LINK INVARIANT AND G_2 WEB SPACE

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Abstract. In this paper, we reconstruct Kuperberg's G_2 web space [5, 6]. We introduce a new web (a trivalent diagram) and new relations between Kuperberg's web diagrams and the new diagram. Using the G_2 webs, we define crossing formulas corresponding to R-matrices associated to some G_2 irreducible representations and calculate G_2 quantum link invariant for some torus links.

1. Introduction

Suppose that $U_q(G_2)$ is the quantum group of type G_2, where $q \in \mathbb{C}$ is neither zero nor a root of unity [1, 3]. An invariant theory of tensor representations of $U_q(G_2)$ fundamental representations is studied in a skein theoretic approach by Kuperberg [6] and in a representation theoretic approach by Lehrer–Zhang [7] (The invariant theory for exceptional Lie group G_2 is studied by Schwarz, Huang–Zhu [2, 11]). As an application of the study, we obtain Reshetikhin–Turaev's quantum link invariant (R-matrix) associated to $U_q(G_2)$ [9] (The G_2 quantum link invariant is also obtained in a planar algebra approach by Morrison–Peters–Snyder [8]).

In the Kuperberg's approach, we introduce diagrams in Figure 1, called G_2 web which is a diagrammatization of intertwiners between tensor representations of $U_q(G_2)$ fundamental representations [5, 6].

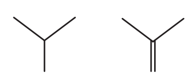

Figure 1. Kuperberg's web diagram

The diagrams correspond ot intertwiners in $\mathrm{Hom}_{U_q(G_2)}(V_{\varpi_1}, V_{\varpi_1} \otimes V_{\varpi_1})$ and $\mathrm{Hom}_{U_q(G_2)}(V_{\varpi_2}, V_{\varpi_1} \otimes V_{\varpi_1})$, where V_{ϖ_1} is the first fundamental representation and V_{ϖ_2} is the second fundamental representation.

The purpose of this work is to reconstruct Kuperberg's web diagram. In Section 2 we introduce a new web diagram in Figure 2 which corresponds to an intertwiner in $\mathrm{Hom}_{U_q(G_2)}(V_{\varpi_2}, V_{\varpi_2} \otimes V_{\varpi_2})$ and show relations between Kuperberg's web diagrams and the new web diagram.

In Section 3 we define G_2 web space W_{G_2} which is a vector space composed of the above web diagrams and show the G_2 web space is isomorphic

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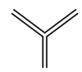
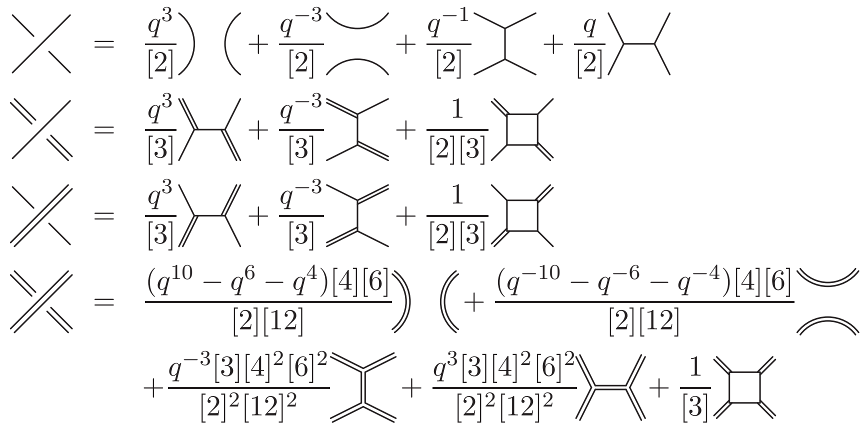


FIGURE 2. New web

to a hom space between tensor representations of $U_q(G_2)$ fundamental representations.

In Sections 4 we have crossing formulas as an expression by G_2 web diagrams[1] which is related to R-matrices associated to $U_q(G_2)$ fundamental representations.

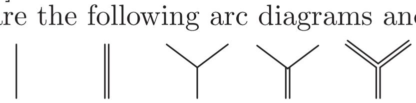


The above crossing formulas induce a braid group action on G_2 web space W_{G_2}. Moreover, we show a relation between G_2 web diagrams and projectors between hom space between tensor representations and we also have crossing formulas as an expression by the projectors in Section 5. The expression is useful for calculating G_2 quantum invariant for typical links. In Section 6, we calculate G_2 quantum invariant of torus links $T(2, n)$.

2. G_2 WEB

First, we introduce G_2 webs for defining G_2 web space.

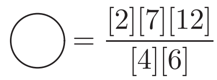
Definition 2.1 (G_2 web)**.** Let $q \in \mathbb{C}$ be neither zero nor a root of unity. Denote by $[n]$ for $n \in \mathbb{Z}_{\geq 0}$ the q-integer $\frac{q^n - q^{-n}}{q - q^{-1}}$ and put $[n]! := [n][n-1] \cdots [1]$ and $\left[{m \atop n} \right] := \dfrac{[m]!}{[n]![m-n]!}$ for $0 \leq n \leq m$.
Elementary G_2 webs are the following arc diagrams and trivalent diagrams



A G_2 *web* is a planar diagram obtained by operations, which are gluing mutual boundaries of two single edges or two double edges of some elementary

[1]Remark that the first three crossing formulas are the same as Kuperberg's formulas [6] but his crossing formula of double edges contains an error.

G_2 webs and taking union of diagrams, with the following relations:
(Loop relation)



$$= \frac{[2][7][12]}{[4][6]}$$

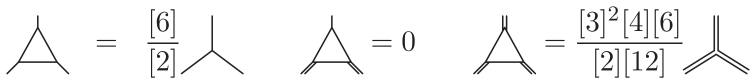
(Monogon relations)

$$= 0 \qquad = 0$$

(Digon relations)

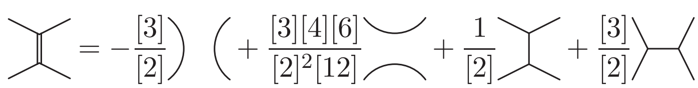
$$= -\frac{[3][8]}{[4]} \qquad = -[2][3]$$

(Triangle relations)



$$= \frac{[6]}{[2]} \qquad = 0 \qquad = \frac{[3]^2[4][6]}{[2][12]}$$

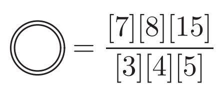
(Double edge elimination)



$$= -\frac{[3]}{[2]} \Big) \Big(+ \frac{[3][4][6]}{[2]^2[12]} + \frac{1}{[2]} + \frac{[3]}{[2]}$$

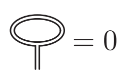
Using the above relations, we obtain the following additional relations.

Propositon 2.2. (Loop relation)



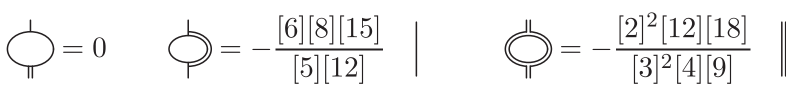
$$= \frac{[7][8][15]}{[3][4][5]}$$

(Monogon relation)



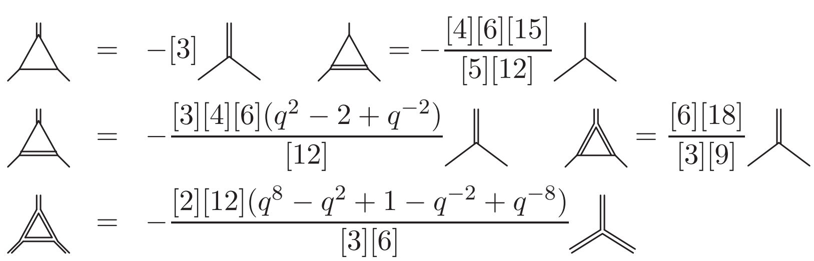
$$= 0$$

(Digon relations)



$$= 0 \qquad = -\frac{[6][8][15]}{[5][12]} \qquad = -\frac{[2]^2[12][18]}{[3]^2[4][9]}$$

(Triangle relations)

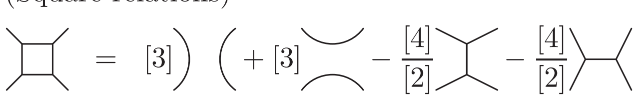


$$= -[3] \qquad = -\frac{[4][6][15]}{[5][12]}$$

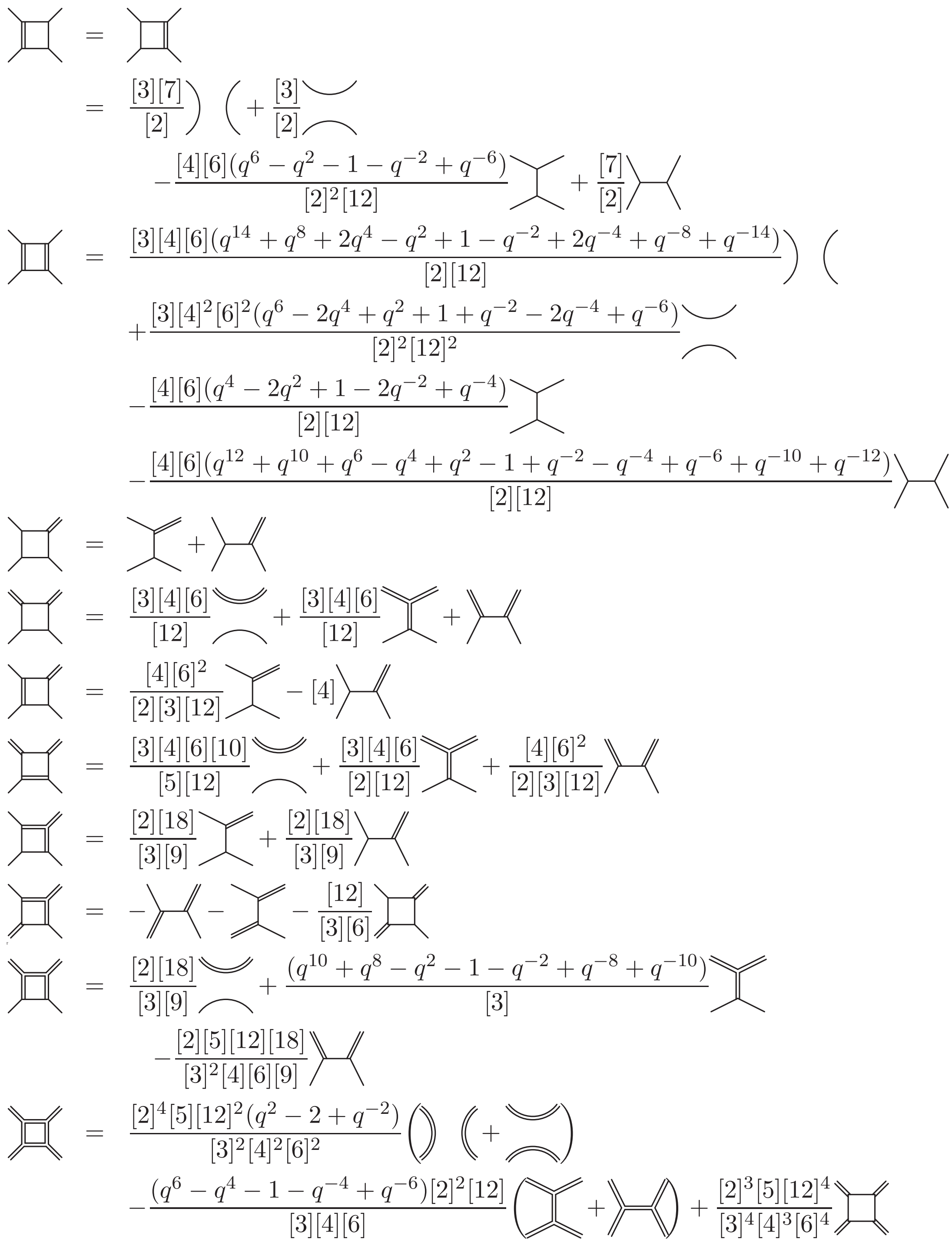
$$= -\frac{[3][4][6](q^2 - 2 + q^{-2})}{[12]} \qquad = \frac{[6][18]}{[3][9]}$$

$$= -\frac{[2][12](q^8 - q^2 + 1 - q^{-2} + q^{-8})}{[3][6]}$$

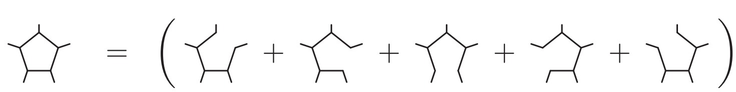
(Square relations)

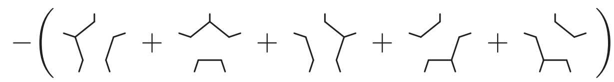


$$= [3] \Big) \Big(+ [3] - \frac{[4]}{[2]} - \frac{[4]}{[2]}$$

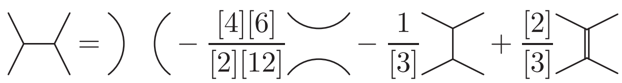


(Pentagon relation)

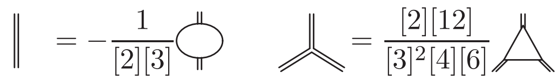




Sketch of proof. Applying the relation (Double edge elimination) or its rearrangement

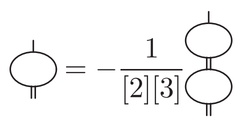


to a diagram, we obtain relations of this proposition by relations of Definition 2.1. If we can not apply the elimination or the rearrangement to a diagram, we first create single edges in the diagram by relations

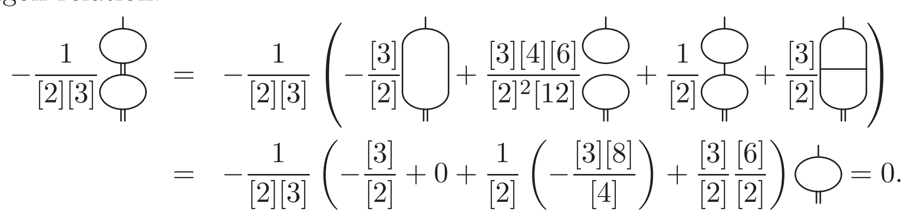


and we apply the elimination or its rearrangement.

For example, to the first digon relation of this proposition, we first create single edges in the diagram:



Applying (Double edge elimination) to the obtained diagram and using monogon, digon and triangle relations of Definition 2.1, we obtain the first digon relation:



□

3. Web space W_{G_2} and invariant space of representation

In this section, we define G_2 web space W_{G_2} which is a $\mathbb{C}$ vector space spanned by G_2 webs embedded on a unit disk.

Let D be a closed unit disk in $\mathbb{R}^2$ with a fixed base point $*$ on the boundary ∂D and P be a G_2 web. A G_2 *web diagram* is the image of an embedding on D of a G_2 web P such that boundaries of P on $\partial D \backslash \{*\}$.

For a given G_2 web diagram W, put the number 1 at intersection of single edges of W and ∂D and put the number 2 at intersection on double edges of W and ∂D. A *coloring* of W is defined by a sequence obtained by reading numbers 1 and 2 on ∂D clockwise from the base point $*$. If W has no boundary point, a coloring of W is defined by the empty $\emptyset$. Denote by $s(W)$ the coloring of W. Two G_2 web diagram W_1 and W_2 are *isotopic* if there exist a base point-preserving isotopy of D which moves W_1 to W_2.

Example of G_2 web diagrams in Figure 3, we find colorings $s(W_1) =$

$(1,1,1,1)$, $s(W_1) = (2,1,1)$, $s(W_1) = (1,1,2,1)$, $s(W_1) = (1,2,2,1,1)$.

FIGURE 3. G_2 web diagrams

Hereafter fix a base point as G_2 web diagrams in Figure 3 and omit the boundary of diagrams.

Write

$$S := \{s = (s_1, s_2, ..., s_n) \mid n \geq 1, s_i \in \{1,2\}\, (i = 1,2,...,n)\} \cup \{\emptyset\}.$$

We define G_2 web space $\mathrm{W}_{G_2}(s)$ for $s \in S$ by a $\mathbb{C}$-linear space spanned by isotopy classes of G_2 web diagrams with the coloring s.

Remark 3.1. The collection of web spaces $\{\mathrm{W}_{G_2}(s)\}_{s\in S}$ has the spader structure in the sense of Kuperberg [6, Section 3]:
(Join)

$$\mu_{s,t} : \mathrm{W}_{G_2}(s) \times \mathrm{W}_{G_2}(t) \to \mathrm{W}_{G_2}(st)$$

(Rotation)

$$\rho_{s,t} : \mathrm{W}_{G_2}(st) \to \mathrm{W}_{G_2}(ts)$$

(Stitch)

$$\sigma_{sst} : \mathrm{W}_{G_2}(sst) \to \mathrm{W}_{G_2}(t).$$

For $s = (s_1, s_2, ..., s_n) \in S$, let V_s be a tensor representation of G_2 quantum group $V_{\varpi_{s_1}} \otimes V_{\varpi_{s_2}} \otimes \cdots V_{\varpi_{s_n}}$, where $V_{\varpi_{s_i}}$ is the s_i-th fundamental representation $(i = 1, ..., n)$.

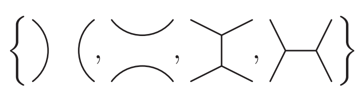
Following is a theorem due to [6, Theorem 6.10]

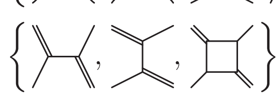
Theorem 3.2 ([6])**.** The vector space $\mathrm{W}_{G_2}(s)$ and $\mathrm{Inv}(V_s)$ have the same dimension.

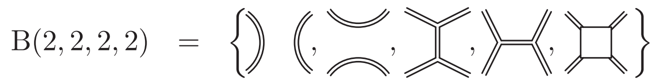
Proof. Replacing numbers 2 in the coloring s into $[1,1]$, we obtain a clasp sequence C (See [6]). Since the web space $\mathrm{W}_{G_2}(s)$ and the clasp web space $\mathrm{W}_{G_2}(C)$ is the same dimension, we have the theorem. □

We denote by $\mathrm{B}(s)$ a basis of the vector space $\mathrm{W}_{G_2}(s)$, called G_2 *web basis*.

Example 3.3. For $s = (1,1,1,1)$, $(1,2,1,2)$ and $(2,2,2,2)$, we have a G_2 web basis $\mathrm{B}(s)$.

$\mathrm{B}(1,1,1,1) =$ 

$\mathrm{B}(1,2,1,2) =$ 



4. Braid action on G_2 web space W_{G_2}

Let $\#(s)$ be a length of a sequence $s \in S$ and define

$$S[n] := \{s \in S | \#(s) = n\}.$$

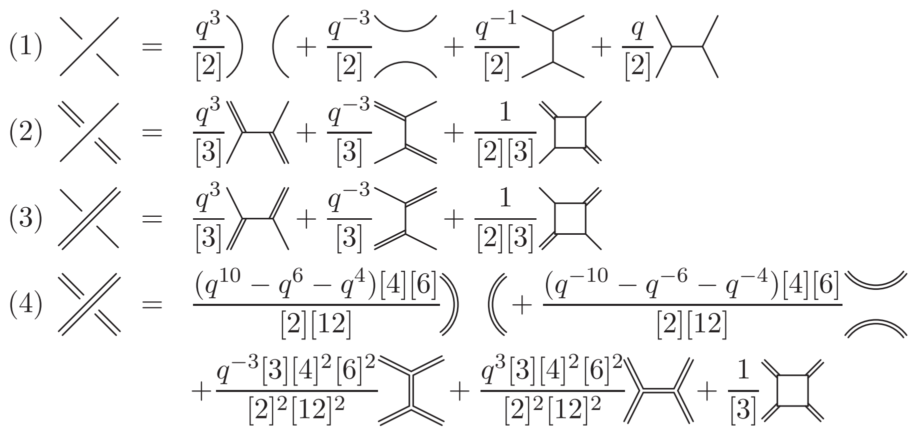
We define an action of the braid group

$$B_n = \left\langle b_i \, (1 \le i \le n-1) \,\middle|\, \begin{array}{ll} b_i b_j = b_j b_i & (|i-j| > 1), \\ b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1} & (1 \le i \le n-2) \end{array} \right\rangle$$

on the web space

$$\mathrm{W}_{G_2}[n] := \bigoplus_{s \in S[n]} \mathrm{W}_{G_2}(s).$$

Definition 4.1. Four types of crossings have the following descriptions in G_2 web diagram:



We have the following theorem by a direct calculation.

Theorem 4.2. Four types of crossing are regular isotopic (i.e. invariant under Reidemeister move 2 and 3. See [4]).

The above crossing formulas (1), (2), (3) and (4) correspond to R-matrices $R_{11} \in \mathrm{End}_{U_q(G_2)}(V_{\varpi_1}^{\otimes 2})$, $R_{12} \in \mathrm{Hom}_{U_q(G_2)}(V_{\varpi_1} \otimes V_{\varpi_2}, V_{\varpi_2} \otimes V_{\varpi_1})$, $R_{21} \in \mathrm{Hom}_{U_q(G_2)}(V_{\varpi_2} \otimes V_{\varpi_1}, V_{\varpi_1} \otimes V_{\varpi_2})$ and $R_{22} \in \mathrm{End}_{U_q(G_2)}(V_{\varpi_2}^{\otimes 2})$

Using the crossing formulas of Definition 4.1, we define an action $b_i \in B_n$ on $\mathrm{W}_{G_2}[n]$ as follows: The braid group action on a direct summand $\mathrm{W}_{G_2}(s) \subset \mathrm{W}_{G_2}[n]$ is

$$b_i : \mathrm{W}_{G_2}(s) \to \mathrm{W}_{G_2}(\sigma_i(s)),$$

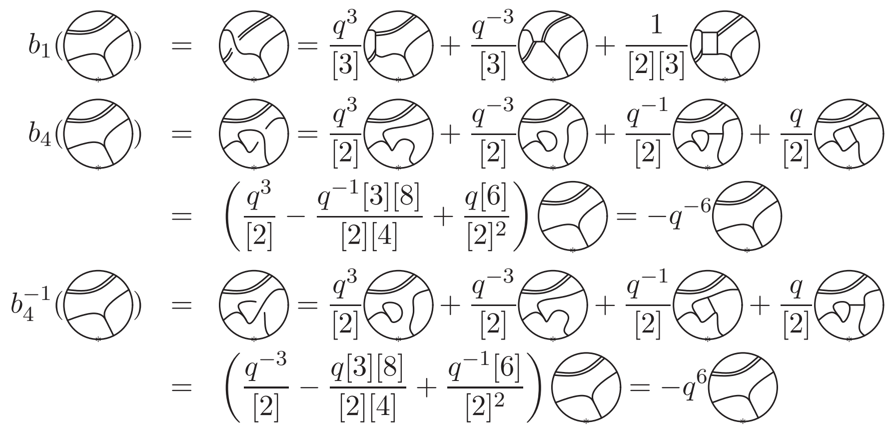
where σ_i is the transpose between an i-th entry and an $i+1$-entry. If $(s_i, s_{i+1}) = (1,1)$ (resp. $(s_i, s_{i+1}) = (1,2)$, $(s_i, s_{i+1}) = (2,1)$, $(s_i, s_{i+1}) = (2,2)$), gluing the boundaries of a G_2 web diagram $W \in \mathrm{W}_{G_2}(s)$ and the crossing in formula (1) of Definition 4.1 (resp. the crossing in (2), (3), (4)) at

(s_i, s_{i+1}) as the boundary at s_i connects to the over arc of the crossing and replacing the crossing into web diagrams by the formula (1) of Definition 4.1 (resp. formulas (2), (3), (4)), we obtain a linear sum of G_2 web diagrams in $\mathrm{W}_{G_2}(\sigma_i(s))$. An action of $b_i^{-1} \in B_n$,

$$b_i^{-1} : \mathrm{W}_{G_2}(s) \to \mathrm{W}_{G_2}(\sigma_i(s)),$$

is defined by gluing the boundaries at (s_i, s_{i+1}) and the crossing as the boundary at s_i connects to the under arc of the crossing and replacing the crossing into the linear sum of G_2 web diagrams.

For example, we have the B_5 action on the G_2 web space $\mathrm{W}_{G_2}(1,2,2,1,1)$. To the G_2 web diagram W_4 in Figure 3, the action of $b_1, b_4, b_4^{-1} \in B_5$ is



5. Relation to projectors and R-matrix of other irreducible representations

In this section, we show a relationship between G_2 web diagrams and projectors in hom set $\mathrm{Hom}_{U_q(G_2)}(V_\varpi \otimes V_{\varpi'})$, where ϖ and ϖ are fundamental weights. Using projectors, we construct the crossing formulas associated to other irreducible representations.

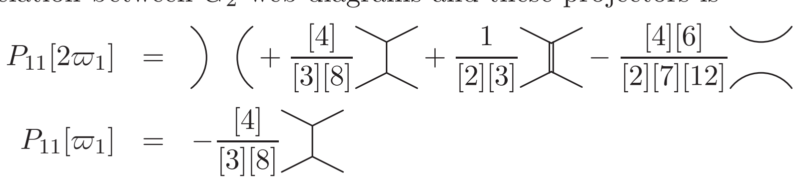
Let $P_{11}[\varpi]$ be a projector in $\mathrm{End}_{U_q(G_2)}(V_{\varpi_1}^{\otimes 2})$ which factors through the irreducible representation with highest weight ϖ and let R_{11} be the R-matrix in $\mathrm{End}_{U_q(G_2)}(V_{\varpi_1}^{\otimes 2})$. Remark that the projectors have idempotency

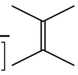
$$P_{11}[\varpi]P_{11}[\varpi'] = \delta_{\varpi,\varpi'}P_{11}[\varpi].$$

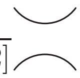
The description of R_{11} by the projectors (see [7]) is

$$R_{11} = q^2 P_{11}[2\varpi_1] - q^{-6}P_{11}[\varpi_1] - P_{11}[\varpi_2] + q^{-12}P_{11}[0].$$

A relation between G_2 web diagrams and these projectors is



$P_{11}[\varpi_2] = -\frac{1}{[2][3]}$ 

$P_{11}[0] = \frac{[4][6]}{[2][7][12]}$ 

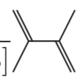
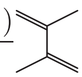
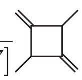
In other R-matrices of $R_{12} \in \mathrm{Hom}_{U_q(G_2)}(V_{\varpi_1} \otimes V_{\varpi_2}, V_{\varpi_2} \otimes V_{\varpi_1})$, $R_{21} \in \mathrm{Hom}_{U_q(G_2)}(V_{\varpi_2} \otimes V_{\varpi_1}, V_{\varpi_1} \otimes V_{\varpi_2})$ and $R_{22} \in \mathrm{End}_{U_q(G_2)}(V_{\varpi_2}^{\otimes 2})$, these descriptions by projectors are

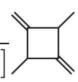
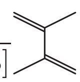
$$\begin{aligned} R_{12} &= q^3 P_{12}[\varpi_1 + \varpi_2] + q^{-4} P_{12}[2\varpi_1] - q^{-12} P_{12}[\varpi_1] \\ R_{21} &= q^3 P_{21}[\varpi_1 + \varpi_2] + q^{-4} P_{21}[2\varpi_1] - q^{-12} P_{21}[\varpi_1] \\ R_{22} &= q^6 P_{22}[2\varpi_2] - P_{22}[3\varpi_1] + q^{-10} P_{22}[2\varpi_1] - q^{-12} P_{22}[\varpi_2] + q^{-24} P_{22}[0], \end{aligned}$$

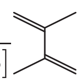
where $P_{ij}[\varpi]$, $i, j \in \{1, 2\}$, is a projector in $\mathrm{Hom}_{U_q(G_2)}(V_{\varpi_i} \otimes V_{\varpi_j}, V_{\varpi_j} \otimes V_{\varpi_i})$ which factors through the representation V_ϖ. Remark that the projectors have a structure

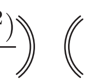
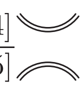
$$P_{12}[\varpi] P_{21}[\varpi'] P_{12}[\varpi] = \delta_{\varpi,\varpi'} P_{12}[\varpi], \quad P_{22}[\varpi] P_{22}[\varpi'] = \delta_{\varpi,\varpi'} P_{22}[\varpi].$$

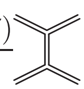
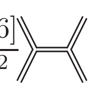
A relation between G_2 web diagrams and projectors $P_{ij}[\varpi]$ is

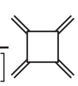
$P_{12}[\varpi_1 + \varpi_2] = \frac{1}{[3]}$  $+ \frac{[5](q^8 + q^2 - 1 + q^{-2} + q^{-8})}{[7][15]}$  $+ \frac{[4]}{[2][3][7]}$ 

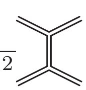
$P_{12}[2\varpi_1] = \frac{1}{[2][7]}$  $+ \frac{[3][4]}{[2][7][8]}$ 

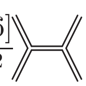
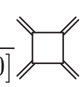
$P_{12}[\varpi_1] = -\frac{[5][12]}{[6][8][15]}$ 

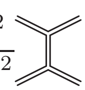
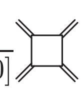
$P_{22}[2\varpi_2] = \frac{[3][4][5](q^2 - 2 + q^{-2})}{[12]}$  $\Big(- \frac{[3]^2[4][5][14]}{[7][8][12][15]}$ 

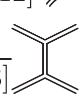
$+ \frac{[3]^2[4]^2[6][9]([4][14] - 7)}{[2]^2[7][8][12]^2[18]}$  $+ \frac{[3]^2[4]^2[6]}{[2]^2[12]^2}$ 

$+ \frac{[5]}{[6][8]}$ 

$P_{22}[3\varpi_1] = \frac{[3][4]}{[12]}$  $\Big(+ \frac{[2][3][4]^2[5]}{[8][10][12]}$  $- \frac{[3]^4[4]^2[5]}{[2]^2[10][12]^2}$ 

$- \frac{[3]^2[4]^2[6]}{[2]^2[12]^2}$  $- \frac{[4][5]}{[2][6][10]}$ 

$P_{22}[2\varpi_1] = -\frac{[2][3]^2[4][5][6]}{[7][8][10][12]}$  $+ \frac{[3]^3[4]^2[5][6]^2}{[2]^3[8][10][12]^2}$  $+ \frac{[5]}{[8][10]}$ 

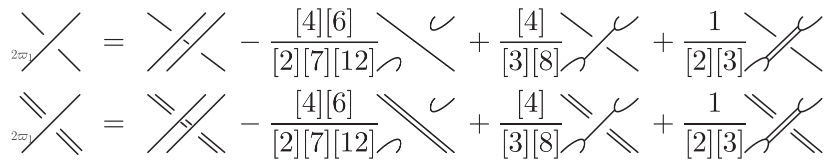
$P_{22}[\varpi_2] = -\frac{[3]^2[4][9]}{[2]^2[12][18]}$ 

$P_{22}[0] = \frac{[3][4][5]}{[7][8][15]}$ 

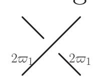
A projector $P_{21}[\varpi]$ is symmetry of the description of $P_{12}[\varpi]$ by G_2 web diagrams. In other words,

$$P_{21}[\varpi] = R_{21} P_{12}[\varpi] R_{12}^{-1}.$$

Using the description of projectors, we obtain a crossing formula associated to the irreducible representation V_ϖ. For example, using the description of $P_{11}[2\varpi_1]$ by G_2 web diagrams, we obtain a crossing formula with coloring $2\varpi_1$



We also have a formula for the following crossing



as a linear sum of 16 diagrams. Similarly, we have crossing formulas with colorings $\varpi_1+\varpi_2$, $2\varpi_2$ and $3\varpi_1$ using the projectors $P_{12}[\varpi_1+\varpi_2]$, $P_{22}[2\varpi_2]$ and $P_{22}[3\varpi_1]$.

An open problem is to construct projectors which factor through other irreducible representations as a linear sum of G_2 web diagrams.

6. G_2 quantum invariant of torus links

We have the evaluations for positive and negative crossings curl (diagrams in Reidemeister move 1) by crossing formulas (1) and (4) of Definition 4.1

Therefore, to obtain G_2 quantum invariant of an oriented link, we need to normalize crossing formulas of Definition 4.1.

Let L be an oriented link with k components $(L_1, L_2, ..., L_k)$, let D be a link diagram of L (forgetting the orientation) and $(D_1, D_2, ..., D_k)$ be the image of $(L_1, L_2, ..., L_k)$ by the projection $L \to D$.

Using crossing formulas, we define a polynomial evaluation for a link diagram D, denoted by $\langle D\rangle_{(\varpi_{i_1},\varpi_{i_2},...,\varpi_{i_k})}$, $i_j \in \{1,2\}$ and $j = 1, ..., k$, as follows: Replacing a diagram component D_j into the double line of G_2 web diagram if $\varpi_{i_j} = \varpi_2$ (we regard a diagram component D_j as the single line of G_2 web diagram if $\varpi_{i_j} = \varpi_1$), applying the crossing formulas of Definition 4.1 to all crossings of the replaced diagram of D and evaluating a linear sum of G_2 web diagrams by relations of Definition 2.1 and Proposition 2.2, we obtain a polynomial.

Theorem 6.1. For an oriented link L,

$$(q^{-12})^{\omega_{11}(D)}(q^{-24})^{\omega_{22}(D)}\langle D\rangle_{(\varpi_{i_1},\varpi_{i_2},\dots,\varpi_{i_k})}$$

is a link invariant of L, where D is a link diagram of L and $\omega_{11}(D)$ (resp. $\omega_{22}(D)$) is the number of positive crossings of single edge in D minus the number of negative crossings of single edge (resp. the number of positive crossings of double edge minus the number of negative crossings of double edge).

The link invariant is called G_2 quantum invariant associated to the G_2 fundamental representations, G_2 quantum invariant for short. Denote by $P_{(\varpi_{i_1},\varpi_{i_2},\dots,\varpi_{i_k})}(L)$ the G_2 quantum invariant of an oriented link L.

We show G_2 quantum invariant of a torus link $T(2,n)$ in Figure 4.

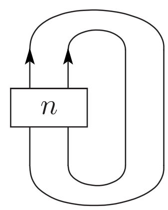
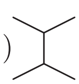
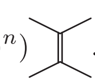


FIGURE 4. Torus link $T(2,n)$

Let $Cr(n)$, where $n\in\mathbb{Z}$, be a tangle diagram with n-crossing in Figure 5. The evaluation $\langle Cr(n)\rangle_{(\varpi_1,\varpi_1)}$ is

$$\langle Cr(n)\rangle_{(\varpi_1,\varpi_1)} = (q^2P_{11}[2\varpi_1]-q^{-6}P_{11}[\varpi_1]-P_{11}[\varpi_2]+q^{-12}P_{11}[0])^n$$

$$= q^{2n}$$  $$+\frac{[4][6]}{[2][7][12]}(-q^{2n}+q^{-12n})$$ 

$$+\frac{[4]}{[3][8]}(q^{2n}-(-q^{-6})^n)$$  $$+\frac{1}{[2][3]}(q^{2n}-(-1)^n)$$  .

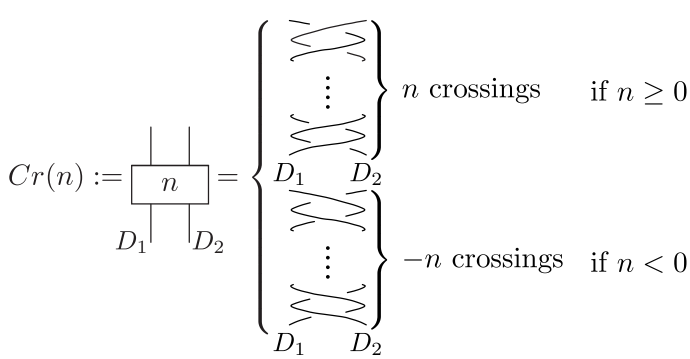


FIGURE 5. n-crossing

Using the evaluation $\langle Cr(n)\rangle_{(\varpi_1,\varpi_1)}$, the G_2 invariant $P_{(\varpi_1,\varpi_1)}(T(2,n))$, where $n\in\mathbb{Z}$, is

$$\begin{aligned} & q^{-12n}\langle T(2,n)\rangle_{(\varpi_1,\varpi_1)} \\ = \; & q^{-10n}\frac{[3][6][12][15]}{[4][5][6]}+(-q^{-12})^n\frac{[7][8][15]}{[3][4][5]}+(-q^{-18})^n\frac{[2][7][12]}{[4][6]}+q^{-24n}. \end{aligned}$$

Similarly, calculating $\langle Cr(n)\rangle_{(\varpi_1,\varpi_2)}$ for an even number n and $\langle Cr(n)\rangle_{(\varpi_2,\varpi_2)}$ for $n\in\mathbb{Z}$ (the details are left to the reader), G_2 invariant $P_{(\varpi_1,\varpi_2)}(T(2,n))$, where n is an even integer, is

$$\langle T(2,n)\rangle_{(\varpi_1,\varpi_2)} = q^{3n}\frac{[2][8][10][12][18]}{[3][4][5][9]}+q^{-4n}\frac{[3][12][15]}{[4][5]}+(-q^{-12})^n\frac{[2][7][12]}{[4][6]}.$$

and G_2 invariant $P_{(\varpi_2,\varpi_2)}(T(2,n))$, where $n\in\mathbb{Z}$, is

$$\begin{aligned} q^{-24n}\langle T(2,n)\rangle_{(\varpi_2,\varpi_2)} &= q^{-18n}\frac{[10][11][12][21]}{[3][4][5][6]}+(-q^{-24})^n\frac{[7][11][15][18]}{[5][6][9]} \\ &\quad +q^{-34n}\frac{[3][12][15]}{[4][5]}+(-q^{-36})^n\frac{[7][8][15]}{[3][4][5]}+q^{-48n}. \end{aligned}$$

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