DETECTING ASYMPTOTIC NON-REGULAR VALUES BY POLAR CURVES

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Abstract. We locate the Malgrange non-regular values of a given polynomial function $f : \mathbb{C}^n \to \mathbb{C}$ by using a series of affine polar curves. We moreover show that all non-trivial Malgrange non-regular values of f are indicated by a single "super-polar curve" which we introduce here, providing also an effective algorithm of detection.

1. Introduction

Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial of degree $d \geq 2$. René Thom proved that f is a $C^\infty-$fibration outside a finite set, where the smallest such set is called *the bifurcation set of* f and is denoted by $B(f)$. Roughly speaking the set $B_\infty(f)$ consists of points at which f is not a locally trivial fibration at infinity (i.e., outside a large ball). Two fundamental questions appear in a natural way: how to characterize the set $B(f)$ and how to estimate the number of points of this set.

Let us recall that the set $B(f)$ contains the set $f(\mathrm{Sing} f)$ of critical values of f and the set $B_\infty(f)$ of bifurcations points at infinity.

In case $n = 2$ there are well-known criteria to detect $B(f)$, see e.g. [Ti4], [Du], and there are also estimations of the upper bound of the number $\#B(f)$ in terms of the degree or other data [Ha1], [Ha2], [Jel6], [LO], [Gw], [JT] etc.

Whenever $n > 2$ one has no exact criteria but one defines regularity conditions at infinity that each yield some finite set of values containing $B(f)$ and thus approaching the problem of estimating $\#B(f)$. To control the set $B_\infty(f)$ one can use the set of *asymptotic critical values of* f:

$$K_\infty(f) := \{y \in \mathbb{C} \mid \exists\ (x_l)_{l\in\mathbb{N}}, \|x_l\| \to \infty,\ \text{such that } f(x_l) \to y \text{ and } \|x_l\|\|df(x_l)\| \to 0\}.$$

If $c \notin K_\infty(f)$, then it is usual to say that f satisfies *Malgrange's condition* at c (or c is Malgrange regular). The set $K_\infty(f)$ naturally decomposes into two pieces: the set $TK_\infty(f)$ of trivial Malgrange non-regular values which come from the critical points of f (i.e. there is a sequence $x_l \to \infty$ such that $x_l \in \mathrm{Sing}(f)$ and $f(x_l) \to y$) and the remaining set $NK_\infty(f) := K_\infty(f) \setminus TK_\infty(f)$ of non-trival Malgrange non-regular values. Of course $TK_\infty(f) \subset f(\mathrm{Sing} f)$. Since the set $f(\mathrm{Sing} f)$ is relatively easy to compute, the problem which remains is how to compute the set $NK_\infty(f)$.

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It was proved (cf [Pa2], [JK1], [Jel4]) that one has the inclusion $B_\infty(f) \subset K_\infty(f)$. Setting $K(f) := f(\mathrm{Sing} f) \cup K_\infty(f)$, we get the inclusion $B(f) \subset K(f)$. Estimations of the number of the Malgrange non-regular values have been given in [JK1]. An algorithmic method for recovering the set $K_\infty(f)$ has been produced more recently [JK2].

We present here two new methods for detecting $K_\infty(f)$ and for estimating the number $\#K_\infty(f)$, together with an effective algorithm. Our first approach, based on the use of a series of *polar curves* and their relation to Malgrange non-regularity via the t-regularity, yields an exact characterisation of the set $NK_\infty(f)$. We shall recall the notions and the relevant preliminary results in §2, but let us introduce already here our first main statement.

Let $\{x_1, \ldots, x_n\}$ be a *generic system of coordinates* of $\mathbb{C}^n$, after Definition 2.8. Let us consider the successive restrictions of f to the affine hyperplanes:

$$f_0 := f, \quad f_1 := f_{|x_1=0}, \ \ldots, \ f_{n-2} := f_{|x_1=\cdots=x_{n-2}=0},$$

and the corresponding generic polar curves $\Gamma(x_i, f_{i-1})$, for $i = 1, \ldots, n-1$.

For a mapping $g : X \to Y$, let J_g denote the *non-properness set* [Jel1], [Jel2] (also called the *Jelonek set*) of the mapping g, see §4, Theorem 4.1. If $A \subset X$, then by $J_g(A)$ we denote the non-properness set of the restriction $g_{|A}$.

We say that an irreducible algebraic variety $S \subset \mathbb{C}^n$ is *horizontal* if $f(S)$ is not a point (i.e. S is not included in some fibre of f). The union of all horizontal components of the polar curve $\Gamma(x_i, f_{i-1})$ will be called the *horizontal part* and will be denoted by $H\Gamma(x_i, f_{i-1})$.

We prove the following characterisation of the set $NK_\infty(f)$:

Theorem 1.1. *The set $NK_\infty(f)$ of non-trivial Malgrange non-regular regular values of f is included in the union of the non-properness sets of the mapping f restricted to a horizontal part of the polar curves $\Gamma(x_i, f_{i-1})$, more precisely we have the equality:*

$$NK_\infty(f) = \bigcup_{i=1}^{n-1} J_f(H\Gamma(x_i, f_{i-1})) \setminus J_f(\mathrm{Sing} f). \tag{1}$$

Note that $J_f(\mathrm{Sing} f)$ equals the the set of critical values of f which are images of fibers containing nonisolated singularities.

Let $\mathrm{Sing} f = S_0 \cup S_1 \cup \cdots \cup S_r$ be the decomposition of the singular locus into irreducible components, where S_0 denotes the union of all point-components (i.e. S_0 is the set of isolated singularities of f). For $i > 1$ we denote by $d_i = \deg S_i$ the degree of the positive dimensional component S_i.

Corollary 1.2. *For $d > 2$ we have:*

$$\#NK_\infty(f) \le \frac{(d-1)^n - 1}{d-2} - \sum_{i=1}^{r} d_i \dim S_i, \tag{2}$$

and for $d = 2$:

$$\#NK_\infty(f) \le n - 1 - \sum_{i=1}^{r} d_i \dim S_i.$$

□

According to their definition, the polar curves of the above statement are affine curves, some of them are maybe empty, and they do not detect, in general, values from $TK_\infty(f)$. The first polar curve $\Gamma(x_1, f)$ detects some Malgrange non-regular value $c \in NK_\infty(f)$ whenever the fiber $f^{-1}(c)$ has only isolated singularities at infinity in the sense of Definition 2.5, cf Theorem 2.9. However, the first polar curve may not detect all values from $NK_\infty(f)$ and that is why we need more polar curves. We explain this phenomenon by the existence of *non-isolated t-singularities at infinity*, cf §3. For example, if $f(x, y, x) = x + x^2y$ then the polar curve of f is empty, but f has a non-trivial Malgrange non-regular value 0. This example also shows that Theorem 3.6 in [Sa] is not correct. More precisely, if we use polar curves, then the problem of detecting non-trivial Malgrange non-regular values cannot be done in a single step (as was wrongly claimed in [Sa]), but turns out to fall into $n - 1$ steps as we describe now in our Theorem 1.1, each step being the detection of the non-properness set of a certain generic polar curve.

However, the question "*is it possible to recover all non-trivial Malgrange non-regular values in just one single step*" subsists as a chalenging problem. We solve it positively in the second part of our paper by introducing a new and different device called "super-polar curve". Let us give here an outline of its construction. We consider the following polynomials:

$$g_i(a, b) = \sum_j a_{ij} \frac{\partial f}{\partial x_j} + \sum_{j,k} b_{ijk} x_k \frac{\partial f}{\partial x_j},\ i = 1, \ldots, n - 1,$$

where a_{ij}, b_{ijk} are complex constants. Let:

$$\Gamma_f(a, b) := \text{closure}\{V(g_1, \ldots, g_{n-1}) \setminus \text{Sing}(f)\}, \tag{3}$$

where we use here the Zariski closure. It turns out that, for general a_{ij}, b_{ijk} the set $\Gamma_f(a, b)$ is a non-empty curve, which we shall call *super-polar curve of f*. We say that a component $S \subset \Gamma_f(a, b)$ is *horizontal* if $f(S)$ is not a single point. The union of all horizontal components of $\Gamma_f(a, b)$ will be called the *horizontal part of* $\Gamma_f(a, b)$ and will be denoted by $H\Gamma_f(a, b)$. We obviously have the inclusion $J_f(H\Gamma_f(a, b)) \subset J_f(\Gamma_f(a, b))$. We prove the following result:

Theorem 1.3. *The set $NK_\infty(f)$ of nontrivial Malgrange non-regular values of f is included in the non-properness set of a mapping f restricted to the horizontal part of a sufficiently general super-polar curve $\Gamma_f(a, b)$, namely one has the following inclusion:*

$$NK_\infty(f) \subset J_f(H\Gamma_f(a, b)). \tag{4}$$

Corollary 1.4. *If $n > 2$ and $NK_\infty(f) \neq \emptyset$ then:*

$$\#NK_\infty(f) \leq d^{n-1} - 1 - \sum_{i=1}^{r} d_i.$$

In particular if $NK_\infty(f) \neq \emptyset$, then:

$$\#K_\infty(f) \leq d^{n-1} - 1 - \sum_{i=1}^{r} (d_i - 1).$$

In case $n = 2$, if $NK_\infty(f) \neq \emptyset$, then:

$$\#NK_\infty(f) \leq d - 2 - \sum_{i=1}^{r} d_i, \quad \textit{and} \quad \#K_\infty(f) \leq d - 2 - \sum_{i=1}^{r}(d_i - 1).$$

The plan of the paper goes as follows: in §2 and §4 we develop some preliminary results in order to prepare the proofs of Theorem 1.1 in §3, and of Theorem 1.3 in §5, together with their corollaries, respectively. In §6 we sketch the algorithm to detect the set $NK_\infty(f)$ effectively.

2. Regularity conditions at infinity

2.1. **Malgrange regularity condition at a point at infinity.** Pham formulated in [Ph, 2.1] a regularity condition which he attributed to Malgrange. We recall the localized version at infinity, after [Ti2], [Ti4].

We identify $\mathbb{C}^n$ to the graph of f, namely $X := \{(x, \tau) \in \mathbb{C}^n \times \mathbb{C} \mid f(x) = \tau\}$, and consider its algebraic closure in $\mathbb{P}^n \times \mathbb{C}$, which is the hypersurface:

$$\mathbb{X} = \{\tilde{f}(x_0, x) - \tau x_0^d = 0\} \subset \mathbb{P}^n \times \mathbb{C}, \tag{5}$$

where x_0 denotes the variable at infinity, $d = \deg f$ and $\tilde{f}(x_0, x)$ denotes the homogenization of degree d of f by the variable x_0. Let $\boldsymbol{t} : \mathbb{X} \to \mathbb{C}$ denote the restriction to $\mathbb{X}$ of the second projection $\mathbb{P}^n \times \mathbb{C} \to \mathbb{C}$, a proper extension of the map f. We denote by $\mathbb{X}^\infty = \mathbb{X} \setminus X$ the divisor at infinity defined in each affine chart by the equation $x_0 = 0$.

Definition 2.1. [Ti4] Let $\{\mathbf{x}_i\}_{i \in \mathbb{N}} \subset \mathbb{C}^n$ be a sequence of points with the following properties:

(L_1) $\|\mathbf{x}_i\| \to \infty$ and $f(\mathbf{x}_i) \to \tau$, as $i \to \infty$.
(L_2) $\mathbf{x}_i \to p \in \mathbb{X}^\infty$, as $i \to \infty$.

One says that the fibre $f^{-1}(\tau)$ verifies the *Malgrange condition* if there is $\delta > 0$ such that, for any sequence of points with property (L_1) one has

(M) $\|\mathbf{x}_i\| \cdot \|\operatorname{grad} f(\mathbf{x}_i)\| > \delta$.

We say that f verifies *Malgrange condition at* $p \in \mathbb{X}^\infty$ if there is $\delta_p > 0$ such that one has (M) for any sequence of points with property (L_2).

Remark 2.2. It follows from the definition that $f^{-1}(\tau)$ verifies the Malgrange condition if and only if f verifies Malgrange condition (M) at any point $p = (z, \tau) \in \mathbb{X}^\infty \cap \boldsymbol{t}^{-1}(\tau)$ and for the same positive constant $\delta_p = \delta$.

2.2. **Characteristic covectors and t-regularity.** We recall the notion of t-regularity from [Ti1], [Ti4]. Let $H^\infty = \{[x_0 : x_1 : \ldots : x_n] \in \mathbb{P}^n \mid x_0 = 0\}$ denote the hyperplane at infinity and let $\mathbb{X}^\infty := \mathbb{X} \cap (H^\infty \times \mathbb{C})$.

We consider the affine charts $U_j \times \mathbb{C}$ of $\mathbb{P}^n \times \mathbb{C}$, where $U_j = \{x_j \neq 0\}$, $j = 0, 1, \ldots, n$. Identifying the chart U_0 with the affine space $\mathbb{C}^n$, we have $\mathbb{X} \cap (U_0 \times \mathbb{C}) = \mathbb{X} \setminus \mathbb{X}^\infty = X$ and $\mathbb{X}^\infty$ is covered by the charts $U_1 \times \mathbb{C}, \ldots, U_n \times \mathbb{C}$.

If g denotes the projection to the variable x_0 in some affine chart $U_j \times \mathbb{C}$, then the *relative conormal* $C_g(\mathbb{X} \setminus \mathbb{X}^\infty \cap U_j \times \mathbb{C}) \subset \mathbb{X} \times \check{\mathbb{P}}^n$ is well defined (see e.g. [Ti3], [Ti5]), with the projection $\pi(y, H) = y$, where $\check{\mathbb{P}}^n$ is identified to the projective space of hyperplanes in $U_j \times \mathbb{C}$. Let us then consider the space $\pi^{-1}(\mathbb{X}^\infty)$ which is well defined for every chart $U_j \times \mathbb{C}$ as a subset of $C_g(\mathbb{X} \setminus \mathbb{X}^\infty \cap U_j \times \mathbb{C})$. By [Ti2, Lemma 3.3], the definitions coincide at the intersections of the charts.

Definition 2.3. We call *space of characteristic covectors at infinity* the well-defined set $\mathcal{C}^\infty := \pi^{-1}(\mathbb{X}^\infty)$. For some $p_0 \in \mathbb{X}^\infty$, we denote $\mathcal{C}^\infty_{p_0} := \pi^{-1}(p_0)$.

Considering now the second projection $\boldsymbol{t} : \mathbb{P}^n \times \mathbb{C} \to \mathbb{C}$ in place of the function g in the above consideration, we obtain the relative conormal space $C_{\boldsymbol{t}}(\mathbb{P}^n \times \mathbb{C})$. Then we have:

Definition 2.4. [Ti2] We say that f is *t-regular* at $p_0 \in \mathbb{X}^\infty$ if $C_{\boldsymbol{t}}(\mathbb{P}^n \times \mathbb{C}) \cap \mathcal{C}^\infty_{p_0} = \emptyset$ or, equivalently, $\mathrm{d}\boldsymbol{t} \notin \mathcal{C}^\infty_{p_0}$.

Definition 2.5. We say that f has *isolated t-singularities at infinity* at the fibre $f^{-1}(t_0)$ if this fibre has isolated singularities in $\mathbb{C}^n$ and if the set

$$\mathrm{Sing}^\infty f := \{p \in \mathbb{X}^\infty \mid f^{-1}(t_0) \text{ is not } t\text{-regular at } p\}$$

is a finite set.

It follows from the definition that $\mathrm{Sing}^\infty f$ is a closed algebraic subset of $\mathbb{X}^\infty$, see e.g. [Ti2], [Ti5], [DRT, §6.1]. By the algebraic Sard Theorem, the image $\boldsymbol{t}(\mathrm{Sing}^\infty f)$ consists of a finite number of points.

We need the following key equivalence in the localized setting (proved initially in [ST, Proposition 5.5] and [Pa1, Theorem 1.3], as explained in [Ti4]):

Theorem 2.6. [Ti5, Prop. 1.3.2] *A polynomial $f : \mathbb{C}^n \to \mathbb{C}$ is t-regular at $p_0 \in \mathbb{X}^\infty$ if and only if f verifies the Malgrange condition at this point.* □

More precisely we have the following relations, cf [Ti3], [Ti5]:

(6) Malgrange regularity $\iff$ t-regularity $\implies$ ρ_E-regularity $\implies$ topological triviality

which also extend to polynomial maps $\mathbb{C}^n \to \mathbb{C}^p$ as shown in [DRT].

2.3. **Polar curves and t-regularity.** We define the affine polar curves of f and show how they are related to the t-regularity condition, after [Ti3].

Given a polynomial $f : \mathbb{C}^n \to \mathbb{C}$ and a linear function $l : \mathbb{C}^n \to \mathbb{C}$, the *polar curve of f with respect to l*, denoted by $\Gamma(l, f)$, is the closure in $\mathbb{C}^n$ of the set $\mathrm{Sing}(l, f) \setminus \mathrm{Sing} f$, where $\mathrm{Sing}(l, f)$ is the critical locus of the map $(l, f) : \mathbb{C}^n \to \mathbb{C}^2$. Denoting by $l_H : \mathbb{C}^n \to \mathbb{C}$ the unique linear form (up to multiplication by a constant) which defines a hyperplane $H \in \mathbb{C}^n$ (also regarded as a point in the projective space $\check{\mathbb{P}}^{n-1}$ of linear hyperplanes in $\mathbb{C}^n$), we have the following genericity result of Bertini-type.

Lemma 2.7. [Ti2, Lemma 1.4]
There exists a Zariski-open set $\Omega_{f,a} \subset \check{\mathbb{P}}^{n-1}$ such that, for any $H \in \Omega_{f,a}$ and some fixed $a \in \mathbb{C}$, the polar set $\Gamma(l_H, f)$ is a curve or it is an empty set, and no component is contained in the fibre $f^{-1}(a)$. □

Definition 2.8. For $H \in \Omega_{f,a}$, we call $\Gamma(l_H, f)$ the *generic affine polar curve* of f with respect to l_H. A system of coordinates $(x_1, \ldots, x_n)$ in $\mathbb{C}^n$ is called *generic* with respect to f iff $\{x_i = 0\} \in \Omega_{f,a}$, $\forall i$.

It follows from Lemma 2.7 that such systems of coordinates are generic among all linear systems of coordinates.

Let $\overline{\Gamma(l_H, f)}$ and $\overline{\mathrm{Sing} f}$ denote the algebraic closure in $\mathbb{X}$ of the respective sets. We then have:

Theorem 2.9. *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial function and let $p \in \mathbb{X}^\infty$, $a := \boldsymbol{t}(p)$.*

(a) *If p is a t-regular point then $p \notin (\overline{\Gamma(l_H, f)} \cup \overline{\mathrm{Sing} f}) \cap \mathbb{X}^\infty$, for any $H \in \Omega_{f,a}$.*

(b) *Let p be either t-regular or an isolated t-singularity at infinity. Then p is a t-singularity at infinity if and only if $p \in \overline{\Gamma(l_H, f)}$ for some $H \in \Omega_{f,a}$.*

Proof. The result and its proof can be actually extracted from [Ti5, §2.1]. More precisely: (a) follows from [Ti5, Prop. 2.1.3] and [Ti5, (2.1), pag.17].
(b) follows by combining Thm. 2.1.7, Thm. 2.1.6 and Prop. 2.1.3 from [Ti5, §2.1]. □

The above theorem means that isolated t-singularities at infinity are precisely detected by the horizontal part of the generic polar curve. In case $p \in \mathbb{X}^\infty$ is a non-isolated t-singularity (which occurs whenever $n > 2$), the general affine polar curve $\Gamma(l_H, f)$ might not contain the chosen point p in its closure at infinity. We shall show in the next section how to deal with this situation.

3. Proof of Theorem 1.1

Let $B := \mathrm{Sing}^\infty f \cap (\mathbb{X} \setminus \cup_{a \in f(\mathrm{Sing} f)} \mathbb{X}_a)$. By Theorem 2.6, we have the equality:

$$NK_\infty(f) = \boldsymbol{t}(B).$$

By Theorem 2.9(a) and Theorem 2.6, if the generic polar curve $\Gamma(l_H, f)$ is nonempty, then it intersects the hypersurface $\mathbb{X}^\infty$ at finitely many points and these points are t-singularities, hence Malgrange non-regular points at infinity.

Let us first assume that $\dim B = 0$. Then, by Theorem 2.9, for $p \in B$ (which by our assumption is an isolated t-singularity), the generic polar curve passes through p, so this point is "detected" by the horizontal part of the polar curve $\overline{\Gamma(x_1, f)}$, for some generic choice of the coordinate x_1 (in the sense of Definition 2.8 and Lemma 2.7). Therefore, in the notations of the Introduction, the corresponding asymptotic non-regular value belongs to $J_f(H\Gamma(x_1, f))$.

Therefore, in our case $\dim B = 0$, the equality (1) follows from Theorem 2.9 and Theorem 2.6.

Let us now treat the case $\dim B > 0$. We will show (1) by a double inclusion.

The inclusion "$\supset$". Let us first prove the inclusion $J_f(H\Gamma(x_i, f_{i-1})) \setminus J_f(\mathrm{Sing} f) \subset NK_\infty(f)$ for each $i = 1, \ldots, n-1$. We proceed by a "reductio ad absurdum" argument. Assume that $a \notin NK_\infty(f)$ and denote $\mathbb{X}^\infty_a := \mathbb{X}^\infty \cap \boldsymbol{t}^{-1}(a)$.

(a). If $a \in J_f(H\Gamma(x_1, f_0)) \setminus J_f(\mathrm{Sing} f)$, then there exist points $p \in \mathbb{X}^\infty_a \cap \overline{\Gamma(x_1, f_0)}$. By Theorem 2.9(a), this means that p is a t-non-regular point, which implies in turn that $a \in NK_\infty(f)$, by Theorem 2.6.

(b). Assume that $a \notin J_f(H\Gamma(x_i, f_{i-1})) \setminus J_f(\mathrm{Sing} f)$ for $i = 1, \ldots, k-1$ (for some $k \geq 2$), and that $a \in J_f(H\Gamma(x_k, f_{k-1})) \setminus J_f(\mathrm{Sing} f)$.

We endow the hypersurface $\mathbb{X} \subset \mathbb{P}^n \times \mathbb{C}$ with a finite complex Whitney stratification $\mathcal{W}$ such that $\mathbb{X}^\infty := \{f_d = 0\} \times \mathbb{C}$ is a union of strata. Our Whitney stratification at infinity is also Thom (a_{x_0})-regular, by [Ti2, Theorem 2.9], where $x_0 = 0$ is some local equation for H^∞ at p.

There exists a Zariski-open set $\Omega' \subset \check{\mathbb{P}}^{n-1}$ of linear forms $\mathbb{C}^n \to \mathbb{C}$ such that, if $H \in \Omega'$, then $(H^\infty \cap \overline{H}) \times \mathbb{C}$ is transversal in $H^\infty \times \mathbb{C}$ to all strata of $\mathcal{W}$ in the neighbourhood of $\mathbb{X}^\infty_a$. Due to the Thom (a_{x_0})-regularity of the stratification, it follows that slicing by $H \in \Omega'$ insures the t-regularity of the restriction $f_{|H}$ at any point $p \in (\overline{H} \times \mathbb{C}) \cap \mathbb{X}^\infty_a$. More precisely, from our hypothesis $\mathrm{d}\boldsymbol{t} \notin \mathcal{C}^\infty_p$ (see Definition 2.4) we deduce that $\mathrm{d}\boldsymbol{t}' \notin \mathcal{C}'^\infty_p$ for $p \in \overline{H} \cap \mathbb{X}^\infty_a$, where $H \in \Omega'$, $\boldsymbol{t}' := \boldsymbol{t}_{|\overline{H} \times \mathbb{C}}$ and $\mathcal{C}'^\infty$ is the space of Definition 2.3 starting with the restriction $f_{|H}$ instead of f. This implies that $a \notin NK_\infty(f_{|H})$.

By taking $H \in \Omega' \cap \Omega_{f,a}$ we get in addition that $a \notin J_f(\mathrm{Sing} f_{|H})$. We denote $f_1 := f_{|H}$.

Now, if $k = 2$ in our first assumption at point (b), we may apply the reasoning (a) to f_1 in place of f and obtain $a \in NK_\infty(f_1)$, hence a contradiction.

In case $k > 2$, after applying the slicing process (b) exactly $k - 2$ more times, namely successively to $f_1, \ldots, f_{k-2}$, we arrive to the similar contradiction for f_{k-1}.

The inclusion "$\subset$". Let $a \in \boldsymbol{t}(B)$ be an asymptotic non-regular value such that the set of t-singularities in $\mathbb{X}^\infty_a$ is not isolated. More precisely, according to Definition 2.5, this set is equal to $\mathbb{X}^\infty_a \cap \mathrm{Sing}^\infty f$. From the remark after Definition 2.5, it follows that $\mathbb{X}^\infty_a \cap \mathrm{Sing}^\infty f$ is an algebraic set. Let therefore $k := \dim \mathbb{X}^\infty_a \cap \mathrm{Sing}^\infty f$ be its dimension, where $k > 0$ by our assumption $\dim B > 0$. We show how to reduce k one by one until zero.

For that we use two facts:

(a). From the above proof of the first inclusion we extract the fact that if $\mathrm{d}\boldsymbol{t} \notin \mathcal{C}^\infty_p$ then $\mathrm{d}\boldsymbol{t}' \notin \mathcal{C}'^\infty_p$, for any $H \in \Omega'$, where $\boldsymbol{t}' := \boldsymbol{t}_{|\overline{H} \times \mathbb{C}}$.

(b). Moreover, by a Bertini type argument[1], there exists a Zariski-open set $\Omega'' \subset \check{\mathbb{P}}^{n-1}$ such that if $H \in \Omega''$ then $H \times \mathbb{C}$ is transversal to any stratum $\mathcal{W}_i \subset \mathbb{X}^\infty$ of the Whitney stratification except at finitely many points.

For some $H \in \Omega' \cap \Omega''$ we consider the restriction $f_{|H}$ and the space similar to $\mathbb{X}$ defined at (5) attached to the polynomial function $f_{|H}$, which we denote by $\mathbb{Y}$. These two facts imply the equality:

$$\dim(\mathbb{Y}^\infty_a \cap \mathrm{Sing}^\infty f_{|H}) = \dim(\mathbb{X}^\infty_a \cap \mathrm{Sing}^\infty f) - 1,$$

as long as $k > 0$ (which is our assumption). This shows the reduction to $k - 1$.

We thus continue to slice by generic hyperplanes and lower one by one the dimension of the set $\mathrm{Sing}^\infty f$ until we reach zero, thus we slice a total number of k times. The restriction of f to these iterated slices identifies to the restriction f_k defined in the Introduction.

After this iterated slicing we have f_k with a nonempty set of isolated t-singularities at infinity over a, each of which are detected by the horizontal part of the polar curve $\Gamma(x_{k+1}, f_k)$, like shown in the first part of the above proof. We therefore get $a \in J_f(H\Gamma(x_{k+1}, f_k)$.

[1]based on the fact that the relative conormal $T^*_{\boldsymbol{t}_{|\mathcal{W}_i}}$ is of dimension $n - 1$, the same as $\check{\mathbb{P}}^{n-1}$.

Altogether this shows the inclusion: $NK_\infty(f) \subset \bigcup_{i=1}^{n-1} J_f(H\Gamma(x_i, f_{i-1})) \setminus J_f(\mathrm{Sing} f)$. Our proof of Theorem 1.1 is now complete. □

3.1. **Proof of Corollary 1.2.** We estimate the number of Malgrange non-regular values $K_\infty(f)$ given by Theorem 1.1. Let us fix a generic system of coordinates $(x_1, \ldots, x_n)$. The following equations:

$$\frac{\partial f_d}{\partial x_2} = 0, \ldots, \frac{\partial f_d}{\partial x_n} = 0 \tag{7}$$

define the algebraic set $\Gamma(x_1, f) \cup \mathrm{Sing} f \subset \mathbb{C}^n$ of degree $(d-1)^{n-1}$. Therefore, if nonempty, $\Gamma(x_1, f)$ is a curve of degree $\leq (d-1)^{n-1}$. After Bezout, the curve $\overline{\Gamma(x_1, f)}$ will meet a non-degenerate hyperplane, and in particular the hyperplane at infinity, at a number of points which is bounded from above by $(d-1)^{n-1} - \sum_{i=1}^{r} d_i$. Repeating this procedure after successively slicing by general hyperplanes like explained in the above proof, we finally add up the numbers of solutions. This gives the following sum:

$$(d-1)^{n-1} + (d-1)^{n-2} + \cdots + (d-1) = \frac{(d-1)^n - 1}{d-2} \tag{8}$$

to which we have to substract the sums of degrees of the positive dimensional irreducible components of $\mathrm{Sing} f$ and their successive slices. It follows that we substract the degree d_i a number of $\dim S_i$ times which corresponds to the number of times we slice S_i and drop its dimension one-by-one until we reach dimension 0. This proves Corollary 1.2. □

3.2. **New bound for the number of atypical values at infinity.** In [JK2, Corollary 1.1] one finds the following upper bound for Malgrange non-regular values:

$$\#K_\infty(f) \leq \frac{d^n - 1}{d+1}. \tag{9}$$

Our estimation (2) yields to the following one for $K_\infty(f)$:

$$\#K_\infty(f) \leq \frac{(d-1)^n - 1}{d-2} - \sum_{i=1}^{r} d_i \dim S_i \; + r. \tag{10}$$

This is somewhat sharper than (9). Both have the highest degree term d^{n-1} and the coefficient of the term d^{n-2} in our formula is smaller for high values of n.

4. The non-properness set and the generalized Noether lemma

In this section we give the preliminary material which will lead to the definition in §5 of the "super-polar curve".

If $f : X \to Y$ is a dominant, generically finite polynomial map of smooth affine varieties, we denote by $\mu(f)$ the number of points in a generic fiber of f. If $\{x\}$ is an isolated component of the fiber $f^{-1}(f(x))$, then we denote by $\mathrm{mult}_x(f)$ the multiplicity of f at x.

Let X, Y be affine varieties, recall that a mapping $f : X \to Y$ is *not proper* at a point $y \in Y$ if there is no neighborhood U of y such that $f^{-1}(\overline{U}))$ is compact. In other words, f is not proper at y if there is a sequence $x_l \to \infty$ such that $f(x_l) \to y$. Let J_f denote the set of points at which the mapping f is not proper. The set J_f has the following properties (see [Jel1], [Jel2], [Jel3]):

Theorem 4.1. *Let $X \subset \mathbb{C}^k$ be an irreducible variety of dimension n and let $f = (f_1, \dots, f_m) : X \to \mathbb{C}^m$ be a generically-finite polynomial mapping. Then the set J_f is an algebraic subset of $\mathbb{C}^m$ and it is either empty or it has pure dimension $n-1$. Moreover, if $n = m$ then*

$$\deg\ J_f \le \frac{\deg X(\prod_{i=1}^n \deg\ f_i) - \mu(f)}{\min_{1\le i\le n} \deg\ \ f_i}.$$

In the case of a polynomial map of normal affine varieties it is easy to show the following:

Proposition 4.2. *Let $f : X \to Y$ be a dominant and quasi-finite polynomial map of normal affine varieties. Let $Z \subset Y$ be an irreducible subvariety which is not contained in J_f. Then every component of the set $f^{-1}(Z)$ has dimension $\dim Z$, and if g denotes the restriction of f to $f^{-1}(Z)$, then*

$$J_g = J_f \cap Z.$$

□

Proof. By the Zariski Main Theorem in version of Grothendieck, there is an affine variety $\overline{X}$, which contains X as a dense subset and a regular finite mapping $F : \overline{X} \to Y$ such that $F_{|X} = f$. Since the mapping F is finite, all components of $F^{-1}(Z)$ have dimension $\dim Z$. Now the condition $Z \not\subset J_f$ implies that all components of $f^{-1}(Z)$ have dimension $\dim Z$. Let $S := \overline{X} \setminus X$. Observe that $J_f = F(S)$. Moreover, $J_g = F(S \cap F^{-1}(Z)) = F(S) \cap Z$. □

Let M_m^n denotes the set of all linear forms $L : \mathbb{C}^m \to \mathbb{C}^n$. We need the following result, which is a modification of [Jel5, Lemma 4.1]:

Proposition 4.3. (Generalized Noether Lemma)
Let $X \subset \mathbb{C}^m$ be an affine variety of dimension n. Let $A \subset \mathbb{C}^m$ be a line and $B \subset X$ be a subvariety such that $A \not\subset B$. Let $x_1 : \mathbb{C}^m \to \mathbb{C}$ be a linear projection and assume that x_1 is non-constant on X and on A. Let $a_1, \dots, a_s \in A \cap X$ be some fixed set of points.

There exist a Zariski open dense subset $U \subset M_m^{n-1}$ such that for every $(n-1)$-tuple $(L_1, \dots, L_{n-1}) \in U$ the mapping $\Pi = (x_1, L_1, \dots, L_{n-1}) : X \to \mathbb{C}^n$ satisfies the following conditions:

(a) *the fibers of Π have dimension at most one,*
(b) *there is a polynomial $\rho \in \mathbb{C}[t_1]$ such that*

$$J_\Pi = \{(t_1, \dots, t_n) \in \mathbb{C}^n \mid \rho(t_1) = 0\},$$

(c) $\Pi(A) \not\subset \Pi(B)$,
(d) *all fibers $\Pi^{-1}(\Pi(a_i))$, $i = 1, \dots, s$ are finite and non-empty.*

Proof. For any $Z \subset \mathbb{C}^m$, denote by $\tilde{Z}$ the projective closure of Z in $\mathbb{P}^m$, and let H^∞ denote the hyperplane at infinity. Then $\dim \tilde{X} \cap H^\infty = n - 1$.

Hence there is a non-empty Zariski open subset $U_1 \subset M_m^{n-1}$ of $(n-1)$-tuples of linear forms such that for any $L = (l_1, \dots, l_{n-1}) \in U_1$ we have $\dim \tilde{X} \cap H^\infty \cap \ker\ L \le 0$.

Let l_n be a general linear form. Since the $(n+1)$ linear forms $(x_1, l_1, \dots, l_n)$ are algebraically dependent on X, there exists a non-zero polynomial $W \in \mathbb{C}[T, T_1, \dots, T_n]$ such that we have $W(x_1, l_1, \dots, l_n) = 0$ on X. Let us define:

$$L_i := l_i - \alpha_i l_n, \text{ for } i = 1, \dots, n-1; \ \alpha_i \in \mathbb{C}^*. \tag{11}$$

Operating on W the linear change of coordinates $l_i \mapsto L_i$, for sufficiently general coefficients $\alpha_i \in \mathbb{C}$, we then get a relation:

$$l_n^N \rho(x_1) + \sum_{j=1}^{N} l_n^{N-j} A_j(x_1, L_1, \dots, L_{n-1}) = 0, \tag{12}$$

where N is some positive integer, ρ and A_j are polynomials, such that $\rho \not\equiv 0$.

The map $P = (x_1, L_1, \dots, L_{n-1}, l_n) : X \to \mathbb{C}^{n+1}$ is finite and proper, since $(L_1, \dots, L_{n-1}, l_n)$ is so. Let $X' := P(X)$ and consider the projection:

$$\pi : X' \to \mathbb{C}^n, \quad (x_1, \dots, x_{n+1}) \mapsto (x_1, \dots, x_n).$$

Note that the mapping π has fibers of dimension at most one. From the above constructions it follows that the non-properness locus of the projection π is:

$$J_\pi = \{(t_1, \dots, t_n) \in \mathbb{C}^n \mid \rho(t_1) = 0\},$$

for the polynomial $\rho \in \mathbb{C}[t_1]$ defined by the relation (12), since J_π is precisely the locus of the values of $(x_1, L_1, \dots, L_{n-1})$ such that the equation (12) has less than N solutions for l_n, counted with multiplicities.

Let us remark that the genericity conditions on $(l_1, \dots, l_{n-1}, l_n) \in M_m^n$ and the condition that $(\alpha_1, \dots, \alpha_{n-1})$ ensure the non-triviality of the polynomial ρ in (12), yield a constructible subset S of $\mathbb{C}^{n-1} \times M_m^n$. The algebraic mapping:

$$\Psi : S \to M_m^{n-1}, \quad (\alpha_1, \dots, \alpha_{n-1}; l_1, \dots, l_{n-1}, l_n) \mapsto (L_1, \dots, L_{n-1})$$

where L_i are defined in (11), has a constructible image $\Psi(S) \subset M_m^{n-1}$ which contains U_1 in its closure, thus $\Psi(S)$ contains a non-empty Zariski-open subset U_2 of M_m^{n-1}.

We thus obtain (a) and (b) for $U := U_2$ and for $\Pi := \pi \circ P$.

Next, let us show that there is a non-empty Zariski open subset included in U_2 such that condition (c) is also satisfied.

Note that $\dim B \le n-1$. Moreover, there is a point $a \in A \setminus B$, such that the dimension of $B_a := B \cap x_1^{-1}(x_1(a))$ is $< n-1$. Let $\Lambda \subset \mathbb{C}^m$ be the Zariski closure of the cone over B_a with vertex a, $C_a B_a := \bigcup_{x \in B_a} \overline{ax}$, which is of dimension $\le n-1$. Hence

$$\dim \tilde{\Lambda} \cap H^\infty < n-1.$$

Consequently, there is a Zariski open subset $U_3 \subset U_2$ such that for $L = (L_1, \dots, L_{n-1}) \in U_3$ we have $\dim \tilde{\Lambda} \cap H^\infty \cap \ker L = \emptyset$. This means that for $\Pi := (x_1, L_1, \dots, L_{n-1})$ we have $\Pi(a) \notin \Pi(B)$, which finishes the proof of (c).

Let us finally show that there is an eventually smaller non-empty Zariski open subset $U \subset U_3$ such that (d) is satisfied too. Let $D_i := x_1^{-1}(x_1(a_i))$, for $i = 1, \dots, s$. Since $\dim D_i = n-1$, the Zariski closure D of $\bigcup_{i=1}^s D_i$ has dimension $n-1$. Hence

$$\dim H^\infty \cap \tilde{D} < n-1.$$

Like in the above argument, there is a Zariski open subset $U \subset U_3$ such that for $L = (L_1, \ldots, L_{n-1}) \in U$ we have dim $\tilde{D} \cap H^\infty \cap \ker L = \emptyset$. Consequently, for any $i = 1, \ldots, s$, the fiber $\Pi^{-1}(\Pi(a_i))$ is finite and non-empty. □

Definition 4.4. In the notations of Proposition 4.3, we call *base-set of non-properness of linear projections* of X with respect to x_1, the set:

$$B(x_1, X) := \bigcap_{L \in U} J_{(x_1, L)}.$$

Remark 4.5. If non-empty, the set $B(x_1, X)$ is a finite union of hyperplanes of the form $\{b_i\} \times \mathbb{C}^{n-1}$, by Proposition 4.3(b).

5. Super-polar curve and proof of Theorem 1.3

We have defined at (3) the *super-polar curve* $\Gamma_f(a, b)$ as the Zariski closure of $V(g_1, \ldots, g_{n-1}) \setminus \mathrm{Sing}(f)$, where

$$g_i(a, b) := \sum_{j=1}^{n} a_{ij} \frac{\partial f}{\partial x_j} + \sum_{j,k=1}^{n} b_{ijk} x_k \frac{\partial f}{\partial x_j}, \quad i = 1, \ldots, n-1. \tag{13}$$

That for general $a_{ij}, b_{ijk} \in \mathbb{C}$ this is indeed a non-degenerate curve follows in particular from the next result, which is equivalent to Theorem 1.3. Let us recall that $H\Gamma_f(a, b)$ denotes the horizontal part of $\Gamma_f(a, b)$.

Theorem 5.1. *There is a Zariski open non-empty set Ω in the space of parameters $(a, b) \in \mathbb{C}^{n(n+1)}$ such that:*

(a) for $(a, b) \in \Omega$ the set $\Gamma_f(a, b)$ is a non-empty curve,
(b) $NK_\infty(f) \subset J_f(H\Gamma_f(a, b))$.

Proof. Let $\Phi : \mathbb{C}^n \to \mathbb{C} \times \mathbb{C}^{n(n+1)}$ be the polynomial mapping defined by:

$$\Phi = \left(f, \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}, h_{11}, h_{12}, \ldots, h_{nn} \right),$$

where $h_{ij} = x_i \frac{\partial f}{\partial x_j}$, $i = 1, \ldots, n$, $j = 1, \ldots, n$.

Let us observe that Φ is a birational mapping (onto its image), in particular it is generically finite, since Φ is injective outside the critical set of f.

Let $A := \mathbb{C} \times \{(0, \ldots, 0)\} \subset \mathbb{C} \times \mathbb{C}^{n(n+1)}$. By the definitions of $K_\infty(f)$ and of Φ, we have the equality:

$$K_\infty(f) = A \cap J_\Phi, \tag{14}$$

where J_Φ denotes the set of points at which the mapping Φ is not proper. Recall that $K_\infty(f)$ is finite, hence the set $A \cap J_\Phi$ is finite too.

Let $X := \overline{\Phi(\mathbb{C}^n)} \subset \mathbb{C} \times \mathbb{C}^{n(n+1)}$ and $B := J_\Phi$. Let $B(x_1, X)$ be a base-set of non-properness of linear projections of X with respect to x_1 (cf Definition 4.4).

In the following we identify the target $\mathbb{C}$ of f with the line $A \subset \mathbb{C} \times \mathbb{C}^{n(n+1)}$.

Let then $\{p_1, \ldots, p_s\} := NK_\infty(f) \cup (B(x_1, X) \cap A) \setminus f(\mathrm{Sing} f) \subset A \cap X$. By Proposition 4.3 and using its notations, for general $(L_1, \ldots, L_{n-1}) \in U \subset M^{n-1}_{1+n(n+1)}$, the mapping:

$$\Pi = (x_1, L_1, \ldots, L_{n-1}) : X \to \mathbb{C}^n$$

satisfies the following conditions:

(a) the fibers of Π have dimension at most one,
(b) there is a polynomial $\rho \in \mathbb{C}[t_1]$ such that

$$J_\Pi = \{(t_1, \ldots, t_n) \in \mathbb{C}^n \mid \rho(t_1) = 0\},$$

(c) $\Pi(A) \not\subset \Pi(B)$,
(d) all fibers $\Pi^{-1}(\Pi(p_j))$, $j = 1, \ldots, s$ are finite and non-empty.

Let us write $L_i = c_i x_1 + l_i(a, b)$, $i = 1, \ldots, n-1$, where the linear form $l_i(a, b)$ does not depend on variable x_1. Note that:

$$\Pi(A) = \{x \in \mathbb{C}^n \mid x_1 = t, x_2 = c_1 t, \ldots, x_n = c_{n-1} t, \quad t \in \mathbb{C}\}.$$

For $\Psi := \Pi \circ \Phi$, we have (see (14)):

$$\Pi(K_\infty(f)) = \Pi(A \cap J_\Phi) \ \subset \ \Pi(A) \cap J_\Psi.$$

Let $V := \{y \in \mathbb{C}^n \mid \dim \Psi^{-1}(y) > 0\}$. Since the fibers $\Psi^{-1}(\Pi(p_j))$, $j = 1, \ldots, s$, are finite and non-empty we have $\Pi(p_j) \notin \overline{V}$ for $j = 1, \ldots, s$. So let S be a hypersurface in $\mathbb{C}^n$ which contains $\overline{V}$ but does not contain the set of points $\{\Pi(p_1), \ldots, \Pi(p_s)\}$ and let

$$R := S \cup \{y \in \mathbb{C}^n \mid \prod^{r}_{c \in \Pi(f(\mathrm{Sing} f))} (y_1 - c) = 0\}.$$

With these notations, the mapping

$$\Psi' : \mathbb{C}^n \setminus \Psi^{-1}(R) \to \mathbb{C}^n \setminus R, \quad x \mapsto \Psi(x)$$

is quasi-finite, and moreover $\Pi(NK_\infty(f)) \subset J_{\Psi'}$.

Let $\Gamma' := \Psi'^{-1}(\Pi(A))$. By Proposition 4.2, Γ' is a curve and $\Pi(NK_\infty(f))$ is contained in the non-properness set of the mapping $\Psi|_{\Gamma'} : \Gamma' \to \Pi(A) \setminus R$. Consequently, the set $\Pi(NK_\infty(f))$ is also contained in the non properness set of the mapping Ψ restricted to $\overline{\Psi^{-1}(\Pi(A) \setminus \Pi(f(\mathrm{Sing} f))}$.

By the definition of Ψ we have $\Psi^{-1}(\Pi(A)) = \Phi^{-1}(\Pi^{-1}(\Pi(A)))$, where:

$$\Pi^{-1}(\Pi(A)) = \{x \in X \mid l_1(a, b)(x_2, \ldots, x_{n(n+1)}) = 0, \ldots, l_{n-1}(a, b)(x_2, \ldots, x_{n(n+1)}) = 0\}.$$

Comparing to the definition (13), we see that the set $\overline{\Phi^{-1}(\Pi^{-1}(\Pi(A))) \setminus \mathrm{Sing}(f)}$ coincides with the super-polar curve $\Gamma_f(a, b)$.

The set $\Gamma_f(a, b)$ is a curve since it is union of the curve $\overline{\Gamma'}$, which actually coincide with the horizontal part $H\Gamma_f(a, b)$, and, eventually, some of the one dimensional fibers of Ψ.

Let us now consider a linear isomorphism:

$$T : \mathbb{C}^n \to \mathbb{C}^n, \quad (x_1, \ldots, x_n) \mapsto (x_1, x_2 - c_2 x_1, \ldots, x_n - c_n x_1).$$

From the above construction we know that $\Pi(NK_\infty(f)) \subset J_{\Psi_{|\Gamma'}}$. We then have the inclusion $T(\Pi(NK_\infty(f))) \subset J_{T \circ \Psi_{|\Gamma'}}$. But $T \circ \Psi_{|\Gamma'}$ coincides with f on $\Gamma' = H\Gamma_f(a, b)$, and

$T(\Pi(NK_\infty(f)))$ coincides with $NK_\infty(f)$. This shows the inclusion $NK_\infty(f) \subset J_f(H\Gamma_f(a,b))$ and ends the proof of point (b) of our theorem. □

5.1. **Proof of Corollary 1.4.**

We use the terminology of the above proof. We have actually shown that if $NK_\infty(f) \neq \emptyset$ then the curve $\overline{\Gamma'}$ is non-empty, and that the set $NK_\infty(f)$ is contained in the non-properness set of the restriction $f_{|\Gamma'}$. The curve $\overline{\Gamma'}$ is a subset of the super-polar curve $\Gamma_f(a,b)$ for general coefficients a and b, and moreover, f is constant on all other components of $\Gamma_f(a,b)$. By the generalized Bezout Theorem we have $\deg \Gamma_f(a,b) \leq d^{n-1} - \sum_{i=1}^r d_i$, thus $\deg \overline{\Gamma'} \leq d^{n-1} - \sum_{i=1}^r d_i$. Note that the cardinality of the non-properness set of $f_{|\Gamma'}$ is estimated by the number of these points at infinity of a curve Γ' which are transformed by f into $\mathbb{C}$. Consequently, the cardinality of the non-properness set of $f_{|\Gamma'}$ is bounded from above by the number $d^{n-1} - 1 - \sum_{i=1}^r d_i$. We can substract 1 in this formula since actually each branch of $\overline{\Gamma'}$ intersects the hyperplane at infinity also at the value infinity of f. Thus we also have $\#NK_\infty(f) \leq d^{n-1} - 1 - \sum_{i=1}^r d_i$. Since every connected positive-dimensional component of the critical set $\mathrm{Sing} f$ is contained in one fiber of f thus indicates a trivial non-regular value, we obtain:

$$\#K_\infty(f) \leq d^{n-1} - 1 - \sum_{i=1}^r (d_i - 1).$$

For $n = 2$, it turns out that the Malgrange condition can be recovered (see [Ha1], [Ha2], [LO]) by the asymptotic behavior of the derivatives of f only. We thus consider, instead of the mapping Φ of the proof of Theorem 5.1, the new mapping $\Phi(x,y) = (f(x,y), \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$. This mapping is generically finite if $NK_\infty(f) \neq \emptyset$. In this case, arguing as above we get the last inequalities of our Corollary 1.4.

6. Algorithm

We present here a fast algorithm which yields a finite set $S \subset \mathbb{C}$ such that $NK_\infty(f) \subset S$, for a given polynomial $f : \mathbb{C}^n \to \mathbb{C}$. By our results, this problem reduces to computing the non-properness set of the mapping $f_{|\Gamma} : \Gamma \to \mathbb{C}$ where Γ is a super-polar curve of f.

Let us first show how to compute the non-properness set J_g of the mapping $g : X \to \mathbb{C}$, where $X \subset \mathbb{C}^n$ is a curve. The following result can be found in [PP]:

Theorem 6.1. *If $\mathcal{B} = (b_1, \ldots, b_t)$ is the Gröbner basis of the ideal $I \subset k[x_1, \ldots, x_n]$ with the lexicographic order in which $x_1 > x_2 > \ldots > x_n$, then for every $0 \leq m \leq n$ the set $\mathcal{B} \cap k[x_{m+1}, \ldots, x_n]$ is the Gröbner basis of the ideal $I \cap k[x_{m+1}, \ldots, x_n]$.* □

Corollary 6.2. *Consider the ring $\mathbb{C}[x_1, \ldots, x_n; y_1, \ldots, y_m]$. Let $V \subset \mathbb{C}^n \times \mathbb{C}^m$ be an algebraic set and let $p : \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^m$ denote the projection. Assume that $\mathcal{B}$ is a Gröbner basis of the ideal $I(V)$ with the lexicographic order. Then $\mathcal{B} \cap \mathbb{C}[y_1, \ldots, y_m]$ is a Gröbner basis of the ideal $I(p(V))$.*

Proof. Observe that $I(p(V)) = I(V) \cap \mathbb{C}[y_1, \ldots, y_m]$ and then use Theorem 6.1. □

Let then $I(X) := (h_1, \dots, h_r)$ be the ideal of our curve X. The graph $G \subset \mathbb{C}^n \times \mathbb{C}$ of the non-constant mapping $f : X \to \mathbb{C}$ is given by the ideal $I = (h_i = 0,\ i = 1, \dots, r;\ f(x) - z) \subset \mathbb{C}[x_1, \dots, x_n, z]$.

Let O be the order in $\mathbb{C}[x_1, \dots, x_n, z]$ such that $x_1 > x_2 > \dots > x_i > x_{i+1} > \dots > x_n > z$. Let $\mathcal{B}$ denote the Gröbner basis of I with respect to the order O. Let $f_i \in \mathcal{B} \cap \mathbb{C}[x_i, z]$ be a non-zero polynomial which depends on x_i. Then:

$$f_i = x_i^{n_i} a_0^i(z) + x_i^{n_i-1} a_1^i(z) + \dots + a_{n_i}^i(z).$$

By [Jel1, Prop. 7], [Jel2, Th. 3.10], for our mapping $f : X \to \mathbb{C}$ we have:

$$J_f = \bigcup_{i=1}^{n} \{z \in \mathbb{C} \mid a_0^i(z) = 0\}.$$

With this preparation, we now state the algorithm:

Special case: $\mathrm{Sing}(f)$ is a finite set.

INPUT: the polynomial $f : \mathbb{C}^n \to \mathbb{C}$

(1) choose random coefficients $\alpha_i^k, \alpha_{ij}^k,\ k = 1, \dots, n-1;\ i, j = 1, \dots, n$.
(2) put $g_k = \sum_j \alpha_j^k \frac{\partial f}{\partial x_j} + \sum_{i,j} \alpha_{ij}^k x_i \frac{\partial f}{\partial x_j}$.
(3) put $W := (g_1, ..., g_{n-1}) \subset \mathbb{C}[x_1, ..., x_n]$, if $\dim W > 1$ then go back to (1).
(4) compute a Gröbner basis $\mathcal{B}$ of the ideal $I = (g_1, \dots, g_{n-1}, f - z) \subset \mathbb{C}[x_1, \dots, x_n, z]$ with respect to order O (as defined above).
(5) let $f_i = x_i^{n_i} a_0^i(z) + x_i^{n_i-1} a_1^i(z) + \dots + a_{n_i}^i(z) \in \mathcal{B}_i \cap \mathbb{C}[x_i, z]$ be a non zero polynomial which depends on x_i.
(6) let $S := \bigcup_{i=1}^n \{z \in \mathbb{C} \mid a_0^i(z) = 0\}$. The set S is the non-properness set of the mapping f restricted to $\{g_1 = 0, \dots, g_{n-1} = 0\}$).

OUTPUT: a finite set $S \subset \mathbb{C}$ such that $NK_\infty(f) \subset S$.

In the general case, in order to grip the super-polar curve, we have to remove from the set $\{g_1 = 0, \dots, g_{n-1} = 0\}$ the singular set $\mathrm{Sing}(f)$. To do this, it is enough to remove the hypersurface $\{\sum \beta_j \frac{\partial f}{\partial x_j} = 0\}$, where the coefficients β_j are sufficiently general. Indeed such a hypersurface does contain $\mathrm{Sing}(f)$ but does not contain any component of $\Gamma(a, b)$.

General case:

INPUT: the polynomial $f : \mathbb{C}^n \to \mathbb{C}$

(1) choose random coefficients $\alpha_i^k, \alpha_{ij}^k, \beta_i,\ k = 1, ..., n-1,\ i, j = 1, \dots, n$.
(2) put $g_k = \sum_j \alpha_j^k \frac{\partial f}{\partial x_j} + \sum_{i,j} \alpha_{ij}^k x_i \frac{\partial f}{\partial x_j}$.
(3) put $h = \sum_{j=1}^n \beta_j \frac{\partial f}{\partial x_j}$.
(4) put $W := (g_1, \dots, g_{n-1}, th - 1) \subset \mathbb{C}[t, x_1, \dots, x_n]$; if $\dim W > 1$, then go back to (1).

(5) compute a Gröbner basis $\mathcal{B}$ of the ideal $I = (th-1, g_1, \ldots, g_{n-1}, f-z) \subset \mathbb{C}[t, x_1, \ldots, x_n, z]$ with respect to the order O such that $t > x_1 > x_2 > \ldots > \hat{x}_i > x_{i+1} > \cdots > x_n >> z$.
(6) let $f_i = x_i^{n_i} a_0^i(z) + x_i^{n_i-1} a_1^i(z) + \cdots + a_{n_l}^i(z) \in \mathcal{B} \cap \mathbb{C}[x_i, z]$ be a non zero polynomial which depends on x_i.
(7) let $S = \bigcup_{i=1}^n \{z \in \mathbb{C} \mid a_0^i(z) = 0\}$. Here S is the non-properness set of the mapping f restricted to $\{g_1 = 0, \ldots, g_{n-1} = 0\} \setminus \{h = 0\}$.

OUTPUT: a finite set $S \subset \mathbb{C}$ such that $NK_\infty(f) \subset S$.

Remark 6.3. The above algorithm is probabilistic (without certification), hence for really random coefficients α and β it gives a good subset $S(\alpha, \beta)$, but for some choices it can produce a bad answer. However generically it produces subsets $S(\alpha, \beta)$ which contains $NK_\infty(f)$ Therefore in practice we must repeat the algorithm several times and select only the subset $S(\alpha, \beta)$ which contains the same fixed subset all times. The final answer should then be the intersection $S := \bigcap_{\alpha,\beta} S(\alpha, \beta)$.

At step (5) (and (4) in the isolated singularity case, respectively) we compute Gröbner bases in polynomial rings of at most $n + 2$ variables.

It is possible to construct also a version of this algorithm with a certification, however in that case we have to compute Gröbner bases in polynomial rings of $2n + 1$ variables.

Remark 6.4. A similar algorithm can be constructed for the iterated polar curves method that we use in the first part of our paper; more steps will be needed. We leave the details to the reader.

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