

# SOME NEW $(H_p, L_p)$ TYPE INEQUALITIES OF MAXIMAL OPERATORS OF VILENKIN-NÖRLUND MEANS WITH NON-DECREASING COEFFICIENTS

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ABSTRACT. In this paper we prove and discuss some new  $(H_p, L_p)$  type inequalities of maximal operators of Vilenkin-Nörlund means with non-decreasing coefficients. We also apply these inequalities to prove strong convergence theorems of such Vilenkin-Nörlund means. These inequalities are the best possible in a special sense. As applications, both some well-known and new results are pointed out.

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## 1. INTRODUCTION

The definitions and notations used in this introduction can be found in our next Section. In the one-dimensional case the weak (1,1)-type inequality for the maximal operator of Fejér means

$$\sigma^* f := \sup_{n \in \mathbb{N}} |\sigma_n f|$$

can be found in Schipp [19] for Walsh series and in Pál, Simon [17] for bounded Vilenkin series. Fujji [9] and Simon [21] verified that  $\sigma^*$  is bounded from  $H_1$  to  $L_1$ . Weisz [31] generalized this result and proved boundedness of  $\sigma^*$  from the martingale space  $H_p$  to the space  $L_p$ , for  $p > 1/2$ . Simon [20] gave a counterexample, which shows that boundedness does not hold for  $0 < p < 1/2$ . A counterexample for  $p = 1/2$  was given by Goginava [6] (see also [23]). Moreover, Weisz [33] proved that the maximal operator of the Fejér means  $\sigma^*$  is bounded from the Hardy space  $H_{1/2}$  to the space *weak*  $-L_{1/2}$ . In [24] and [25] it was proved that the weighted maximal operator of Fejér means

$$\tilde{\sigma}_p^* f := \sup_{n \in \mathbb{N}_+} \frac{|\sigma_n f|}{(n+1)^{1/p-2} \log^{2[1/2+p]}(n+1)}$$

is bounded from the Hardy space  $H_p$  to the space  $L_p$ , when  $0 < p \leq 1/2$ . Moreover, the rate of the weights  $\left\{ 1/(n+1)^{1/p-2} \log^{2[p+1/2]}(n+1) \right\}_{n=1}^{\infty}$  in  $n$ -th Fejér mean was given exactly.

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Móricz and Siddiqi [13] investigated the approximation properties of some special Nörlund means of Walsh-Fourier series of  $L_p$  function in norm. In the two-dimensional case approximation properties of Nörlund means was considered by Nagy (see [14]-[16]). In [18] it was proved that the maximal operator of Nörlund means

$$t^* f := \sup_{n \in \mathbb{N}} |t_n f|$$

with non-decreasing coefficients is bounded from the Hardy space  $H_{1/2}$  to the space *weak* -  $L_{1/2}$ . Moreover, there exists a martingale and Nörlund means, with non-decreasing coefficients, such that it is not bounded from the Hardy space  $H_p$  to the space *weak* -  $L_p$ , when  $0 < p < 1/2$ .

It is well-known that Vilenkin systems do not form bases in the space  $L_1$ . Moreover, there is a function in the Hardy space  $H_1$ , such that the partial sums of  $f$  are not bounded in  $L_1$ -norm. Simon [22] proved that there exists an absolute constant  $c_p$ , depending only on  $p$ , such that the inequality

$$\frac{1}{\log^{[p]} n} \sum_{k=1}^n \frac{\|S_k f\|_p^p}{k^{2-p}} \leq c_p \|f\|_{H_p}^p \quad (0 < p \leq 1)$$

holds for all  $f \in H_p$  and  $n \in \mathbb{N}_+$ , where  $[p]$  denotes the integer part of  $p$ . For  $p = 1$  analogous results with respect to more general systems were proved in [2] and [4] and for  $0 < p < 1$  another proof can be found in [27].

In [3] it was proved that there exists an absolute constant  $c_p$ , depending only on  $p$ , such that the inequality

$$(1) \quad \frac{1}{\log^{[1/2+p]} n} \sum_{k=1}^n \frac{\|\sigma_k f\|_p^p}{k^{2-2p}} \leq c_p \|f\|_{H_p}^p \quad (0 < p \leq 1/2, n = 2, 3, \dots).$$

holds. An analogous result for the Walsh system can be found in [28].

In this paper we derive some new  $(H_p, L_p)$ -type inequalities for weighted maximal operators of Nörlund means with non-decreasing coefficients. Moreover, we prove strong convergence theorems of such Nörlund means.

This paper is organized as follows: In order not to disturb our discussions later on some definitions and notations are presented in Section 2. The main results and some of its consequences can be found in Section 3. For the proofs of the main results we need some auxiliary Lemmas, some of them are new and of independent interest. These results are presented in Section 4. The detailed proofs are given in Section 5.

## 2. DEFINITIONS AND NOTATION

Denote by  $\mathbb{N}_+$  the set of the positive integers,  $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$ . Let  $m := (m_0, m_1, \dots)$  be a sequence of the positive integers not less than 2. Denote by

$$Z_{m_k} := \{0, 1, \dots, m_k - 1\}$$

the additive group of integers modulo  $m_k$ .

Define the group  $G_m$  as the complete direct product of the groups  $Z_{m_i}$  with the product of the discrete topologies of  $Z_{m_j}$ 's.

The direct product  $\mu$  of the measures

$$\mu_k(\{j\}) := 1/m_k \quad (j \in Z_{m_k})$$

is the Haar measure on  $G_m$  with  $\mu(G_m) = 1$ .

In this paper we discuss bounded Vilenkin groups, i.e. the case when  $\sup_n m_n < \infty$ .

The elements of  $G_m$  are represented by sequences

$$x := (x_0, x_1, \dots, x_j, \dots), \quad (x_j \in Z_{m_j}).$$

Set  $e_n := (0, \dots, 0, 1, 0, \dots) \in G$ , the  $n$ -th coordinate of which is 1 and the rest are zeros ( $n \in \mathbb{N}$ ).

It is easy to give a basis for the neighborhoods of  $G_m$  :

$$I_0(x) := G_m, \quad I_n(x) := \{y \in G_m \mid y_0 = x_0, \dots, y_{n-1} = x_{n-1}\},$$

where  $x \in G_m$ ,  $n \in \mathbb{N}$ .

If we define  $I_n := I_n(0)$ , for  $n \in \mathbb{N}$  and  $\overline{I}_n := G_m \setminus I_n$ , then

$$(2) \quad \overline{I}_N = \left( \bigcup_{k=0}^{N-2} \bigcup_{l=k+1}^{N-1} I_N^{k,l} \right) \cup \left( \bigcup_{k=1}^{N-1} I_N^{k,N} \right),$$

where

$$I_N^{k,l} := \begin{cases} I_N(0, \dots, 0, x_k \neq 0, 0, \dots, 0, x_l \neq 0, x_{l+1}, \dots, x_{N-1}, \dots), & \text{for } k < l < N, \\ I_N(0, \dots, 0, x_k \neq 0, 0, \dots, x_{N-1} = 0, x_N, \dots), & \text{for } l = N. \end{cases}$$

If we define the so-called generalized number system based on  $m$  in the following way :

$$M_0 := 1, \quad M_{k+1} := m_k M_k \quad (k \in \mathbb{N}),$$

then every  $n \in \mathbb{N}$  can be uniquely expressed as  $n = \sum_{j=0}^{\infty} n_j M_j$ , where  $n_j \in Z_{m_j}$  ( $j \in \mathbb{N}_+$ ) and only a finite number of  $n_j$ 's differ from zero.

We introduce on  $G_m$  an orthonormal system which is called the Vilenkin system. At first, we define the complex-valued function  $r_k(x) : G_m \rightarrow \mathbb{C}$ , the generalized Rademacher functions, by

$$r_k(x) := \exp(2\pi i x_k / m_k), \quad (i^2 = -1, x \in G_m, k \in \mathbb{N}).$$

Next, we define the Vilenkin system  $\psi := (\psi_n : n \in \mathbb{N})$  on  $G_m$  by:

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x), \quad (n \in \mathbb{N}).$$

Specifically, we call this system the Walsh-Paley system when  $m \equiv 2$ .

The norms (or quasi-norms) of the spaces  $L_p(G_m)$  and *weak* -  $L_p(G_m)$  ( $0 < p < \infty$ ) are respectively defined by

$$\|f\|_p^p := \int_{G_m} |f|^p d\mu, \quad \|f\|_{\text{weak-L}_p}^p := \sup_{\lambda > 0} \lambda^p \mu(f > \lambda) < +\infty.$$

The Vilenkin system is orthonormal and complete in  $L_2(G_m)$  (see [29]).

Now, we introduce analogues of the usual definitions in Fourier-analysis. If  $f \in L_1(G_m)$  we can define Fourier coefficients, partial sums of the Fourier series and Dirichlet kernels with respect to the Vilenkin system in the usual manner:

$$\widehat{f}(n) := \int_{G_m} f \overline{\psi}_n d\mu \quad (n \in \mathbb{N}),$$

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}(k) \psi_k, \quad D_n := \sum_{k=0}^{n-1} \psi_k, \quad (n \in \mathbb{N}_+),$$

respectively.

The  $\sigma$ -algebra generated by the intervals  $\{I_n(x) : x \in G_m\}$  will be denoted by  $F_n$  ( $n \in \mathbb{N}$ ). Denote by  $f = (f^{(n)}, n \in \mathbb{N})$  a martingale with respect to  $F_n$  ( $n \in \mathbb{N}$ ). (for details see e.g. [30]).

The maximal function of a martingale  $f$  is defined by

$$f^* = \sup_{n \in \mathbb{N}} |f^{(n)}|.$$

For  $0 < p < \infty$  the Hardy martingale spaces  $H_p$  consist of all martingales  $f$  for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

If  $f = (f^{(n)}, n \in \mathbb{N})$  is a martingale, then the Vilenkin-Fourier coefficients must be defined in a slightly different manner:

$$\widehat{f}(i) := \lim_{k \rightarrow \infty} \int_{G_m} f^{(k)} \overline{\psi}_i d\mu.$$

Let  $\{q_k : k \geq 0\}$  be a sequence of nonnegative numbers. The  $n$ -th Nörlund mean  $t_n$  for a Fourier series of  $f$  is defined by

$$(3) \quad t_n f = \frac{1}{Q_n} \sum_{k=1}^n q_{n-k} S_k f,$$

where  $Q_n := \sum_{k=0}^{n-1} q_k$ .

We always assume that  $q_0 > 0$  and  $\lim_{n \rightarrow \infty} Q_n = \infty$ . In this case it is well-known that the summability method generated by  $\{q_k : k \geq 0\}$  is regular if and only if

$$\lim_{n \rightarrow \infty} \frac{q_{n-1}}{Q_n} = 0.$$

Concerning this fact and related basic results we refer to [12]. In this paper we consider regular Nörlund means only.

If  $q_k \equiv 1$ , we respectively define the Fejér means  $\sigma_n$  and Kernels  $K_n$  as follows:

$$\sigma_n f := \frac{1}{n} \sum_{k=1}^n S_k f, \quad K_n := \frac{1}{n} \sum_{k=1}^n D_k.$$

It is well-known that (see [1])

$$(4) \quad n |K_n| \leq c \sum_{l=0}^{|n|} M_l |K_{M_l}|$$

and

$$(5) \quad \|K_n\|_1 \leq c < \infty.$$

Denote

$$\log^{(0)} x = x \quad \text{and} \quad \log^{(\beta)} x := \overbrace{\log \dots \log x}^{\beta \text{ times}}, \quad \text{for } \beta \in \mathbb{N}_+.$$

Let  $\alpha \in \mathbb{R}_+$ ,  $\beta \in \mathbb{N}_+$  and  $\{q_k = \log^{(\beta)} k^\alpha : k \geq 0\}$ . Then we get the class of Nörlund means, with non-decreasing coefficients:

$$\theta_n f := \frac{1}{Q_n} \sum_{k=1}^n \log^{(\beta)} (n-k)^\alpha S_k f,$$

where

$$\begin{aligned} Q_n &= \sum_{k=1}^{n-1} \log^{(\beta)} (n-k)^\alpha = \sum_{k=1}^{n-1} \log^{(\beta)} k^\alpha = \log \prod_{k=1}^{n-1} \log^{(\beta-1)} k^\alpha \\ &\geq \log \left( \log^{(\beta-1)} \left( \frac{n-1}{2} \right)^\alpha \right)^{\frac{(n-1)}{2}} \geq \frac{n}{4} \log \log^{(\beta-1)} \left( \frac{n-1}{2} \right)^\alpha \sim n \log^{(\beta)} n^\alpha. \end{aligned}$$

It follows that

$$\frac{q_{n-1}}{Q_n} \leq \frac{c \log^{(\beta)} (n-1)^\alpha}{n \log^{(\beta)} n^\alpha} = O\left(\frac{1}{n}\right) \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Finally, we say that a bounded measurable function  $a$  is a  $p$ -atom, if there exists a interval  $I$ , such that

$$\int_I a d\mu = 0, \quad \|a\|_\infty \leq \mu(I)^{-1/p}, \quad \text{supp}(a) \subset I.$$

### 3. THE MAIN RESULTS AND APPLICATIONS

Our first main result reads:

**Theorem 1.** *a) Let  $0 < p < 1/2$ ,  $f \in H_p$  and  $\{q_k : k \geq 0\}$  be a sequence of non-decreasing numbers. Then there exists an absolute constant  $c_p$ , depending only on  $p$ , such that the inequality*

$$\sum_{k=1}^{\infty} \frac{\|t_k f\|_p^p}{k^{2-2p}} \leq c_p \|f\|_{H_p}^p$$

*holds.*

*b) Let  $f \in H_{1/2}$  and  $\{q_k : k \geq 0\}$  be a sequence of non-decreasing numbers, satisfying the condition*

$$(6) \quad \frac{q_{n-1}}{Q_n} = O\left(\frac{1}{n}\right), \quad \text{as } n \rightarrow \infty.$$

*Then there exists an absolute constant  $c$ , such that the inequality*

$$(7) \quad \frac{1}{\log n} \sum_{k=1}^n \frac{\|t_k f\|_{1/2}^{1/2}}{k} \leq c \|f\|_{H_{1/2}}^{1/2}$$

*holds.*

**Example 1.** Let  $0 < p \leq 1/2$ ,  $f \in H_p$  and  $\{q_k : k \geq 0\}$  be a sequence of non-decreasing numbers, such that

$$\sup_n q_n < c < \infty.$$

Then

$$\frac{q_{n-1}}{Q_n} \leq \frac{c}{Q_n} \leq \frac{c}{q_0 n} = \frac{c_1}{n} = O\left(\frac{1}{n}\right), \text{ as } n \rightarrow 0,$$

i.e. condition (6) is satisfied and for such Nörlund means there exists an absolute constant  $c$ , such that the inequality (7) holds.

**Example 2.** Let  $0 < p \leq 1/2$  and  $f \in H_p$ . Then there exists absolute constant  $c_p$ , depending only on  $p$ , such that the following inequality holds:

$$\frac{1}{\log^{[1/2+p]} n} \sum_{k=1}^n \frac{\|\sigma_k f\|_p^p}{k^{2-2p}} \leq c_p \|f\|_{H_p}^p.$$

**Remark 1.** This result for the Walsh system can be found in [28] and for any bounded Vilenkin system in [3].

We have already considered the case when the sequence  $\{q_k : k \geq 0\}$  is bounded. Now, we consider some Nörlund means, which are generated by a unbounded sequence  $\{q_k : k \geq 0\}$ .

**Example 3.** Let  $0 < p \leq 1/2$  and  $f \in H_p$ . Then there exists an absolute constant  $c_p$ , depending only on  $p$ , such that the following inequality holds:

$$\frac{1}{\log^{[1/2+p]} n} \sum_{k=1}^n \frac{\|\theta_k f\|_p^p}{k^{2-2p}} \leq c_p \|f\|_{H_p}^p.$$

Up to now we have considered strong convergence theorems in the case  $0 < p \leq 1/2$ , but in our next main result we consider boundedness of weighed maximal operators of Nörlund means when  $0 < p \leq 1/2$ , and now without any restriction like (6).

**Theorem 2.** Let  $0 < p \leq 1/2$ ,  $f \in H_p$  and  $\{q_k : k \geq 0\}$  be a sequence of non-decreasing numbers. Then the maximal operator

$$\tilde{t}_p^* f := \sup_{n \in \mathbb{N}_+} \frac{|t_n f|}{(n+1)^{1/p-2} \log^{2[1/2+p]}(n+1)}$$

is bounded from the Hardy space  $H_p$  to the space  $L_p$ .

**Example 4.** Let  $0 < p \leq 1/2$ ,  $f \in H_p$  and  $\{q_k : k \geq 0\}$  be a sequence of non-decreasing numbers. Then the maximal operator

$$\tilde{\sigma}_p^* f := \sup_{n \in \mathbb{N}_+} \frac{|\sigma_n f|}{(n+1)^{1/p-2} \log^{2[1/2+p]}(n+1)}$$

is bounded from the Hardy space  $H_p$  to the space  $L_p$ .

**Remark 2.** This result for the Walsh system when  $p = 1/2$  can be found in [7]. Later on, it was generalized for bounded Vilenkin systems in [24]. The case  $0 < p < 1/2$  can be found in [25]. Analogous results with respect to Walsh-Kachmarz systems were considered in [8] for  $p = 1/2$  and in [26] for  $0 < p < 1/2$ .

**Example 5.** Let  $0 < p \leq 1/2$ ,  $f \in H_p$  and  $\{q_k : k \geq 0\}$  be a sequence of non-decreasing numbers. Then the maximal operator

$$\tilde{\theta}_p^* f := \sup_{n \in \mathbb{N}_+} \frac{|\theta_n f|}{(n+1)^{1/p-2} \log^{2[1/2+p]}(n+1)}$$

is bounded from the Hardy space  $H_p$  to the space  $L_p$ .

#### 4. AUXILIARY LEMMAS

We need the following auxiliary Lemmas:

**Lemma 1** (see e.g. [32]). A martingale  $f = (f^{(n)}, n \in \mathbb{N})$  is in  $H_p$  ( $0 < p \leq 1$ ) if and only if there exists a sequence  $(a_k, k \in \mathbb{N})$  of  $p$ -atoms and a sequence  $(\mu_k, k \in \mathbb{N})$  of real numbers such that, for every  $n \in \mathbb{N}$ ,

$$(8) \quad \sum_{k=0}^{\infty} \mu_k S_{M_n} a_k = f^{(n)}, \quad a.e.$$

and

$$\sum_{k=0}^{\infty} |\mu_k|^p < \infty.$$

Moreover,

$$\|f\|_{H_p} \sim \inf \left( \sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p}$$

where the infimum is taken over all decompositions of  $f$  of the form (8).

**Lemma 2** (see e.g. [32]). Suppose that an operator  $T$  is  $\sigma$ -sublinear and for some  $0 < p \leq 1$

$$\int_I |Ta|^p d\mu \leq c_p < \infty,$$

for every  $p$ -atom  $a$ , where  $I$  denotes the support of the atom. If  $T$  is bounded from  $L_\infty$  to  $L_\infty$ , then

$$\|Tf\|_p \leq c_p \|f\|_{H_p}, \quad 0 < p \leq 1.$$

**Lemma 3** (see [5]). Let  $n > t$ ,  $t, n \in \mathbb{N}$ . Then

$$K_{M_n}(x) = \begin{cases} \frac{M_t}{1-r_t(x)}, & x \in I_t \setminus I_{t+1}, \quad x - x_t e_t \in I_n, \\ \frac{M_n-1}{2}, & x \in I_n, \\ 0, & \text{otherwise.} \end{cases}$$

For the proof of our main results we also need the following new Lemmas of independent interest:

**Lemma 4.** Let  $\{q_k : k \geq 0\}$  be a sequence of non-decreasing numbers, satisfying condition (6). Then

$$|F_n| \leq \frac{c}{n} \left\{ \sum_{j=0}^{|n|} M_j |K_{M_j}| \right\},$$

for some positive constant  $c$ .

*Proof.* By using Abel transformation we obtain that

$$(9) \quad Q_n := \sum_{j=0}^{n-1} q_j = \sum_{j=1}^n q_{n-j} \cdot 1 = \sum_{j=1}^{n-1} (q_{n-j} - q_{n-j-1}) j + q_0 n$$

and

$$(10) \quad F_n = \frac{1}{Q_n} \left( \sum_{j=1}^{n-1} (q_{n-j} - q_{n-j-1}) j K_j + q_0 n K_n \right).$$

Since  $\{q_k : k \geq 0\}$  be a non-decreasing sequence, satisfying condition (6) we obtain that

$$(11) \quad \frac{1}{Q_n} \left( \sum_{j=1}^{n-1} |q_{n-j} - q_{n-j-1}| + q_0 \right) \leq \frac{1}{Q_n} \left( \sum_{j=1}^{n-1} (q_{n-j} - q_{n-j-1}) + q_0 \right) \\ = \frac{q_{n-1}}{Q_n} \leq \frac{c}{n}.$$

By combining (4) with equalities (10) and (11) we immediately get that

$$|F_n| \leq \left( \frac{1}{Q_n} \left( \sum_{j=1}^{n-1} |q_{n-j} - q_{n-j-1}| + q_0 \right) \right) \sum_{i=0}^{|n|} M_i |K_{M_i}| \leq \frac{c}{n} \sum_{i=0}^{|n|} M_i |K_{M_i}|.$$

The proof is complete by combining the estimates above.  $\square$

**Lemma 5.** *Let  $n \geq M_N$  and  $\{q_k : k \geq 0\}$  be a sequence of non-decreasing numbers. Then*

$$\left| \frac{1}{Q_n} \sum_{j=M_N}^n q_{n-j} D_j \right| \leq \frac{c}{M_N} \left\{ \sum_{j=0}^{|n|} M_j |K_{M_j}| \right\},$$

for some positive constant  $c$ .

*Proof.* Let  $M_N \leq j \leq n$ . By using (4) we get that

$$|K_j| \leq \frac{1}{j} \sum_{l=0}^{|j|} M_l |K_{M_l}| \leq \frac{1}{M_N} \sum_{l=0}^{|n|} M_l |K_{M_l}|.$$

Let the sequence  $\{q_k : k \geq 0\}$  be non-decreasing. Then

$$M_N q_{n-M_N-1} \leq q_{n-M_N-1} + q_{n-M_N} + \dots + q_{n-1} \leq Q_n.$$

If we apply (9) we obtain that

$$\sum_{j=M_N}^{n-1} |q_{n-j} - q_{n-j-1}| j + q_0 n \leq \sum_{j=0}^{n-1} |q_{n-j} - q_{n-j-1}| j + q_0 n \\ = \sum_{j=1}^{n-1} (q_{n-j} - q_{n-j-1}) j + q_0 n = Q_n.$$

By using Abel transformation we find that

$$\left| \frac{1}{Q_n} \sum_{j=M_N}^n q_{n-j} D_j \right| = \left| \frac{1}{Q_n} \left( \sum_{j=M_N}^{n-1} (q_{n-j} - q_{n-j-1}) j K_j + q_0 n K_n - M_N q_{n-M_N-1} \right) \right|$$



$$\left( \frac{1}{Q_n} \left( \sum_{j=M_N}^{n-1} |q_{n-j} - q_{n-j-1}| j + q_0 n + M_N q_{n-M_N-1} \right) \right) \frac{1}{M_N} \sum_{i=0}^{|n|} M_i |K_{M_i}| \leq \frac{1}{M_N} \sum_{i=0}^{|n|} M_i |K_{M_i}|.$$

The proof is complete.  $\square$

**Lemma 6.** *Let  $\{q_k : k \geq 0\}$  be a sequence of non-decreasing numbers, satisfying condition (6). Let  $x \in I_N^{k,l}$ ,  $k = 0, \dots, N-2$ ,  $l = k+1, \dots, N-1$ . Then*

$$\int_{I_N} |F_n(x-t)| d\mu(t) \leq \frac{cM_l M_k}{nM_N}.$$

Let  $x \in I_N^{k,N}$ ,  $k = 0, \dots, N-1$ . Then

$$\int_{I_N} |F_n(x-t)| d\mu(t) \leq \frac{cM_k}{M_N}.$$

Here  $c$  is a positive constant.

*Proof.* Let  $x \in I_N^{k,l}$ . Then, by applying Lemma 3, we have that

$$(12) \quad K_{M_n}(x) = 0, \text{ when } n > l.$$

Let  $k < n \leq l$ . Then we get that

$$(13) \quad |K_{M_n}(x)| \leq cM_k.$$

Let  $x \in I_N^{k,l}$ , for  $0 \leq k < l \leq N-1$  and  $t \in I_N$ . Since  $x-t \in I_N^{k,l}$  and  $n \geq M_N$ , by combining Lemma 4 with (12) and (13), we obtain that

$$(14) \quad \begin{aligned} \int_{I_N} |F_n(x-t)| d\mu(t) &\leq \frac{c}{n} \sum_{i=0}^{|n|} M_i \int_{I_N} |K_{M_i}(x-t)| d\mu(t) \\ &\leq \frac{c}{n} \int_{I_N} M_i \sum_{i=0}^l M_k d\mu(t) \leq \frac{cM_k M_l}{nM_N} \end{aligned}$$

and the first estimate is proved.

Now, let  $x \in I_N^{k,N}$ . Since  $x-t \in I_N^{k,N}$  for  $t \in I_N$ , by applying Lemma 3, we obtain that

$$|K_{M_i}(x-t)| \leq cM_k, \quad (k \in \mathbb{N}).$$

Hence, according to Lemma 4, we have that

$$(15) \quad \begin{aligned} \int_{I_N} |F_n(x-t)| d\mu(t) &\leq \frac{c}{n} \sum_{i=0}^{|n|} M_i \int_{I_N} |K_{M_i}(x-t)| d\mu(t) \\ &\leq \frac{c}{n} \sum_{i=0}^{|n|-1} M_i \int_{I_N} M_k d\mu(t) \leq \frac{cM_k}{M_N}. \end{aligned}$$

By combining (14) and (15) we complete the proof of Lemma 6.  $\square$

Analogously we can prove the similar estimation, but now without any restriction like (6).

**Lemma 7.** *Let  $x \in I_N^{k,l}$ ,  $k = 0, \dots, N-1$ ,  $l = k+1, \dots, N$  and  $\{q_k : k \geq 0\}$  be a sequence of non-decreasing sequence. Then*

$$\int_{I_N} \left| \frac{1}{Q_n} \sum_{j=M_N}^n q_{n-j} D_j(x-t) \right| d\mu(t) \leq \frac{cM_l M_k}{M_N^2},$$

for some positive constant  $c$ .

## 5. PROOFS OF THE THEOREMS

*Proof of Theorem 1.* By Lemma 1, the proof of Theorem 1 will be complete, if we show that

$$(16) \quad \frac{1}{\log^{[1/2+p]} n} \sum_{m=1}^n \frac{\|t_m a\|_{H_p}^p}{m^{2-2p}} \leq c_p,$$

for every  $p$ -atom  $a$ , with support  $I$ ,  $\mu(I) = M_N^{-1}$ . We may assume that  $I = I_N$ . It is easy to see that  $S_n(a) = t_n(a) = 0$ , when  $n \leq M_N$ . Therefore, we can suppose that  $n > M_N$ .

Let  $x \in I_N$ . Since  $t_n$  is bounded from  $L_\infty$  to  $L_\infty$  (the boundedness follows from (5)) and  $\|a\|_\infty \leq M_N^{1/p}$  we obtain that

$$\int_{I_N} |t_m a|^p d\mu \leq \frac{\|a\|_\infty^p}{M_N} \leq c < \infty, \quad 0 < p \leq 1/2.$$

Hence,

$$(17) \quad \frac{1}{\log^{[1/2+p]} n} \sum_{m=1}^n \frac{\int_{I_N} |t_m a|^p d\mu}{m^{2-2p}} \leq \frac{1}{\log^{[1/2+p]} n} \sum_{k=1}^n \frac{1}{m^{2-2p}} \leq c < \infty.$$

It is easy to see that

$$(18) \quad |t_m a(x)| = \int_{I_N} |a(t) F_n(x-t)| d\mu(t) = \int_{I_N} \left| a(t) \frac{1}{Q_n} \sum_{j=M_N}^n q_{n-j} D_j(x-t) \right| d\mu(t) \\ \leq \|a\|_\infty \int_{I_N} \left| \frac{1}{Q_n} \sum_{j=M_N}^n q_{n-j} D_j(x-t) \right| d\mu(t) \leq M_N^{1/p} \int_{I_N} \left| \frac{1}{Q_n} \sum_{j=M_N}^n q_{n-j} D_j(x-t) \right| d\mu(t)$$

Let  $t_n$  be Nörlund means, with non-decreasing coefficients  $\{q_k : k \geq 0\}$  and  $x \in I_N^{k,l}$ ,  $0 \leq k < l \leq N$ . Then, in the view of Lemma 7 we get that

$$(19) \quad |t_m a(x)| \leq cM_l M_k M_N^{1/p-2}, \quad \text{for } 0 < p \leq 1/2.$$

First, we consider the case  $0 < p < 1/2$ . By using (2), (18), (19) we find that

$$(20) \quad \int_{I_N} |t_m a|^p d\mu = \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \sum_{x_j=0, j \in \{l+1, \dots, N-1\}}^{m_{j-1}} \int_{I_N^{k,l}} |t_m a|^p d\mu + \sum_{k=0}^{N-1} \int_{I_N^{k,N}} |t_m a|^p d\mu \\ \leq c \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \frac{m_{l+1} \cdots m_{N-1}}{M_N} (M_l M_k)^p M_N^{1-2p} + \sum_{k=0}^{N-1} \frac{1}{M_N} M_k^p M_N^{1-p} \\ \leq c M_N^{1-2p} \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \frac{(M_l M_k)^p}{M_l} + \sum_{k=0}^{N-1} \frac{M_k^p}{M_N^p} \leq c M_N^{1-2p}.$$

Moreover, according to (20), we get that

$$\sum_{m=M_N+1}^{\infty} \frac{\int_{I_N} |t_m a|^p d\mu}{m^{2-2p}} \leq \sum_{m=M_N+1}^{\infty} \frac{cM_N^{1-2p}}{m^{2-2p}} < c < \infty, \quad (0 < p < 1/2).$$

Now, by combining this estimate with (17) we obtain (16) so the proof of part a) is complete.

Let  $p = 1/2$  and  $t_n$  be Nörlund means, with non-decreasing coefficients  $\{q_k : k \geq 0\}$ , satisfying condition (6). We can write that

$$(21) \quad \begin{aligned} |t_m a(x)| &\leq \int_{I_N} |a(t)| |F_m(x-t)| d\mu(t) \\ &\leq \|a\|_{\infty} \int_{I_N} |F_m(x-t)| d\mu(t) \leq M_N^2 \int_{I_N} |F_m(x-t)| d\mu(t). \end{aligned}$$

Let  $x \in I_N^{k,l}$ ,  $0 \leq k < l < N$ . Then, in the view of Lemma 6 we get that

$$(22) \quad |t_m a(x)| \leq \frac{cM_l M_k M_N}{m}.$$

Let  $x \in I_N^{k,N}$ . Then, according to Lemma 6 we obtain that

$$(23) \quad |t_m a(x)| \leq cM_k M_N.$$

By combining (2), (21), (22) and (23) we obtain that

$$\begin{aligned} &\int_{I_N} |t_m a(x)|^{1/2} d\mu(x) \\ &\leq c \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \frac{m_{l+1} \cdots m_{N-1}}{M_N} \frac{(M_l M_k)^{1/2} M_N^{1/2}}{m^{1/2}} + \sum_{k=0}^{N-1} \frac{1}{M_N} M_k^{1/2} M_N^{1/2} \\ &\leq M_N^{1/2} \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \frac{(M_l M_k)^{1/2}}{m^{1/2} M_l} + \sum_{k=0}^{N-1} \frac{M_k^{1/2}}{M_N^{1/2}} \leq \frac{cM_N^{1/2} N}{m^{1/2}} + c. \end{aligned}$$

It follows that

$$(24) \quad \frac{1}{\log n} \sum_{m=M_N+1}^n \frac{\int_{I_N} |t_m a(x)|^{1/2} d\mu(x)}{m} \leq \frac{1}{\log n} \sum_{m=M_N+1}^n \left( \frac{cM_N^{1/2} N}{m^{3/2}} + \frac{c}{m} \right) < c < \infty.$$

The proof of part b) is completed by just combining (17) and (24).  $\square$

**Proof of Theorem 2.** Since  $t_n$  is bounded from  $L_{\infty}$  to  $L_{\infty}$  (the boundedness follows from (5)), by Lemma 2, the proof of Theorem 2 will be complete, if we show that

$$\int_{I_N} \left( \sup_{n \in \mathbb{N}} \frac{|t_n a|}{\log^{2[1/2+p]}(n+1) (n+1)^{1/p-2}} \right)^p d\mu \leq c < \infty$$

for every p-atom  $a$ , where  $I$  denotes the support of the atom. Let  $a$  be an arbitrary p-atom, with support  $I$  and  $\mu(I) = M_N^{-1}$ . Analogously to in the proof of Theorem 1 we may assume that  $I = I_N$  and  $n > M_N$ .

Let  $x \in I_N^{k,l}$ ,  $0 \leq k < l \leq N$ . Then, by combining (18) and Lemma 7, (see also (19)) we get that

$$(25) \quad \frac{|t_n(a(x))|}{(n+1)^{1/p-2} \log^{2[1/2+p]}(n+1)} \leq \frac{M_N^{1/p}}{M_N^{1/p-2} N^{2[1/2+p]}} \int_{I_N} \left| \frac{1}{Q_n} \sum_{j=M_N}^n q_{n-j} D_j(x-t) \right| d\mu(t) \\ \leq \frac{cM_N^{1/p}}{M_N^{1/p-2} N^{2[1/2+p]}} \frac{M_l M_k}{M_N^2} = \frac{cM_l M_k}{N^{2[1/2+p]}}.$$

By combining (2) and (25) we obtain that (see [24] and [25])

$$\int_{I_N} |t^* a|^p d\mu = \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \sum_{x_j=0, j \in \{l+1, \dots, N-1\}}^{m_{j-1}} \int_{I_N^{k,l}} |t^* a|^p d\mu + \sum_{k=0}^{N-1} \int_{I_N^{k,N}} |t^* a|^p d\mu \\ \leq \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \frac{m_{l+1} \dots m_{N-1}}{M_N} \left( \frac{M_l M_k}{N^{2[1/2+p]}} \right)^p + \sum_{k=0}^{N-1} \frac{1}{M_N} \left( \frac{M_N M_k}{N^{2[1/2+p]}} \right)^p \\ \leq \frac{c}{N^{2[1/2+p]}} \sum_{k=0}^{N-2} \sum_{l=k+1}^{N-1} \frac{(M_l M_k)^p}{M_l} + \frac{c}{M_N^{1-2p} N^{2p[1/2+p]}} \sum_{k=0}^{N-1} \frac{M_k^p}{M_N^p} < \infty.$$

The proof is complete.  $\square$

**A final remark:** Several of the operators considered in this paper, e.g. those described by the Nörlund means are called Hardy type operators in the literature. The mapping properties of such operators, especially between weighted Lebesgue spaces, is much studied in the literature, see e.g. the books [10] and [11] and the references there. Such complimentary information can be of interest for further studies of the inequalities considered in this paper.

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