\omega}_\epsilon \le -\mathrm{Ric}(\tilde{\omega}_\epsilon) - \tilde{\omega}_\epsilon + C\tilde{\omega}.$$

Lemma 4.1. *Let $\tilde{\varphi}_\epsilon$ be the solution to (4.2), then there exists a constant $C > 0$ such that for any $t \geq 0$, $\epsilon \in (0,1)$, we have*

$$\sup_{\tilde{X}} \tilde{\varphi}_\epsilon(t, \cdot) \leq C, \qquad \sup_{\tilde{X}} \frac{\partial \tilde{\varphi}_\epsilon}{\partial t}(t, \cdot) \leq C.$$

Proof. Let

$$V_\epsilon = \int_{\tilde{X}} (|\sigma_E|_{h_E}^{2(n-1)} + \epsilon)(|\sigma_F|_{h_F}^2 + \epsilon)\tilde{\Omega}$$

be the volume with respect to the volume form $(|\sigma_E|_{h_E}^{2(n-1)} + \epsilon)(|\sigma_F|_{h_F}^2 + \epsilon)\tilde{\Omega}$. We see that

$$V_1 \geq V_\epsilon \geq V_0 = \int_X \Omega,$$

hence V_ϵ is uniformly bounded. We consider (for simplicity we denote $\tilde{\Omega}_\epsilon = (|\sigma_E|_{h_E}^{2(n-1)} + \epsilon)(|\sigma_F|_{h_F}^2 + \epsilon)\tilde{\Omega}$)

$$\frac{\partial}{\partial t}\left(\frac{1}{V_\epsilon}\int_{\tilde{X}} \tilde{\varphi}_\epsilon \tilde{\Omega}_\epsilon\right)$$

$$= \frac{1}{V_\epsilon}\int_{\tilde{X}} \log \frac{(\tilde{\chi} + e^{-t}(\pi^*\omega_0 - \tilde{\chi}) + \epsilon\tilde{\omega} + i\partial\bar{\partial}\tilde{\varphi}_\epsilon)^n}{(|\sigma_E|_{h_E}^{2(n-1)} + \epsilon)(|\sigma_F|_{h_F}^2 + \epsilon)\tilde{\Omega}}\tilde{\Omega}_\epsilon - \frac{1}{V_\epsilon}\int_{\tilde{X}} \tilde{\varphi}_\epsilon \tilde{\Omega}_\epsilon$$

$$\leq \log\left(\int_{\tilde{X}} (\tilde{\chi} + e^{-t}(\pi^*\omega_0 - \tilde{\chi}) + \epsilon\tilde{\omega} + i\partial\bar{\partial}\tilde{\varphi}_\epsilon)^n\right) - \frac{1}{V_\epsilon}\int_{\tilde{X}} \tilde{\varphi}_\epsilon \tilde{\Omega}_\epsilon$$

$$\leq C - \frac{1}{V_\epsilon}\int_{\tilde{X}} \tilde{\varphi}_\epsilon \tilde{\Omega}_\epsilon,$$

where for the first inequality we use Jensen's inequality. From the above we see that

$$\frac{1}{V_\epsilon}\int_{\tilde{X}} \tilde{\varphi}_\epsilon \tilde{\Omega}_\epsilon \leq C.$$

Since $\tilde{\varphi}_\epsilon \in PSH(\tilde{X}, \tilde{\chi} + e^{-t}(\pi^*\omega_0 - \tilde{\chi}) + \epsilon\tilde{\omega})$, the mean value inequality implies the uniform upper bound of $\tilde{\varphi}_\epsilon$

$$\sup_{\tilde{X}} \tilde{\varphi}_\epsilon(t) \leq C.$$

Direct calculations show that (we will denote $\dot{\tilde{\varphi}}_\epsilon = \frac{\partial}{\partial t}\tilde{\varphi}_\epsilon$)

$$\frac{\partial}{\partial t}\dot{\tilde{\varphi}}_\epsilon = \Delta_{\tilde{\omega}_\epsilon}\dot{\tilde{\varphi}}_\epsilon - tr_{\tilde{\omega}_\epsilon}e^{-t}(\pi^*\omega_0 - \tilde{\chi}) - \dot{\tilde{\varphi}}_\epsilon$$

$$= \Delta_{\tilde{\omega}_\epsilon}\dot{\tilde{\varphi}}_\epsilon - n + tr_{\tilde{\omega}_\epsilon}\tilde{\chi} + \epsilon tr_{\tilde{\omega}_\epsilon}\tilde{\omega} + \Delta_{\tilde{\omega}_\epsilon}\tilde{\varphi}_\epsilon - \dot{\tilde{\varphi}}_\epsilon.$$

Hence

$$(4.5) \qquad (\frac{\partial}{\partial t} - \Delta_{\tilde{\omega}_\epsilon})\left((e^t - 1)\dot{\tilde{\varphi}}_\epsilon - \tilde{\varphi}_\epsilon\right) = -tr_{\tilde{\omega}_\epsilon}\pi^*\omega_0 + n - \epsilon tr_{\tilde{\omega}_\epsilon}\tilde{\omega} \leq n,$$

then maximum principle implies

$$(4.6) \qquad \dot{\tilde{\varphi}}_\epsilon(t) \leq \frac{\tilde{\varphi}_\epsilon + nt}{e^t - 1} \leq C, \quad \forall t > 0.$$

$\square$

Lemma 4.2. *For any* $\delta \in (0,1)$, *there is a constant* $C = C_\delta$ *such that for any* $t \geq 0$, *we have*

$$\tilde{\varphi}_\epsilon(t) \geq \delta \log |\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}} - C_\delta.$$

Proof. We will apply the maximum principle. For any small $\delta > 0$ such that

$$\tilde{\chi} - \delta \mathrm{Ric}(h_{\tilde{D}}) > 0,$$

where we may also assume $|\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}} \leq 1$ by rescaling the metric $h_{\tilde{D}}$. We consider the function $H := \tilde{\varphi}_\epsilon - \delta \log |\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}}$. On $\tilde{X} \backslash \tilde{D}$, H satisfies the equation

$$(4.7) \qquad \frac{\partial H}{\partial t} = \log \frac{(\tilde{\chi} + e^{-t}(\pi^*\omega_0 - \tilde{\chi}) + \epsilon\tilde{\omega} - \delta \mathrm{Ric}(h_{\tilde{D}}) + i\partial\bar{\partial}H)^n}{(|\sigma_E|^{2(n-1)h_E} + \epsilon)(|\sigma_F|^2_{h_F} + \epsilon)\tilde{\Omega}} - H - \delta \log |\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}}.$$

By choosing δ even smaller, we may obtain $\omega'_\epsilon := \tilde{\chi} - \delta \mathrm{Ric}(h_{\tilde{D}}) + e^{-t}(\pi^*\omega_0 - \tilde{\chi}) + \epsilon\tilde{\omega}$ is a Kähler metric on $\tilde{X}$ for all $t \geq 0$, and these metrics are uniformly equivalent to $\tilde{\omega}$, i.e., there exists $C_0 > 0$ such that

$$C_0^{-1}\tilde{\omega}_\epsilon \leq \omega'_\epsilon \leq C_0\tilde{\omega}.$$

Consider the Monge-Ampere equations

$$(4.8) \quad (\tilde{\chi} - \delta \mathrm{Ric}(h_{\tilde{D}}) + e^{-t}(\pi^*\omega_0 - \tilde{\chi}) + \epsilon\tilde{\omega} + i\partial\bar{\partial}\psi_\epsilon)^n = e^{\psi_\epsilon}(|\sigma_E|^{2(n-1)}_{h_E} + \epsilon)(|\sigma_F|^2_{h_F} + \epsilon)\tilde{\Omega},$$

where $\psi_\epsilon \in PSH(\tilde{X}, \omega'_\epsilon)$. By the Aubin-Yau theorem, (4.8) admits a unique smooth solution ψ_ϵ for any $\epsilon \in (0,1)$ and $t \geq 0$. It can be seen that

$$\frac{1}{V_\epsilon} \int_{\tilde{X}} e^{\psi_\epsilon}\tilde{\Omega}_\epsilon \leq \frac{1}{V_\epsilon} \int_{\tilde{X}} (\omega'_\epsilon)^n \leq C.$$

Hence mean value inequality implies $\sup_{\tilde{X}} \psi_\epsilon \leq C$. Then by [9], we have $\inf_{\tilde{X}} \psi_\epsilon \geq -C$, hence

$$\|\psi_\epsilon\|_{L^\infty} \leq C,$$

for some C independent of $\epsilon \in (0,1)$ and $t \geq 0$.

Denote $\omega'_\epsilon(t) = \omega'_\epsilon + i\partial\bar{\partial}\psi_\epsilon$. Taking derivative with respective to t on both sides of (4.8), we get

$$(4.9) \qquad\qquad \Delta_{\omega'_\epsilon(t)}\dot{\psi}_\epsilon - tr_{\omega'_\epsilon(t)}e^{-t}(\pi^*\omega_0 - \tilde{\chi}) = \dot{\psi}_\epsilon.$$

$$\Delta_{\omega'_\epsilon(t)}\psi_\epsilon = n - tr_{\omega'_\epsilon(t)}\Big(\tilde{\chi} - \delta \mathrm{Ric}(h_{\tilde{D}}) + e^{-t}(\pi^*\omega_0 - \tilde{\chi}) + \epsilon\tilde{\omega}\Big),$$

Hence

$$\Delta_{\omega'_\epsilon(t)}(\dot{\psi}_\epsilon - \psi_\epsilon) = \dot{\psi}_\epsilon - n + tr_{\omega'_\epsilon(t)}\Big(\tilde{\chi} - \delta \mathrm{Ric}(h_{\tilde{D}}) + 2e^{-t}(\pi^*\omega_0 - \tilde{\chi}) + \epsilon\tilde{\omega}\Big)$$

Noting that $\tilde{\chi} - \delta \mathrm{Ric}(h_{\tilde{D}}) + 2e^{-t}(\pi^*\omega_0 - \tilde{\chi}) + \epsilon\tilde{\omega} > 0$ is δ is chosen appropriately, hence at the maximum point of $\dot{\psi}_\epsilon - \psi_\epsilon$, we have $\dot{\psi}_\epsilon \leq n$, thus

$$\dot{\psi}_\epsilon \leq C + n \leq C.$$

Let $G = H - \psi_\epsilon = \tilde{\varphi}_\epsilon - \delta \log |\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}} - \psi_\epsilon$. On $\tilde{X} \backslash \tilde{D}$ it satisfies the equation

$$(4.10)$$
$$\frac{\partial G}{\partial t} = \log \frac{(\tilde{\chi} - \delta \mathrm{Ric}(h_{\tilde{D}}) + e^{-t}(\pi^*\omega_0 - \tilde{\chi}) + \epsilon\tilde{\omega} + i\partial\bar{\partial}\psi_\epsilon + i\partial\bar{\partial}G)^n}{(\tilde{\chi} - \delta \mathrm{Ric}(h_{\tilde{D}}) + e^{-t}(\pi^*\omega_0 - \tilde{\chi}) + \epsilon\tilde{\omega} + i\partial\bar{\partial}\psi_\epsilon)^n} - G - \delta \log |\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}} - \dot{\psi}_\epsilon.$$

The minimum of G cannot be at $\tilde{D}$ for any $t \geq 0$, since it tends to $+\infty$ when approaching $\tilde{D}$. For any $T > 0$, suppose $(p_0, t_0) \in \tilde{X} \backslash \tilde{D} \times [0, T]$ is the minimum point of G, then we have at this point $\frac{\partial G}{\partial t} \leq 0$ and $i\partial\bar{\partial}G \geq 0$, hence maximum principle implies at this point

$$G \geq -\delta \log |\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}} - \dot{\psi}_\epsilon \geq -C_\delta,$$

combining with L^∞ bound of ψ_ϵ, we have

$$\tilde{\varphi}_\epsilon \geq \delta \log |\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}} - C_\delta.$$

$\square$

Lemma 4.3. *There exist two constants $C > 0, \lambda > 0$ such that for all $t \geq 0$,*

$$(4.11) \qquad \tilde{\omega}_\epsilon \leq C|\sigma_{\tilde{D}}|^{-2\lambda}_{h_{\tilde{D}}}\tilde{\omega}.$$

Proof. By the classical C^2 estimate for Monge-Ampere equations, there is a constant C_1 depending on the lower bound of bisectional curvature of $\tilde{\omega}$, such that

$$(4.12) \qquad \Delta_{\tilde{\omega}_\epsilon} \log tr_{\tilde{\omega}}\tilde{\omega}_\epsilon \geq -C_1 tr_{\tilde{\omega}_\epsilon}\tilde{\omega} - C_1 - \frac{tr_{\tilde{\omega}}\mathrm{Ric}(\tilde{\omega}_\epsilon)}{tr_{\tilde{\omega}}\tilde{\omega}_\epsilon}.$$

By the inequality (4.4), we have on $\tilde{X}\backslash\tilde{D}$

$$(4.13) \qquad \frac{\partial}{\partial t} \log tr_{\tilde{\omega}}\tilde{\omega}_\epsilon = \frac{tr_{\tilde{\omega}}\frac{\partial}{\partial t}\tilde{\omega}_\epsilon}{tr_{\tilde{\omega}}\tilde{\omega}_\epsilon} \leq \frac{-tr_{\tilde{\omega}}\mathrm{Ric}(\tilde{\omega}_\epsilon)}{tr_{\tilde{\omega}}\tilde{\omega}_\epsilon} - 1 + \frac{C}{tr_{\tilde{\omega}}\tilde{\omega}_\epsilon}.$$

So we have on $\tilde{X}\backslash\tilde{D}$,

$$\begin{aligned}
&(\frac{\partial}{\partial t} - \Delta_{\tilde{\omega}_\epsilon})\left(\log tr_{\tilde{\omega}}\tilde{\omega}_\epsilon - A\tilde{\varphi}_\epsilon + A\delta \log |\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}}\right)\\
\leq& \frac{C}{tr_{\tilde{\omega}}\tilde{\omega}_\epsilon} + Ctr_{\tilde{\omega}_\epsilon}\tilde{\omega} - A\log\frac{\tilde{\omega}^n_\epsilon}{\tilde{\Omega}_\epsilon} + A\tilde{\varphi}_\epsilon + C\\
&\quad - Atr_{\tilde{\omega}_\epsilon}(\tilde{\chi} - \delta\mathrm{Ric}(h_{\tilde{D}}) + e^{-t}(\pi^*\omega_0 - \tilde{\chi}) + \epsilon\tilde{\omega})\\
\leq& \frac{C}{tr_{\tilde{\omega}}\tilde{\omega}_\epsilon} - 2tr_{\tilde{\omega}_\epsilon}\tilde{\omega} - A\log\frac{\tilde{\omega}^n_\epsilon}{\tilde{\omega}^n} - A\log\frac{\tilde{\omega}^n}{\tilde{\Omega}_\epsilon} + C\\
\leq& \frac{C}{tr_{\tilde{\omega}}\tilde{\omega}_\epsilon} - tr_{\tilde{\omega}_\epsilon}\tilde{\omega} + C,
\end{aligned}$$

if we choose A sufficiently large and δ suitably small, and in the last inequality we use the facts that

$$\log\frac{\tilde{\omega}^n}{\tilde{\Omega}_\epsilon} \leq C,$$

and

$$-tr_{\tilde{\omega}_\epsilon}\tilde{\omega} - A\log\frac{\tilde{\omega}^n_\epsilon}{\tilde{\omega}^n} \leq -tr_{\tilde{\omega}_\epsilon}\tilde{\omega} + An\log tr_{\tilde{\omega}_\epsilon}\tilde{\omega} \leq C,$$

since the function $x(\in \mathbb{R}^+) \mapsto -x + An\log x$ is bounded above.

Using the inequality

$$tr_{\tilde{\omega}}\tilde{\omega}_\epsilon \leq (tr_{\tilde{\omega}_\epsilon}\tilde{\omega})^{n-1}\frac{\tilde{\omega}^n}{\tilde{\omega}^n_\epsilon} \leq (tr_{\tilde{\omega}_\epsilon}\tilde{\omega})^{n-1}e^{\tilde{\varphi}_\epsilon + \dot{\tilde{\varphi}}_\epsilon}\frac{\tilde{\Omega}_\epsilon}{\tilde{\omega}^n} \leq C(tr_{\tilde{\omega}_\epsilon}\tilde{\omega})^{n-1}.$$

The maximum of $\log tr_{\tilde{\omega}}\tilde{\omega}_\epsilon - A\tilde{\varphi}_\epsilon + A\delta \log |\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}}$ cannot be at $\tilde{D}$, then maximum principle implies that at the maximum of $\log tr_{\tilde{\omega}}\tilde{\omega}_\epsilon - A\tilde{\varphi}_\epsilon + A\delta \log |\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}}$, we have

$$0 \leq \frac{C}{tr_{\tilde{\omega}}\tilde{\omega}_\epsilon} - C(tr_{\tilde{\omega}}\tilde{\omega}_\epsilon)^{\frac{1}{n-1}} + C,$$

that is, at the maximum point,

$$tr_{\tilde{\omega}}\tilde{\omega}_\epsilon \leq C.$$

Thus we get the desired estimate. $\square$

By standard Schauder estimate ([15]) for parabolic equations, we have

Lemma 4.4. *For any integer $\ell \in \mathbb{Z}_+$, on any compact subset $K \subset\subset \tilde{X}\backslash\tilde{D}$, there is a constant $C_{\ell,K}$ such that for any $t \geq 0$*

$$\|\tilde{\varphi}_\epsilon\|_{C^\ell(K)} \leq C_{\ell,K}.$$

From Lemma 4.4, we see that $\tilde{\varphi}_\epsilon(t)$ converge to a smooth function φ_∞ on $\tilde{X}\backslash\tilde{D}$ as $t \to \infty$ and $\epsilon \to 0$, which satisfies the estimates

$$\delta \log |\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}} - C_\delta \leq \varphi_\infty \leq C, \text{ on } \tilde{X}\backslash\tilde{D}, \text{ for any } \delta \in (0,1)$$

$$\tilde{\omega}_\infty := \tilde{\chi} + i\partial\bar{\partial}\varphi_\infty \leq C|\sigma_{\tilde{D}}|^{-2\lambda}_{h_{\tilde{D}}}.$$

Moreover, from (4.6) we see that

$$\frac{\partial}{\partial t}(\tilde{\varphi}_\epsilon + Ce^{-t/2}) \leq 0.$$

Hence on any compact subset $K \subset \tilde{X}\backslash\tilde{D}$, the function $\tilde{\varphi}_\epsilon(t) + Ce^{-t/2}$ decreases to a function $\varphi_{\infty,\epsilon}$ as $t \to \infty$. Hence $\dot{\tilde{\varphi}}_\epsilon(t)|_K$ approaches zero as $t \to \infty$ and $\epsilon \to 0$. Thus the metric $\tilde{\omega}_\infty$ satisfies the equation

$$\tilde{\omega}_\infty^n = (\tilde{\chi} + i\partial\bar{\partial}\varphi_\infty)^n = e^{\varphi_\infty}|\sigma_E|^{2(n-1)}_{h_E}|\sigma_F|^2_{h_F}\tilde{\Omega}, \text{ on } \tilde{X}\backslash\tilde{D}.$$

Let $\epsilon \to 0$, $\tilde{\varphi}_\epsilon(t)$ tends to a function $\tilde{\varphi}_0(t) \in PSH(\tilde{X}, \tilde{\chi} + e^{-t}(\pi^*\omega_0 - \tilde{\chi})) \cap C^\infty(\tilde{X}\backslash\tilde{D})$ in $C^\infty_{loc}(\tilde{X}\backslash\tilde{D} \times [0,\infty))$-topology, which satisfies the degenerate parabolic Monge-Ampere equation

$$(4.14) \qquad \begin{cases} \dfrac{\partial\tilde{\varphi}_0}{\partial t} = \log \dfrac{(\tilde{\chi} + e^{-t}(\pi^*\omega_0 - \tilde{\chi}) + i\partial\bar{\partial}\tilde{\varphi}_0)^n}{|\sigma_E|^{2(n-1)}_{h_E}|\sigma_F|^2_{h_F}\tilde{\Omega}} - \tilde{\varphi}_0, \\ \tilde{\varphi}_0(0) = 0 \end{cases}$$

with the estimates

$$(4.15) \qquad \delta \log |\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}} - C_\delta \leq \tilde{\varphi}_0 \leq C, \text{ on } \tilde{X}\backslash\tilde{D}, \forall \delta \in (0,1),$$

$$(4.16) \qquad \tilde{\omega}_0(t) := \tilde{\chi} + e^{-t}(\pi^*\omega_0 - \tilde{\chi}) + i\partial\bar{\partial}\tilde{\varphi}_0 \leq C|\sigma_{\tilde{D}}|^{-2\lambda}_{h_{\tilde{D}}}, \text{ on } \tilde{X}\backslash\tilde{D}.$$

When the solutions $\tilde{\varphi}_0$ to (4.14) are in $PSH(\tilde{X}, \tilde{\chi} + e^{-t}(\pi^*\omega_0 - \tilde{\chi})) \cap L^\infty(\tilde{X})$, the uniqueness of such solutions has been proved in [20]. In the following, we will adapt their method to prove the uniqueness when solutions satisfy (4.15), instead of global L^∞-bound.

Proposition 4.1. *Let $\varphi' \in PSH(\tilde{X}, \tilde{\chi} + e^{-t}(\pi^*\omega_0 - \tilde{\chi})) \cap C^\infty(\tilde{X}\backslash\tilde{D} \times [0,\infty))$ be a solution to the equation (4.14) with the estimate (4.15), then*

$$\varphi' = \tilde{\varphi}_0.$$

Proof. We consider the following perturbed equation

$$\begin{cases} \dfrac{\partial}{\partial t}\varphi_{\epsilon,\gamma} = \log \dfrac{(\tilde{\chi} + e^{-t}(\pi^*\omega_0 - \tilde{\chi}) + \epsilon\tilde{\omega} + i\partial\bar{\partial}\varphi_{\epsilon,\gamma})^n}{(|\sigma_E|_{h_E}^{2(n-1)} + \gamma)(|\sigma_F|_{h_F}^2 + \gamma)\tilde{\Omega}} - \varphi_{\epsilon,\gamma} \\ \varphi_{\epsilon,\gamma}(0) = 0, \end{cases}$$

for any $\epsilon \in (0,1)$, $\gamma \in (0,1)$. By similar arguments as in Lemmas 4.1, 4.2, 4.3, we can get the following estimates for $\varphi_{\epsilon,\gamma}$,

$$(4.17) \qquad\qquad \sup_{\tilde{X}} \varphi_{\epsilon,\gamma} \le C, \quad \sup_{\tilde{X}} \dot{\varphi}_{\epsilon,\gamma} \le C.$$

For any $\delta \in (0,1)$, there is a constant C_δ such that

$$(4.18) \qquad\qquad \varphi_{\epsilon,\gamma} \ge \delta \log |\sigma_{\tilde{D}}|_{h_{\tilde{D}}}^2 - C_\delta.$$

$$(4.19) \qquad \omega_{\epsilon,\gamma}(t) := \tilde{\chi} + e^{-t}(\pi^*\omega_0 - \tilde{\chi}) + \epsilon\tilde{\omega} + i\partial\bar{\partial}\varphi_{\epsilon,\gamma} \le C|\sigma_{\tilde{D}}|_{h_{\tilde{D}}}^{-2\lambda}\tilde{\omega}.$$

$$(4.20) \qquad \|\varphi_{\epsilon,\gamma}\|_{C^\ell(K)} \le C_{\ell,K}, \text{ for any compact } K \subset\subset \tilde{X}\backslash\tilde{D}.$$

Moreover, by maximum principle, we have the following monotonicity properties

$$\varphi_{\epsilon,\gamma_1} \ge \varphi_{\epsilon,\gamma_2}, \text{ for any } \gamma_1 \le \gamma_2, \forall\epsilon \in (0,1);$$

$$\varphi_{\epsilon_1,\gamma} \le \varphi_{\epsilon_2,\gamma}, \text{ for any } \epsilon_1 \le \epsilon_2, \forall\gamma \in (0,1).$$

We can define a function

$$\varphi_\epsilon := (\lim_{\gamma\to 0}\varphi_{\epsilon,\gamma})^*,$$

where $f^*(z) = \lim_{r\to 0}\sup_{w\in B(z,r)\backslash\{z\}} f(w)$ is the upper regularization of a function. Then φ_ϵ satisfies the equation

$$\begin{cases} \dfrac{\partial}{\partial t}\varphi_\epsilon = \log \dfrac{(\tilde{\chi} + e^{-t}(\pi^*\omega_0 - \tilde{\chi}) + \epsilon\tilde{\omega} + i\partial\bar{\partial}\varphi_\epsilon)^n}{|\sigma_E|_{h_E}^{2(n-1)}|\sigma_F|_{h_F}^2\tilde{\Omega}} - \varphi_\epsilon, \text{ on } \tilde{X}\backslash\tilde{D} \\ \varphi_\epsilon(0) = 0, \end{cases}$$

And we have the monotonicity

$$\varphi_{\epsilon_1} \le \varphi_{\epsilon_2}, \text{ for any } \epsilon_1 \le \epsilon_2, \text{ on } \tilde{X}\backslash\tilde{D}.$$

So we can define $\varphi_0 := \lim_{\epsilon\to 0}\varphi_\epsilon$, which satisfies the equation

$$(4.21) \qquad \begin{cases} \dfrac{\partial}{\partial t}\varphi_0 = \log \dfrac{(\tilde{\chi} + e^{-t}(\pi^*\omega_0 - \tilde{\chi}) + i\partial\bar{\partial}\varphi_0)^n}{|\sigma_E|_{h_E}^{2(n-1)}|\sigma_F|_{h_F}^2\tilde{\Omega}} - \varphi_0, \text{ on } \tilde{X}\backslash\tilde{D} \\ \varphi_0(0) = 0, \end{cases}$$

The estimates (4.17), (4.18), (4.19) and (4.20) implies that

$$\varphi_\epsilon \xrightarrow{C^\infty(K)} \varphi_0, \text{ as } \epsilon \to 0,$$

for any compact $K \subset\subset \tilde{X}\backslash\tilde{D}$, and φ_0 satisfies similar estimates as in (4.17), (4.18), (4.19) and (4.20).

For any φ' as in Proposition 4.1, define a function

$$\psi := \varphi_\epsilon - \varphi' - \varepsilon_0\epsilon \log |\sigma_{\tilde{D}}|_{h_{\tilde{D}}}^2$$

on $\tilde{X}\backslash\tilde{D}$, where ε_0 is a small number such that $\tilde{\omega} - \varepsilon_0\mathrm{Ric}(h_{\tilde{D}}) > 0$. For any ϵ,

$$\psi \geq \delta\log|\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}} - C_\delta - C - \varepsilon_0\epsilon\log|\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}} \to +\infty,$$

as the point approaching $\tilde{D}$, if δ is small enough, say, $\delta \leq \varepsilon_0\epsilon/2$. Hence the minimum of $\psi(\cdot, t)$ can only be at $\tilde{X}\backslash\tilde{D}$. And on $\tilde{X}\backslash\tilde{D}$, ψ satisfies the equation

$$\frac{\partial\psi}{\partial t} = \log\frac{(\tilde{\chi} + e^{-t}(\pi^*\omega_0 - \tilde{\chi}) + i\partial\bar{\partial}\varphi' + \epsilon(\tilde{\omega} - \omega_0\mathrm{Ric}(h_{\tilde{D}})) + i\partial\bar{\partial}\psi)^n}{(\tilde{\chi} + e^{-t}(\pi^*\omega_0 - \tilde{\chi}) + i\partial\bar{\partial}\varphi')^n} - \psi - \varepsilon_0\epsilon\log|\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}}.$$

Maximum principle argument implies that $\psi_{\min} = \inf_{\tilde{X}\backslash\tilde{D}}\psi(\cdot, t) \geq 0$. (Recall we assume $|\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}} \leq 1$.) Hence

$$\varphi_\epsilon \geq \varphi' + \varepsilon_0\epsilon\log|\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}}, \quad \text{on } \tilde{X}\backslash\tilde{D}.$$

On any compact $K \subset\subset \tilde{X}\backslash\tilde{D}$, letting $\epsilon \to 0$, we get

$$\varphi_0 \geq \varphi', \quad \text{on } K,$$

then let $K \to \tilde{X}\backslash\tilde{D}$, we see that

(4.22) $$\varphi_0 \geq \varphi'.$$

To show the uniqueness, we only need to show $\varphi_0 \leq \varphi'$, and this will be done by another perturbed equation, as Song-Tian do in [20].

$$\begin{cases} \dfrac{\partial}{\partial t}\varphi^{(r)}_{\epsilon,\gamma} = \log\dfrac{(\tilde{\chi} + e^{-t}((1-r)\pi^*\omega_0 - \tilde{\chi}) + \epsilon\tilde{\omega} + i\partial\bar{\partial}\varphi_{\epsilon,\gamma})^n}{(|\sigma_E|^{2(n-1)}_{h_E} + \gamma)(|\sigma_F|^2_{h_F} + \gamma)\tilde{\Omega}} - \varphi^{(r)}_{\epsilon,\gamma} \\ \varphi^{(r)}_{\epsilon,\gamma}(0) = 0, \end{cases}$$

It's not hard to see that $\varphi^{(r)}_{\epsilon,\gamma} \to \varphi_{\epsilon,\gamma}$ as $r \to 0$. Denote $\hat{\omega} = \tilde{\chi} + e^{-t}((1-r)\pi^*\omega_0 - \tilde{\chi}) + \epsilon\tilde{\omega} + i\partial\bar{\partial}\varphi_{\epsilon,\gamma}$.

Lemma 4.5. *For some constant $C > 0$, we have*

$$\sup_{\tilde{X}}\varphi^{(r)}_{\epsilon,\gamma} \leq C, \quad C\log|\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}} - C \leq \frac{\partial}{\partial r}\varphi^{(r)}_{\epsilon,\gamma} \leq 0$$

Proof. The upper bound of $\varphi^{(r)}_{\epsilon,\gamma}$ follows similarly as the proof in Lemma 4.1.

$$\frac{\partial}{\partial t}\left(\frac{\partial\varphi^{(r)}_{\epsilon,\gamma}}{\partial r}\right) = \Delta_{\hat{\omega}}\frac{\partial\varphi^{(r)}_{\epsilon,\gamma}}{\partial r} - e^{-t}tr_{\hat{\omega}}\pi^*\omega_0 - \frac{\partial\varphi^{(r)}_{\epsilon,\gamma}}{\partial r} \leq \Delta_{\hat{\omega}}\frac{\partial\varphi^{(r)}_{\epsilon,\gamma}}{\partial r} - \frac{\partial\varphi^{(r)}_{\epsilon,\gamma}}{\partial r}.$$

Maximum principle argument implies $\frac{\partial\varphi^{(r)}_{\epsilon,\gamma}}{\partial r} \leq 0$.

Let $H := \frac{\partial\varphi^{(r)}_{\epsilon,\gamma}}{\partial r} + A\varphi^{(r)}_{\epsilon,\gamma} - A\varepsilon_0\log|\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}}$, where $\varepsilon_0 > 0$ is a small number such that

$$\tilde{\chi} + e^{-t}((1-r)\pi^*\omega_0 - \tilde{\chi}) + \epsilon\tilde{\omega} - \varepsilon_0\mathrm{Ric}(h_{\tilde{D}}) \geq c_0\tilde{\omega},$$

for all $t \geq 0$ and $c_0 > 0$ is a uniform constant.

On $\tilde{X}\backslash\tilde{D}$, if we choose A sufficiently large, we have

$$(\frac{\partial}{\partial t} - \Delta_{\hat{\omega}})H = -e^{-t}tr_{\hat{\omega}}\pi^*\omega_0 - \frac{\partial\varphi^{(r)}_{\epsilon,\gamma}}{\partial r} + A\log\frac{\hat{\omega}^n}{\tilde{\Omega}_\gamma} - A\varphi^{(r)}_{\epsilon,\gamma} - An$$

$$+ Atr_{\hat{\omega}}(\tilde{\chi} + e^{-t}((1-r)\pi^*\omega_0 - \tilde{\chi}) + \epsilon\tilde{\omega} - \varepsilon_0\mathrm{Ric}(h_{\tilde{D}}))$$

$$\geq -H - A\varepsilon_0\log|\sigma_{\tilde{D}}|^2_{h_{\tilde{D}}} - C$$

$$\geq -H - C.$$

Since the minimum of H cannot occur at $\tilde{D}$, maximum principle argument implies that $H \geq -C$, combing with the uniform upper bound of $\varphi_{\epsilon,\gamma}^{(r)}$, we conclude that

$$\frac{\partial}{\partial r}\varphi_{\epsilon,\gamma}^{(r)} \geq C\log|\sigma_{\tilde{D}}|_{h_{\tilde{D}}}^2 - C.$$

$\square$

Let

$$\varphi_\epsilon^{(r)} := (\lim_{\gamma\to 0}\varphi_{\epsilon,\gamma}^{(r)})^*,$$

then it satisfies the equation

$$\begin{cases} \dfrac{\partial}{\partial t}\varphi_\epsilon^{(r)} = \log\dfrac{(\tilde{\chi} + e^{-t}((1-r)\pi^*\omega_0 - \tilde{\chi}) + \epsilon\tilde{\omega} + i\partial\bar{\partial}\varphi_\epsilon^{(r)}}{|\sigma_E|_{h_E}^{2(n-1)}|\sigma_F|_{h_F}^2\tilde{\Omega}} - \varphi_\epsilon^{(r)}, \text{ on } \tilde{X}\backslash\tilde{D} \\ \varphi_\epsilon^{(r)}(0) = 0. \end{cases}$$

We have the monotonicity $\varphi_{\epsilon_1}^{(r)} \leq \varphi_{\epsilon_2}^{(r)}$ for any $\epsilon_1 \leq \epsilon_2$. Define

$$\varphi^{(r)} = \lim_{\epsilon\to 0}\varphi_\epsilon^{(r)}.$$

From Lemma 4.5 it's not hard to see that

$$|\varphi^{(r_1)} - \varphi^{(r_2)}| \leq C(1 - \log|\sigma_{\tilde{D}}|_{h_{\tilde{D}}}^2)|r_1 - r_2|, \text{ on } \tilde{X}\backslash\tilde{D},$$

hence on any compact subset $K \subset\subset \tilde{X}\backslash\tilde{D}$, $\varphi^{(r)} \to \varphi_0$ in the C^∞ sense as $r \to 0$, where φ_0 is the solution constructed in (4.21).

Now we are ready to finish the proof of Proposition 4.1. Define $G := \varphi' - \varphi^{(r)} - e^{-t}r\varepsilon_0\log|\sigma_{\tilde{D}}|_{h_{\tilde{D}}}^2$. By the assumption on φ', for any fixed $t \geq 0$, $r \in (0,1)$

$$G \geq \delta\log|\sigma_{\tilde{D}}|_{h_{\tilde{D}}}^2 - C_\delta - C - e^{-t}r\varepsilon_0\log|\sigma_{\tilde{D}}|_{h_{\tilde{D}}}^2 \to +\infty,$$

as approaching $\tilde{D}$, if δ is smaller than $e^{-t}r\varepsilon_0$, hence the minimum of G cannot be at $\tilde{D}$. On the other hand, on $\tilde{X}\backslash\tilde{D}$, we have

$$\frac{\partial}{\partial t}G = \log\frac{(\tilde{\chi} + e^{-t}((1-r)\pi^*\omega_0 - \tilde{\chi}) + i\partial\bar{\partial}\varphi^{(r)} + re^{-t}(\pi^*\omega_0 - \varepsilon_0\mathrm{Ric}(h_{\tilde{D}})) + i\partial\bar{\partial}G)^n}{(\tilde{\chi} + e^{-t}((1-r)\pi^*\omega_0 - \tilde{\chi}) + i\partial\bar{\partial}\varphi^{(r)})^n} - G$$

$$\geq \log\frac{(\tilde{\chi} + e^{-t}((1-r)\pi^*\omega_0 - \tilde{\chi}) + i\partial\bar{\partial}\varphi^{(r)} + i\partial\bar{\partial}G)^n}{(\tilde{\chi} + e^{-t}((1-r)\pi^*\omega_0 - \tilde{\chi}) + i\partial\bar{\partial}\varphi^{(r)})^n} - G,$$

by maximum principle, we have $G \geq 0$, i.e.,

$$\varphi' \geq \varphi^{(r)} + e^{-t}r\varepsilon_0\log|\sigma_{\tilde{D}}|_{h_{\tilde{D}}}^2.$$

On any compact subset $K \subset\subset \tilde{X}\backslash\tilde{D}$, letting $r \to 0$, we get

$$\varphi' \geq \varphi_0, \quad \text{on } K.$$

Then let $K \to \tilde{X}\backslash\tilde{D}$, we see that $\varphi' \geq \varphi_0$ on $\tilde{X}\backslash\tilde{D}$, combing with (4.22), we show that $\varphi' = \varphi_0$. Hence we finish the proof of uniqueness of solutions.

$\square$

From the uniqueness of solutions to (4.14) and estimates of $\pi^*\varphi$, we see that

$$(4.23) \qquad \tilde{\omega}_\epsilon(t) \xrightarrow{C^\infty_{loc}(\tilde{X}\backslash\tilde{D})} \pi^*\omega(t), \quad \text{as } \epsilon \to 0,$$

where $\omega(t)$ is the solution to the Kähler Ricci flow (1.1) on X.

We will come back to equation (4.2).

Let $O \in B_O \subset Z$ be a small Euclidean ball, $\tilde{B}_O = \pi_2^{-1}(B_O) \subset \tilde{X}$. The divisors $\tilde{D}'$ and $\pi^{-1}(D) - E$ (the proper transform of D) lie in the zero set of of a local holomorphic function w in $\tilde{B}_O$. By Lemma 4.3, we have

Lemma 4.6.

$$\tilde{\omega}_\epsilon(t) \leq \frac{C}{|w|^{2\lambda}}\tilde{\omega}, \qquad \text{on } \partial\tilde{B}_O.$$

Let $\hat{\omega} := \pi_2^*\omega_{Eucl}$, where ω_{Eucl} is the Euclidean metric on B_O, then local calculation shows that (see [22, 17])

$$(4.24) \qquad C_0^{-1}\hat{\omega} \leq \tilde{\omega} \leq \frac{C_0}{|\sigma_E|_{h_E}^2}\hat{\omega}, \quad \text{in } \tilde{B}_O,$$

and

$$\tilde{\chi} - \varepsilon_0 \text{Ric}(h_E) > 0, \quad \text{in } \tilde{B}_O.$$

Proposition 4.2. *There exist a small $\delta \in (0,1)$ and $\lambda > 0$ such that for any $t \geq 0, \epsilon > 0$, we have*

$$(4.25) \qquad \tilde{\omega}_\epsilon(t) \leq \frac{C}{|\sigma_E|_{h_E}^{2(1-\delta)}|w|^{2\lambda}}\tilde{\omega}, \quad \text{in } \tilde{B}_O.$$

Proof. We will do the calculation in $\tilde{B}_O\backslash E \cup \{w = 0\}$. Since $\hat{\omega}$ has flat curvature in $\tilde{B}_O\backslash E \cup \{w = 0\}$, we have

$$\Delta_{\tilde{\omega}_\epsilon(t)} \log tr_{\hat{\omega}}\tilde{\omega}_\epsilon(t) \geq -\frac{tr_{\hat{\omega}}\text{Ric}(\tilde{\omega}_\epsilon(t))}{tr_{\hat{\omega}}\tilde{\omega}_\epsilon(t)},$$

and by (4.4)

$$\frac{\partial}{\partial t} \log tr_{\hat{\omega}}\tilde{\omega}_\epsilon(t) \leq \frac{tr_{\hat{\omega}}(-\text{Ric}(\tilde{\omega}_\epsilon(t)) - \tilde{\omega}_\epsilon(t) + C\tilde{\omega})}{tr_{\hat{\omega}}\tilde{\omega}_\epsilon(t)}.$$

So

$$(\frac{\partial}{\partial t} - \Delta_{\tilde{\omega}_\epsilon(t)}) \log tr_{\hat{\omega}}\tilde{\omega}_\epsilon(t) \leq -1 + C\frac{tr_{\hat{\omega}}\tilde{\omega}}{tr_{\hat{\omega}}\tilde{\omega}_\epsilon(t)} \leq \frac{C}{|\sigma_E|_{h_E}^2 tr_{\hat{\omega}}\tilde{\omega}_\epsilon(t)},$$

where we have used (4.24).

So we have (r is a sufficiently small number)

$$(\frac{\partial}{\partial t} - \Delta_{\tilde{\omega}_\epsilon(t)}) \log(|\sigma_E|_{h_E}^{2(1+r)}|w|^{2\lambda}tr_{\hat{\omega}}\tilde{\omega}_\epsilon(t)) \leq \frac{C}{|\sigma_E|_{h_E}^2 tr_{\hat{\omega}}\tilde{\omega}_\epsilon(t)} + (1+r)tr_{\tilde{\omega}_\epsilon(t)}\text{Ric}(h_E).$$

$$(\frac{\partial}{\partial t} - \Delta_{\tilde{\omega}_\epsilon(t)})\left(\log|\sigma_E|_{h_E}^{2(1+r)}|w|^{2\lambda}tr_{\hat{\omega}}\tilde{\omega}_\epsilon(t) - A\tilde{\varphi}_\epsilon \right)$$

$$\leq C - A\log\frac{\tilde{\omega}_\epsilon(t)^n}{\tilde{\omega}^n} + \frac{C}{|\sigma_E|_{h_E}^2 tr_{\hat{\omega}}\tilde{\omega}_\epsilon(t)} + (1+r)\text{Ric}(h_E)$$

$$- Atr_{\tilde{\omega}_\epsilon(t)}\tilde{\chi} - Ae^{-t}tr_{\tilde{\omega}_\epsilon(t)}(\pi^*\omega_0 - \tilde{\chi}) - A\epsilon tr_{\tilde{\omega}_\epsilon(t)}\tilde{\omega}$$

$$\leq C - tr_{\tilde{\omega}_\epsilon(t)}\tilde{\omega} + \frac{C}{|\sigma_E|^2_{h_E} tr_{\hat{\omega}}\tilde{\omega}_\epsilon(t)},$$

if A is sufficiently large, and in the last inequality we have used the fact that $\tilde{\chi} - \varepsilon \mathrm{Ric}(h_E)$ is a Kähler metric on $\tilde{B}_O$ when ε is small.

On the other hand, similar calculation shows that

$$(\frac{\partial}{\partial t} - \Delta_{\tilde{\omega}_\epsilon(t)})\log tr_{\tilde{\omega}}\tilde{\omega}_\epsilon(t) \leq C_1 tr_{\tilde{\omega}_\epsilon(t)}\tilde{\omega} + C_1 + \frac{C_1}{tr_{\tilde{\omega}}\tilde{\omega}_\epsilon(t)},$$

where C_1 depends on the lower bound of the bisectional curvature of $\tilde{\omega}$.

Define

$$G = \log |\sigma_E|^{2(1+r)}_{h_E} |w|^{2\lambda} tr_{\hat{\omega}}\tilde{\omega}_\epsilon(t) - A\tilde{\varphi}_\epsilon + \frac{1}{2C_1}\log |w|^{2\lambda+2} tr_{\tilde{\omega}}\tilde{\omega}_\epsilon(t),$$

by the calculations above, we have

$$(\frac{\partial}{\partial t} - \Delta_{\tilde{\omega}_\epsilon(t)})G \leq C - \frac{1}{2}tr_{\tilde{\omega}_\epsilon(t)}\tilde{\omega} + \frac{C}{|\sigma_E|^2_{h_E} tr_{\hat{\omega}}\tilde{\omega}_\epsilon(t)} + \frac{1}{tr_{\tilde{\omega}}\tilde{\omega}_\epsilon(t)}$$

$$\leq C_2 - \frac{1}{2}tr_{\tilde{\omega}_\epsilon(t)}\tilde{\omega} + \frac{C}{|\sigma_E|^2_{h_E} tr_{\hat{\omega}}\tilde{\omega}_\epsilon(t)},$$

where in the last inequality we use (4.24).

For any small positive r, $|\sigma_E|^{2(1+r)}_{h_E} |w|^{2\lambda} tr_{\hat{\omega}}\tilde{\omega}_\epsilon(t)$ tends to 0 as approaching E and $\{w = 0\}$, so for any $t \geq 0$, G cannot obtain its maximum at $\tilde{B}_O \cap E \cap \{w = 0\}$. Moreover, we know

$$\tilde{\varphi}_\epsilon \geq \delta \log |w| - C_\delta, \text{ on } \partial\tilde{B}_O,$$

for any small $\delta > 0$. Hence by Lemma 4.6, we have

$$\sup_{\partial\tilde{B}_O} G \leq C.$$

For any $T > 0$, assume $(p_0, t_0) \in \overline{\tilde{B}_O}\backslash E \cup \{w = 0\} \times [0, T]$ is the maximum point of G. If $p_0 \in \partial\tilde{B}_O$, then we are done. Otherwise, we have at this maximum point

$$|\sigma_E|^2_{h_E} tr_{\hat{\omega}}\tilde{\omega}_\epsilon(t)(tr_{\tilde{\omega}_\epsilon(t)}\tilde{\omega} - 2C_2) \leq C.$$

By the inequality

$$tr_{\tilde{\omega}}\tilde{\omega}_\epsilon(t) \leq \frac{\tilde{\omega}_\epsilon(t)^n}{\tilde{\omega}^n}(tr_{\tilde{\omega}_\epsilon(t)}\tilde{\omega})^{n-1} = (tr_{\tilde{\omega}_\epsilon(t)}\tilde{\omega})^{n-1} e^{\tilde{\varphi}_\epsilon + \dot{\tilde{\varphi}}_\epsilon} \frac{\tilde{\Omega}_\epsilon}{\tilde{\omega}^n} \leq C_3(tr_{\tilde{\omega}_\epsilon(t)}\tilde{\omega})^{n-1}.$$

So

$$|\sigma_E|^2_{h_E} tr_{\hat{\omega}}\tilde{\omega}_\epsilon(t)((tr_{\tilde{\omega}}\tilde{\omega}_\epsilon(t))^{1/(n-1)} - C_4) \leq C, \quad \text{at } (p_0, t_0).$$

If $tr_{\tilde{\omega}}\tilde{\omega}_\epsilon(t)(p_0, t_0) \leq 2^{n-1}C_4^{n-1}$, then $|\sigma_E|^2_{h_E} tr_{\hat{\omega}}\tilde{\omega}_\epsilon(t)(p_0, t_0) \leq 2^{n-1}CC_4^{n-1}$. Noting that in $\tilde{B}_O$,

$$\tilde{\varphi}_\epsilon \geq \delta \log |\sigma_E|^2_{h_E} + \delta \log |w|^2 - C_\delta,$$

hence G is bounded above by a uniform constant, if we choose δ small enough in the above inequality.

If $tr_{\tilde{\omega}}\tilde{\omega}_\epsilon(t)(p_0, t_0) \geq 2^{n-1}C_4^{n-1}$, then we have

$$|\sigma_E|^2_{h_E} tr_{\hat{\omega}}\tilde{\omega}_\epsilon(t)(p_0, t_0) \leq C.$$

Then for δ small enough,

$$G(p_0, t_0) \leq r\log |\sigma_E|^2_{h_E} - \delta \log |\sigma_E|^2_{h_E} - \delta \log |w|^2 + (\lambda + 1)\log |w|^2 + C \leq C.$$

In sum, in all cases, we have $\sup_{\tilde{B}_O \times [0,T]} G \leq C$. Then

$$\log \left(|\sigma_E|_{h_E}^{2(1+r)} tr_{\hat{\omega}}\tilde{\omega}_\epsilon(t) |w|^{2\lambda + (2+2\lambda)(2C_1)^{-1}} (tr_{\tilde{\omega}}\tilde{\omega}_\epsilon(t))^{(2C_1)^{-1}} \right) \leq \tilde{\varphi}_\epsilon + C \leq C,$$

noting that $tr_{\hat{\omega}}\tilde{\omega}_\epsilon(t) \geq C_0^{-1} tr_{\tilde{\omega}}\tilde{\omega}_\epsilon(t)$, we have

$$\left(tr_{\tilde{\omega}}\tilde{\omega}_\epsilon(t) \right)^{1+(2C_1)^{-1}} \leq \frac{C}{|\sigma_E|_{h_E}^{2(1+r)} |w|^{2\Lambda}}.$$

If we choose r sufficiently small, say, $r \leq (10C_1)^{-1}$, then $\frac{1+r}{1+(2C_1)^{-1}} = 1 - \delta$ for some $\delta \in (0,1)$, and hence

$$tr_{\tilde{\omega}}\tilde{\omega}_\epsilon(t) \leq \frac{C}{|\sigma_E|_{h_E}^{2(1-\delta)} |w|^{2\Lambda}}, \quad \text{in } \tilde{B}_O \backslash E \cup \{w = 0\}.$$

$\square$

Corollary 4.1. *By letting $\epsilon \to 0$ in (4.25) and the convergence (4.23), we have for any $t \geq 0$*

$$(4.26) \qquad \pi^*\omega(t) \leq \frac{C}{|\sigma_E|_{h_E}^{2(1-\delta)} |w|^{2\lambda}} \tilde{\omega}, \quad \text{in } \tilde{B}_O \backslash (E \cup \{w = 0\}).$$

Letting $t \to \infty$, we have

$$\pi^*\omega_\infty \leq \frac{C}{|\sigma_E|_{h_E}^{2(1-\delta)} |w|^{2\lambda}} \tilde{\omega}, \quad \text{in } \tilde{B}_O \backslash (E \cup \{w = 0\}).$$

Lemma 4.7. *For any $q \in D \subset X$, there exists a smooth curve $\gamma(s) :\in [0,1] \to X$ such that*
  (1) $\gamma([0,1)) \subset X \backslash D$, and $\gamma(1) = q$;
  (2) γ is transversal to D;
  (3) for any $\varepsilon > 0$, there exists an $s_0 > 0$, such that for all $s \in [s_0, 1)$

$$d_{g(t)}(q, \gamma(s)) \leq \varepsilon, \quad \forall t \geq 0.$$

Proof. We take the resolution $\pi_1 : Z \to X$ and choose a point O in a smooth component of $\pi_1^{-1}(D)$ with $\pi_1(O) = q$, and blow up O, $\pi_2 : \tilde{X} \to Z$, and $\pi = \pi_1 \circ \pi_2 : \tilde{X} \to X$. We choose an appropriate smooth path $\tilde{\gamma}([0,1)) \subset \tilde{B}_O \backslash E \cup \{w = 0\}$ which keeps away from $\{w = 0\}$ and $\tilde{\gamma}(1) \subset E$. Then $\gamma = \pi(\tilde{\gamma})$ is the desired path, and last item follows from the uniform estimate (4.26). $\square$

Corollary 4.2. *For a fixed $p \in X \backslash D$, any $q \in D$, there exists a constant $C = C_q$ such that for any $t \geq 0$*

$$d_{g(t)}(p, q) \leq C_q.$$

Hence along the convergent sequence $(X, g(t_i), p) \xrightarrow{d_{GH}} (X_\infty, d_\infty, p_\infty)$, $q \in (X, g(t_i))$ converges (up to a subsequence) to some $q_\infty \in X_\infty$ in the Gromov-Hausdorff sense.

Since we aim to give a purely analytic proof of our main results, without using of Kawamata's base point free theorem, we need the local freeness of some power of the canonical line bundle K_X as proved in [18], for completeness we give a sketched proof.

Proposition 4.3. [18] *For any $q \in D$, there exists $\sigma \in H^0(X, mK_X)$ for some $m \in \mathbb{Z}_+$ such that*

$$\sigma(q) \neq 0.$$

Proof. By Corollary 4.2, we can take q_∞ as the limit point of q. By [18], there exists a $\sigma \in H^0(X, mK_X)$ such that

$$|\sigma|^2_{h^m_{KE}}(q_\infty) > 1,$$

where $|\sigma|^2_{h^m_{KE}}$ is a Lipschitz continuous function on X_∞.

By Lemma 4.7, there exists a sequence of points $\{q_k\} \subset \mathcal{R}_X$ which transversely tend to q and

$$d_{g(t_i)}(q, q_k) \leq k^{-1}, \quad \forall i.$$

We may assume $q_k \in \mathcal{R}_X$ converge to the same point $q_k \in \mathcal{R} = \mathcal{R}_X$. Hence

$$d_\infty(q_\infty, q_k) = \lim_{i \to \infty} d_{g(t_i)}(q, q_k) \leq k^{-1}.$$

By the continuity of $|\sigma|^2_{h^m_{KE}}$, for k large enough,

$$e^{-m\varphi_{KE}}|\sigma|^2_{h^m_\chi}(q_k) = |\sigma|^2_{h^m_{KE}}(q_k) \geq \frac{1}{2},$$

i.e.

$$\varphi_{KE}(q_k) \leq C + C\log|\sigma|^2_{h^m_\chi}(q_k).$$

On the other hand, for any $\delta \in (0, 1)$, we have

$$\delta \log|\sigma_D|^2_{h_D}(q_k) - C_\delta \leq \varphi_{KE}(q_k),$$

hence

$$|\sigma_D|^{2\delta}_{h_D}(q_k) \leq C|\sigma|^2_{h^m_\chi}(q_k).$$

Since q_k approaches D transversely, and δ is any arbitrarily small number, we see that σ cannot vanish at q. Thus complete the proof. $\qquad\square$

By a compactness argument and the previous proposition, we have:

Proposition 4.4. *There exists an integer $m \in \mathbb{Z}_+$ such that for any $q \in X$, there exists a holomorphic section $\sigma \in H^0(X, mK_X)$ such that $\sigma(q) \neq 0$, i.e., mK_X is base point free. Thus a basis $\{\sigma_0, \ldots, \sigma_{N_m}\}$ of $H^0(X, mK_X)$ gives a morphism*

$$\Phi_m : X \to X_{can} \subset \mathbb{CP}^{N_m},$$

where X_{can} is the image of X under Φ_m.

Remark 4.1. *Proposition 4.4 is well-known from Kawamata's base point free theorem. It follows from algebraic geometry [13] that when mK_X is base point free the maps Φ_m stabilize when m is sufficiently large, i.e., Φ_m is independent of m when m is large enough and we will denote this map by Φ.*

For the given basis $\{\sigma_0, \ldots, \sigma_N\}$ of $H^0(X, mK_X)$, we have

$$\sum_{i=0}^N |\sigma_i|^2_{h^m_t} = \sum_{i=0}^N |\sigma_i|^2_{h^m_\chi} e^{-\varphi - \dot\varphi} \geq c_0 \sum_{i=0}^N |\sigma_i|_{h^m_\chi} \geq c_1 > 0.$$

Moreover, by Proposition 3.1, we know

$$\sup_X \sum_{i=1}^N |\sigma_i|^2_{h^m_t} \leq C, \quad \sup_X |\nabla_{\omega(t)}\sigma_i|^2_{h^m_t} \leq C, \forall i, \ \forall t \geq 0,$$

thus the map

$$\Phi_i : (X, \omega(t_i)) \to (X_{can}, \omega_{FS}), \quad x \mapsto [\sigma_0(x) : \ldots : \sigma_N(x)] \in \mathbb{CP}^N$$

has uniformly bounded derivatives (see e.g. [8]). Since the target space (X_{can}, ω_{FS}) is compact, by Arzela-Ascoli theorem, the map extends to the Gromov-Hausdorff limit space

$$\Phi_\infty : (X_\infty, d_\infty) \to (X_{can}, \omega_{FS}),$$

which is Lipschitz continuous.

Under our assumption that the Ricci curvature is uniformly bounded below, Tian-Wang's theory ([24]) on the structure of limit of almost Kähler Einstein manifolds implies that the singular set is closed and of Hausdorff codimension at least 4, which also implies that any tangent cone in the limit space is good (see [8]), in the sense that there exists a tangent cone $C(Y)$, such that for any $\eta > 0$ there exists a cut-off function β which is 1 on a small neighborhood of the singular set $S_Y \subset Y$, and vanishes outside the η-neighborhood of S_Y, and $\|\nabla \beta\|_{L^2(Y)} \leq \eta$, then following Donaldson-Sun's idea ([8]) on partial C^0 estimates (see also [23]), by similar arguments as in [17], for any two distinct points $p, q \in X_\infty$, one can construct two holomorphic sections $\sigma_1, \sigma_2 \in H^0(X_\infty, mK_{X_\infty})$ which separate p, q, hence we have

Proposition 4.5. [17] Φ_∞ *is injective.*

5. Proof of Theorems

Proof of Theorem 1.1. To prove Theorem 1.1, we will argue by contradiction. Following the ideas in [18], we need the following lemma:

Lemma 5.1. *Suppose* $\operatorname{diam}(X, g(t_i)) \to \infty$, *then we have* $\operatorname{diam}(X_\infty, d_\infty) = \infty$,
  (1) $\Phi_\infty : (X_\infty, d_\infty) \to (X_{can}, \omega_{FS})$ *is not surjective;*
  (2) *For* $p \in X_{can} \backslash \Phi_\infty(X_\infty)$ *and any sequence of points* $q_j \in X_\infty$ *with* $d_{\omega_{FS}}(\Phi_\infty(q_j), p) \to 0$ *as* $j \to \infty$, *we have*

$$d_\infty(p_\infty, q_j) \to \infty.$$

Proof. (1) Suppose Φ_∞ is surjective. Since $\operatorname{diam}(X_\infty, d_\infty) = \infty$, there exists a sequence of points $q_j \in \mathcal{R} \subset X_\infty$ with $d_\infty(p_\infty, q_j) \to \infty$. (X_{can}, ω_{FS}) is a compact metric space, hence there exists a convergent subsequence of $\{\Phi_\infty(q_j)\}$ which converge to some $q'_\infty \in X_{can}$ with respect to the metric ω_{FS}. Then there is a point $q_\infty \in X_\infty$ such that $\Phi_\infty(q_\infty) = q'_\infty$. We claim that the ball $B_{d_\infty}(q_\infty, 1)$ contains all but finitely many q_j's. Assuming this claim, we get that the distance of q_j and p_∞ is bounded by $d_\infty(p_\infty, q_\infty) + 1$ contradicting the choice of q_j which converge to ∞ under d_∞ as $j \to \infty$. To see the claim, suppose not, there exists a subsequence $q_{j_l} \subset \{q_j\}$ such that $d_\infty(x_{j_l}, q_{j_l}) \geq 1$, where x_j is a sequence of points contained in $\mathcal{R}$ which converge to q_∞ under the metric d_∞. Since $\Phi_\infty(q_{j_l})$ and $\Phi_\infty(x_{j_l})$ are both in X_{can}^{reg} which is connected and these points both converge to q'_∞, so we can choose a curve $\gamma_{j_l} \subset X_{can}^{reg}$ whose length under ω_{FS} tend to 0 as $j_l \to \infty$. Then $\Phi_\infty^{-1}(\gamma_{j_l}) \subset \mathcal{R} \subset X_\infty$ is a connected curve connecting x_{j_l} and q_{j_l} which has d_∞-length greater than 1, so we can take a point $y_{j_l} \in \Phi_\infty^{-1}(\gamma_{j_l})$ such that $1/2 \leq d_\infty(x_{j_l}, y_{j_l}) \leq 1$. Then by compactness we may assume that up to a subsequence y_{j_l} converge to a point $y_\infty \in X_\infty$ which satisfies $1/2 \leq d_\infty(q_\infty, y_\infty) \leq 1$. It's not hard to see by triangle inequality that $d_{\omega_{FS}}(\Phi_\infty(y_{j_l}), q'_\infty) \to 0$, hence $d_{\omega_{FS}}(\Phi_\infty(y_\infty), q'_\infty) = 0$ and $\Phi_\infty(y_\infty) = q'_\infty$, and this contradicts the property that Φ_∞ is injective. Hence we prove the claim.

(2) Suppose $d_\infty(p_\infty, q_j) \leq A$ for some constant $A > 0$. By compactness we can assume a subsequence of q_j converges to a point q_∞, with $d_\infty(p_\infty, q_\infty) \leq A$.

Then we have $d_{\omega_{FS}}(p, \Phi_\infty(q_j)) \to d_{\omega_{FS}}(p, \Phi_\infty(q_\infty)) = 0$ as $j \to \infty$, thus $p = \Phi_\infty(q_\infty)$ and this contradicts the choice of p. $\qquad\square$

Since Φ_∞ is not surjective, there exists a $q' \in X_{can} \backslash \Phi_\infty(X_\infty)$. Consider a point $q \in D \subset X$ with $\Phi(q) = q'$, for a sequence of points $\{q_j\} \subset \mathcal{R}_X = \mathcal{R}$ in the path constructed in Lemma 4.7 with $q_j \to q$, i.e. $d_{\omega_{FS}}(q', \Phi_\infty(q_j)) \to 0$, we have

$$\sup_j d_\infty(p_\infty, q_j) < \infty.$$

This contradicts item (2) in Lemma 5.1. Hence the diameter of $(X, g(t_i))$ is uniformly bounded. And we finish the proof of Theorem 1.1 $\qquad\square$

Proof of Corollary 1.1. We will show $\Phi_\infty : X_\infty \to X_{can}$ is surjective. Suppose not, there is $p \notin \Phi_\infty(X_\infty)$. Since $\Phi_\infty(\mathcal{R})$ is dense in X_{can}, there exists a sequence of points $q_j \in \mathcal{R}$ such that $d_{\omega_{FS}}(p, \Phi_\infty(q_j)) \to 0$ as $j \to \infty$. We have shown $\text{diam}(X_\infty, d_\infty)$ is bounded, hence q_j would converge to some point $q_\infty \in X_\infty$ under d_∞. Hence

$$d_{\omega_{FS}}(p, \Phi_\infty(q_\infty)) = \lim_{j\to\infty} d_{\omega_{FS}}(p, \Phi_\infty(q_j)) = 0,$$

and we conclude that $p = \Phi_\infty(q_\infty)$, and thus a contradiction. Hence Φ_∞ is surjective. Combining with Song's result that Φ_∞ is also injective, we see that Φ_∞ is a Lipschitz continuous homeomorphism of (X_∞, d_∞) and X_{can}, since (X_∞, d_∞) is a compact space. Moreover, $\Phi_\infty|_{\mathcal{R}} : (\mathcal{R}, d_\infty) \to (\mathcal{R}_X, g_\infty)$ is an isometry so Φ_∞ induces an isometry between (X_∞, d_∞) and $\overline{(\mathcal{R}_X, g_\infty)} = (X_{can}, g_\infty)$. Hence the Gromov-Hausdroff limit of the Kähler Ricci flow (1.1) is the canonical model of X, with the limit metric of the flow, under the assumption of bounded Ricci curvature along the flow. $\qquad\square$

Proof of Theorem 1.2. Suppose the flow (1.1) is of Type III, i.e. $|Rm|(g(t))$ is uniformly bounded, by Shi's derivative estimates all derivatives of Rm are bounded. Fix a point $p \in X \backslash D$, for any sequence $t_i \to \infty$, by the smooth convergence of $\omega(t_i)$ on $X \backslash D$ (Lemma 3.1), the volumes of unit balls $B_{g(t_i)}(p, 1) \subset (X, g(t_i), p)$ are bounded below by a uniform positive constant, the limit space $(X_\infty, d_\infty, p_\infty)$ is smooth. Hence $X_\infty = \mathcal{R}$ and $\mathcal{S} = \mathcal{S}_X = \emptyset$. Then $\mathcal{R}_X = X$, otherwise, if there exists $q \in X \backslash \mathcal{R}_X$, then by Corollary 4.2 we have $d_{g(t_i)}(p, q) \leq C_q$ for any t_i and a uniform constant C_q depending only on q, hence q must converge to some point $q_\infty \in X_\infty$ along the Gromov-Hausdorff convergence, by the definition of $\mathcal{S}_X$, $q \in \mathcal{S}_X \neq \emptyset$, thus a contradiction. So we have X is a compact Kähler manifold admitting a smooth Kähler Einstein metric ω_{KE} with $\text{Ric}(\omega_{KE}) = -\omega_{KE}$, hence K_X is ample. $\qquad\square$

APPENDIX

In this appendix, we will show that along the Kähler-Ricci flow (1.1), assume Ricci curvature is uniformly bounded below for all $t \geq 0$, then for any sequence $t_i \to \infty$, $(X, \omega(t_i), p)$ is a sequence of almost Kähler-Einstein manifolds in the sense of Tian-Wang ([24]), where $p \in X \backslash D$ is a fixed point. Recall a sequence of Kähler manifolds (X_i, ω_i, p_i) is called almost Kähler Einstein if the following conditions are satisfied.

(1) $\text{Ric}(\omega_i) \geq -\omega_i$
(2) $Vol_{\omega_i}(B(p_i, r_0)) \geq v_0 > 0$, for two fixed constants $r_0 > 0$ and v_0.

(3) The flow $\frac{\partial}{\partial t}\omega = -\mathrm{Ric}(\omega) + \lambda_i\omega$ has a solution $\omega(t)$ with $\omega(0) = \omega_i$ on $X_i \times [0, 1]$, where $\lambda_i \in [-1, 1]$ is a constant. Moreover, $\int_0^1 \int_{X_i} |R(\omega(t)) - n\lambda_i|\omega(t)^n dt \to 0$ as $i \to \infty$.

We may assume $\mathrm{Ric}(\omega(t)) \geq -K$ for a constant $K > 0$ (we may assume $K \geq 1$) and any $t \geq 0$. Let $\tilde{\omega}_i = K\omega(t_i)$, then $\mathrm{Ric}(\tilde{\omega}_i) \geq -1$. Since $(X, \omega(t_i))$ is non-collapsed at the point $p \in X \backslash D$ due to the smooth convergence, we have $(X, \tilde{\omega}_i)$ is also non-collapsed at p, i.e., there exists $v_0 > 0$ such that $Vol_{\tilde{\omega}_i}(B_{\tilde{\omega}_i}(p, r_0)) \geq v_0$ for some small $r_0 > 0$.

$\tilde{\omega}_i(t) := K\omega(t_i + K^{-1}t)$ with $t \in [0, 1]$ satisfies the (normalized) Kähler Ricci flow equation

$$\frac{\partial}{\partial t}\tilde{\omega}_i(t) = -\mathrm{Ric}(\tilde{\omega}_i(t)) - K^{-1}\tilde{\omega}_i(t),$$

with the initial $\tilde{\omega}_i(0) = \tilde{\omega}_i$. From the evolution equation for the scalar curvature $R(\omega(t))$ ($\omega(t)$ is the solution to (1.1))

$$\frac{\partial}{\partial t}R = \Delta_{\omega(t)}R + |\mathrm{Ric}|^2 + R,$$

by maximum principle, at the minimum point of $R(\omega(t))$ for each t, $R_{\min} = \min_X R(\omega(t))$, we have

$$\frac{d}{dt}R_{\min}(t) \geq |\mathrm{Ric}|^2 + R_{\min}(t) \geq \frac{R_{\min}(t)^2}{n} + R_{\min}(t).$$

Standard comparison theorem of ODE implies that

$$R_{\min}(t) \geq -n - \frac{R_{\min}(0)n + n^2}{R_{\min}(0)e^t - R_{\min}(0) - n} \geq -n - O(e^{-t}).$$

Hence for $t \in [0, 1]$, we have

$$R(\tilde{\omega}_i(t)) = K^{-1}R(\omega(t_i + K^{-1}t)) \geq -K^{-1}n - O(e^{-t_i}).$$

Then

$$\int_0^1 \int_X |R(\tilde{\omega}_i(t)) + K^{-1}n|\tilde{\omega}_i(t)^n dt \leq \int_0^1 \int_X \left((R(\tilde{\omega}_i(t)) + K^{-1}n) + O(e^{-t_i})\right)\tilde{\omega}_i(t)^n dt$$

$$= \int_0^1 \int_X n(\mathrm{Ric}(\tilde{\omega}_i(t)) + K^{-1}\tilde{\omega}_i(t)) \wedge \tilde{\omega}_i(t)^{n-1} dt + O(e^{-t_i})$$

$$= \int_0^1 \int_X n(e^{-t_i - K^{-1}t}(\omega_0 - \chi) - i\partial\bar{\partial}\dot{\varphi}) \wedge \tilde{\omega}_i(t)^{n-1} dt + O(e^{-t_i})$$

$$= \int_0^1 \int_X ne^{-t_i - K^{-1}t}(\omega_0 - \chi) \wedge \tilde{\omega}_i(t)^{n-1} dt + O(e^{-t_i})$$

$$\leq O(e^{-t_i}) \to 0, \quad \text{as } t_i \to \infty.$$

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