

# HOMOTOPY CATEGORY OF $N$ -COMPLEXES OF PROJECTIVE MODULES

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ABSTRACT. In this paper, we show that the homotopy category of  $N$ -complexes of projective  $R$ -modules is triangle equivalent to the homotopy category of projective  $\mathbb{T}_{N-1}(R)$ -modules where  $\mathbb{T}_{N-1}(R)$  is the ring of triangular matrices of order  $N - 1$  with entries in  $R$ . We also define the notions of  $N$ -singularity category and  $N$ -totally acyclic complexes. We show that the category of  $N$ -totally acyclic complexes of finitely generated projective  $R$ -modules embeds in the  $N$ -singularity category, which is a result analogous to the case of ordinary chain complexes.

## 1. INTRODUCTION

Given an associative unitary ring  $R$ , by an  $N$ -complex  $X^\bullet$ , we mean a sequence of  $R$ -modules and  $R$ -linear maps  $\cdots \rightarrow X^{n-1} \rightarrow X^n \rightarrow X^{n+1} \rightarrow \cdots$  such that composition of any  $N$  consecutive maps gives the zero map. The notion of  $N$ -complexes first appeared in the paper [Kap96] by Kapranov. Besides their applications in theoretical physics [CSW07], [Hen08], the homological properties of  $N$ -complexes have become a subject of study for many authors as in [DV98], [Est07], [Gil12], [GH10], [Tik02]. Iyama and et. al. studied the homotopy category  $\mathbb{K}_N(\mathcal{B})$  of  $N$ -complexes of an additive category  $\mathcal{B}$  as well as the derived category  $\mathbb{D}_N(\mathcal{A})$  of an abelian category  $\mathcal{A}$ . Recall that an abelian category  $\mathcal{A}$  is an  $(Ab4)$ -category (resp.  $(Ab4)^*$ -category) provided that it has any coproduct (resp. product) of objects, and that the coproduct (resp. product) of monomorphisms (resp. epimorphisms) is monic (resp. epic). In the paper, [IKM14], they showed that the well known equivalences between homotopy category of chain complexes and their derived categories also generalize to the case of  $N$ -complexes. More precisely, if  $\mathcal{A}$  is an abelian category satisfying the condition  $(Ab4)$ , then we have triangle equivalence

$$\mathbb{K}_N^{\natural}(\text{Prj-}\mathcal{A}) \cong \mathbb{D}_N^{\natural}(\mathcal{A}).$$

where  $(\natural, \sharp) = (proj, nothing), (-, -), ((-, b), b)$  and  $\text{Prj-}\mathcal{A}$  is the category of projective objects of  $\mathcal{A}$ . As for chain complexes a similar statement is also true for the category  $\text{Inj-}\mathcal{A}$  of injective objects of  $\mathcal{A}$  provided that  $\mathcal{A}$  satisfies the condition  $(Ab4)^*$ . They also showed that there exists a triangle equivalence

$$(1.1) \quad \mathbb{D}_N(\mathcal{A}) \cong \mathbb{D}(\mathbb{T}_{N-1}(\mathcal{A})).$$

As a consequence of this equivalence they showed that there exists the following triangle equivalences between derived and homotopy categories.

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**Corollary 1.1.** *For a ring  $R$ , we have the following triangle equivalences.*

$$\mathbb{D}_N^{\natural}(\text{Mod-}R) \cong \mathbb{D}^{\natural}(\text{Mod-}\mathbb{T}_{N-1}(R)),$$

where  $\natural = -, b$ .

$$\mathbb{K}_N^{\natural}(\text{Prj-}R) \cong \mathbb{K}^{\natural}(\text{Prj-}\mathbb{T}_{N-1}(R)),$$

where  $\natural = -, b, (-, b)$  and also

$$\mathbb{K}_N^{\natural}(\text{prj-}R) \cong \mathbb{K}^{\natural}(\text{prj-}\mathbb{T}_{N-1}(R)),$$

where  $\natural = -, b, (-, b)$ .

In this paper, we show that the homotopy category  $\mathbb{K}_N(\text{Prj-}(R))$  of  $N$ -complexes is embedded in the ordinary homotopy category  $\mathbb{K}(\text{Prj-}\mathbb{T}_{N-1}(R))$ . Having this embedding in hand we are able recover (1.1) by using different techniques than those in [IKM14]. We also show that  $\mathbb{K}_N(\text{Prj-}(R))$  is equivalent to  $\mathbb{K}(\text{Prj-}\mathbb{T}_{N-1}(R))$  whenever  $R$  is a left coherent ring.

The explicit construction of such triangle equivalence allows us to prove an  $N$ -complex version of the following equivalence of triangulated categories given in [Buc87], [Hap91], [BJO14].

$$\mathbb{K}_{\text{tac}}(\text{prj-}R) \rightarrow \mathbb{D}_{\text{sg}}^b(R)$$

where  $\mathbb{K}_{\text{tac}}(\text{prj-}R)$  is the homotopy category of totally acyclic complexes of finitely generated projective  $R$ -modules and  $\mathbb{D}_{\text{sg}}^b(R)$  is the singularity category.

The paper is organized as follows. In section 2, we recall some generalities on  $N$ -complexes and provide any background information needed through this paper. Our main result appears in section 3 as Theorem 3.17. In that section, we show that the category  $\mathbb{K}_N(\text{Prj-}R)$  embeds as a triangulated subcategory in the category  $\mathbb{K}(\text{Prj-}\mathbb{T}_{N-1}(R))$  see proposition 3.9. As an application of this embedding we provide a different proof for the triangle equivalence in (1.1). At the end of this section we show that this embedding is also dense, hence an equivalence.

In section 4 we define an  $N$ -totally acyclic complex as a complex  $X^\bullet$  in  $\text{prj-}R$  satisfying the property that for all  $P^\bullet \in \mathbb{K}_N^b(\text{prj-}R)$ ,  $\text{Hom}_{\mathbb{K}_N(\text{prj-}R)}(P^\bullet, X^\bullet) = \text{Hom}_{\mathbb{K}_N(\text{prj-}R)}(X^\bullet, P^\bullet) = 0$ . Then we show that the homotopy category  $\mathbb{K}_N^{\text{tac}}(\text{prj-}R)$  of  $N$ -totally acyclic complexes in  $\text{prj-}R$  in this sense is triangle equivalent to the homotopy category of ordinary totally acyclic complexes in  $\text{prj-}\mathbb{T}_{N-1}(R)$ , i.e.  $\mathbb{K}_{\text{tac}}(\text{prj-}\mathbb{T}_{N-1}(R))$ . We also define a similar notion of singularity category for  $N$ -complexes  $\mathbb{D}_N^{\text{sg}}(R)$  and show that it contains  $\mathbb{K}_N^{\text{tac}}(\text{prj-}R)$  as a triangulated subcategory. Furthermore, the embedding

$$\mathbb{K}_N^{\text{tac}}(\text{prj-}R) \rightarrow \mathbb{D}_N^{\text{sg}}(R)$$

is an equivalence of triangulated categories, when  $R$  is a Gorenstein ring.

## 2. PRELIMINARIES

**2.1. The category of  $N$ -complexes.** Throughout,  $R$  is an associative ring with identity.  $\text{Mod-}R$  denotes the category of all right  $R$ -modules. We fix a positive integer  $N \geq 2$ . An  $N$ -complex  $X^\bullet$  is a diagram

$$\dots \xrightarrow{d_{X^\bullet}^{i-1}} X^i \xrightarrow{d_{X^\bullet}^i} X^{i+1} \xrightarrow{d_{X^\bullet}^{i+1}} \dots$$

with  $X^i \in \text{Mod-}R$  and morphisms  $d_{X^\bullet}^i \in \text{Hom}_R(X^i, X^{i+1})$  satisfying  $d^N = 0$ . That is, composition of any  $N$ -consecutive maps is 0. A morphism between  $N$ -complexes is a commutative

diagram

$$\begin{array}{ccccccc}
 \cdots & \xrightarrow{d_{X^\bullet}^{i-1}} & X^i & \xrightarrow{d_{X^\bullet}^i} & X^{i+1} & \xrightarrow{d_{X^\bullet}^{i+1}} & \cdots \\
 & & \downarrow f^i & & \downarrow f^{i+1} & & \\
 \cdots & \xrightarrow{d_{Y^\bullet}^{i-1}} & Y^i & \xrightarrow{d_{Y^\bullet}^i} & Y^{i+1} & \xrightarrow{d_{Y^\bullet}^{i+1}} & \cdots
 \end{array}$$

We denote by  $\mathbb{C}_N(R)$  (resp.  $\mathbb{C}_N^-(R)$ ,  $\mathbb{C}_N^+(R)$ ,  $\mathbb{C}_N^b(R)$ ) the category of unbounded (resp. bounded above, bounded below, bounded)  $N$ -complexes over  $\text{Mod-}R$ .

For any object  $M$  of  $\text{Mod-}R$ ,  $j \in \mathbb{Z}$  and  $1 \leq i \leq N$ , let

$$D_i^j(M) : \cdots \longrightarrow 0 \longrightarrow X^{j-i+1} \xrightarrow{d_{X^\bullet}^{j-i+1}} \cdots \xrightarrow{d_{X^\bullet}^{j-2}} X^{j-1} \xrightarrow{d_{X^\bullet}^{j-1}} X^j \longrightarrow 0 \longrightarrow \cdots$$

be an  $N$ -complex satisfying  $X^n = M$  for all  $j - i + 1 \leq n \leq j$  and  $d_{X^\bullet}^n = 1_M$  for all  $j - i + 1 \leq n \leq j - 1$ .

For  $0 \leq r < N$  and  $i \in \mathbb{Z}$ , we define

$$d_{X^\bullet, \{r\}}^i := d_{X^\bullet}^{i+r-1} \cdots d_{X^\bullet}^i.$$

In this notation  $d_{X^\bullet, \{1\}}^i = d_{X^\bullet}^i$  and  $d_{X^\bullet, \{0\}}^i = 1_{X^i}$ .

**Definition 2.1.** Let  $f : X^\bullet \rightarrow Y^\bullet$  be a morphism in  $\mathbb{C}_N(R)$ . The mapping cone  $C(f)$  of  $f$  is defined as follows

$$C(f)^m = Y^m \oplus \coprod_{i=m+1}^{m+N-1} X^i, \quad d_{C(f)}^m = \begin{bmatrix} d_{Y^\bullet}^m & f^{m+1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ & 0 & \cdots & & 1 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 \\ 0 & -d_{X^\bullet, \{N-1\}}^{m+1} & -d_{X^\bullet, \{N-2\}}^{m+2} & \cdots & \cdots & -d_{X^\bullet}^{m+N-1} \end{bmatrix}$$

Let  $\mathcal{S}_N(R)$  be the collection of short exact sequences in  $\mathbb{C}_N(R)$  of which each term is split short exact in  $R$ . Then it is easy to see that a category  $(\mathbb{C}_N(R), \mathcal{S}_N(R))$  is an exact category such that for every  $M \in \text{Mod-}R$  and every  $i \in \mathbb{Z}$ ,  $D_N^{-i+N-1}(M)$  is an  $\mathcal{S}_N$ -projective and  $\mathcal{S}_N$ -injective object of this category. Hence this category is a Frobenius category, See [IKM14, Proposition 1.5].

**Definition 2.2.** A morphism  $f : X^\bullet \rightarrow Y^\bullet$  of  $N$ -complexes is called null-homotopic if there exists  $s^i \in \text{Hom}_R(X^i, Y^{i-N+1})$  such that

$$f^i = \sum_{j=0}^{N-1} d_{Y^\bullet, \{N-1-j\}}^{i-(N-1-j)} s^{i+j} d_{X^\bullet, \{j\}}^i$$

We denote the homotopy category of unbounded  $N$ -complexes by  $\mathbb{K}_N(R)$ .

**Definition 2.3.** For  $X^\bullet = (X^i, d^i) \in \mathbb{C}_N(R)$ , we define a shift functor  $\Theta : \mathbb{C}_N(R) \rightarrow \mathbb{C}_N(R)$  by

$$\Theta(X^\bullet)^i = X^{i+1}, \quad \Theta(d)^i = d^{i+1}.$$

We also define suspension functor  $\Sigma : \mathbb{K}_N(R) \longrightarrow \mathbb{K}_N(R)$  as follows

$$(\Sigma X^\bullet)^m = \coprod_{i=m+1}^{m+N-1} X^i, \quad d_{\Sigma X^\bullet}^m = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ & 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ & & & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 1 \\ -d_{\{N-1\}}^{m+1} & -d_{\{N-2\}}^{m+2} & \cdots & \cdots & \cdot & -d^{m+N-1} \end{bmatrix}$$

$$(\Sigma^{-1} X^\bullet)^m = \coprod_{i=m-N+1}^{i=m-1} X^i, \quad d_{\Sigma^{-1} X^\bullet}^m = \begin{bmatrix} -d^{m-1} & 1 & 0 & \cdots & \cdots & 0 \\ -d_{\{2\}}^{m-1} & 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0 \\ & & & \ddots & \ddots & 0 \\ -d_{\{N-2\}}^{m-1} & 0 & \cdots & \cdots & 0 & 1 \\ -d_{\{N-1\}}^{m-1} & 0 & \cdots & \cdots & \cdot & 0 \end{bmatrix}$$

It is known that  $\mathbb{K}_N(R)$  together with this suspension functor is a triangulated category, see [IKM14, Theorem 1.7].

Let  $X^\bullet$ ,

$$\cdots \xrightarrow{d_{X^\bullet}^{i-1}} X^i \xrightarrow{d_{X^\bullet}^i} X^{i+1} \xrightarrow{d_{X^\bullet}^{i+1}} \cdots$$

be an  $N$ -complex of  $R$ -modules. We define

$$Z_r^i(X^\bullet) := \text{Ker} d_{X^\bullet, \{r\}}^i, \quad B_r^i(X^\bullet) := \text{Im} d_{X^\bullet, \{r\}}^{i-r}$$

$$C_r^i(X^\bullet) := \text{Coker} d_{X^\bullet, \{r\}}^{i-r}, \quad H_r^i(X^\bullet) := Z_r^i(X^\bullet)/B_{N-r}^i(X^\bullet).$$

In each degree we have  $N-1$  cycles and clearly  $Z_N^n(X^\bullet) = X^n$ .

**Remark 2.4.** For any  $X^\bullet \in \mathbb{C}_N(R)$  if  $H_1^i(X^\bullet) = 0$  for any  $i \in \mathbb{Z}$ , then we have  $H_r^i(X^\bullet) = 0$  for any  $i \in \mathbb{Z}$  and  $0 < r < N$ .

**Definition 2.5.** Let  $X^\bullet \in \mathbb{K}_N(R)$ . We say  $X^\bullet$  is  $N$ -exact if  $H_r^i(X^\bullet) = 0$  for each  $i \in \mathbb{Z}$  and all  $r = 1, 2, \dots, N-1$ . We denote the full subcategory of  $\mathbb{K}_N(R)$  consisting of  $N$ -exact complexes by  $\mathbb{K}_N^{\text{ac}}(R)$ .

For a full subcategory  $\mathcal{B}$  of  $\text{Mod-}R$ , we denote by  $\mathbb{K}_N^{\natural, \flat}(\mathcal{B})$  the full subcategory of  $\mathbb{K}_N^{\natural}(\mathcal{B})$  consisting of  $N$ -complexes  $X^\bullet$  satisfying  $H_r^i(X^\bullet) = 0$  for almost all but finitely many  $i$  and  $r$ , where  $\natural = \text{nothing}, -, +$ .

**Definition 2.6.** A morphism  $f : X^\bullet \rightarrow Y^\bullet$  is called quasi-isomorphism if the induced morphism  $H_r^i(f) : H_r^i(X^\bullet) \rightarrow H_r^i(Y^\bullet)$  is an isomorphism for any  $i$  and  $1 \leq r \leq N-1$ , or equivalently if the mapping cone  $C(f)$  belongs to  $\mathbb{K}_N^{\text{ac}}(R)$ . The derived category  $\mathbb{D}_N(R)$  of  $N$ -complexes is defined as the quotient category  $\mathbb{K}_N(R)/\mathbb{K}_N^{\text{ac}}(R)$ .

**2.2. Triangular matrix ring.** Let  $\mathbb{M}_n(R)$  be the set of all  $n \times n$  square matrices with coefficients in  $R$  for  $n \in \mathbb{N}$ .  $\mathbb{M}_n(R)$  is a ring with respect to the usual matrix addition and multiplication. The identity of  $\mathbb{M}_n(R)$  is the matrix  $E = \text{diag}(1, \dots, 1) \in \mathbb{M}_n(R)$  with 1 on the main diagonal and zeros elsewhere. The subset

$$\mathbb{T}_n(R) = \begin{bmatrix} R & 0 & \cdots & 0 \\ R & R & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ R & R & \cdots & R \end{bmatrix}$$

of  $\mathbb{M}_n(R)$  consisting of all triangular matrices  $[a_{ij}]$  in  $\mathbb{M}_n(R)$  with zeros over the main diagonal is a subring of  $\mathbb{M}_n(R)$ . It is well known that if  $Q$  is the quiver

$$A_n = 1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots \longrightarrow n$$

then  $RQ \cong \mathbb{T}_n(R)$ , where  $RQ$  is a path algebra of quiver  $Q$ .

Let  $Q = (V, E)$  be a quiver. A representation of  $Q$  by a ring  $R$  is a correspondence which associates an object  $M_v$  to each vertex  $v$  and a morphism  $\varphi_a : M_{s(a)} \rightarrow M_{t(a)}$  to each arrow  $a \in E$ . Let  $\mathcal{X}$  and  $\mathcal{Y}$  be two representations by left  $R$ -modules of the quiver  $Q$ . A morphism  $f : \mathcal{X} \rightarrow \mathcal{Y}$  is a family of homomorphisms  $f_v : \mathcal{X}_v \rightarrow \mathcal{Y}_v$  such that  $\mathcal{Y}_a \circ f_v = f_w \circ \mathcal{X}_a$  for any arrow  $a : v \rightarrow w$ . The representations of  $Q$  by  $R$ -modules and  $R$ -homomorphisms form a category denoted by  $\text{Rep}(Q, R)$ .

It is known that the category  $\text{Rep}(Q, R)$  is equivalent to the category of modules over path algebra  $RQ$  whenever  $Q$  is finite quiver.

Set  $Q = A_n$ . For  $1 \leq i \leq n$ , let  $e^i : \text{Rep}(Q, R) \rightarrow \text{Mod-}R$  be the evaluation functor defined by  $e^i(\mathcal{X}) = \mathcal{X}_i$ , for any  $\mathcal{X} \in \text{Rep}(Q, R)$ . It is proved in [EH99] that  $e^i$  has a right adjoint  $e_\rho^i : \text{Mod-}R \rightarrow \text{Rep}(Q, R)$ , where  $e_\rho^i(M)$  is the following representation

$$M \longrightarrow M \longrightarrow \cdots \longrightarrow M \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0$$

for the  $R$ -module  $M$ , where  $M$  ends in  $i$ -th position with identity morphisms beforehand. Moreover, for  $1 \leq j \leq n$ , it is shown that  $e^j$  also admits a left adjoint  $e_\lambda^j$ , defined by  $e_\lambda^j(M)$  as follows:

$$0 \longrightarrow 0 \longrightarrow \cdots \longrightarrow M \longrightarrow M \longrightarrow \cdots \longrightarrow M$$

for the  $R$ -module  $M$ , where  $M$  starts in  $j$ -th position with identity morphisms afterward. It is proved in [EER09] that any projective (resp. injective) representation  $\mathcal{X}$  in  $\text{Rep}(Q, R)$  is of the form  $\bigoplus_{i=1}^n e_\lambda^i(P^i)$  (resp.  $\bigoplus_{i=1}^n e_\rho^i(I^i)$ ), where for any  $1 \leq i \leq n$ ,  $P^i$  (resp.  $I^i$ ), is the cokernel (resp. kernel) of the split monomorphism  $\mathcal{X}^{i-1} \rightarrow \mathcal{X}^i$  (resp. epimorphism  $\mathcal{X}^i \rightarrow \mathcal{X}^{i+1}$ ). Hence any projective object in  $\text{Mod-}\mathbb{T}_n(R)$  is of the form

$$P^1 \rightarrow P^1 \oplus P^2 \rightarrow P^1 \oplus P^2 \oplus P^3 \rightarrow \cdots \rightarrow P^1 \oplus P^2 \oplus \cdots \oplus P^n$$

and an injective object in  $\text{Mod-}\mathbb{T}_n(R)$  is of the form

$$I^1 \oplus I^2 \oplus \cdots \oplus I^n \rightarrow I^1 \oplus I^2 \oplus \cdots \oplus I^{n-1} \rightarrow \cdots \rightarrow I^1 \oplus I^2 \rightarrow I^1$$

### 3. SOME TRIANGLE EQUIVALENCES BETWEEN HOMOTOPY CATEGORIES

In this section, we show that the homotopy category  $\mathbb{K}_N(\text{Prj-}(R))$  of  $N$ -complexes is embedded in the ordinary homotopy category  $\mathbb{K}(\text{Prj-}\mathbb{T}_{N-1}(R))$ . As a result of this embedding we show that there exists a triangle equivalence between derived category of  $N$ -complexes and ordinary derived category of complexes of  $\text{Mod-}\mathbb{T}_{N-1}(R)$ . At the end of this section we

show that  $\mathbb{K}_N(\text{Prj-}(R)) \cong \mathbb{K}(\text{Prj-}\mathbb{T}_{N-1}(R))$ .

Let  $\mathcal{S}_N(R)$  be the collection of short exact sequence in  $\mathbb{C}_N(R)$  of which each term is split exact then it is shown in [IKM14] that  $(\mathbb{C}_N(R), \mathcal{S}_N(R))$  is a Frobenius category. We need the following definition and lemma from [Hap88].

**Definition 3.1.** Let  $(\mathcal{B}, \mathcal{S})$  and  $(\mathcal{B}', \mathcal{S}')$  be Frobenius categories. An additive functor  $F : \mathcal{B} \rightarrow \mathcal{B}'$  is called exact if  $0 \rightarrow F(X) \xrightarrow{F(u)} F(Y) \xrightarrow{F(v)} F(Z) \rightarrow 0$  is contained in  $\mathcal{S}'$  whenever  $0 \rightarrow X \xrightarrow{u} Y \xrightarrow{v} Z \rightarrow 0$  is contained in  $\mathcal{S}$ .

If  $F$  transforms  $\mathcal{S}$ -injectives into  $\mathcal{S}'$ -injectives then  $F$  induces a functor  $\underline{F} : \underline{\mathcal{B}} \rightarrow \underline{\mathcal{B}'}$ . Denote by  $T$  (resp.  $T'$ ) the translation functor on stable category  $\underline{\mathcal{B}}$  (resp.  $\underline{\mathcal{B}'}$ ).

**Lemma 3.2.** Let  $F$  be an exact functor between Frobenius categories  $\mathcal{B}$  and  $\mathcal{B}'$  such that  $F$  transforms  $\mathcal{S}$ -injectives into  $\mathcal{S}'$ -injectives. If there exists an invertible natural transformation  $\alpha : FT \rightarrow T'F$  then  $F$  is an exact functor of triangulated categories.

In order to show that  $\mathbb{K}_N(\text{Prj-}(R))$  embeds in  $\mathbb{K}(\text{Prj-}\mathbb{T}_{N-1}(R))$ , we explicitly construct the embedding functor.

**Construction 3.3.** Define the functor  $\mathbf{F} : \mathbb{C}_N(\text{Prj-}R) \rightarrow \mathbb{C}(\text{Prj-}\mathbb{T}_{N-1}(R))$  by the following rules.

**On objects:** Let  $(P^\bullet, d^\bullet)$  be an object in  $\mathbb{C}_N(\text{Prj-}R)$ . Define  $i$ -th term of  $\mathbf{F}(P^\bullet)$  as follows

- For  $i = 2r$ , let  $m = Nr$  and define  $\mathbf{F}(P^\bullet)^i$  as the following projective representation of  $A_{N-1}$ :

$$P^m \longrightarrow P^m \oplus P^{m+1} \longrightarrow \dots \longrightarrow P^m \oplus P^{m+1} \oplus \dots \oplus P^{m+N-3} \longrightarrow P^m \oplus \dots \oplus P^{m+N-2}$$

- For  $i = 2r+1$ , let  $m = Nr$  and define  $\mathbf{F}(P^\bullet)^i$  as the following projective representation of  $A_{N-1}$ :

$$P^{m+N-1} \longrightarrow P^{m+N-1} \oplus P^{m+N} \longrightarrow \dots \longrightarrow P^{m+N-1} \oplus P^{m+N} \oplus \dots \oplus P^{m+2N-4} \longrightarrow P^{m+N-1} \oplus \dots \oplus P^{m+2N-3}$$

For the definition of differential of  $\mathbf{F}(P^\bullet)$ , we consider the following two cases:

- (i) For  $i = 2r$ , define  $\mu^i : \mathbf{F}(P^\bullet)^i \rightarrow \mathbf{F}(P^\bullet)^{i+1}$  by  $\mu^i = (\mu_j^i)_{1 \leq j \leq N-1}$  where

$$\mu_j^i = \begin{bmatrix} d_{\{N-1\}}^m & d_{\{N-2\}}^{m+1} & \dots & d_{\{N-j\}}^{m+j-1} \\ 0 & d_{\{N-1\}}^{m+1} & \dots & d_{\{N-j+1\}}^{m+j-1} \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_{\{N-1\}}^{m+j-1} \end{bmatrix}$$

- (ii) For  $i = 2r + 1$ , define  $\lambda^i : \mathbf{F}(P^\bullet)^i \rightarrow \mathbf{F}(P^\bullet)^{i+1}$  by  $\lambda^i = (\lambda_j^i)_{1 \leq j \leq N-1}$  where

$$\lambda_j^i = \begin{bmatrix} d^{m+N-1} & -1 & 0 & \cdots & \cdots & 0 \\ 0 & d^{m+N} & -1 & 0 & \cdots & 0 \\ 0 & 0 & d^{m+N+1} & -1 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ \vdots & \vdots & & & & -1 \\ 0 & 0 & \cdots & \cdots & 0 & d^{m+N+j-2} \end{bmatrix}$$

**On morphisms:** Let  $f^\bullet : Q^\bullet \longrightarrow P^\bullet$  be a morphism in  $\mathbb{C}_N(\text{Prj-}R)$ . We define  $\mathbf{F}(f^\bullet)$  as follows:

(i) For  $i = 2r$ , let  $m = Nr$ . Define  $\varphi^i : \mathbf{F}(Q^\bullet)^i \longrightarrow \mathbf{F}(P^\bullet)^i$  by  $\varphi^i = (\varphi_j^i)_{0 \leq j \leq N-2}$  where

$$\varphi_j^i = \text{diag}(f^m, \dots, f^{m+j})$$

(ii) For  $i = 2r + 1$ , let  $m = Nr$ . Define  $\varphi^i : \mathbf{F}(Q^\bullet)^i \longrightarrow \mathbf{F}(P^\bullet)^i$  by  $\varphi^i = (\varphi_j^i)_{0 \leq j \leq N-2}$  where

$$\varphi_j^i = \text{diag}(f^{m+N-1}, \dots, f^{m+N-j}).$$

It is straightforward to show that this construction defines covariant functor from  $\mathbb{C}_N(\text{Prj-}R)$  to  $\mathbb{C}(\text{Prj-}\mathbf{T}_{N-1}(R))$ .

**Example 3.4.** Let  $N = 3$ . Let  $P^\bullet$

$$P^\bullet = \cdots \xrightarrow{d^{-2}} P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} P^1 \xrightarrow{d^1} P^2 \xrightarrow{d^2} P^3 \xrightarrow{d^3} \cdots$$

be a 3-complex in  $\mathbb{C}_3(\text{Prj-}R)$ . The functor  $\mathbf{F}$  maps  $P^\bullet$  in to the following complex in  $\mathbb{C}(\text{Prj-}\mathbf{T}_2(R))$

$$\begin{array}{ccccccccc} P^{-1} & \xrightarrow{d^{-1}} & P^0 & \xrightarrow{d^1 d^0} & P^2 & \xrightarrow{d^2} & P^3 & \xrightarrow{d^4 d^3} & P^5 \\ \downarrow (1,0) & & \downarrow (1,0) & & \downarrow (1,0) & & \downarrow (1,0) & & \downarrow (1,0) \\ P^{-1} \oplus P^0 & \xrightarrow{\begin{bmatrix} d^{-1} & -1 \\ 0 & d^0 \end{bmatrix}} & P^0 \oplus P^1 & \xrightarrow{\begin{bmatrix} d^1 d^0 & d^1 \\ 0 & d^2 d^1 \end{bmatrix}} & P^2 \oplus P^3 & \xrightarrow{\begin{bmatrix} d^2 & -1 \\ 0 & d^3 \end{bmatrix}} & P^3 \oplus P^4 & \xrightarrow{\begin{bmatrix} d^4 d^3 & d^4 \\ 0 & d^5 d^4 \end{bmatrix}} & P^5 \oplus P^6 \end{array}$$

Now consider the morphism  $f^\bullet : (Q^\bullet, e^\bullet) \longrightarrow (P^\bullet, d^\bullet)$  in  $\mathbb{C}_3(\text{Prj-}R)$  as follows:

$$\begin{array}{ccccccccccc} \cdots & \xrightarrow{e^{-2}} & Q^{-1} & \xrightarrow{e^{-1}} & Q^0 & \xrightarrow{e^0} & Q^1 & \xrightarrow{e^1} & Q^2 & \xrightarrow{e^2} & Q^3 & \xrightarrow{e^3} & \cdots \\ & & \downarrow f^{-1} & & \downarrow f^0 & & \downarrow f^1 & & \downarrow f^2 & & \downarrow f^3 & & \\ \cdots & \xrightarrow{d^{-2}} & P^{-1} & \xrightarrow{d^{-1}} & P^0 & \xrightarrow{d^0} & P^1 & \xrightarrow{d^1} & P^2 & \xrightarrow{d^2} & P^3 & \xrightarrow{d^3} & \cdots \end{array}$$

The image of  $f^\bullet$  under  $\mathbf{F}$  is the following diagram in  $\mathbb{C}(\text{Prj-}\mathbb{T}_2(R))$ :

$$\begin{array}{ccccccc}
 & & Q^{-1} & \xrightarrow{\quad} & Q^0 & \xrightarrow{\quad} & Q^2 & \xrightarrow{\quad} & Q^3 \\
 & \swarrow & \vdots & & \vdots & & \vdots & & \vdots \\
 Q^{-1} \oplus Q^0 & \xrightarrow{\quad} & Q^0 \oplus Q^1 & \xrightarrow{\quad} & Q^2 \oplus Q^3 & \xrightarrow{\quad} & Q^3 \oplus Q^4 & & \\
 \downarrow \begin{bmatrix} f^{-1} & 0 \\ 0 & f^0 \end{bmatrix} & & \downarrow \begin{bmatrix} f^0 & 0 \\ 0 & f^1 \end{bmatrix} & & \downarrow \begin{bmatrix} f^2 & 0 \\ 0 & f^3 \end{bmatrix} & & \downarrow \begin{bmatrix} f^3 & 0 \\ 0 & f^4 \end{bmatrix} & & \\
 P^{-1} \oplus P^0 & \xrightarrow{\quad} & P^0 \oplus P^1 & \xrightarrow{\quad} & P^2 \oplus P^3 & \xrightarrow{\quad} & P^3 \oplus P^4 & & \\
 & \swarrow & \vdots & & \vdots & & \vdots & & \vdots \\
 & & P^{-1} & \xrightarrow{\quad} & P^0 & \xrightarrow{\quad} & P^2 & \xrightarrow{\quad} & P^3
 \end{array}$$

**Lemma 3.5.** *The functor  $\mathbf{F}$ , defined above, induces a functor from the category  $\mathbb{K}_N(\text{Prj-}R)$  to the category  $\mathbb{K}(\text{Prj-}\mathbb{T}_{N-1}(R))$  which we denote it again by  $\mathbf{F}$ .*

*Proof.* We show that if  $f^\bullet : (Q^\bullet, e^\bullet) \rightarrow (P^\bullet, d^\bullet)$  is a null homotopic map in  $\mathbb{C}_N(\text{Prj-}R)$  then  $\mathbf{F}(f^\bullet)$  is a null homotopic map in  $\mathbb{C}(\text{Prj-}\mathbb{T}_{N-1}(R))$ . Since  $f^\bullet \sim 0^\bullet$ , by definition there exists  $s^m \in \text{Hom}_R(Q^m, P^{m-N+1})$  such that

$$f^m = \sum_{k=0}^{N-1} d_{\{N-1-k\}}^{m-(N-1-k)} s^{m+k} e_{\{k\}}^m$$

We want to construct  $(t^i)_{i \in \mathbb{Z}}$ , such that  $(\mathbf{F}(f^\bullet))^i = \lambda_{P^\bullet}^{i-1} t^i + t^{i+1} \mu_{Q^\bullet}^i$  (resp.  $(\mathbf{F}(f^\bullet))^i = \mu_{P^\bullet}^{i-1} t^i + t^{i+1} \lambda_{Q^\bullet}^i$ ) when  $i$  is even (resp. odd). We consider the following two cases:

(i) Let  $i = 2r$  and  $m = Nr$ . We define  $t^i = (t_j^i)_{1 \leq j \leq N-1}$  where

$$t_j^i = \begin{bmatrix} \sum_{k=0}^{N-2} d_{\{N-2-k\}}^{m-(N-1-k)} s^{m+k} e_{\{k\}}^m & \sum_{k=1}^{N-2} d_{\{N-2-k\}}^{m-(N-1-k)} s^{m+k} e_{\{k-1\}}^{m+1} & \cdots & \sum_{k=j-1}^{N-2} d_{\{N-2-k\}}^{m-(N-1-k)} s^{m+k} e_{\{k-j+1\}}^{m+j} \\ 0 & \sum_{k=0}^{N-2} d_{\{N-2-k\}}^{m-(N-2-k)} s^{m+k+1} e_{\{k\}}^{m+1} & \cdots & \sum_{k=l-2}^{N-2} d_{\{N-2-k\}}^{m-(N-2-k)} s^{m+k+1} e_{\{k-j+2\}}^{m+j} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sum_{k=0}^{N-2} d_{\{N-2-k\}}^{m-(N-j-k)} s^{m+k+j-1} e_{\{k\}}^{m+j} \end{bmatrix}$$

(ii) Let  $i = 2r + 1$  and  $m = Nr$ . We define:

$$t^i = (t_j^i)_{1 \leq j \leq N-1} \quad \text{where} \quad t_j^{i+1} = \begin{bmatrix} s^{m+N-1} & 0 & \cdots & 0 \\ 0 & s^{m+N} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s^{m+N+j-2} \end{bmatrix}$$

□

**Lemma 3.6.** *The functor  $\mathbf{F} : \mathbb{K}_N(\text{Prj-}(R)) \rightarrow \mathbb{K}(\text{Prj-}\mathbb{T}_{N-1}(R))$  is a fully faithful functor.*

*Proof.* Let  $P^\bullet$  and  $Q^\bullet$  be two objects in  $\mathbb{C}_N(\text{Prj-}R)$ . We want to show that:

$$\text{Hom}_{\mathbb{K}_N(\text{Prj-}R)}(Q^\bullet, P^\bullet) \cong \text{Hom}_{\mathbb{K}(\text{Prj-}\mathbb{T}_{N-1}(R))}(\mathbf{F}(Q^\bullet), \mathbf{F}(P^\bullet))$$



Let  $f^\bullet : (Q^\bullet, e^\bullet) \longrightarrow (P^\bullet, d^\bullet)$  be a morphism in  $\mathbb{C}_N(\text{Prj-}R)$  such that  $\mathbf{F}(f^\bullet) = 0$  in  $\mathbb{K}(\text{Prj-}\mathbb{T}_{N-1}(R))$ . We want to show that  $f^\bullet = 0$  in  $\mathbb{K}_N(\text{Prj-}R)$ . We construct  $s^m : Q^m \rightarrow P^{m-N+1}$  such that

$$(3.1) \quad f^m = \sum_{k=0}^{N-1} d_{\{N-1-k\}}^{m-(N-1-k)} s^{m+k} e_{\{k\}}^m$$

for all  $m \in \mathbb{Z}$ .

Suppose  $i = Nr$  for some  $r \in \mathbb{Z}$ . Consider the following diagram:

$$\begin{array}{ccccccccccc} Q^i & \xrightarrow{e^i} & Q^{i+1} & \xrightarrow{e^{i+1}} & Q^{i+2} & \xrightarrow{e^{i+2}} & \dots & \longrightarrow & Q^{i+N-2} & \xrightarrow{e^{i+N-2}} & Q^{i+N-1} \\ \downarrow f^i & & \downarrow f^{i+1} & & \downarrow f^{i+2} & & & & \downarrow f^{i+N-2} & & \downarrow f^{i+N-1} \\ P^i & \xrightarrow{d^i} & P^{i+1} & \xrightarrow{d^{i+1}} & P^{i+2} & \xrightarrow{d^{i+2}} & \dots & \longrightarrow & P^{i+N-2} & \xrightarrow{d^{i+N-2}} & P^{i+N-1} \end{array}$$

Since  $\mathbf{F}(f^\bullet) \sim 0^\bullet$  there exists

$$t^n = (t_j^n)_{1 \leq j \leq N-1} \quad \text{where} \quad t_j^n = \begin{bmatrix} \alpha_{11}^n & \alpha_{12}^n & \cdots & \alpha_{1j}^n \\ 0 & \alpha_{22}^n & \cdots & \alpha_{2j}^n \\ 0 & 0 & & \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_{jj}^n \end{bmatrix}$$

such that

$$(\mathbf{F}(f^\bullet))_j^n = \begin{cases} (\lambda_{P^\bullet}^{n-1})_j t_j^n + t_j^{n+1} (\mu_{Q^\bullet}^n)_j & \text{if } n \text{ is even} \\ (\mu_{P^\bullet}^{n-1})_j t_j^n + t_j^{n+1} (\lambda_{Q^\bullet}^n)_j & \text{if } n \text{ is odd} \end{cases}$$

By setting  $n$  equal to  $i-1, i, i+1$  we have the following equations:

$$(3.2) \quad f^{i+j-1} = d^{i+j-2} \alpha_{jj}^i + \alpha_{jj}^{i+1} e_{\{N-1\}}^{i+j-1} = d_{\{N-1\}}^{i-N+j} \alpha_{(j+1)(j+1)}^{i-1} + \alpha_{(j+1)(j+1)}^i e^{j-1}$$

for any  $1 \leq j \leq N-1$ ,

$$(3.3) \quad \alpha_{xy}^i = \sum_{k=x-1}^y d_{\{N-1-k\}}^{i-N+x-1+k} \alpha_{(k+1)(y+1)}^{i-1} + \alpha_{x(y+1)}^i e^{y-1}$$

and

$$(3.4) \quad \alpha_{pq}^i = d^{i-p+1} \alpha_{(p-1)q}^i + \sum_{k=p-1}^q \alpha_{1k}^{i+1} e_{\{N-1-q+k\}}^{q-1}$$

for any  $1 \leq x \leq y \leq N-2$  and  $2 \leq p \leq q \leq N-1$ . We define homotopy maps  $s^m$ ,  $i \leq m \leq i+N-1$  as

$$s^m = \begin{cases} \alpha_{11}^{i+1} & \text{if } m = i+N-1, \\ \alpha_{1(N-1)}^i & \text{if } m = i+N-2, \\ \alpha_{(m-i+2)(m-i+2)}^{i-1} + \sum_{k=1}^{N-2-m+i} d^{m-N} \alpha_{(m-i+1)(k+1)}^{i-1} e_{\{k-1\}}^i & \text{if } i \leq m \leq i+N-3. \end{cases}$$

As  $r$  varies in  $\mathbb{Z}$ , the numbers  $i = Nr$  give us a collection of morphisms as above. So we can construct homotopy maps  $(s^m)_{m \in \mathbb{Z}}$ . If  $i = Nr$ , then it is easy to show that  $f^i, f^{i+1}, \dots, f^{i+N-1}$  and the homotopy maps  $(s^m)_{i \leq m \leq i+2N-1}$  satisfy relation (3.1).

Now let  $\varphi^\bullet \in \text{Hom}_{\mathbb{K}(\text{Prj-T}_{N-1}(R))}(\mathbf{F}(Q^\bullet), \mathbf{F}(P^\bullet))$ . We want to find  $f^\bullet \in \text{Hom}_{\mathbb{K}_N(\text{Prj-R})}(Q^\bullet, P^\bullet)$  such that  $\mathbf{F}(f^\bullet) = \varphi^\bullet$ . Suppose that  $i = Nr$  and  $\varphi^i = (\varphi_j^i)_{1 \leq j \leq N-1}$  where

$$\varphi_j^i = \begin{bmatrix} \beta_{11}^i & \beta_{12}^i & \cdots & \beta_{1j}^i \\ 0 & \beta_{22}^i & \cdots & \beta_{2j}^i \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \beta_{jj}^i \end{bmatrix}$$

We again consider the above diagram. Our goal is to construct  $(f^m)_{i \leq m \leq i+N-1}$ . By assumption we have  $\varphi_j^i(\lambda_{Q^\bullet}^{i-1})_j = (\lambda_{P^\bullet}^{i-1})_j \varphi_j^{i-1}$  and  $\varphi_j^{i+1}(\mu_{Q^\bullet}^i)_j = (\mu_{P^\bullet}^i)_j \varphi_j^i$  for any  $1 \leq j \leq N-1$ . Hence we have the following equations:

$$(3.5) \quad d^{i-1} \beta_{pq}^{i-1} - \beta_{(p+1)q}^{i-1} = -\beta_{p(q-1)}^i + \beta_{pq}^i e^{i+q-2}, \quad 1 \leq p \leq N-1, \quad p < q \leq N-1$$

$$(3.6) \quad d^{i+p-2} \beta_{pp}^{i-1} = \beta_{pp}^i e^{i+p-2}, \quad 1 \leq p \leq N-1$$

and

$$(3.7) \quad \sum_{k=2}^N d_{\{k-1\}}^{i+N-k} \beta_{(N-k+1)(N-1)}^i = \sum_{k=1}^{N-1} \beta_{1k}^{i+1} e_{\{k\}}^{i+N-2}.$$

We define

$$f^i = \sum_{k=2}^{N-1} \beta_{2k}^{i-1} e_{\{k-2\}}^i + \beta_{1(N-1)}^i e_{\{N-2\}}^i.$$

For  $i+1 \leq j \leq i+N-2$  define

$$f^j = \sum_{k=2+j}^{N-1} \beta_{(2+j)k}^{j-2} e_{\{k-(j+2)\}}^{j-2} + \sum_{k=1}^{j+1} d_{\{j+1-k\}}^{j+k-2} \beta_{k(N-1)}^{j-1} e_{\{N-j-2\}}^j,$$

and

$$f^{i+N-1} = \sum_{k=1}^{N-1} \beta_{1k}^{i+1} e_{\{k-1\}}^{i+N-1}.$$

By (3.7) we have

$$d^{i+N-1} f^{i+N-1} = f^{i+N} e^{i+N-1}$$

It is not so hard to see that  $\mathbf{F}(f^\bullet) \sim \varphi^\bullet$ . □

**Lemma 3.7.** *Let  $(P^\bullet, d^\bullet) \in \mathbb{K}_N^{\text{ac}}(\text{Prj-R})$ . The image of  $P^\bullet$  under  $\mathbf{F}$  is an exact complex in  $\mathbb{K}(\text{Prj-T}_{N-1}(R))$ .*

*Proof.* Suppose  $i = 2r$  and  $m = Nr$ . The following diagram shows the image of  $P^\bullet$  under  $\mathbf{F}$  in degree  $i, i + 1, i + 2$  and  $i + 3$ .

$$\begin{array}{ccccccc}
 P^m & \longrightarrow & P^m \oplus P^{m+1} & \longrightarrow & \dots & \longrightarrow & P^m \oplus \dots \oplus P^{m+N-2} \\
 \downarrow \mu_1^i & & \downarrow \mu_2^i & & & & \downarrow \mu_{N-1}^i \\
 P^{m+N-1} & \longrightarrow & P^{m+N-1} \oplus P^{m+N} & \longrightarrow & \dots & \longrightarrow & P^{m+N-1} \oplus \dots \oplus P^{m+2N-3} \\
 \downarrow \lambda_1^{i+1} & & \downarrow \lambda_2^{i+1} & & & & \downarrow \lambda_{N-1}^{i+1} \\
 P^{m+N} & \longrightarrow & P^{m+N} \oplus P^{m+N+1} & \longrightarrow & \dots & \longrightarrow & P^{m+N} \oplus \dots \oplus P^{m+2N-2} \\
 \downarrow \mu_1^{i+2} & & \downarrow \mu_2^{i+2} & & & & \downarrow \mu_{N-1}^{i+2} \\
 P^{m+2N-1} & \longrightarrow & P^{m+2N-1} \oplus P^{m+2N} & \longrightarrow & \dots & \longrightarrow & P^{m+2N-1} \oplus \dots \oplus P^{m+3N-3}
 \end{array}$$

We want to show that  $\text{Im}\mu_j^i = \text{Ker}\lambda_j^{i+1}$  for any  $1 \leq j \leq N - 1$ . Clearly  $\text{Im}\mu_j^i \subseteq \text{Ker}\lambda_j^{i+1}$ . Let  $(x_1, x_2, \dots, x_j) \in \text{Ker}\lambda_j^{i+1}$ . It is easy to show that there exists  $y_t \in P^{m+t-1}$  for all  $1 \leq t \leq j$  such that

$$x_p = \sum_{k=1}^{j-p+1} d_{\{N-k\}}^{i+p-1+(k-1)}(y_{p+k-1}),$$

for all  $1 \leq p \leq t$ . Hence  $(x_1, x_2, \dots, x_j) = \mu_j^i(y_1, y_2, \dots, y_j)$ .

Likewise suppose that  $i = 2r + 1$  and  $m = Nr$ . We show that  $\text{Im}\lambda_j^{i+1} = \text{Ker}\mu_j^{i+2}$  for any  $1 \leq j \leq N - 1$ . Clearly  $\text{Im}\lambda_j^{i+1} \subseteq \text{Ker}\mu_j^{i+2}$ . Let  $(x_1, x_2, \dots, x_j) \in \text{Ker}\mu_j^{i+2}$ . It is easy to show that there exists  $y_t \in P^{m+N-2+t}$  for all  $1 \leq t \leq j$  such that

$$x_j = d^{m+2N-3}(y_j),$$

and

$$x_q = -y_{q+1} + d^{m+N+q-2}(y_q)$$

for any  $1 \leq q \leq t$  and  $q \neq j$ . Hence  $(x_1, x_2, \dots, x_j) = \lambda_j^{i+1}(y_1, y_2, \dots, y_j)$ .  $\square$

**Lemma 3.8.** *The functor  $\mathbf{F} : \mathbb{K}_N(\text{Prj-}(R)) \longrightarrow \mathbb{K}(\text{Prj-}\mathbb{T}_{N-1}(R))$  is an exact triangulated functor.*

*Proof.* Clearly  $\mathbf{F}$  is an exact functor. Since  $\mathbf{F}$  preserves direct sum it is enough to show that  $\mathbf{F}$  transforms  $D_N^j(P)$  to a projective object of  $\mathbb{C}(\text{Prj-}\mathbb{T}_{N-1}(R))$ . By lemma 3.7  $\mathbf{F}(D_N^j(P))$  is a bounded exact complex in  $\mathbb{C}(\text{Prj-}\mathbb{T}_{N-1}(R))$ , Since  $D_N^j(P)$  is an  $N$ -exact complex. Hence  $\mathbf{F}(D_N^j(P))$  is a projective object in  $\mathbb{C}(\text{Prj-}\mathbb{T}_{N-1}(R))$ . Now we show that

$$\mathbf{F}(\Sigma P^\bullet) \cong \mathbf{F}(P^\bullet)[1]$$

Let  $P^\bullet$  be a complex in  $\mathbb{K}_N(\text{Prj-}(R))$ . For  $i \equiv 0 \pmod{2}$  let  $m = \frac{iN}{2}$ . For  $1 \leq j \leq N-1$  and  $1 \leq k \leq j$  let  $\alpha_{j,k}^i : (\Sigma P^\bullet)^{m+k-1} \rightarrow \bigoplus_{l=m+N-1}^{m+N-2+j} P^l$  be a morphism given as

$$\alpha_{j,k}^i = \begin{matrix} 1 \\ 2 \\ \vdots \\ k \\ k+1 \\ \vdots \\ j \end{matrix} \begin{pmatrix} d_{\{N-k-1\}}^{m+k} & d_{\{N-k-2\}}^{m+k+1} & \cdots & d_{\{-k+1\}}^{m+k+N-2} \\ d_{\{N-k\}}^{m+k} & d_{\{N-k-1\}}^{m+k+1} & \cdots & d_{\{-k+2\}}^{m+k+N-2} \\ \vdots & \vdots & & \vdots \\ d_{\{N-2\}}^{m+k} & d_{\{N-3\}}^{m+k+1} & \cdots & d_{\{0\}}^{m+k+N-2} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

These morphisms define a map  $\alpha_j^i : F(\Sigma P^\bullet)_j^i \rightarrow (\mathbf{F}(P^\bullet)[1])_j^i$  as

$$\alpha_j^i = [\alpha_{j,1}^i \quad \alpha_{j,2}^i \quad \cdots \quad \alpha_{j,j}^i].$$

For  $i \equiv 1 \pmod{2}$  let  $m = \frac{(i+1)N}{2}$ . For  $1 \leq j \leq N-1$  and  $1 \leq k \leq j$  let  $\alpha_{j,k}^i : (\Sigma P^\bullet)^{m+k-1} \rightarrow \bigoplus_{l=m}^{m+j-1} P^l$  be a morphism given as

$$\alpha_{j,k}^i = \begin{matrix} 1 \\ \vdots \\ k-1 \\ k \\ k+1 \\ \vdots \\ j \end{matrix} \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & 0 \\ 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

Likewise these morphisms give a map  $\alpha_j^i : F(\Sigma P^\bullet)_j^i \rightarrow (\mathbf{F}(P^\bullet)[1])_j^i$  defined by

$$\alpha_j^i = [\alpha_{j,1}^i \quad \alpha_{j,2}^i \quad \cdots \quad \alpha_{j,j}^i]$$

In the other direction for  $i \equiv 0 \pmod{2}$ , define  $\beta_j^i : (\mathbf{F}(P^\bullet)[1])_j^i \rightarrow F(\Sigma P^\bullet)_j^i$  as

$$\beta_j^i = \begin{bmatrix} \beta_{j,1}^i \\ \beta_{j,2}^i \\ \vdots \\ \beta_{j,j}^i \end{bmatrix}$$

where,

$$\beta_{j,k}^i = \begin{pmatrix} 1 & \cdots & k-1 & k & k+1 & \cdots & j \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \vdots & & \vdots & 0 & \vdots & & \vdots \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}$$

For  $i \equiv 1 \pmod{2}$  define  $\beta_j^i : (\mathbf{F}(P^\bullet)[1])_j^i \rightarrow F(\Sigma P^\bullet)_j^i$  as

$$\beta_j^i = \begin{bmatrix} \beta_{j,1}^i \\ \beta_{j,2}^i \\ \vdots \\ \beta_{j,j}^i \end{bmatrix}$$

where,

$$\beta_{j,k}^i = \begin{bmatrix} 0_{j-k+1 \times k-1} & I_{j-k+1} \\ 0_{k-1 \times k-1} & 0_{k-1 \times j-k+1} \end{bmatrix},$$

in which,  $I_{j-k+1}$  is the identity matrix of order  $j-k+1$ , and the other three entries are zero matrices of given size.

It is not so hard to see that the composition  $\alpha^i \circ \beta^i : (\mathbf{F}(P^\bullet)[1])^i \rightarrow (\mathbf{F}(P^\bullet)[1])^i$  is the identity morphism. One can show that  $\beta \circ \alpha - 1$  is null-homotopic where the homotopy maps are defined as follows: For  $i \equiv 0 \pmod{2}$ ,  $1 \leq j \leq N-1$  and  $1 \leq k \leq j$  let

$$\psi_k^i = \begin{bmatrix} 0_{k \times N-k-1} & 0_{k \times k} \\ -I_{N-k-1} & 0_{N-k-1 \times k} \end{bmatrix}$$

and define

$$s_j^i = \begin{bmatrix} \psi_1^i & \psi_2^i & \cdots & \psi_j^i \\ 0 & \psi_1^i & \ddots & \psi_{j-1}^i \\ \vdots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & \psi_1^i \end{bmatrix}$$

For  $i \equiv 1 \pmod{2}$  and  $1 \leq j \leq N-1$  define  $s_j^i = 0$ . It is quite tedious to show that the morphisms  $\alpha : F(\Sigma P^\bullet) \rightarrow F(P^\bullet)[1]$  and  $\beta : F(P^\bullet)[1] \rightarrow F(\Sigma P^\bullet)$  are natural in  $P^\bullet$ .  $\square$

Putting it all together, we have

**Proposition 3.9.** *The functor  $\mathbf{F} : \mathbb{K}_N(\text{Prj-}R) \longrightarrow \mathbb{K}(\text{Prj-}\mathbb{T}_{N-1}(R))$  is a fully faithful triangle functor.*

**Definition 3.10.** We call an  $N$ -complex  $P^\bullet$ ,  $K$ -projective, if  $P^\bullet \in \mathbb{C}_N(\text{Prj-}R)$  and

$$\text{Hom}_{\mathbb{K}_N(R)}(P^\bullet, Y^\bullet) = 0$$

for all  $Y^\bullet \in \mathbb{K}_N^{\text{ac}}(R)$ . We denote by  $\mathbb{K}_N^{\text{Proj}}(\text{Prj-}R)$  the category of all  $K$ -projective  $N$ -complexes. Dually we define the triangulated full subcategory  $\mathbb{K}_N^{\text{Inj}}(\text{Inj-}R)$  consisting of complexes  $I^\bullet$  of injectives such that  $\text{Hom}_{\mathbb{K}_N(R)}(X^\bullet, I^\bullet) = 0$  for all  $X^\bullet \in \mathbb{K}_N^{\text{ac}}(R)$

**Theorem 3.11.** *We have the following triangle equivalences:*

$$\mathbb{K}_N^{\text{Proj}}(\text{Prj-}R) \cong \mathbb{D}_N(R) \quad , \quad \mathbb{K}_N^{\text{Inj}}(\text{Inj-}R) \cong \mathbb{D}_N(R)$$

*Proof.* See [IKM14, Theorem 2.22.].  $\square$

**Definition 3.12.** For an additive category  $\mathcal{A}$  with arbitrary coproducts, an object  $C$  is called compact in  $\mathcal{A}$  if the canonical morphism  $\coprod_i \text{Hom}_{\mathcal{A}}(C, X_i) \rightarrow \text{Hom}_{\mathcal{A}}(C, \coprod_i X_i)$  is an isomorphism for any coproduct  $\coprod_i X_i$  in  $\mathcal{A}$ . We denote by  $\mathcal{A}^c$  the subcategory of  $\mathcal{A}$  consisting of all compact objects.

**Definition 3.13.** Let  $\mathcal{T}$  be a triangulated category. A non-empty subcategory  $\mathcal{S}$  of  $\mathcal{T}$  is said to be thick if it is a triangulated subcategory of  $\mathcal{T}$  that is closed under retracts. If, in addition,  $\mathcal{S}$  is closed under all coproducts allowed in  $\mathcal{T}$ , then it is localizing; if it is closed under all products in  $\mathcal{T}$  it is colocalizing.

The following remark gives us a better understanding of the objects in a thick subcategory, see [Kra06].

**Remark 3.14.** Let  $\mathcal{S}$  be a class of objects of a triangulated category  $\mathcal{T}$ . Then

- $\text{Thick}(\mathcal{S}) = \bigcup_{n \in \mathbb{N}} \langle \mathcal{S} \rangle_n$ , where
  - $\langle \mathcal{S} \rangle_1$  is the full subcategory of  $\mathcal{T}$  containing  $\mathcal{S}$  and closed under finite direct sums, direct summands and shifts.
  - For  $n > 1$ ,  $\langle \mathcal{S} \rangle_n$  is the full subcategory of  $\mathcal{T}$  consisting of all objects  $S$  such that there is a distinguished triangle  $Y \rightarrow X \rightarrow Z \rightsquigarrow$  in  $\mathcal{T}$  with  $Y \in \langle \mathcal{S} \rangle_i$ , and  $Z \in \langle \mathcal{S} \rangle_j$  such that  $i, j < n$  and  $S$  is a direct summand of shifting of  $X$ .

The following theorem has been proved by Iyama and et. al. in [IKM14]. As a result of proposition 3.9, we present another proof for this theorem.

**Theorem 3.15.** For a ring  $R$ , we have the following triangle equivalence:

$$\mathbb{D}_N(R) \cong \mathbb{D}(\mathbb{T}_{N-1}(R))$$

*Proof.* By theorem 3.11  $\mathbb{K}_N^{\text{Proj}}(\text{Prj-}R) \cong \mathbb{D}_N(R)$  and  $\mathbb{K}_{\text{Proj}}(\text{Prj-}\mathbb{T}_{N-1}(R)) \cong \mathbb{D}(\mathbb{T}_{N-1}(R))$ , and we have the following diagram:

$$\begin{array}{ccc} \mathbb{K}_N(\text{Prj-}R) & \xrightarrow{\mathbf{F}} & \mathbb{K}(\text{Prj-}\mathbb{T}_{N-1}(R)) \\ \uparrow & & \uparrow \\ \mathbb{K}_N^{\text{Proj}}(\text{Prj-}R) & & \mathbb{K}_{\text{Proj}}(\text{Prj-}\mathbb{T}_{N-1}(R)) \end{array}$$

In addition  $\mathbb{D}(\mathbb{T}_{N-1}(R))^c \cong K^{\text{b}}(\text{prj-}\mathbb{T}_{N-1}(R))$ . For  $1 \leq i \leq N-1$  let  $\mathcal{R}_i$  be the following projective representation of  $A_{N-1}$ :

$$0 \rightarrow 0 \rightarrow \cdots \rightarrow R \rightarrow R \rightarrow \cdots \rightarrow R$$

where  $R$  start in  $i$ -th position with identity morphisms afterward.

We can show that  $\mathbb{K}^{\text{b}}(\text{prj-}\mathbb{T}_{N-1}(R)) = \text{Thick}(\{\mathcal{R}_1^\bullet, \dots, \mathcal{R}_{N-1}^\bullet\})$  whenever  $\mathcal{R}_i^\bullet$  is a complex  $\cdots \rightarrow 0 \rightarrow \mathcal{R}_i \rightarrow 0 \rightarrow 0 \rightarrow \cdots$  concentrated in degree 0. Now we show that each  $\mathcal{R}_i^\bullet$  belong to  $\text{Im}\mathbf{F}$ . Let  $R^\bullet$  be a complex  $\cdots \rightarrow 0 \rightarrow R \rightarrow 0 \rightarrow 0 \rightarrow \cdots$  concentrated in degree 0.  $\mathbf{F}(\Sigma(\Theta^{N-2}R^\bullet)) = \mathcal{R}_{N-1}^\bullet$  and  $\mathbf{F}(\Theta^{N-1}R^\bullet) = \mathcal{R}_1^\bullet$ , hence  $\mathcal{R}_{N-1}^\bullet, \mathcal{R}_1^\bullet \in \text{Im}\mathbf{F}$ . On the other hand  $\mathbf{F}(\Theta^{N-3}R^\bullet) = \cdots \rightarrow 0 \rightarrow \mathcal{R}_{N-1} \rightarrow \mathcal{R}_{N-2} \rightarrow 0 \rightarrow \cdots$  and therefore there exists a short exact sequence  $0^\bullet \rightarrow \mathcal{R}_{N-2}^\bullet \rightarrow \mathbf{F}(\Theta^{N-3}R^\bullet) \rightarrow \mathcal{R}_{N-1}^\bullet[1] \rightarrow 0^\bullet$  with degree-wise split exact sequences. So there exists a triangle  $\mathcal{R}_{N-2}^\bullet \rightarrow \mathbf{F}(\Theta^{N-3}R^\bullet) \rightarrow \mathcal{R}_{N-1}^\bullet[1] \rightsquigarrow$  in  $\mathbb{K}(\text{Prj-}\mathbb{T}_{N-1}(R))$ , hence  $\mathcal{R}_{N-2}^\bullet \in \text{Im}\mathbf{F}$ , since  $\mathbf{F}(\Theta^{N-3}R^\bullet), \mathcal{R}_{N-1}^\bullet[1] \in \text{Im}\mathbf{F}$ . Similarly by induction we can say that  $\mathcal{R}_i^\bullet \in \text{Im}\mathbf{F}$  for  $2 \leq i \leq N-3$ . Hence  $\text{Thick}(\{\mathcal{R}_1^\bullet, \dots, \mathcal{R}_{N-1}^\bullet\}) \subseteq \text{Im}\mathbf{F} \subseteq \mathbb{D}(\mathbb{T}_{N-1}(R))$ . But  $\text{Im}\mathbf{F}$  is closed under coproduct and contains compact objects, therefore the restriction of functor  $\mathbf{F}$  to  $\mathbb{D}_N(R)$  is dense hence  $\mathbb{D}_N(R) \cong \mathbb{D}(\mathbb{T}_{N-1}(R))$ .  $\square$

According to the above theorem, we have a triangle equivalence

$$\mathbb{D}_N^-(R) \cong \mathbb{D}^-(\mathbb{T}_{N-1}(R))$$

thus  $F|_{\mathbb{K}_N^-(\text{Prj-}R)} : \mathbb{K}_N^-(\text{Prj-}R) \xrightarrow{\cong} \mathbb{K}^-(\text{Prj-}\mathbb{T}_{N-1}(R))$ , see [IKM14, Corollaries 4.11, 2.17].

Moreover  $F|_{\mathbb{K}_N^-(\text{Prj-}R)}$  induces some triangle equivalences between subcategories of  $\mathbb{K}_N^-(\text{Prj-}R)$  and subcategories of  $\mathbb{K}^-(\text{Prj-}\mathbb{T}_{N-1}(R))$ , see [IKM14, Corollary 4.15]. We summarize all of these equivalences in the following diagram. Note that the existence of the first row follows from proposition 3.9.

$$\begin{array}{ccc}
 \mathbb{K}_N(\text{Prj-}R) & \xrightarrow{\mathbf{F}} & \mathbb{K}(\text{Prj-}\mathbb{T}_{N-1}(R)) \\
 \uparrow & & \uparrow \\
 \mathbb{K}_N^-(\text{Prj-}R) & \xrightarrow{\cong} & \mathbb{K}^-(\text{Prj-}\mathbb{T}_{N-1}(R)) \\
 \uparrow & & \uparrow \\
 \mathbb{K}_N^{-,b}(\text{Prj-}R) & \xrightarrow{\cong} & \mathbb{K}^{-,b}(\text{Prj-}\mathbb{T}_{N-1}(R)) \\
 \uparrow & & \uparrow \\
 \mathbb{K}_N^b(\text{Prj-}R) & \xrightarrow{\cong} & \mathbb{K}^b(\text{Prj-}\mathbb{T}_{N-1}(R)) \\
 \uparrow & & \uparrow \\
 \mathbb{K}_N^b(\text{prj-}R) & \xrightarrow{\cong} & \mathbb{K}^b(\text{prj-}\mathbb{T}_{N-1}(R))
 \end{array}$$

At the end of this section we show that the functor  $\mathbf{F}$  is dense, hence there exist an triangle equivalence between  $\mathbb{K}_N(\text{Prj-}R)$  and  $\mathbb{K}(\text{Prj-}\mathbb{T}_{N-1}(R))$ . Before we give the proof we need to introduce a another functor.

Let  $Q$  be the quiver of type  $A_n$ . Any projective representation  $\mathcal{P}$  of  $Q$  is of the form  $\mathcal{P} = \bigoplus_{i=1}^n e_\lambda^i(P^i)$ , where  $P^i$  is the cokernel of split monomorphism  $\mathcal{P}_{i-1} \rightarrow \mathcal{P}_i$ . For any projective representation  $\mathcal{P} = \bigoplus_{i=1}^n e_\lambda^i(P^i)$  of  $Q$ , set  $\widehat{\mathcal{P}} = \bigoplus_{i=1}^n e_\rho^i(P^i)$ . Clearly  $\widehat{\mathcal{P}}$  is an object in the category  $\text{Prj-}^{op}(A_n)$ , where  $\text{Prj-}^{op}(A_n)$  is the category of all representations by projective modules with split epimorphism maps.

Now we define a functor  $\widehat{\cdot} : \text{Prj-}A_n \rightarrow \text{Prj-}^{op}A_n$  such that any  $\mathcal{P} \in \text{Prj-}A_n$  is mapped under  $\widehat{\cdot}$  to  $\widehat{\mathcal{P}}$ , as defined above, and for any morphism  $\varphi = (\varphi_t)_{1 \leq t \leq n}$  in  $\text{Hom}(e_\lambda^i(P^i), e_\lambda^j(P^j))$  define  $\widehat{\varphi}$  as follows:

$$\widehat{\varphi} = \begin{cases} 0 & i < j \\ (\varphi_t)_t & i \geq j \end{cases}$$

It is easy to check that  $\widehat{\cdot}$  is in fact an equivalence of categories. We also see that for a finite quiver  $Q$ ,  $\widehat{\cdot}$  is an equivalence of categories, see [AEHS11]. The functor  $\widehat{\cdot}$  can be naturally extended to a functor  $\mathbb{K}(\text{Prj-}A_n) \rightarrow \mathbb{K}(\text{Prj-}^{op}A_n)$  which we denote again by  $\widehat{\cdot}$ . So for any  $X^\bullet \in \mathbb{K}(\text{Prj-}A_n)$ , let  $\widehat{X}^\bullet$  be the complex with  $\widehat{X}_i$  as its  $i$ -th term and  $\widehat{d}_i$  as its  $i$ -th differential. The functor  $\widehat{\cdot}$  also is an equivalence of homotopy categories.

we also need the following description of compact objects in  $\mathbb{K}_N(\text{Prj-}R)$ . Neeman [Nee08] showed that an object  $X^\bullet$  of  $\mathbb{K}(\text{Prj-}R)$  is compact if and only if it is isomorphic, in  $\mathbb{K}(\text{Prj-}R)$ , to a complex  $\mathbf{Y}$  satisfying

- (i)  $\mathbf{Y}$  is a complex of finitely generated projective modules.
- (ii)  $Y^i = 0$  if  $i \ll 0$ .
- (iii)  $H^i(\mathbf{Y}^*) = 0$  if  $i \ll 0$ , where  $\mathbf{Y}^* = \text{Hom}(\mathbf{Y}^\bullet, R)$ .

He also showed that when the compact objects generate the category  $K(\text{Prj-}R)$ .

**Proposition 3.16.** *If  $R$  is a left coherent ring, then the category  $\mathbb{K}(\text{Prj-}R)$  is compactly generated.*

This idea enables us to prove our main theorem:

**Theorem 3.17.** *For a left coherent ring  $R$ , we have triangle equivalence*

$$\mathbb{K}_N(\text{Prj-}R) \cong \mathbb{K}(\text{Prj-}\mathbb{T}_{N-1}(R)).$$

*Proof.* In view of proposition 3.9, the functor  $\mathbf{F}$  is full and faithful. Now we show that  $\mathbf{F}$  is dense. Let  $\mathcal{T}$  be a triangle subcategory of  $\mathbb{K}_N(\text{Prj-}R)$  such that for all  $\mathbf{T} \in \mathcal{T}$  we have the following conditions

- (1)  $\mathbf{T} \in \mathbb{K}_N^+(\text{prj-}R)$ ;
- (2) There exists an integer  $n \in \mathbb{Z}$  such that for every  $i < n$  and  $1 \leq r \leq N-1$ ,  $H_r^i \mathbf{T}^* = 0$  where  $\mathbf{T}^*$  denotes the induced complex  $\text{Hom}(\mathbf{T}, R)$ .

Clearly the duality  $\text{Hom}(-, R) : \text{prj-}R \rightarrow \text{prj-}R^{op}$  induces an equivalence  $\mathcal{T}^{op} \cong \mathbb{K}_N^{-,b}(\text{prj-}R^{op})$  of triangulated categories. On the other hand, by restricting the functor  $\mathbf{F}$  to  $\mathcal{T}$ , we have

$$\begin{array}{ccc} \mathbb{K}_N(\text{Prj-}R) & \xrightarrow{\mathbf{F}} & \mathbb{K}(\text{Prj-}\mathbb{T}_{N-1}(R)) \\ \uparrow & & \uparrow \\ \mathcal{T} & \xrightarrow{\mathbf{F}|_{\mathcal{T}}} & \mathbb{K}(\text{prj-}\mathbb{T}_{N-1}(R))^c \end{array}$$

We want to show that  $\mathbf{F}|_{\mathcal{T}}$  is an equivalence. It is easy to check that we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{T} & \xrightarrow{\mathbf{F}|_{\mathcal{T}}} & \mathbb{K}(\text{prj-}\mathbb{T}_{N-1}(R))^c \\ \cong \downarrow \text{Hom}(-, R) & \sim \circ \text{Hom}(-, R) \uparrow \cong & \\ \mathbb{K}_N^{-,b}(\text{prj-}R^{op}) & \xrightarrow[\cong]{\mathbf{F}^{op}} & \mathbb{K}_N^{-,b}(\text{prj-}\mathbb{T}_{N-1}(R^{op})) \end{array}$$

where  $\mathbf{F}^{op}$  is the functor  $\mathbf{F}$  in construction 3.3 whenever we define it on  $\mathbb{C}_N(\text{Prj-}R^{op})$  and  $\sim$  is the quasi-inverse of the functor  $\hat{\cdot}$ . Therefore  $\mathbf{F}|_{\mathcal{T}} = \sim \circ \text{Hom}(-, R) \circ \mathbf{F}^{op} \circ \text{Hom}(-, R)$ , hence it is an equivalence. It shows that  $\mathbb{K}(\text{prj-}\mathbb{T}_{N-1}(R))^c \subseteq \text{Im} \mathbf{F}$ . On the other hand  $\text{Im} \mathbf{F}$  is closed under coproduct and contains compact objects, therefore  $\text{Im} \mathbf{F} = \mathbb{K}(\text{Prj-}\mathbb{T}_{N-1}(R))$ .  $\square$

**Corollary 3.18.** *If  $R$  is a left coherent ring, then the category  $\mathbb{K}_N(\text{Prj-}R)$  is compactly generated.*

In a dual manner, in view of construction 3.3 and lemmas 3.5, 3.6 and 3.8 we can embed the category  $\mathbb{K}_N(\text{Inj-}R)$  into the category  $\mathbb{K}(\text{Inj-}\mathbb{T}_{N-1}(R))$ . Since the compact objects of  $\mathbb{K}(\text{Inj-}\mathbb{T}_{N-1}(R))$  are different from  $\mathbb{K}(\text{Prj-}\mathbb{T}_{N-1}(R))$ , the proof of theorem 3.17 dose not work. However, when  $R$  is an artin algebra the embedding is dense.

**Proposition 3.19.** *Let  $\Lambda$  be an artin algebra. We have a triangle equivalence*

$$\mathbb{K}_N(\text{Inj-}\Lambda) \cong \mathbb{K}(\text{Inj-}\mathbb{T}_{N-1}(\Lambda)).$$



*Proof.* Let  $D$  denote the duality between right and left  $\Lambda$ -modules. The adjoint pair of functors  $-\otimes_{\Lambda} D(\Lambda)$  and  $\text{Hom}_{\Lambda}(D(\Lambda), -)$  induces an equivalence between  $\text{Prj-}\Lambda$  and  $\text{Inj-}\Lambda$ , which restricts to an equivalence between  $\text{prj-}\Lambda$  and  $\text{inj-}\Lambda$ . Therefore the adjoint pair induces an equivalence between  $\mathbb{K}_N(\text{Prj-}\Lambda)$  and  $\mathbb{K}_N(\text{Inj-}\Lambda)$ . So if we denote the embedding  $\mathbb{K}_N(\text{Inj-}R) \hookrightarrow \mathbb{K}(\text{Inj-}\mathbb{T}_{N-1}(R))$  by  $\mathbf{G}$ , then we have the following diagram:

$$\begin{array}{ccc} \mathbb{K}_N(\text{Prj-}\Lambda) & \xrightarrow{\cong} & \mathbb{K}_N(\text{Inj-}\Lambda) \\ \downarrow \cong & & \downarrow \mathbf{G} \\ K(\text{Prj-}\mathbb{T}_{N-1}(\Lambda)) & \xrightarrow{\cong} & K(\text{Inj-}\mathbb{T}_{N-1}(\Lambda)) \end{array}$$

Hence  $\mathbf{G}$  is an equivalence of categories.  $\square$

As a final remark we will discuss about  $N$ -dualizing complex.

**Remark 3.20.** Let  $R$  be a commutative noetherian ring with a dualizing complex  $D$  and  $Q$  be a finite quiver. In [AEHS11] they showed that Grothendieck duality is extendable to path algebra  $RQ$ . Therefore when  $Q = A_{N-1}$ , we have

$$\mathbb{D}^b(\text{mod-}\mathbb{T}_{N-1}(R))^{op} \cong \mathbb{D}^b(\text{mod-}\mathbb{T}_{N-1}(R)).$$

On the other hand by [IKM14] we have

$$\mathbb{D}_N^b(\text{mod-}R) \cong \mathbb{D}^b(\text{mod-}\mathbb{T}_{N-1}(R)).$$

Hence

$$\mathbb{D}_N^b(\text{mod-}R)^{op} \cong \mathbb{D}_N^b(\text{mod-}R).$$

So in case  $R$  has a dualizing complex, there exists a Grothendieck duality for derived category of  $N$ -complexes and the question is "What could be the definition of an  $N$ -dualizing complex?".

#### 4. N-TOTALLY ACYCLIC COMPLEXES:

Let  $\mathcal{A}$  be an additive category. We say that a complex  $X^\bullet$  in  $\mathcal{A}$  is acyclic if the complex  $\text{Hom}_{\mathcal{A}}(A, X^\bullet)$  of abelian groups is acyclic for all  $A \in \mathcal{A}$ . If in addition  $\text{Hom}_{\mathcal{A}}(X^\bullet, A)$  is acyclic for all  $A \in \mathcal{A}$ , then  $X^\bullet$  is called totally acyclic. Let  $\mathbb{C}_{\text{tac}}(\mathcal{A})$  denote the full subcategory of  $\mathbb{C}(\mathcal{A})$  consisting of totally acyclic complexes. Note that these definitions are up to isomorphism in  $\mathbb{K}(\mathcal{A})$ . The full triangulated subcategory of  $\mathbb{K}(\mathcal{A})$  consisting of totally acyclic complexes, will be denoted by  $\mathbb{K}_{\text{tac}}(\mathcal{A})$ . For instance if  $\mathcal{A} = \text{Prj-}R$  is the class of projective objects in  $\text{Mod-}R$  then the object  $X^\bullet$  of  $\mathbb{K}_{\text{tac}}(\text{Prj-}R)$  is an exact complex such that  $\text{Hom}_R(X^\bullet, P)$  is acyclic for all  $P \in \text{Prj-}R$ . The objects of  $\mathbb{K}_{\text{tac}}(\text{Prj-}R)$  will be called totally acyclic complexes of projectives

**Remark 4.1.** It is easy to see that  $X^\bullet \in \mathbb{K}_{\text{tac}}(\text{Prj-}R)$  if and only if  $\text{Hom}_{\mathbb{K}(\text{Prj-}R)}(P^\bullet, X^\bullet) = 0$  and  $\text{Hom}_{\mathbb{K}(\text{Prj-}R)}(X^\bullet, P^\bullet) = 0$  for all  $P^\bullet \in \mathbb{K}^b(\text{Prj-}R)$ .

Let  $R$  be a Noetherian ring. For the rest of this section we are only considering the category  $\text{prj-}R$ , i.e. the category of finitely generated projective left  $R$ -modules. Given an integer  $n$  and a complex

$$\dots \xrightarrow{d^{n-3}} X^{n-2} \xrightarrow{d^{n-2}} X^{n-1} \xrightarrow{d^{n-1}} X^n \xrightarrow{d^n} X^{n+1} \xrightarrow{d^{n+1}} X^{n+2} \xrightarrow{d^{n+2}} \dots$$

in  $\text{Mod-}R$ , we denote its brutal truncation at degree  $n$  by

$$\beta_{\leq n}(X^\bullet) : \cdots \xrightarrow{d^{n-3}} X^{n-2} \xrightarrow{d^{n-2}} X^{n-1} \xrightarrow{d^{n-1}} X^n \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

If  $X^\bullet \in \mathbb{K}_{\text{tac}}(\text{prj-}R)$ , then  $\beta_{\leq n}(X^\bullet) \in \mathbb{K}^{-,b}(\text{prj-}R)$ . The brutal truncation at degree zero induces a map from the category  $\mathbb{K}_{\text{tac}}(\text{prj-}R)$  to the category  $\mathbb{K}^{-,b}(\text{prj-}R)$ . However, this map is not a functor. Now consider the singularity category

$$\mathbb{D}_{\text{sg}}^b(R) = \mathbb{K}^{-,b}(\text{prj-}R)/\mathbb{K}^b(\text{prj-}R).$$

The brutal truncation induces a triangle functor

$$\beta_{\text{proj}} : \mathbb{K}_{\text{tac}}(\text{prj-}R) \longrightarrow \mathbb{D}_{\text{sg}}^b(R).$$

This functor is always full and faithful. If  $R$  is either an Artin ring or commutative Noetherian local ring, then the functor  $\beta_{\text{proj}}$  is dense if and only if  $R$  is Gorenstein, see [Buc87], [Hap91] and [BJO14].

Motivated by the discussion above, in this section we want to introduce  $N$ -totally acyclic complexes and  $N$ -singularity category. We show that the restriction of functor  $\mathbf{F}$  to this category is an equivalence. As a result of this equivalence, we show that there exists equivalences between  $N$ -singularity category of  $\text{Mod-}R$  and usual singularity category of  $\text{Mod-}\mathbb{T}_{N-1}(R)$ . It is easy to see that an  $N$ -complex  $X^\bullet \in \mathbb{K}_N(\text{prj-}R)$  is  $N$ -acyclic if and only if

$$\text{Hom}_{\mathbb{K}_N(\text{prj-}R)}(P^\bullet, X^\bullet) = 0$$

for all  $P^\bullet \in \mathbb{K}^b(\text{prj-}R)$ .

**Definition 4.2.** An  $N$ -complex  $X^\bullet \in \mathbb{K}_N(\text{prj-}R)$  is called  $N$ -totally acyclic if and only if  $\text{Hom}_{\mathbb{K}_N(\text{prj-}R)}(P^\bullet, X^\bullet) = 0$  and  $\text{Hom}_{\mathbb{K}_N(\text{prj-}R)}(X^\bullet, P^\bullet) = 0$  for all  $P^\bullet \in \mathbb{K}^b(\text{prj-}R)$ . We denote by  $\mathbb{K}_N^{\text{tac}}(\text{prj-}R)$  the category of all  $N$ -totally acyclic complexes in  $\text{prj-}R$ .

Clearly, if  $X^\bullet \in \mathbb{K}_N^{\text{tac}}(\text{prj-}R)$  then  $X^\bullet$  is an  $N$ -acyclic complex.

**Proposition 4.3.** For a left coherent ring  $R$ , we have the following triangle equivalences.

- (i)  $\mathbb{K}_N^{\text{ac}}(\text{prj-}R) \cong \mathbb{K}_{\text{ac}}(\text{prj-}\mathbb{T}_{N-1}(R))$ .
- (ii)  $\mathbb{K}_N^{\text{tac}}(\text{prj-}R) \cong \mathbb{K}_{\text{tac}}(\text{prj-}\mathbb{T}_{N-1}(R))$ .

*Proof.* (i) By theorem 3.17, the functor  $\mathbf{F}$  induced an equivalence

$$\begin{array}{ccc} \mathbb{K}_N(\text{Prj-}R) & \xrightarrow[\cong]{\mathbf{F}} & \mathbb{K}(\text{Prj-}\mathbb{T}_{N-1}(R)) \\ \uparrow & & \uparrow \\ \mathbb{K}_N^{\text{ac}}(\text{prj-}R) & & \mathbb{K}_{\text{ac}}(\text{prj-}\mathbb{T}_{N-1}(R)) \end{array}$$

Let  $P^\bullet$  be an object of  $\mathbb{K}_N^{\text{ac}}(\text{prj-}R)$ . Hence  $\text{Hom}_{\mathbb{K}_N(\text{prj-}R)}(Q^\bullet, P^\bullet) = 0$  for all  $Q^\bullet \in \mathbb{K}_N^b(\text{prj-}R)$ . Now let  $P^\bullet \in \mathbb{K}^b(\text{prj-}\mathbb{T}_{N-1}(R))$ . There exists an object  $Q^\bullet \in \mathbb{K}_N^b(\text{prj-}R)$  such that  $\mathbf{F}(Q^\bullet) = P^\bullet$ . We have

$$\text{Hom}_{\mathbb{K}(\text{prj-}\mathbb{T}_{N-1}(R))}(P^\bullet, \mathbf{F}(P^\bullet)) = \text{Hom}_{\mathbb{K}(\text{prj-}\mathbb{T}_{N-1}(R))}(\mathbf{F}(Q^\bullet), \mathbf{F}(P^\bullet)) \cong \text{Hom}_{\mathbb{K}_N(\text{prj-}R)}(Q^\bullet, P^\bullet) = 0.$$

It shows that  $\mathbf{F}(P^\bullet) \in \mathbb{K}_{\text{ac}}(\text{prj-}\mathbb{T}_{N-1}(R))$ . Hence the functor  $\mathbf{F}$  sends any object of the subcategory  $\mathbb{K}_N^{\text{ac}}(\text{prj-}R)$  of  $\mathbb{K}_N(\text{Prj-}R)$  to an object of the subcategory  $\mathbb{K}_{\text{ac}}(\text{prj-}\mathbb{T}_{N-1}(R))$  of  $\mathbb{K}(\text{Prj-}\mathbb{T}_{N-1}(R))$ .

Let  $P^\bullet \in \mathbb{K}_{\text{ac}}(\text{prj-}\mathbb{T}_{N-1}(R))$ . In a similar way there exists  $P^\bullet \in \mathbb{K}_N^{\text{ac}}(\text{prj-}R)$  such that  $\mathbf{F}(P^\bullet) =$

$\mathcal{P}^\bullet$ ; Hence  $\mathbf{F}|_{\mathbb{K}_N^{\text{ac}}(\text{prj-}R)}$  is dense.

(ii) It is similar to (i) by use of remark 4.1.  $\square$

We define the  $N$ -singularity category  $\mathbb{D}_N^{\text{sg}}(R)$  of  $R$  as a Verdier quotient

$$\mathbb{K}_N^{-,b}(\text{prj-}R)/\mathbb{K}_N^b(\text{prj-}R).$$

**Remark 4.4.** The brutal truncation at degree zero induces a triangle functor

$$\mathbb{K}_N^{\text{tac}}(\text{prj-}R) \longrightarrow \mathbb{D}_N^{\text{sg}}(R),$$

and the following diagram shows that this functor is always full and faithful. Moreover it is a triangle equivalence of categories when  $R$  is a Gorenstein ring.

$$\begin{array}{ccc} \mathbb{K}_N^{\text{tac}}(\text{prj-}R) & \xrightarrow{\quad} & \mathbb{D}_N^{\text{sg}}(R) = \mathbb{K}_N^{-,b}(\text{prj-}R)/\mathbb{K}_N^b(\text{prj-}R) \\ \cong \downarrow \mathbf{F} & & \cong \downarrow \tilde{\mathbf{F}} \\ \mathbb{K}_{\text{tac}}(\text{prj-}\mathbb{T}_{N-1}(R)) & \xrightarrow{\beta_{\text{proj}}} & \mathbb{D}_{\text{sg}}^b(\mathbb{T}_{N-1}(R)) = \mathbb{K}^{-,b}(\text{prj-}\mathbb{T}_{N-1}(R))/\mathbb{K}^b(\text{prj-}R) \end{array}$$

The functor  $\tilde{\mathbf{F}}$  is induced from equivalences in diagram after theorem 3.15.

Remark 4.4 provides us with another interpretation of quotient category

$$\mathbb{K}^{\infty,b}(\text{prj-}R)/\mathbb{K}^b(\text{prj-}R)$$

where  $\mathbb{K}^{\infty,b}(\text{prj-}R)$  is the homotopy category of unbounded complexes with bounded homologies. Iyama and et. al. showed that there is a triangle equivalence between the above quotient category and  $\mathbb{D}_{\text{sg}}^b(\mathbb{T}_2(R))$ , whenever  $R$  is a Gorenstein ring, see [IKM11]. Hence by remark 4.4 we have the following equivalence of categories

$$\mathbb{K}^{\infty,b}(\text{prj-}R)/\mathbb{K}^b(\text{prj-}R) \cong \mathbb{D}_3^{\text{sg}}(R).$$

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