

# Casimir force for geometrically confined ideal Bose gas in a harmonic-optical potential

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In this study, we have derived close form of the Casimir force for the non-interacting ideal Bose gas between two slabs in harmonic-optical lattice potential by using Ketterle and van Druten approximation. We find that Bose-Einstein condensation temperature  $T_c$  is a critical point for different physical behavior of the Casimir force. We have shown that Casimir force of confined Bose gas in the presence of the harmonic-optical potential decays with inversely proportional to  $d^5$  when  $T \leq T_c$ . However, in the case of  $T > T_c$ , it decays exponentially depends on separation  $d$  of the slabs. Additionally we have discussed temperature dependence of Casimir force and importance of the harmonic-optical lattice potential on quantum critical systems, quantum phase transition and nano-devices.

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## I. INTRODUCTION

The Casimir effect is an attractive or repulsive long-range interaction between conducting boundaries [1], which originates from the quantum mechanical fluctuations of the electromagnetic field in vacuum. The electromagnetic field is confined in between plates with distance  $d$ , the fluctuations modes of electromagnetic fields need to have a node at the plate surfaces, so that only waves with wavelength  $\lambda = nd/2$  and integer  $n$  are permitted in the gap between plates. As a consequence the total field outside the gap produces a pressure on the plates which is higher than the one produced from inside, so the surfaces are pushed together by force. This effect has been experimentally measured in 1997 [2] and 1998 [3] which confirms quantum field theory. It is shown that the force depends on size, temperature, geometry, surface roughness and electronic properties of the materials [4–6]. Its importance for practical applications is now becoming more widely appreciated in quantum field theory, gravitation and cosmology, Bose-Einstein condensation (BEC), atomic and molecular systems, mathematical physics and nano-technology. A good review on new developments in the Casimir effect can be found in Ref. [6].

Now it is well-known that the fluctuations which can emerge from different physical mechanisms can lead to Casimir-like effect if it confined in the boundaries. For example, the thermal fluctuation in confined quantum critical systems can leads to Casimir-like effect. This emerges upon approaching a second-order phase transition point which is characterized by the fact that due to the emerging collective behavior of the particles depends on thermal fluctuation. The effective force resulting from the confinement of the fluctuations of the order parameter is called Casimir force. This phenomena firstly was predicted by Fisher and de Gennes [7]. They showed that the fluctuation-induced force arise in critical systems which are close to the critical point, where dynamics are

governed by the thermal fluctuations of the order parameter upon approach to the thermal phase transition. This effect later is called as the critical Casimir effect for quantum particle systems [8]. Firstly in Ref. [9], the Casimir force between two slabs immersed in a perfect gas was calculated for various boundary conditions. It was found that the Casimir force has the standard asymptotic form with universal Casimir terms below the BEC critical temperature  $T_c$  and vanishes exponentially above  $T_c$ . After these seminal works, a great deal of effort has been devoted to calculation of the Casimir force of the free and trapped Bose gas for different geometries and different boundary conditions [10–30].

Another mechanism of Casimir-like effect arises from quantum size effect. It is expected that Casimir-like effects may occur in geometrically confined quantum gas because of the wave character of the gas atoms at nano scale. Indeed it is shown that in the presence of boundaries inside quantum particles can cause Casimir-like effect if the bounded space is smaller than the maximum thermal de Broglie wavelength of the particles [31–42]. Therefore, nowadays it has been attracted to Casimir force at nano scale because of the potential applications in nano-technology area since Casimir phenomena is a typically quantum size effect which appears at nano scale. In this area, researchers aim to produce many nano-device structures such as gas turbines, pumps, mixers, heat exchangers, valves, etc [38–42]. Therefore understanding the Casimir effect on the thermodynamic behavior of quantum gases confined within boundaries at nano scale in nano-devices is very significant problem. Because new nano-devices and technologies can be developed based on the effects appearing at this scale.

The subject of Casimir effect on thermodynamic behaviors of gases at nano scale and in quantum critical systems is a new research area and may have many potential applications in nano technology and other area in physics. In this regard, discussing on the Casimir effect in the previous theoretical studies in Refs. [7–42] have potential to enlighten the thermodynamic behavior of confined quantum gases. So far Casimir force has been cal-

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culated for either free or trapped Bose gas with harmonic potential [9–30]. As it is known that the harmonic-optical potential which consists of combining harmonic and optical lattice potentials plays very important role in BEC systems as well as in other area of physics. The effects of harmonic-optical lattice potential on the BEC and its vortices and quantum phase transition have been considered in Refs. [43–55]. But to our knowledge Casimir force of Bose gas trapped with harmonic-optical potential has never been studied. In this study we investigate Casimir force of the Bose gas trapped with harmonic-optical potential between two infinite parallel slabs in the  $x - y$  direction that are separated by a distance  $d$  in the  $z$ -direction. We obtain explicit form of the Casimir force of the ideal Bose gas trapped in harmonic-optical potential and we show that Casimir force decays with inversely proportional to  $d^5$  when  $T \leq T_c$  while it decays exponentially for  $T > T_c$ .

## II. MODEL AND ANALYTICAL RESULTS

### A. Theoretical framework

In the original paper [1] Casimir force of vacuum fluctuation of electromagnetic field at zero temperature ( $T = 0$ ) was defined as

$$F_C = -\frac{\partial}{\partial d} [E(d) - E(\infty)] \quad (1)$$

where  $E(d)$  is the ground-state energy (i.e. the vacuum energy) of the electromagnetic field in between the two conducting plates separated at a distance  $d$ .  $E(\infty)$  is the energy at infinite. We consider the Casimir effect for the thermodynamical system of Bose gas between two infinite slabs. The geometry of the system on which some external boundary condition can be imposed is responsible for the Casimir effect. In recent studies it has been shown that the Casimir force for critical particle systems can be obtained from Casimir potential at finite temperature [9–22] as

$$F_C(T, \mu, d) = -\frac{\partial}{\partial d} [\varphi_C(T, \mu, d) - \varphi_C(T, \mu, \infty)] \quad (2)$$

where  $\varphi_C(T, \mu, d)$  and  $\varphi_C(T, \mu, \infty)$  are the Casimir potential corresponds to fluctuations of thermally excited states of Bose gas at distance  $d$  and at infinite, respectively [8–13]. The Casimir potentials in Eq. (2) can be obtained from grand canonical potential of the particle system

$$\varphi(T, \mu, d) = -\beta^{-1} \sum_n^{\infty} \ln \left\{ 1 - e^{-\beta(\varepsilon_n - \mu)} \right\} \quad (3)$$

where  $\varepsilon_n$  is the energy of Bose gas, which corresponds to energy eigenvalue of the particles. Additionally  $\beta = (kT)^{-1}$ ,  $k$  is Boltzmann's constant,  $T$  is temperature, and  $\mu$  is the chemical potential of the system. For simplicity we consider single particle energy instead of  $N$  particle Bose system.

### B. Single particle energy

In present model, we assume that ideal Bose gas is trapped in two dimensional harmonic-optical potential between two slabs separated at a distance  $d$ . If quantum particles move in three dimensional space in the absence external potential, single particle energy at finite temperature can be given by  $\varepsilon(p_x, p_y, p_z) = p_x^2/2m + p_y^2/2m + p_z^2/2m$  where  $m$  is the mass of the particle, and  $p_x, p_y$  and  $p_z$  are the momentum along respectively  $x, y$  and  $z$  axis. However in our model particles are trapped in between quasi-two-dimensional slabs with two dimensional potential. Hence the energy eigenvalues of particles at  $x$  and  $y$ -directions depend on the harmonic-optical potentials. For a simplicity, it can be assumed for present model that particle energy  $\varepsilon$  depends on quantum number  $n_{x,y}$  in  $x$  and  $y$ -directions and in the  $z$ -direction, energy of the particle is quantized as  $\varepsilon_{n_z} = (a\pi^2\hbar^2 n_z^2)/(2md^2)$  where  $n_z = 0, 1, 2, 3, \dots$  and  $a$  indicates type of the boundary condition of the system. To define boundary conditions, it can be set as  $a = 1$  and  $n_z = 1, 2, 3, \dots$  for Dirichlet,  $a = 1$  and  $n_z = 0, 1, 2, \dots$  for Neumann, and  $a = 2$  and  $n_z = 0, \pm 1, \pm 2, \dots$  for periodic. Therefore, the single particle energy for this model is arranged in the form

$$\varepsilon_{n_{x,y,z}} = \varepsilon_{n_{x,y}} + \frac{a\hbar^2\pi^2}{2md^2} n_z^2. \quad (4)$$

where  $\varepsilon_{n_{x,y}}$  corresponds to two dimensional quantized energy of the particle in the  $x - y$  directions,  $\varepsilon_{n_{x,y,z}}$  corresponds to energy of the particles for all degrees of freedom. In order to obtain grand canonical potential of an trapped ideal Bose gas we have to compute single particle energy in Eq. (4) for the Bose gas.

The first term of rhs in Eq. (4) must be determined to obtain exact expression for single particle energy  $\varepsilon_{n_{x,y,z}}$ . The particle energy  $\varepsilon_{n_{x,y}}$  in  $x - y$  plane can be determined depending on harmonic-optical potential

$$V(x, y) = V_{har}(x, y) + V_{lat}(x, y) \quad (5)$$

where harmonic potential and optical lattice potential are respectively given

$$V_{har}(x, y) = \frac{1}{2}m(\omega_x^2 x^2 + \omega_y^2 y^2) \quad (6)$$

$$V_{lat}(x, y) = V_0(\sin^2 kx + \sin^2 ky) \quad (7)$$

With  $m$  is the mass of the particle,  $\{\omega_x, \omega_y\}$  are the frequencies of the harmonic potential along the coordinate directions  $x$  and  $y$ .  $V_0$  is the lattice potential depth and  $k = 2\pi/\lambda'$  is the wave vector of the laser light,  $\lambda'$  is the laser wavelength. In terms of  $k$  we can write the recoil energy  $\varepsilon_R = \hbar^2 k^2/2m \{\equiv \hbar\omega_R\}$  as an energy scale for specifying the lattice depth. It is defined as the recoil energy that one atom requires when it absorbs one lattice photon [55].

For the potential (5) it is impossible to find an exact analytical expression for the energy levels  $\varepsilon_{n_x, y}$ . An approximate expression can be readily obtained by extending the standard harmonic approximation for the optical lattice potential [52, 55]. Taylor expansion of Eq. (7) about the lattice site minimum at  $x = y = 0$  leads to

$$V_{lat}(x, y) = \frac{V_0}{2} \{(1 - \cos 2kx) + (1 - \cos 2ky)\} \quad (8)$$

which is approximately equal to

$$V_{lat}(x, y) = V_0 k^2 (x^2 + y^2) - \frac{V_0}{2} k^4 (x^4 + y^4) . \quad (9)$$

As it can be seen from Eq. (9) that the first term can be organized in the form of the harmonic oscillator potential  $\frac{1}{2}m\omega_{lat}^2(x^2 + y^2)$  which yields an effective harmonic oscillator frequency

$$\omega_{lat} = \sqrt{\frac{2V_0 k^2}{m}} = \frac{2\sqrt{E_R V_0}}{\hbar} \quad (10)$$

with equivalent harmonic oscillator length  $l = \sqrt{\hbar/m\omega_{lat}}$ . In this case the localized states in the optical lattice can be approximated by a harmonic oscillator states,  $\hbar\omega_{lat}$ . Under this consideration the effective confining potential can be given by

$$V_{eff}(x, y) = \frac{1}{2}m [(\omega_x^2 + \omega_{lat}^2)x^2 + (\omega_y^2 + \omega_{lat}^2)y^2] - \frac{V_0}{2}k^4(x^4 + y^4) . \quad (11)$$

This potential is characterized by a single particle energy levels with Dirichlet boundary condition given by

$$\varepsilon_{n_x, y} = n_x \hbar \omega'_x + n_y \hbar \omega'_y + \hbar \bar{\omega}_1 - \Delta \varepsilon_n \quad (12)$$

where  $\omega'_i = \sqrt{\omega_i^2 + \omega_{lat}^2}$  with  $i$  stands for  $x$  and  $y$ , on the other hand,  $n_x, n_y = 0, 1, 2, \dots$  and  $\bar{\omega}_1 = \frac{1}{2}(\omega'_x + \omega'_y)$  is the mean of combined frequencies. In deriving Eq. (12), the second term in Eq. (11), quartic term is treated perturbatively by using the normal ladder operators [52]. This term provides a shift in the oscillator state energies. Up to the first order term this shift is given by

$$\Delta \varepsilon_n = \frac{\hbar \omega_R}{2} [(n_x^2 + n_y^2) + (n_x + n_y) + 1] \quad (13)$$

where  $\omega_R = \hbar k^2/2m$  is the recoil frequency. Note that for a fixed value of  $\varepsilon_R = \hbar \omega_R$  the quantity  $\Delta \varepsilon_n$  grows rapidly with  $n$ 's. This is not surprising, since the higher excited states of the harmonic oscillator extend to larger and larger values of  $x$  and  $y$ . In such case the perturbation term can have important effect for higher excited

state. Also from Eq. (13) we see that the higher the value of  $n_x$  and  $n_y$  are smaller than the values of  $\varepsilon_R$  for which reliable values may be obtained from the experimental set-up conditions,  $\varepsilon_R = \hbar^2 k^2/2m$ , where  $k$  is determined from the laser beam wavelength. Thus in order to keep the value of  $\varepsilon_R$  in its experimental range one has to neglect the term which contains  $(n_x^2 + n_y^2)$  in Eq. (13). Under this approximation the single particle energy level depends on  $n_x$  and  $n_y$  is given

$$\varepsilon_{n_x, y} = n_x \hbar \left( \omega'_x - \frac{\omega_R}{2} \right) + n_y \hbar \left( \omega'_y - \frac{\omega_R}{2} \right) + \varepsilon_0 \quad (14)$$

where  $\varepsilon_0 = \hbar \bar{\omega}$  is the single particle ground state energy with  $\bar{\omega} = \bar{\omega}_1 - \frac{\omega_R}{2}$  is the meaning of the combined frequencies. Eq. (14) means that particle oscillates in quasi-two dimensional space. On the other hand, there is an energy contribution to particle in the  $z$ -direction. For Dirichlet boundary condition this contribution can be given as  $\frac{\hbar^2 \pi^2}{2md^2} n_z^2$ . Hence the single particle energy in all directions can be defined by

$$\varepsilon_{n_x, y, z} = n_x \hbar \left( \omega'_x - \frac{\omega_R}{2} \right) + n_y \hbar \left( \omega'_y - \frac{\omega_R}{2} \right) + \frac{\hbar^2 \pi^2}{2md^2} n_z^2 + \varepsilon_0 \quad (15)$$

Now with help Eq. (15) grand canonical potential for two dimensional trapped Bose gas in harmonic-optical potential in between two slabs can be obtained.

### C. Thermodynamical Potential

Grand canonical potential (3) can be written as

$$\varphi(T, \mu, d) = -\varphi_0 - kT \sum_{n=1}^{\infty} \ln(1 - ze^{-\beta \varepsilon_{n_x, y, z}}) \quad (16)$$

where first term  $\varphi_0$  is ground state potential and second term corresponds to the potential of excited particles. We have neglected the ground-state term which does not give any contribution to Casimir potential at finite temperature

since the Casimir effect was caused by thermal or near-critical fluctuations of the order parameter upon approaching the condensation transition. The second term in Eq. (16) for finite particle can be presented as

$$\varphi(T, \mu, d) = -kT \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \frac{z^j}{j} e^{-j\beta\varepsilon_{n,x,y,z}} \quad (17)$$

If Eq. (15) is put into Eq. (17), grand canonical potential for two dimensional trapped Bose gas can be written in generalized form

$$\varphi(T, \mu, d) = -kT \sum_{n_x}^{\infty} \sum_{n_y}^{\infty} \sum_{n_z=1}^{\infty} \sum_{j=1}^{\infty} \frac{z^j}{j} \exp \left\{ -j\beta \left( n_x \hbar \left( \omega'_x - \frac{\omega_R}{2} \right) + n_y \hbar \left( \omega'_y - \frac{\omega_R}{2} \right) + \frac{\hbar^2 \pi^2}{2md^2} n_z^2 \right) \right\} \quad (18)$$

where  $z = e^{\beta\mu}$ . Here we have neglected the energy  $\varepsilon_0$  in Eq. (15) since ground state does not contribute to Casimir potential. Eq. (18) can be simplified in the form

$$\varphi(T, \mu, d) = -kT \sum_{n_z=1}^{\infty} \sum_{j=1}^{\infty} \frac{\frac{z^j}{j} \exp \left\{ -j\beta \left( \frac{\hbar^2 \pi^2}{2md^2} n_z^2 \right) \right\}}{\left( 1 - \exp \left\{ j\beta \hbar \left( \omega'_x - \frac{\omega_R}{2} \right) \right\} \right) \left( 1 - \exp \left\{ j\beta \hbar \left( \omega'_y - \frac{\omega_R}{2} \right) \right\} \right)} \quad (19)$$

To evaluate the sum over  $j$ , we need an approximation due to Ketterle and van Druten [56] which yields near exact results. This approximation uses the thermodynamic limit  $\hbar\omega \ll kT$  to write  $(1 - \exp \{j\beta\hbar(\omega'_i - \frac{\omega_R}{2})\}) = j\beta\hbar(\omega'_i - \frac{\omega_R}{2})$  where  $i = x, y$ . The sum over  $j$  has the factor  $1/j^2$  and is convergent. Thus, Eq. (19) is written as

$$\varphi(T, \mu, d) = -kT \frac{\left(\frac{kT}{\hbar}\right)^2}{\left(\omega'_x - \frac{\omega_R}{2}\right) \left(\omega'_y - \frac{\omega_R}{2}\right)} \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \frac{z^j}{j^3} \exp \left\{ -j\beta \left( \frac{\hbar^2 \pi^2}{2md^2} n^2 \right) \right\}. \quad (20)$$

where we set  $n_z = n$  for simplicity. This analytical expression corresponds to the grand canonical potential of trapped Bose gas in quasi-two-dimensional harmonic-optical potential in between two slabs due to Ketterle and van Druten approximation.

#### D. Casimir potential

Casimir potential can be obtained from Eq. (20). However, Eq. (20) includes contributions of the bulk, surface and fluctuation of excited states of the Bose gas between two slabs separated by  $d$ . Bulk and surface potentials do not contribute to the Casimir force, because the force due to the bulk term is counterbalanced by the same contribution acting from outside the slabs when they are immersed in the critical medium [9, 13, 17], on the other hand, the surface potential caused from two slab-boson gas interfaces, does not change with changing  $d$ . However

thermal fluctuations of excited states of Bose gas leads to Casimir potential. Therefore, to obtain Casimir potential, these contributions must be separated into components. By using the Jacobi identity [57]

$$\sum_{n=1}^{\infty} e^{-\pi n^2 b} = \left( \frac{1}{2\sqrt{b}} - \frac{1}{2} \right) + \frac{1}{\sqrt{b}} \sum_{n=1}^{\infty} e^{-\pi n^2 / b} \quad (21)$$

where  $b = j\pi(\lambda/d)^2/2$  with  $b > 0$  and  $\lambda = \hbar\sqrt{\beta/m}$  is thermal de Broglie wavelength of the particles, these contributions can be separated as

$$\varphi(T, \mu, d) = -\frac{(kT)^3}{\hbar^2 \left(\omega'_x - \frac{\omega_R}{2}\right) \left(\omega'_y - \frac{\omega_R}{2}\right)} \sum_{j=1}^{\infty} \frac{z^j}{j^3} \left[ \left( \frac{1}{2\sqrt{b}} - \frac{1}{2} \right) + \frac{1}{\sqrt{b}} \sum_{n=1}^{\infty} e^{-\pi n^2 / b} \right]. \quad (22)$$

In Eq. (22) the first term corresponds to bulk potential  $\varphi_{bulk}(T, \mu, d)$  and second term is the surface potential  $\varphi_{surf}(T, \mu, d)$ . However, the last term in Eq. (22) corresponds to the Casimir potential for combined potential

$$\varphi_C(T, \mu, d) = -\frac{(kT)^3}{\hbar^2 \left(\omega'_x - \frac{\omega_R}{2}\right) \left(\omega'_y - \frac{\omega_R}{2}\right)} \sqrt{\frac{2}{\pi}} \frac{d}{\lambda} \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \frac{e^{j\beta\mu}}{j^{7/2}} e^{-2(nd/\lambda)^2/j}. \quad (23)$$

where sum over  $j$  and  $n$ . The summation in right hand side of Eq. (23) can be converted to integral in the limit  $d/\xi \ll 1$ ,

$$\sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \frac{e^{j\beta\mu}}{j^{7/2}} e^{-2(nd/\lambda)^2/j} = 2 \left( \frac{\lambda}{d} \right)^5 \int_0^{\infty} x^{-6} e^{-px^2 - q/x^2} dx. \quad (24)$$

where  $p = u^2/2 = (-2\beta\mu)^{1/2} d/\lambda \sim d/\xi$ ,  $\xi$  is the correlation length,  $q = 2n^2$  and  $x^2 = (\frac{\lambda}{d})^2 r$ . Thus, Casimir potential is

$$\varphi_C(T, \mu, d) = -\frac{2(kT)^3}{\hbar^2(\omega'_x - \frac{\omega_R}{2})(\omega'_y - \frac{\omega_R}{2})} \sqrt{\frac{2}{\pi}} \left(\frac{\lambda}{d}\right)^4 \int_0^\infty x^{-6} e^{-px^2 - q/x^2} dx \quad (25)$$

After some mathematical algebra, Casimir potential can be obtained as follows

$$\varphi_C(T, \mu, d) = -\frac{3(kT)^3}{64\hbar^2\pi^2(\omega'_x - \frac{\omega_R}{2})(\omega'_y - \frac{\omega_R}{2})} \left(\frac{\lambda}{d}\right)^4 \sum_{n=1}^\infty \left(\frac{1 + 2un + 4u^2n^2/3}{n^5}\right) e^{-2un}. \quad (26)$$

If  $\lambda = \hbar\sqrt{\beta/m}$  is inserted into Eq. (26), Casimir potential is given in a simple form

$$\varphi_C(T, \mu, d) = -\frac{3\hbar^2 kT}{64\pi^2 m^2 (\omega'_x - \frac{\omega_R}{2})(\omega'_y - \frac{\omega_R}{2})} \frac{1}{d^4} \sum_{n=1}^\infty \left(\frac{1 + 2un + 4u^2n^2/3}{n^5}\right) e^{-2un} \quad (27)$$

which originates from fluctuations of thermally excited states of Bose gas in between slabs.

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### E. Casimir Force

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Now we can compute Casimir force for tapped ideal Bose gas in harmonic-optical potential between two slabs by using of Eqs. (2) and (27). Casimir potential in Eq. (27) goes to zero i.e.,  $\varphi_C(T, \mu, \infty) \rightarrow 0$  for  $d \rightarrow \infty$ . Therefore Casimir force in Eq. (2) reduces to simple form

$$F_C(T, \mu, d) = -\frac{\partial}{\partial d} \varphi_C(T, \mu, d) \quad (28)$$

which implies that Casimir force of Bose gas changes due to the changing of distance between slabs. From Casimir potential (27), an general analytical form for the force is given simply by

$$F_C(T, \mu, d) = \frac{3kT\hbar^2}{16\pi^2 m^2 (\omega'_x - \frac{\omega_R}{2})(\omega'_y - \frac{\omega_R}{2})} \frac{1}{d^5} \sum_{n=1}^\infty \left(\frac{1 + 2un + 4u^2n^2/3}{n^5}\right) e^{-2un} \quad (29)$$

However this expression does draw a physical picture of the Casimir force for the Bose gas since it depends on chemical potential  $\mu$  of the gas. It is well known that  $\mu$  has a curious behavior depends on temperature. In the Bose gas, thermal fluctuations cause second order phase transition which is called BEC occur at critical temperature  $T_c$ . Chemical potential  $\mu$  equal to zero at the  $T = T_c$  and  $\mu < 0$  when  $T > T_c$ . It is known that correlation length  $\xi$  of the thermal fluctuations diverge to infinite at the second order phase transition temperature  $T = T_c$  where most of the particles begin to collapse to ground state and occur condensation. However contribution to Casimir potential does not come from ground state, it is meaning that BEC does not cause Casimir effect. The origin of the Casimir effect in the critical particle systems is excited states. The correlation length  $\xi$  of thermal fluctuation of the excited states of the quantum gas is finite when  $T \gtrsim T_c$ . If the geometry is restricted to be that of two parallel plates separated by distance  $d$  in the  $z$ -direction to be  $k = \pi n/d$  (for the Dirichlet boundary condition), correlated fluctuations become dominate and

the free energy of the Bose gas is altered in the mediate vicinity of the critical point  $d/\xi < 1$  leading to the Casimir force. Therefore, it is suggested that Casimir effect for critical particle systems is caused by thermal or near-critical fluctuations of the order parameter upon approaching the condensation transition. In this picture, we assume that in the near  $T \gtrsim T_c$ , the chemical potential to be  $\mu = 0$ . In this case, when  $T \simeq T_c$ ,  $z = 1$  behaves as the bulk critical point for Casimir force. Hence for  $\mu = 0$ , the sum in Eq. (29) takes  $\zeta(5) = \sum_{n=1}^\infty \frac{1}{n^5}$ . Therefore, the Casimir force in the vicinity the BEC transition temperature  $T_c$  is obtained in terms of Riemann's Zeta function as follow

$$F_C(T, \mu, d) = -\frac{3kT\hbar^2\zeta(5)}{16\pi^2 m^2 (\omega'_x - \frac{\omega_R}{2})(\omega'_y - \frac{\omega_R}{2})} \frac{1}{d^5} \quad (30)$$

where  $\omega'_x = \sqrt{\omega_x^2 + \omega_{lat}^2}$  and  $\omega'_y = \sqrt{\omega_y^2 + \omega_{lat}^2}$ . For isotropic case, the Casimir force in combined potential

reduce to

$$F_C(T, \mu, d) = -\frac{3kT\hbar^2\zeta(5)}{16\pi^2m^2(\omega' - \frac{\omega_B}{2})^2} \frac{1}{d^5} \quad (31)$$

where  $\omega' = \sqrt{\omega^2 + \omega_{lat}^2}$ . It can be seen that Casimir force in BEC phase or  $T \simeq T_c$  decays inversely proportional to  $d^5$  for harmonic-optical potential as well as in the case of harmonic potential. However, as it can be seen from Eqs. (30) and also (31) Casimir force of Bose gas clearly depends on frequencies of harmonic and optical lattice potentials.

On the other hand, in the case of  $T > T_c$ , the chemical potential is  $\mu \neq 0$  and the double sums in Eq. (23) for  $d/\xi \gg 1$  satisfy the condition

$$\sum_{j=1}^{\infty} \frac{e^{\beta\mu j}}{j^{7/2}} \sum_{n=1}^{\infty} e^{-2(nd/\lambda)^2/j} < \frac{\zeta(7/2)}{e^{\sqrt{-8\mu}d/\lambda} - 1}. \quad (32)$$

Therefore in the limit of large  $d$ , Casimir potential for harmonic-optical potential can be given

$$\varphi_C(T, \mu, d) \simeq e^{-\sqrt{-8\mu}d/\lambda}. \quad (33)$$

As it can be expected that for  $d/\xi \gg 1$  the Casimir potential has exponential form which indicates that Casimir force also decays exponentially with  $d$ . Additionally, at the high-temperature limit; the correlated fluctuations between the two slabs are no longer long-ranged therefore the confining boundaries are subject to a vanishing Casimir force [10, 17].

### III. CONCLUSION

In this study, we have derived close form of the Casimir force for the ideal Bose gas between two slabs

in harmonic-optical potential using by Ketterle and van Druten approximation. We have shown that Casimir force of confined Bose gas in the presence of the harmonic-optical potential decays with inversely proportional to  $d^5$  when  $T \leq T_c$ . However, in the case of  $T > T_c$ , it decays exponentially depending on separation  $d$  of the slabs. Analytical results indicate that BEC critical temperature  $T_c$  plays important role on the defining of the Casimir force in confined critical particle systems. As discussed above, when fluctuations are long ranges (comparable to smallest dimension of the system) near the BEC transition in Bose gas there will be a Casimir force between the confining surfaces. Based on above discussing we can suggest that similar behavior may happen for any confined system near a critical point if the order parameter has critical fluctuations on a scale  $\xi$  which becomes large, diverging in the bulk/thermodynamic limit. Finally, we state that the role of the Casimir effect in confined systems such as BEC, quantum phase transition at ultra cold Bose systems and nano structures is still open problems. Harmonic-optical lattice potential has a significant role on these systems. We hope that the present analytical results may contribute to theoretical and experimental studies in these area.

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