

# THE CHERN COEFFICIENT AND COHEN-MACAULAY RINGS

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ABSTRACT. The purpose of this paper is to investigate a relationship between the index of reducibility and the Chern coefficient for primary ideals. Therefore, the main result of this paper gives a characterization of a Cohen-Macaulay ring in terms of its the index of reducibility, its Cohen-Macaulay type, and the Chern coefficient for parameter ideals. As corollaries to the main theorem we obtained the characterizations of a Gorenstein ring in term of its Chern coefficient for parameter ideals.

## 1. INTRODUCTION

Let  $(R, \mathfrak{m})$  be a Noetherian local ring with the maximal ideal  $\mathfrak{m}$  and  $I$  an  $\mathfrak{m}$ -primary ideal of  $R$ . One of our goals is to study the set of  $I$ -good filtrations of  $R$ . More concretely, we will consider the set of multiplicative, decreasing filtrations of  $R$  ideals,  $\mathcal{A} = \{I_n \mid I_0 = R, I_{n+1} = II_n, n \gg 0\}$ , integral over the  $I$ -adic filtration, conveniently coded in the corresponding Rees algebra and its associated graded ring

$$\mathcal{R}(\mathcal{A}) = \bigoplus_{n \geq 0} I_n t^n, \text{gr}_{\mathcal{A}}(R) = \bigoplus_{n \geq 0} I^n / I^{n+1}.$$

We will study certain strata of these algebras. For that we will focus on the role of the Hilbert polynomial of the Hilbert function  $\ell(R/I_{n+1})$ .

$$P_{\mathcal{A}}(n) = \sum_{i=0}^d (-1)^i e_i(\mathcal{A}) \binom{n+d-i}{d-i}.$$

These integers  $e_i(\mathcal{A})$  are called the Hilbert coefficients of  $\mathcal{A}$ . In particularity, the leading coefficient  $e_0(\mathcal{A})$  is called the multiplicity of  $R$  with respect to  $\mathcal{A}$ . On occasionally, the first Hilbert coefficient  $e_1(\mathcal{A})$  is referred to the Chern coefficient of  $\mathcal{A}[V2]$ . For Cohen-Macaulay rings, many penetrating relationships among these coefficients have been proved, beginning with Northcotts [No1]. More recently, similar questions have been examined in general Noetherian local rings. For example, at the conference in Yokohama 2008, W. V. Vasconcelos [V2] (see also) posed the following conjecture:

**The Vanishing Conjecture.** Assume that  $R$  is an unmixed, that is  $\dim(\hat{R}/P) = \dim R$  for all  $P \in \text{Ass } \hat{R}$ , where  $\hat{R}$  is the  $\mathfrak{m}$ -adic completion of  $R$ . Then  $R$  is a Cohen-Macaulay local ring if and only if  $e_1(\mathfrak{q}) \geq 0$  for some parameter ideal  $\mathfrak{q}$  of  $R$ .

Recently, this conjecture has been settled by L. Ghezzi, S. Goto, J.-Y. Hong. K. Ozeki, T. T. Phuong, W. V. Vasconcelos in [GGHOPV]. Moreover, S. Goto showed

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how one can use Hilbert coefficients of parameter ideals in order to study many classes about non-unmixed modules such as Buchsbaum modules, generalized Cohen-Macaulay modules, Vasconcelos modules, sequentially Cohen-Macaulay modules and so on (see [G], [CGT]). The goal of our paper is to continue this research direction. Concretely, we will give characterizations of a Cohen-Macaulay ring in term of its Chern coefficient and its index of reducibility. Let  $M$  be a finitely generated  $R$ -module of dimension  $s$ . Then we say that an  $R$ -submodule  $N$  of  $M$  is irreducible if  $N$  is not written as the intersection of two larger  $R$ -submodules of  $M$ . Every  $R$ -submodule  $N$  of  $M$  can be expressed as an irredundant intersection of irreducible  $R$ -submodules of  $M$  and the number of irreducible  $R$ -submodules appearing in such an expression depends only on  $N$  and not on the expression. Let us call, for each  $\mathfrak{m}$ -primary ideal  $I$  of  $M$ , the number  $\mathcal{N}(I; M)$  of irreducible  $R$ -submodules of  $M$  that appearing in an irredundant irreducible decomposition of  $IM$  the index of reducibility of  $M$  with respect to  $I$ .

In history, Northcott and Rees [NR] proved that every parameter ideals of a Noetherian local ring  $R$  is irreducible if and only if  $R$  is Gorenstein. Addition, they showed that if every parameter ideals of a Noetherian local ring  $R$  is irreducible then  $R$  is Cohen-Macaulay. After that, D. G. Northcott [No, Theorem 3] proved that for parameter ideals  $\mathfrak{q}$  in a Cohen-Macaulay local ring  $R$ , the index  $\mathcal{N}(\mathfrak{q}; R)$  of reducibility is constant and independent on the choice of  $\mathfrak{q}$ . However, the property of constant index of reducibility for parameter ideals does not characterize Cohen-Macaulay rings. The example of a non-Cohen-Macaulay local ring  $R$  with  $\mathcal{N}(\mathfrak{q}; R) = 2$  for every parameter ideal  $\mathfrak{q}$  was firstly given in 1964 by S. Endo and M. Narita [EN]. It seems now natural to ask whether characterize Cohen-Macaulay rings by index of reducibility for parameter ideals. Recently, author gave characterize Cohen-Macaulay rings by index of reducibility for parameter ideals and the irreducible multiplicity of  $M$  with respect to  $I$ . So the aim of our paper is to continue this research direction.

With this notation the main results of this paper are summarized into the following. We denote by  $r(R) = \ell_R(\text{Ext}_R^d(R/\mathfrak{m}, R))$  the Cohen-Macaulay type.

**Theorem 1.1.** *Let  $R$  be a Noetherian local ring with maximal ideal  $\mathfrak{m}$ ,  $d = \dim R \geq 2$ . Assume that  $R$  is unmixed, that is  $\dim \hat{R}/\mathfrak{p} = d$  for all  $\mathfrak{p} \in \text{Ass}(\hat{R})$ . The following statements are equivalent.*

- (i)  $R$  is Cohen-Macaulay.
- (ii) For all parameter ideals  $\mathfrak{q} \subseteq \mathfrak{m}^2$ , we have

$$\mathcal{N}(\mathfrak{q}; R) = e_1(I) - e_1(\mathfrak{q}),$$

where  $I = \mathfrak{q} : \mathfrak{m}$ .

- (iii) For all parameter ideals  $\mathfrak{q} \subseteq \mathfrak{m}^2$ , we have

$$\mathcal{N}(\mathfrak{q}; R) \leq e_1(I) - e_1(\mathfrak{q}),$$

where  $I = \mathfrak{q} : \mathfrak{m}$ .

- (iii) For all parameter ideals  $\mathfrak{q} \subseteq \mathfrak{m}^2$ , we have

$$e_1(I) - e_1(\mathfrak{q}) = r(R),$$

where  $I = \mathfrak{q} : \mathfrak{m}$ .

- (iv) For all parameter ideals  $\mathfrak{q} \subseteq \mathfrak{m}^2$ , we have

$$e_1(I) - e_1(\mathfrak{q}) \leq r(R),$$

where  $I = \mathfrak{q} : \mathfrak{m}$ .

From the main result, we get the following results.

**Corollary 1.2.** *Let  $R$  be a Noetherian local ring with  $d = \dim R \geq 2$ . Assume that  $R$  is unmixed, that is  $\dim \hat{R}/\mathfrak{p} = d$  for all  $\mathfrak{p} \in \text{Ass}(\hat{R})$ . Then for all integers  $n$  there exists a parameter ideal  $\mathfrak{q} \subseteq \mathfrak{m}^n$ , we have*

$$r(R) \leq e_1(I; R) - e_1(\mathfrak{q}; R) \leq \mathcal{N}(\mathfrak{q}; R),$$

where  $I = \mathfrak{q} : \mathfrak{m}$ .

Let us explain how this paper is organized. This paper is divided into 3 sections. In the next section, we give the characterizations of a Cohen-Macaulay ring in term of its Chern coefficient and the irreducible multiplicity for parameter ideals. The section 3 of the paper is devoted to prove a parts of the main result and its consequences. In the last section, we give the characterizations of a Cohen-Macaulay ring in term of its Chern coefficient and the Cohen-Macaulay type. As corollaries, we obtained the characterizations of a Gorenstein ring in term of its Chern coefficient for parameter ideals.

## 2. IRREDUCIBLE MULTIPLICITY

Throughout this paper we fix the following standard notations: Let  $R$  be a Noetherian local ring with maximal ideal  $\mathfrak{m}$ ,  $d = \dim R > 0$ ,  $k = R/\mathfrak{m}$  the infinite residue field. Let  $I$  be an  $\mathfrak{m}$ -primary ideal of  $R$ . The associated graded ring  $\text{gr}_I(R) = \bigoplus_{n \geq 0} I^n/I^{n+1}$  is a standard graded ring with  $[\text{gr}_I(R)]_0 = R/I$  Artinian. Let  $M$  be a finitely generated  $R$ -module of dimension  $s$ . Therefore the associated graded module  $\text{gr}_I(M) = \bigoplus_{n \geq 0} I^n M/I^{n+1} M$  of  $I$  with respect to  $M$  is a finitely generated graded  $\text{gr}_I(R)$ -module. The Hilbert-Samuel function of  $M$  with respect to  $I$  is

$$H(n) = \ell_R(M/I^{n+1}M) = \sum_{i=0}^n \ell_R(I^i M/I^{i+1}M),$$

where  $\ell_R(*)$  stands for the length. For sufficiently large  $n$ , the Hilbert-Samuel function of  $M$  with respect to  $I$   $H(n)$  is of polynomial type,

$$\ell_R(M/I^{n+1}M) = \sum_{i=0}^s (-1)^i e_i(I, M) \binom{n+s-i}{s-i}.$$

These integers  $e_i(I, M)$  are called the Hilbert coefficients of  $M$  with respect to  $I$ . In the particular case, the leading coefficient  $e_0(I, M)$  is said to be the multiplicity of  $M$  with respect to  $I$  and  $e_1(I, M)$  is called by Vasconcelos([V2]) the Chern coefficient of  $I$  with respect to  $M$ . When  $M = R$ , we abbreviate  $e_i(I, M)$  to  $e_i(I)$  for all  $i = 1, \dots, s$ . A lot of results are known on the Chern coefficient in the case where  $R$  is a Cohen-Macaulay ring. For example, as was proved by Northcott [No1], we always have  $e_0(I) - \ell(R/I) \leq e_1(I)$ . After Goto and Nishida in [GNi] gave to extend Northcotts inequality without assuming that  $R$  is a Cohen-Macaulay ring. Suppose that  $I$  contains a parameter ideal  $\mathfrak{q}$  as a reduction. Then  $e_0(I) - \ell(R/I) \leq e_1(I) - e_1(\mathfrak{q})$ . The purpose of this paper is to investigate upper bound of  $e_1(I) - e_1(\mathfrak{q})$  by the index of reducibility of  $\mathfrak{q}$ , when  $I = \mathfrak{q} : \mathfrak{m}$ . Recall, we say that an  $R$ -submodule  $N$  of  $M$  is irreducible if  $N$  is not written as the intersection of two larger  $R$ -submodules of  $M$ . Every  $R$ -submodule  $N$  of

$M$  can be expressed as an irredundant intersection of irreducible  $R$ -submodules of  $M$  and the number of irreducible  $R$ -submodules appearing in such an expression depends only on  $N$  and not on the expression. Let us call, for each  $\mathfrak{m}$ -primary ideal  $I$  of  $M$ , the number  $\mathcal{N}(I; M)$  of irreducible  $R$ -submodules of  $M$  that appearing in an irredundant decomposition of  $IM$  the index of reducibility of  $M$  with respect to  $I$ . Remember that

$$\mathcal{N}(I; M) = \ell_R([IM :_M \mathfrak{m}]/IM).$$

Moreover, by Proposition 2.1 [CQT], it is well known that there exists a polynomial  $p_{I,M}(n)$  of degree  $s - 1$  with rational coefficients such that

$$\mathcal{N}(I^n; M) = \ell_R([I^n M :_M \mathfrak{m}]/I^n M) = p_{I,M}(n)$$

for all large enough  $n$ . Then, there are integers  $f_i(I; M)$  such that

$$p_{I,M}(n) = \sum_{i=0}^{s-1} (-1)^i f_i(I; M) \binom{n+d-1-i}{d-1-i}.$$

The leading coefficient  $f_0(I; M)$  is called the irreducible multiplicity of  $M$  with respect to  $I$ . When  $M = R$ , we abbreviate  $f_0(I, M)$  to  $f_0(I)$ . From above notations, in this section, the main result is stated as follows.

**Theorem 2.1.** *Let  $R$  be a Noetherian local ring with maximal ideal  $\mathfrak{m}$ ,  $d = \dim R \geq 2$ . Assume that  $R$  is unmixed, that is  $\dim \hat{R}/\mathfrak{p} = d$  for all  $\mathfrak{p} \in \text{Ass}(\hat{R})$ . The following statements are equivalent.*

- (i)  *$R$  is Cohen-Macaulay and  $R$  is not a regular local ring.*
- (ii) *For all parameter ideals  $\mathfrak{q} \subseteq \mathfrak{m}^2$ , we have*

$$f_0(\mathfrak{q}) = e_1(I),$$

where  $I = \mathfrak{q} : \mathfrak{m}$ .

- (iii) *For all parameter ideals  $\mathfrak{q} \subseteq \mathfrak{m}^2$ , we have*

$$f_0(\mathfrak{q}) \leq e_1(I),$$

where  $I = \mathfrak{q} : \mathfrak{m}$ .

- (iv) *There exists a parameter ideal  $\mathfrak{q}$  such that  $I^2 = \mathfrak{q}I$  and  $f_0(\mathfrak{q}) \leq e_1(I)$ , where  $I = \mathfrak{q} : \mathfrak{m}$ .*

In our proof of Theorem 2.1, the key of problem is described by under facts. Let  $x_1, x_2, \dots, x_s \in R (s \geq 1)$ . Then  $x_1, x_2, \dots, x_s$  is called a  $d$ -sequence if

$$(x_1, \dots, x_{i-1}) :_R x_j = (x_1, \dots, x_{i-1}) :_R x_i x_j$$

for all  $1 \leq i \leq j \leq s$ . We say that  $x_1, x_2, \dots, x_s$  forms a strong  $d$ -sequence in  $R$  if  $x_1^{n_1}, x_2^{n_2}, \dots, x_s^{n_s}$  is a  $d$ -sequence in  $R$  for all integers  $n_i \geq 1 (1 \leq i \leq s)$ . See [Hu] for basic but deep results on  $d$ -sequences.

Now, for a while, let us assume that  $R$  is a homomorphic image of a Gorenstein ring and  $\dim R/\mathfrak{p} = d$  for all  $\mathfrak{p} \in \text{Ass}(R)$ . Hence  $R$  contains a system  $x_1, x_2, \dots, x_d$  of parameters which forms a strong  $d$ -sequence in  $R$  (see [Cu, Theorem 2.6] or [Kw, Theorem 4.2] for the existence of such systems of parameters). The following result was given by Goto and Sakurai (see [GSa, Theorem 2.1]) about the existence of equality  $I^2 = \mathfrak{q}I$ , where  $I = \mathfrak{q} : \mathfrak{m}$ .

**Lemma 2.2.** *Let  $R$  be a Noetherian local ring with  $d = \dim R \geq 2$ . Assume that  $R$  is unmixed, that is  $\dim \hat{R}/\mathfrak{p} = d$  for all  $\mathfrak{p} \in \text{Ass}(\hat{R})$ . Then there exists a system  $x_1, x_2, \dots, x_d$  of parameters such that for all integer  $n_i$ ,  $1 \leq i \leq d$ , the equality  $I^2 = \mathfrak{q}I$  holds true, where  $I = \mathfrak{q} : \mathfrak{m}$  and  $\mathfrak{q} = (x_1^{n_1}, x_2^{n_2}, \dots, x_d^{n_d})$ .*

Now we will apply the following result to the equality  $I^2 = \mathfrak{q}I$  of Goto and Sakurai (see [GSa, Theorem 2.1]).

**Proposition 2.3.** *Assume that  $\mathfrak{q} = (x_1, \dots, x_d)$  be a parameter ideal such that  $I^2 = \mathfrak{q}I$ , where  $I = \mathfrak{q} : \mathfrak{m}$ . Then we have*

$$e_1(I) - e_1(\mathfrak{q}) \leq f_0(\mathfrak{q}).$$

*Proof.* Since  $I^2 = \mathfrak{q}I$ , we have  $I^{n+1} = \mathfrak{q}^n I$  for all  $n \geq 1$ . Thus, we have

$$\ell(R/\mathfrak{q}^{n+1}) - \ell(R/I^{n+1}) = \ell((\mathfrak{q}^n(\mathfrak{q} : \mathfrak{m}))/\mathfrak{q}^{n+1}) \leq \ell((\mathfrak{q}^{n+1} : \mathfrak{m})/\mathfrak{q}^{n+1}).$$

Since  $I^2 = \mathfrak{q}I$ , we have  $e_0(\mathfrak{q}) = e_0(I)$ . Therefore  $e_1(I) - e_1(\mathfrak{q}) \leq f_0(\mathfrak{q})$ . □

*of Theorem 2.1.* (i)  $\Rightarrow$  (ii). Let  $\mathfrak{q}$  be a parameter ideal of  $R$  such that  $\mathfrak{q} \subseteq \mathfrak{m}^2$ . Put  $I = \mathfrak{q} : \mathfrak{m}$ . Since  $R$  is Cohen-Macaulay, we have  $f_0(\mathfrak{q}) = \mathcal{N}(\mathfrak{q}; N) = \ell(I/\mathfrak{q})$  by Theorem 1.1 in [Tr]. Since  $\mathfrak{q} \subseteq \mathfrak{m}^2$ , by [CP, Theorem 2.2], we have  $I^2 = \mathfrak{q}I$ . It follows from  $R/\mathfrak{m}$  is infinite that  $e_1(I) = \ell(R/I) - e_0(\mathfrak{q})$  by Huneke and Ooishi ([Hu], [O] or cf. [CGT, Theorem 6.1]). Since  $R$  is Cohen-Macaulay, we have  $f_0(\mathfrak{q}) = \ell(I/\mathfrak{q}) = e_1(I)$ .

(ii)  $\Rightarrow$  (iii) is trivial.

(iii)  $\Rightarrow$  (iv) follows from Lemma 2.2.

(iv)  $\Rightarrow$  (i) Since  $I^2 = \mathfrak{q}I$ , by Proposition 2.3 we have  $e_1(I) - e_1(\mathfrak{q}) \leq f_0(\mathfrak{q})$ . Thus we get that  $0 \leq e_1(I) - f_0(\mathfrak{q}) \leq e_1(\mathfrak{q})$ . It follows from  $R$  is unmixed and the Theorem 1.1 of [GGHOPV] that  $R$  is Cohen-Macaulay, as required. □

**Corollary 2.4.** *Let  $R$  be a Noetherian local ring with  $d = \dim R \geq 2$ . Assume that  $R$  is unmixed, that is  $\dim \hat{R}/\mathfrak{p} = d$  for all  $\mathfrak{p} \in \text{Ass}(\hat{R})$ . Then for all integers  $n$  there exists a parameter ideal  $\mathfrak{q} \subseteq \mathfrak{m}^n$ , we have*

$$e_1(I; R) \leq f_0(\mathfrak{q}; R),$$

where  $I = \mathfrak{q} : \mathfrak{m}$ .

*Proof.* The result follows from Theorem 2.1. □

### 3. CHERN COEFFICIENT

In this section, we denote by  $\mathfrak{q}_i$  the ideal  $(x_1, \dots, x_i)R$  for  $i = 1, \dots, d$  and stipulate that  $\mathfrak{q}_0$  is the zero ideal of  $R$ . For a module  $M$  over a ring  $R$ , we denote by  $H_{\mathfrak{a}}^i(M)$  the  $i$ -th local cohomology module of  $M$  with respect to  $\mathfrak{a}$ . Then an  $R$ -module  $M$  is said to be a *generalized Cohen-Macaulay module* if  $H_{\mathfrak{m}}^i(M)$  are of finite length for all  $i = 0, 1, \dots, d-1$  (see [CST]). This condition is equivalent to say that there exists a parameter ideal  $\mathfrak{q} = (x_1, \dots, x_d)$  of  $M$  such that  $\mathfrak{q}H_{\mathfrak{m}}^i(M/\mathfrak{q}_j M) = 0$  for all  $0 \leq i+j < d$  (see [T]), and such a parameter ideal was called a *standard parameter ideal* of  $M$ . It is well-known that if  $M$  is a generalized Cohen-Macaulay module, then every parameter ideal of  $M$  in a high enough power of the maximal ideal  $\mathfrak{m}$  is standard. The following lemma can be easily derived from the basic properties of generalized Cohen-Macaulay modules (see [CT, Theorem 1.1 and Theorem 1.2] and [Tr, Proposition 3.4]).

**Fact 3.1.** Let  $(R, \mathfrak{m})$  be a generalized Cohen-Macaulay ring of dimension  $d \geq 1$ . Set  $r_i(R) = \dim_{R/\mathfrak{m}}(H_{\mathfrak{m}}^i(R))$ . Then the following statements hold true.

- (1) There exists an integer  $n$  such that for all parameter ideal  $\mathfrak{q} \subseteq \mathfrak{m}^n$ , we have  $I^2 = \mathfrak{q}I$ , where  $I = \mathfrak{q} : \mathfrak{m}$  and

$$\mathcal{N}(\mathfrak{q}; R) = \sum_{i=0}^d \binom{d}{i} r_i(R),$$

- (2) Let  $\mathfrak{q} = (x_1, x_2, \dots, x_d)$  be a standard parameter ideal such that

$$\mathcal{N}(\mathfrak{q}; R) = \sum_{i=0}^d \binom{d}{i} r_i(R).$$

Then we have

(a)  $r_i(R/(x_1)) = r_i(R) + r_{i+1}(R)$  for all  $i \geq 0$ .

(b)

$$f_0(\mathfrak{q}; R) \leq \begin{cases} f_0(\mathfrak{q}'; R') - (r_0(R) + r_1(R)) & \text{if } \dim R = 2, \\ f_0(\mathfrak{q}'; R') & \text{if } \dim R \geq 3. \end{cases}$$

(c)

$$f_0(\mathfrak{q}; R) \leq \sum_{j=1}^d \binom{d-1}{j-1} r_j(R).$$

Now we will apply Proposition 2.3 to generalized Cohen-Macaulay rings. From there we get the following result.

**Corollary 3.2.** Let  $R$  be a generalized Cohen-Macaulay ring of dimension  $d$ . Then there exists an integer  $n$  such that for all parameter ideals  $\mathfrak{q} \subseteq \mathfrak{m}^n$ , we have

$$e_1(I) - e_1(\mathfrak{q}) \leq f_0(\mathfrak{q}) \leq \mathcal{N}(\mathfrak{q}; R),$$

where  $I = \mathfrak{q} : \mathfrak{m}$ . Moreover,  $e_1(I) - e_1(\mathfrak{q}) = \mathcal{N}(\mathfrak{q}; R)$  if and only if  $R$  is Cohen-Macaulay.

*Proof.* Choose an integer  $n$  as in Fact 3.1 1). Let  $\mathfrak{q}$  be a parameter ideal such that  $\mathfrak{q} \subseteq \mathfrak{m}^n$ . Then by Fact 3.1 2) we have  $I^2 = \mathfrak{q}I$  and  $f_0(\mathfrak{q}) \leq \mathcal{N}(\mathfrak{q}; R)$  for all parameter ideals  $\mathfrak{q} \subseteq \mathfrak{m}^n$ . It follows from Proposition 2.3 that

$$e_1(I) - e_1(\mathfrak{q}) \leq f_0(\mathfrak{q}) \leq \mathcal{N}(\mathfrak{q}; R).$$

Now assume that  $e_1(I) - e_1(\mathfrak{q}) = \mathcal{N}(\mathfrak{q}; R)$ . Then  $f_0(\mathfrak{q}) = \mathcal{N}(\mathfrak{q}; R)$ . Since  $\mathfrak{q} \subseteq \mathfrak{m}^n$ , we have  $f_0(\mathfrak{q}; R) \leq \sum_{j=1}^d \binom{d-1}{j-1} r_j(R) \leq \sum_{i=0}^d \binom{d}{i} r_i(R) = \mathcal{N}(\mathfrak{q}; R)$ . Therefore  $H_{\mathfrak{m}}^i(R) = 0$  for all  $i \neq d$ . Hence  $R$  is Cohen-Macaulay.

If  $R$  is Cohen-Macaulay then by Theorem 2.1 and Theorem 1.1 in [Tr] we have  $e_1(\mathfrak{q}) = 0$ ,  $e_1(I) = f_0(\mathfrak{q})$  and  $f_0(\mathfrak{q}) = \mathcal{N}(\mathfrak{q}; R)$ . Hence  $e_1(I) - e_1(\mathfrak{q}) = \mathcal{N}(\mathfrak{q}; R)$ , as required.  $\square$

Now we set  $W = H_{\mathfrak{m}}^0(R)$ . When we investigate in the case of  $W \neq 0$ , we reduce  $W = 0$  using the next result (See [CGT] and [Tr]), which is well known, plays a key role.

**Fact 3.3.** Set  $\overline{R} = R/W$ . Then the following statements holds true.

- (1)  $e_1(I; \overline{R}) = e_1(I; R)$  provided  $d \geq 2$  (see [CGT]).

- (2) There exists a positive integer  $n_0$  such that for all  $\mathfrak{m}$ -primary ideals  $I \subseteq \mathfrak{m}^{n_0}$ , we have

$$\mathcal{N}(I; R) = \mathcal{N}(I; \overline{R}) + \ell((0) :_R \mathfrak{m}),$$

and

$$(\mathfrak{q} + W) : \mathfrak{m} = \mathfrak{q} : \mathfrak{m} + W.$$

- (3) We have

$$f_0(I; R) = \begin{cases} f_0(\overline{I}; \overline{R}) + \ell((0) :_R \mathfrak{m}) & \text{if } \dim R = 1, \\ f_0(\overline{I}; \overline{R}) & \text{if } \dim R \geq 2, \end{cases}$$

where  $\overline{I} = (I + W)/W$ .

The next lemma shows the existence of a special superficial element which is useful in many inductive proofs in the sequel.

**Fact 3.4** ([GNi]). Suppose  $\mathfrak{q}$  is a reduction of  $I$ . Then there exists an element  $x \in \mathfrak{q}$  which is superficial for both  $I$  and  $\mathfrak{q}$ . Moreover, for such element  $x \in \mathfrak{q}$ , setting  $R' = R/xR$ , we have  $e_1(I) - e_1(\mathfrak{q}) = e_1(IR') - e_1(\mathfrak{q}R')$  provided  $d \geq 2$ .

**Proposition 3.5.** *Let  $R$  be a Noetherian local ring with  $d = \dim R \geq 2$ . Assume that  $R$  is unmixed, that is  $\dim \hat{R}/\mathfrak{p} = d$  for all  $\mathfrak{p} \in \text{Ass}(\hat{R})$ . Assume that there exists an integer  $n$  such that for all parameter ideals  $\mathfrak{q} = (x_1, x_2, \dots, x_d) \subseteq \mathfrak{m}^n$  we have*

$$\mathcal{N}(\mathfrak{q}; R) \leq e_1(I) - e_1(\mathfrak{q}),$$

where  $I = \mathfrak{q} : \mathfrak{m}$ . Then  $R$  is Cohen-Macaulay.

In our proof of Proposition 3.5 the following facts are the key. See [GN, Section 3] for the proof.

**Lemma 3.6.** *Let  $R$  be a homomorphic image of a Cohen-Macaulay local ring and assume that  $\text{Ass}(R) \subseteq \text{Assh}(R) \cup \{\mathfrak{m}\}$ . Then*

$$\mathcal{F} = \{\mathfrak{p} \in \text{Spec}(R) \mid \text{ht}_R(\mathfrak{p}) > 1 = \text{depth}(R_{\mathfrak{p}})\}$$

is a finite set.

*of Proposition 3.5.* We shall now show the our result by induction on the dimension of  $R$ . In the case  $\dim R = 2$ ,  $R$  is a generalized Cohen-Macaulay ring since  $R$  is unmixed. It follows from Corollary 3.2 and  $\mathcal{N}(\mathfrak{q}; R) \leq e_1(I; R) - e_1(\mathfrak{q}; R)$  that  $R$  is Cohen-Macaulay.

Suppose that  $\dim R > 2$  and that our assertion holds true for  $\dim R - 1$ . Let

$$\mathcal{F} = \{\mathfrak{p} \in \text{Spec}R \mid \mathfrak{p} \neq \mathfrak{m}, \dim R_{\mathfrak{p}} > \text{depth}R_{\mathfrak{p}} = 1\}.$$

Then by Lemma 3.6,  $\mathcal{F}$  is a finite set. We choose  $x \in \mathfrak{m}$  such that

$$x \notin \bigcup_{\mathfrak{p} \in \text{Ass}R} \mathfrak{p} \cup \bigcup_{\mathfrak{p} \in \mathcal{F}} \mathfrak{p}.$$

Let  $n_1 > n$  be an integer such that  $x^{n_1}H_{\mathfrak{m}}^1(R) = 0$ . Put  $y = x^{n_1}$ . Let  $A = R/(y)$ . Then  $\dim A = d - 1$  and  $\text{Ass}A \setminus \{\mathfrak{m}\} = \text{Assh}A$ . Therefore the unmixed component  $U_A(0)$  of 0 in  $B$  has finite length, so that  $U_A(0) = H_{\mathfrak{m}}^0(A)$ . We now take a system  $y_2, y_3, \dots, y_d$  of parameters of  $R$ -module  $A$  and assume that  $y_2, y_3, \dots, y_d$  form a  $d$ -sequence in  $A$ .

Then since  $y$  is an  $R$ -regular, sequence  $y = y_1, y_2, \dots, y_d$  form  $d$ -sequence in  $R$ , whence  $y_1$  is a superficial element of  $R$  with respect to  $\mathfrak{q} = (y_1, y_2, \dots, y_d)$ . Since  $\dim R \geq 3$ , therefore for all parameter ideal  $\mathfrak{q}' = (y_2, y_3, \dots, y_d) \subseteq \mathfrak{m}^n$  of  $A$  which  $y_2, y_3, \dots, y_d$  is  $d$ -sequence, it follows from Fact 3.4 we have  $\mathcal{N}(\mathfrak{q}'A; A) = \mathcal{N}(\mathfrak{q}; R) \leq e_1(I; R) - e_1(\mathfrak{q}; R) = e_1(I'; A) - e_1(\mathfrak{q}'; A)$ , where  $I' = \mathfrak{q}'A :_A \mathfrak{m}A = (\mathfrak{q} :_R \mathfrak{m})A$ . Let  $W = H_{\mathfrak{m}}^0(A)$  and  $\bar{A} = A/W$ , and  $\mathfrak{n} = \mathfrak{m}A$ . By Fact 3.3 2), that we can choose an integer  $n_0 > n$  such that for all parameters ideals  $\mathfrak{q}' \subseteq \mathfrak{n}^{n_0}$ , we have  $\mathcal{N}(\mathfrak{q}'; A) = \mathcal{N}(\mathfrak{q}'; \bar{A}) + \ell(0 :_A \mathfrak{n})$  and  $e_1(I'; A) - e_1(\mathfrak{q}'; A) = e_1(\mathfrak{q}'\bar{A} : \mathfrak{n}; \bar{A}) - e_1(\mathfrak{q}'\bar{A}; \bar{A})$ . Let  $n' > n_0$  be an integer such that  $\mathfrak{m}A \cap H_{\mathfrak{m}}^0(A) = 0$ . Let  $y_2, y_3, \dots, y_d$  be a system of parameters of  $R$ -module  $\bar{A}$  such that  $(y_2, y_3, \dots, y_d) \subseteq \mathfrak{m}^{n'}$  and assume that  $y_2, y_3, \dots, y_d$  is a  $d$ -sequence in  $\bar{A}$ . Then because  $(y_2, y_3, \dots, y_d)A \cap W = 0$ , we have  $y_2, y_3, \dots, y_d$  form a  $d$ -sequence in  $A$ . Therefore, we have

$$\mathcal{N}(\mathfrak{q}'\bar{A}; \bar{A}) \leq e_1(\mathfrak{q}'\bar{A} : \mathfrak{n}; \bar{A}) - e_1(\mathfrak{q}'\bar{A}; \bar{A}),$$

where  $\mathfrak{q}' = (y_2, y_3, \dots, y_d)$ . By hypothesis of induction on  $d$ , we have  $\bar{A}$  is Cohen-Macaulay. Thus  $H_{\mathfrak{m}}^i(A) = 0$  for all  $i \neq 1, d$ . It follows from the following sequence

$$0 \longrightarrow R \xrightarrow{\cdot y} R \longrightarrow A \longrightarrow 0$$

that we have the long exact sequence

$$\dots \longrightarrow H_{\mathfrak{m}}^1(R) \xrightarrow{\cdot y} H_{\mathfrak{m}}^1(R) \longrightarrow H_{\mathfrak{m}}^1(A) \longrightarrow \dots$$

$$\dots \longrightarrow H_{\mathfrak{m}}^i(R) \xrightarrow{\cdot y} H_{\mathfrak{m}}^i(R) \longrightarrow H_{\mathfrak{m}}^i(A) \longrightarrow \dots$$

Then we have  $H_{\mathfrak{m}}^i(R) = 0$  for all  $2 \leq i \leq d-1$  and  $H_{\mathfrak{m}}^1(R) = yH_{\mathfrak{m}}^1(R)$ . Thus  $H_{\mathfrak{m}}^1(R) = 0$  because  $H_{\mathfrak{m}}^1(R)$  is a finite generated  $R$ -module. Therefore  $R$  is Cohen-Macaulay.  $\square$

**Theorem 3.7.** *Let  $R$  be a Noetherian local ring with maximal ideal  $\mathfrak{m}$ ,  $d = \dim R \geq 2$ . Assume that  $R$  is unmixed, that is  $\dim \hat{R}/\mathfrak{p} = d$  for all  $\mathfrak{p} \in \text{Ass}(\hat{R})$ . The following statements are equivalent.*

- (i)  $R$  is Cohen-Macaulay.
- (ii) For all parameter ideals  $\mathfrak{q} \subseteq \mathfrak{m}^2$ , we have

$$\mathcal{N}(\mathfrak{q}; R) = e_1(I) - e_1(\mathfrak{q}),$$

where  $I = \mathfrak{q} : \mathfrak{m}$ .

- (iii) For all parameter ideals  $\mathfrak{q} \subseteq \mathfrak{m}^2$ , we have

$$\mathcal{N}(\mathfrak{q}; R) \leq e_1(I) - e_1(\mathfrak{q}),$$

where  $I = \mathfrak{q} : \mathfrak{m}$ .

*Proof.* (i)  $\Rightarrow$  (ii). Let  $\mathfrak{q}$  be a parameter ideal of  $R$  such that  $\mathfrak{q} \subseteq \mathfrak{m}^2$ . Put  $I = \mathfrak{q} : \mathfrak{m}$ . Since  $R$  is Cohen-Macaulay, we have  $\mathcal{N}(\mathfrak{q}; R) = \ell(I/\mathfrak{q})$  and  $e_1(\mathfrak{q}) = 0$ . Note that  $\mathfrak{q} \subseteq \mathfrak{m}^2$ , by [CP, Theorem 2.2], we have  $I^2 = \mathfrak{q}I$ . It follows from  $R/\mathfrak{m}$  is infinite that  $e_1(I) = \ell(R/I) - e_0(\mathfrak{q})$  by Huneke and Ooishi([Hu], [O] or cf. [CGT, Theorem 6.1]). Since  $R$  is Cohen-Macaulay, we have  $\mathcal{N}(\mathfrak{q}; R) = \ell(I/\mathfrak{q}) = e_1(I) - e_1(\mathfrak{q})$ .

(ii)  $\Rightarrow$  (iii) and (iii)  $\Rightarrow$  (iv) are trivial. (iv)  $\Rightarrow$  (i) follows from Proposition 3.5.  $\square$



**Corollary 3.8.** *Let  $R$  be a Noetherian local ring with  $d = \dim R \geq 2$ . Assume that  $R$  is unmixed, that is  $\dim \hat{R}/\mathfrak{p} = d$  for all  $\mathfrak{p} \in \text{Ass}(\hat{R})$ . Then for all integers  $n$  there exists a parameter ideal  $\mathfrak{q} \subseteq \mathfrak{m}^n$ , we have*

$$e_1(I; R) - e_1(\mathfrak{q}; R) \leq \mathcal{N}(\mathfrak{q}; R),$$

where  $I = \mathfrak{q} : \mathfrak{m}$ .

*Proof.* The result follows from Theorem 3.7. □

Let us note the following example of parameter ideals  $\mathfrak{q}$  in non-Cohen-Macaulay local rings  $R$  with  $\text{depth } R = d - 1$ , for which one has  $e_1(I) - e_1(\mathfrak{q}) = \mathcal{N}(\mathfrak{q}; R)$ , where  $I = \mathfrak{q} : \mathfrak{m}$ .

**Example 3.9.** ([GSa1, Section 4])

Let  $2 \leq d \leq m$  be integers. Let  $A = k[X_1, X_2, \dots, X_m, V, Z_1, Z_2, \dots, Z_d]$  be the polynomial ring with  $m + d + 1$  indeterminates over a field  $k$  and let

$$\mathfrak{b} = (X_i \mid 1 \leq i \leq m - 1)^2 + (X_2^m) + (X_i V \mid 1 \leq i \leq m) + (V^2 - \sum_{i=1}^d X_i Z_i).$$

We put  $C = A/\mathfrak{b}$ . Let  $M = C_+ = (x_1, x_2, \dots, x_m) + (v) + (a_1, a_2, \dots, a_d)$  be the graded maximal ideal in  $C$ , where  $x_i, v$ , and  $a_j$  denote the images  $X_i, V$ , and  $Z_j$  in  $C$ , respectively. Then  $C$  is a  $d$ -dimensional graded non-Cohen-Macaulay ring with  $\text{depth } C = d - 1$  and  $\ell(H_{\mathfrak{m}}^{d-1}(C)) = 1$  ([GSa1, Theorem 4.5]). We put  $\mathfrak{q} = (a_1, a_2, \dots, a_d)$ . Then  $M^2 = \mathfrak{q}M$ , whence  $\mathfrak{q}$  is a reduction of  $M$  and  $a_1, a_2, \dots, a_d$  is a homogeneous system of parameters for the graded ring  $C$ . Let  $J = \mathfrak{q} : M$ . We then have  $J^3 = \mathfrak{q}J^2$  and  $\ell_C(J^2/\mathfrak{q}J) = 1$  ([GSa1, Proposition 4.7]). Let  $R = C_M$ ,  $I = JR$ , and  $Q = \mathfrak{q}R$ . Then since  $\ell(H_{\mathfrak{m}}^{d-1}(R)) = 1$  and  $\text{depth } R = d - 1$  we have

$$e_i(\mathfrak{q}; R) = \begin{cases} 2m & \text{if } i = 0, \\ -1 & \text{if } i = 1, \\ 0 & \text{if } 2 \leq i \leq d. \end{cases}$$

Moreover, we have

$$\ell(R/I^{n+1}) = 2m \binom{n+d}{d} - (m-2) \binom{n+d-1}{d-1},$$

and so that  $e_1(I) = m - 2$  and  $\mathcal{N}(\mathfrak{q}; R) = \ell(I/\mathfrak{q}) = \ell(R/\mathfrak{q}) - \ell(R/I) = 2m + 1 - (m + 2) = m - 1$ . Therefore we have

$$e_1(I) - e_1(\mathfrak{q}) = \mathcal{N}(\mathfrak{q}; R),$$

as required.

#### 4. THE COHEN-MACAULAY TYPE

In this section, we give the characterizations of a Cohen-Macaulay ring in term of its Chern coefficient and the Cohen-Macaulay type. As corollaries, we obtained the characterizations of a Gorenstein ring in term of its Chern coefficient. In order to give the proof of the main theorem, we begin the following result.

**Lemma 4.1.** *Let  $R$  be a Noetherian local ring with  $\dim R = 1$ . Assume that  $\mathfrak{q} = (x)$  be a standard parameter ideal of  $R$  such that  $I^2 = \mathfrak{q}I$ , where  $I = \mathfrak{q} : \mathfrak{m}$  and*

$$\mathcal{N}(\mathfrak{q}; R) = r_1(R) + r_0(R).$$

Then we have

$$e_1(I) - e_1(\mathfrak{q}) = f_0(\mathfrak{q}) - r_0(R),$$

where  $r_0(R) = \ell((0) :_R \mathfrak{m})$ .

*Proof.* First, we shall show that  $\mathfrak{q}^{n+1} :_R \mathfrak{m} = \mathfrak{q}^n(\mathfrak{q} :_R \mathfrak{m}) + ((0) :_R \mathfrak{m})$  for all  $n \geq 0$ . Indeed, the case  $n = 0$  is trivial so we can assume that  $n \geq 1$ . Let  $a \in (x^{n+1}) : \mathfrak{m}$ . Since  $(x^{n+1}) : \mathfrak{m} \subseteq (x^{n+1}) : x = (x^n) + H_{\mathfrak{m}}^0(R)$ , we have  $a = x^n b + c$  for some  $b \in R$  and  $c \in H_{\mathfrak{m}}^0(R)$ . Since  $\mathfrak{m}a \subseteq (x^{n+1})$  and  $\mathfrak{m}x^n b \subseteq (x^n)$  we have  $\mathfrak{m}c \subseteq (x) \cap H_{\mathfrak{m}}^0(R) = 0$ . Thus  $c \in (0) :_R \mathfrak{m}$ . Therefore  $x^n \mathfrak{m}b = \mathfrak{m}a \subseteq (x^{n+1})$ . Hence  $\mathfrak{m}b \subseteq (x) + H_{\mathfrak{m}}^0(R)$ . Since  $(x) \cap H_{\mathfrak{m}}^0(R) = 0$ , we have the following exact sequence

$$0 \rightarrow H_{\mathfrak{m}}^0(R) \rightarrow R/(x) \rightarrow R/((x) + H_{\mathfrak{m}}^0(R)) \rightarrow 0.$$

It follows from  $\mathcal{N}(\mathfrak{q}; R) = r_1(R) + r_0(R)$  that the sequence

$$0 \rightarrow (0) :_R \mathfrak{m} \rightarrow ((x) : \mathfrak{m})/(x) \rightarrow (((x) + H_{\mathfrak{m}}^0(R)) : \mathfrak{m})/((x) + H_{\mathfrak{m}}^0(R)) \rightarrow 0$$

is exact. Therefore  $b \in ((x) + H_{\mathfrak{m}}^0(R)) :_R \mathfrak{m} = ((x) :_R \mathfrak{m}) + H_{\mathfrak{m}}^0(R)$ . Thus  $b = d + e$  with some  $d \in (x) :_R \mathfrak{m}$  and  $e \in H_{\mathfrak{m}}^0(R)$ . In conclusion  $a = x^n(d + e) + c = x^n d + c \in x^n((x) :_R \mathfrak{m}) + (0) :_R \mathfrak{m}$ . Hence we have  $(x^{n+1}) :_R \mathfrak{m} \subseteq x^n((x) :_R \mathfrak{m}) + (0) :_R \mathfrak{m}$  as desired.

Since  $I^2 = \mathfrak{q}I$ , we have  $I^{n+1} = \mathfrak{q}^n I$  for all  $n \geq 1$ . Since  $\mathfrak{q} \cap H_{\mathfrak{m}}^0(R) = 0$  and  $\mathfrak{q}^{n+1} :_R \mathfrak{m} = \mathfrak{q}^n(\mathfrak{q} :_R \mathfrak{m}) + ((0) :_R \mathfrak{m})$ , we have the following exact sequence

$$0 \rightarrow (0) :_R \mathfrak{m} \rightarrow \mathfrak{q}^{n+1} :_R \mathfrak{m} / \mathfrak{q}^{n+1} \rightarrow (\mathfrak{q}^n(\mathfrak{q} :_R \mathfrak{m})) / \mathfrak{q}^{n+1} \rightarrow 0.$$

Thus, we have

$$\ell(R/\mathfrak{q}^{n+1}) - \ell(R/I^{n+1}) = \ell((\mathfrak{q}^n(\mathfrak{q} : \mathfrak{m})) / \mathfrak{q}^{n+1}) = \ell((\mathfrak{q}^{n+1} : \mathfrak{m}) / \mathfrak{q}^{n+1}) - \ell((0) :_R \mathfrak{m}).$$

Since  $I^2 = \mathfrak{q}I$ , we have  $e_0(\mathfrak{q}) = e_0(I)$ . Therefore  $e_1(I) - e_1(\mathfrak{q}) = f_0(\mathfrak{q}) - r_0(R)$ . □

**Lemma 4.2.** *Let  $R$  be a generalized Cohen-Macaulay ring of dimension  $d \geq 2$ . Assume that  $\mathfrak{q} = (x_1, x_2, \dots, x_d)$  be a standard parameter ideal of  $R$  such that  $I^2 = \mathfrak{q}I$ , where  $I = \mathfrak{q} : \mathfrak{m}$  and*

$$\mathcal{N}(\mathfrak{q}; R) = \sum_{i=0}^d \binom{d}{i} r_i(R).$$

Then we have

$$e_1(I) - e_1(\mathfrak{q}) = f_0(\mathfrak{q}) = \sum_{j=1}^d \binom{d-1}{j-1} r_j(R).$$

*Proof.* Let  $R' = R/(x_1)$ ,  $\mathfrak{q}' = \mathfrak{q}/(x_1)$ ,  $I' = I/(x_1)$  and  $\mathfrak{m}' = \mathfrak{m}/(x_1)$ . We shall now show the our result by induction on the dimension of  $R$ . In the case  $\dim R = 2$ . Since  $\dim R' = 1$  and  $\mathfrak{q}'$  is a parameter ideal of  $R'$ , we have

$$e_1(I') - e_1(\mathfrak{q}') = f_0(\mathfrak{q}') - r_0(R').$$

Because of Fact 3.3 (3), we have  $f_0(\mathfrak{q}') = r_0(R') + r_1(R')$ . It follows from Fact 3.1 (2), Fact 3.4 and Proposition 2.3 that we have

$$e_1(I') - e_1(\mathfrak{q}') = e_1(I) - e_1(\mathfrak{q}) \leq f_0(\mathfrak{q}) \leq f_0(\mathfrak{q}') - (r_0(R) + r_1(R)) = f_0(\mathfrak{q}') - r_0(R) = r_1(R').$$

Hence we have  $e_1(I) - e_1(\mathfrak{q}) = f_0(\mathfrak{q}) = r_1(R') = r_1(R) + r_2(R)$ .

Suppose that  $\dim R > 2$  and our assertion holds true for  $\dim R - 1$ . By Fact 3.4 (2), we have  $f_0(\mathfrak{q}; R) \leq f_0(\mathfrak{q}'; R')$ . By the inductive hypothesis and Proposition 2.3, we have

$$f_0(\mathfrak{q}') = e_1(I') - e_1(\mathfrak{q}') = e_1(I) - e_1(\mathfrak{q}) \leq f_0(\mathfrak{q}) \leq f_0(\mathfrak{q}').$$

and  $f_0(\mathfrak{q}') = \sum_{j=1}^{d-1} \binom{d-2}{j-1} r_j(R') = \sum_{j=1}^d \binom{d-1}{j-1} r_j(R)$  Hence we get

$$e_1(I) - e_1(\mathfrak{q}) = f_0(\mathfrak{q}) = \sum_{j=1}^d \binom{d-1}{j-1} r_j(R),$$

as required. □

**Corollary 4.3.** *Let  $R$  be a generalized Cohen-Macaulay ring of dimension  $d \geq 2$ . Then there exists an integer  $n$  such that for all parameter ideals  $\mathfrak{q} \subseteq \mathfrak{m}^n$ , we have*

$$e_1(I) - e_1(\mathfrak{q}) = f_0(\mathfrak{q}) = \sum_{j=1}^d \binom{d-1}{j-1} r_j(R),$$

where  $I = \mathfrak{q} : \mathfrak{m}$ .

*Proof.* The result follows from Theorem 4.2 and Fact 3.1. □

Let  $\mathfrak{q} = (x_1, x_2, \dots, x_d)$  be a parameter ideal in  $R$  and let  $M$  be an  $R$ -module. For each integer  $n \geq 1$  we denote by  $\underline{x}^n$  the sequence  $x_1^n, x_2^n, \dots, x_d^n$ . Let  $K^\bullet(x^n)$  be the Koszul complex of  $R$  generated by the sequence  $\underline{x}^n$  and let  $H^\bullet(\underline{x}^n; M) = H^\bullet(\text{Hom}_R(K^\bullet(\underline{x}^n), M))$  be the Koszul cohomology module of  $M$ . Then for every  $p \in \mathbb{Z}$  the family  $\{H^p(\underline{x}^n; M)\}_{n \geq 1}$  naturally forms an inductive system of  $R$ -modules, whose limit

$$H_{\mathfrak{q}}^p = \lim_{n \rightarrow \infty} H^p(\underline{x}^n; M)$$

is isomorphic to the local cohomology module

$$H_{\mathfrak{m}}^p(M) = \lim_{n \rightarrow \infty} \text{Ext}_R^p(R/\mathfrak{m}^n, M)$$

For each  $n \geq 1$  and  $p \in \mathbb{Z}$  let  $\phi_{\underline{x}, M}^{p, n} : H^p(\underline{x}^n; M) \rightarrow H_{\mathfrak{m}}^p(M)$  denote the canonical homomorphism into the limit. With this notation we have the following.

**Lemma 4.4** ([GSa1] Lemma 3.12). *Let  $R$  be a Noetherian local ring with the maximal ideal  $\mathfrak{m}$  and  $\dim R = d \geq 1$ . Let  $M$  be a finitely generated  $R$ -module. Then there exists an integer  $\ell$  such that for all systems of parameters  $\underline{x} = x_1, \dots, x_d$  for  $R$  contained in  $\mathfrak{m}^\ell$  and for all  $p \in \mathbb{Z}$ , the canonical homomorphisms*

$$\phi_{\underline{x}, M}^{p, 1} : H^p(\underline{x}, M) \rightarrow H_{\mathfrak{m}}^p(M)$$

*into the inductive limit are surjective on the socles.*

With this notation we have the following.

**Lemma 4.5** ([GS1], Lemma 1.7). *Let  $M$  be a finitely generated  $R$ -module and  $x$  be an  $M$ -regular element and  $\underline{x} = x_1, \dots, x_r$  be a system of elements in  $R$  with  $x_1 = x$ . Then there exists a splitting exact sequence for each  $p \in \mathbb{Z}$ ,*

$$0 \rightarrow H^p(\underline{x}; M) \rightarrow H^p(\underline{x}; M/xM) \rightarrow H^{p+1}(\underline{x}; M) \rightarrow 0.$$

Let  $L$  be an arbitrary finitely generated  $R$ -module of dimension  $s \geq 0$ . We put

$$r_R(L) = \ell_R(\text{Ext}_R^s(R/\mathfrak{m}, L))$$

and call it the Cohen-Macaulay type of  $L$ . (Let us simply write  $r(R)$  for  $L = R$ .) We then have

$$\mathcal{N}(\mathfrak{q}; L) = r_R(L/\mathfrak{q}L)$$

for a parameter ideal  $\mathfrak{q}$  of  $L$ . As is well known, if  $L$  is a Cohen-Macaulay  $R$ -module, then for every parameter ideal  $\mathfrak{q}$  of  $L$ , we have

$$\mathcal{N}(\mathfrak{q}; L) = \ell_R(\text{Ext}_R^s(R/\mathfrak{m}, L)) = \ell_R((0) :_{\text{H}_{\mathfrak{m}}^s(L)} \mathfrak{m}).$$

The following result give the characterizations of a Cohen-Macaulay ring in term of its Chern coefficient and its Cohen-Macaulay type.

**Proposition 4.6.** *Let  $R$  be a Noetherian local ring with  $d = \dim R \geq 2$ . Assume that  $R$  is unmixed, that is  $\dim \hat{R}/\mathfrak{p} = d$  for all  $\mathfrak{p} \in \text{Ass}(\hat{R})$ . Assume that there exists an integer  $n$  such that for all parameter ideals  $\mathfrak{q} = (x_1, x_2, \dots, x_d) \subseteq \mathfrak{m}^n$  we have*

$$e_1(I) - e_1(\mathfrak{q}) \leq r(R),$$

where  $I = \mathfrak{q} : \mathfrak{m}$ . Then  $R$  is Cohen-Macaulay.

*Proof.* We shall now show the our result by induction on the dimension of  $R$ . In the case  $\dim R = 2$ ,  $R$  is a generalized Cohen-Macaulay ring since  $R$  is unmixed. By Corollary 4.3, we have  $e_1(I) - e_1(\mathfrak{q}) = f_0(\mathfrak{q}) = \sum_{j=1}^d \binom{d-1}{j-1} r_j(R)$ . Since  $e_1(I) - e_1(\mathfrak{q}) \leq r(R) = r_d(R)$ ,  $r_j(R) = 0$  for all  $j \neq d$ . Therefore  $R$  is Cohen-Macaulay.

Suppose that  $\dim R > 2$  and our assertion holds true for  $\dim R - 1$ . Let

$$\mathcal{F} = \{\mathfrak{p} \in \text{Spec}R \mid \mathfrak{p} \neq \mathfrak{m}, \dim R_{\mathfrak{p}} > \text{depth}R_{\mathfrak{p}} = 1\}.$$

Then by Lemma 3.6,  $\mathcal{F}$  is a finite set. We choose  $x \in \mathfrak{m}$  such that

$$x \notin \bigcup_{\mathfrak{p} \in \text{Ass}R} \mathfrak{p} \cup \bigcup_{\mathfrak{p} \in \mathcal{F}} \mathfrak{p}.$$

By Lemma 4.4 there exists an integer  $\ell$  such that for all systems of parameters  $\underline{x} = x_1, \dots, x_d$  for  $R$  contained in  $\mathfrak{m}^{\ell}$  and for all  $p \in \mathbb{Z}$ , the canonical homomorphisms

$$H^p(\underline{x}, R) \rightarrow H_{\mathfrak{m}}^p(R)$$

into the inductive limit are surjective on the socles. Let  $n_1 > \max\{n, \ell\}$  be an integer such that  $x^{n_1} H_{\mathfrak{m}}^1(R) = 0$ . Put  $y = x^{n_1}$ . Let  $A = R/(y)$ . Then  $\dim A = d - 1$  and  $\text{Ass}A \setminus \{\mathfrak{m}\} = \text{Ass}A$ . Therefore the unmixed component  $U_A(0)$  of 0 in  $B$  has finite length, so that  $U_A(0) = H_{\mathfrak{m}}^0(A)$ . We now take a system  $y_2, y_3, \dots, y_d$  of parameters of  $R$ -module  $A$  and assume that  $y_2, y_3, \dots, y_d$  form a  $d$ -sequence in  $A$ . Then since  $y$  is an  $R$ -regular, sequence  $y = y_1, y_2, \dots, y_d$  form  $d$ -sequence in  $R$ , whence  $y_1$  is a superficial element of  $R$  with respect to  $\mathfrak{q} = (y_1, y_2, \dots, y_d)$ . Since  $\dim R \geq 3$ , therefore for all parameter ideal  $\mathfrak{q}' = (y_2, y_3, \dots, y_d) \subseteq \mathfrak{m}^n$  of  $A$  which  $y_2, y_3, \dots, y_d$  is  $d$ -sequence, it

follows from Fact 3.4 we have  $e_1(I; R) - e_1(\mathfrak{q}; R) = e_1(I'; A) - e_1(\mathfrak{q}'; A)$ , where  $I' = \mathfrak{q}'A :_A \mathfrak{m}A = (\mathfrak{q} :_R \mathfrak{m})A$ .

On the other hand, by Lemma 4.4, we have the canonical homomorphism

$$H^i(\underline{y}, R) \rightarrow H_{\mathfrak{m}}^i(R)$$

into the inductive limit are surjective on the socles, for each  $i \in \mathbb{Z}$  where  $\underline{y} = y_1, y_2, \dots, y_d$ . By the regularity of  $y = y_1$  on  $R$ , it follows from the following sequence

$$0 \longrightarrow R \xrightarrow{\cdot y} R \longrightarrow A \longrightarrow 0$$

that there are induced the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^i(\underline{y}; R) & \longrightarrow & H^i(\underline{y}, A) & \longrightarrow & H^{i+1}(\underline{y}; R) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & H_{\mathfrak{m}}^i(R) & \longrightarrow & H_{\mathfrak{m}}^i(A) & \longrightarrow & H_{\mathfrak{m}}^{i+1}(R) \longrightarrow \end{array}$$

commutes, for all  $i \in \mathbb{Z}$ . It follows from the above commutative diagrams and Lemma 4.5 that after applying the functor  $\text{Hom}(k, *)$ , we obtain the commutative diagram

$$\begin{array}{ccccc} \text{Hom}(k, H^i(\underline{y}, A)) & \longrightarrow & \text{Hom}(k, H^{i+1}(\underline{y}; R)) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ \text{Hom}(k, H_{\mathfrak{m}}^i(A)) & \longrightarrow & \text{Hom}(k, H_{\mathfrak{m}}^{i+1}(R)) & & \end{array}$$

for all  $i \in \mathbb{Z}$ . Since the map  $\text{Hom}(k, H^{i+1}(\underline{y}; R)) \rightarrow \text{Hom}(k, H_{\mathfrak{m}}^{i+1}(R))$  is surjective, so is the map  $\text{Hom}(k, H_{\mathfrak{m}}^i(A)) \rightarrow \text{Hom}(k, H_{\mathfrak{m}}^{i+1}(R))$ . In particular,  $\text{Hom}(k, H_{\mathfrak{m}}^{d-1}(A)) \rightarrow \text{Hom}(k, H_{\mathfrak{m}}^d(R))$  is surjective and so that  $r(R) \leq r(A)$ . Hence we have

$$e_1(I'; A) - e_1(\mathfrak{q}'; A) \leq r(A).$$

Let  $W = H_{\mathfrak{m}}^0(A)$  and  $\overline{A} = A/W$ , and  $\mathfrak{n} = \mathfrak{m}A$ . By Fact 3.3 1) and 2), that we can choose an integer  $n_0 > n$  such that for all parameters ideals  $\mathfrak{q}' \subseteq \mathfrak{n}^{n_0}$ ,  $\mathfrak{q}'A :_A \mathfrak{m}A + W = (\mathfrak{q}'A + W) :_A \mathfrak{m}A$  and so that we have  $e_1(I'; A) - e_1(\mathfrak{q}'; A) = e_1(\mathfrak{q}'\overline{A} : \mathfrak{n}; \overline{A}) - e_1(\mathfrak{q}'\overline{A}; \overline{A})$ . Let  $n' > n_0$  be an integer such that  $\mathfrak{m}A \cap H_{\mathfrak{m}}^0(A) = 0$ . Let  $y_2, y_3, \dots, y_d$  be a system of parameters of  $R$ -module  $\overline{A}$  such that  $(y_2, y_3, \dots, y_d) \subseteq \mathfrak{m}^{n'}$  and assume that  $y_2, y_3, \dots, y_d$  is a  $d$ -sequence in  $\overline{A}$ . Then because  $(y_2, y_3, \dots, y_d)A \cap W = 0$ , we have  $y_2, y_3, \dots, y_d$  form a  $d$ -sequence in  $A$ . Therefore, since  $d \geq 3$ , we have

$$e_1(\mathfrak{q}'\overline{A} : \mathfrak{n}; \overline{A}) - e_1(\mathfrak{q}'\overline{A}; \overline{A}) \leq r(A) = r(\overline{A}),$$

where  $\mathfrak{q}' = (y_2, y_3, \dots, y_d)$ . By hypothesis of induction on  $d$ , we have  $\overline{A}$  is Cohen-Macaulay. Thus  $H_{\mathfrak{m}}^i(A) = 0$  for all  $i \neq 1, d$ . It follows from the following sequence

$$0 \longrightarrow R \xrightarrow{\cdot y} R \longrightarrow A \longrightarrow 0$$

that we have the long exact sequence

$$\begin{array}{ccccccc} \dots & \longrightarrow & H_{\mathfrak{m}}^1(R) & \xrightarrow{\cdot y} & H_{\mathfrak{m}}^1(R) & \longrightarrow & H_{\mathfrak{m}}^1(A) \longrightarrow \dots \\ \dots & \longrightarrow & H_{\mathfrak{m}}^i(R) & \xrightarrow{\cdot y} & H_{\mathfrak{m}}^i(R) & \longrightarrow & H_{\mathfrak{m}}^i(A) \longrightarrow \dots \end{array}$$

Then we have  $H_{\mathfrak{m}}^i(R) = 0$  for all  $2 \leq i \leq d-1$  and  $H_{\mathfrak{m}}^1(R) = yH_{\mathfrak{m}}^1(R)$ . Thus  $H_{\mathfrak{m}}^1(R) = 0$  because  $H_{\mathfrak{m}}^1(R)$  is a finite generated  $R$ -module. Therefore  $R$  is Cohen-Macaulay.  $\square$

**Theorem 4.7.** *Let  $R$  be a Noetherian local ring with maximal ideal  $\mathfrak{m}$ ,  $d = \dim R \geq 2$ . Assume that  $R$  is unmixed, that is  $\dim \hat{R}/\mathfrak{p} = d$  for all  $\mathfrak{p} \in \text{Ass}(\hat{R})$ . The following statements are equivalent.*

- (i)  $R$  is Cohen-Macaulay.
- (ii) For all parameter ideals  $\mathfrak{q} \subseteq \mathfrak{m}^2$ , we have

$$e_1(I) - e_1(\mathfrak{q}) = r(R),$$

where  $I = \mathfrak{q} : \mathfrak{m}$ .

- (iii) For all parameter ideals  $\mathfrak{q} \subseteq \mathfrak{m}^2$ , we have

$$e_1(I) - e_1(\mathfrak{q}) \leq r(R),$$

where  $I = \mathfrak{q} : \mathfrak{m}$ .

*Proof.* (i)  $\Rightarrow$  (ii). Let  $\mathfrak{q}$  be a parameter ideal of  $R$  such that  $\mathfrak{q} \subseteq \mathfrak{m}^2$ . Put  $I = \mathfrak{q} : \mathfrak{m}$ . Since  $R$  is Cohen-Macaulay, we have  $\mathcal{N}(\mathfrak{q}; R) = \ell(I/\mathfrak{q})$  and  $e_1(\mathfrak{q}) = 0$ . Note that  $\mathfrak{q} \subseteq \mathfrak{m}^2$ , by [CP, Theorem 2.2], we have  $I^2 = \mathfrak{q}I$ . It follows from  $R/\mathfrak{m}$  is infinite that  $e_1(I) = \ell(R/I) - e_0(\mathfrak{q})$  by Huneke and Ooishi ([Hu], [O] or cf. [CGT, Theorem 6.1]). Since  $R$  is Cohen-Macaulay, we have

$$e_1(I) - e_1(\mathfrak{q}) = \ell(I/\mathfrak{q}) = \mathcal{N}(\mathfrak{q}; R) = r(R).$$

(ii)  $\Rightarrow$  (iii) are trivial.

(iii)  $\Rightarrow$  (i) follows from Proposition 4.6.  $\square$

**Corollary 4.8.** *Let  $R$  be a Noetherian local ring with  $d = \dim R \geq 2$ . Assume that  $R$  is unmixed, that is  $\dim \hat{R}/\mathfrak{p} = d$  for all  $\mathfrak{p} \in \text{Ass}(\hat{R})$ . Then for all integers  $n$  there exists a parameter ideal  $\mathfrak{q} \subseteq \mathfrak{m}^n$ , we have*

$$r(R) \leq e_1(I; R) - e_0(\mathfrak{q}; R),$$

where  $I = \mathfrak{q} : \mathfrak{m}$ .

*Proof.* The result follows from Theorem 4.7.  $\square$

**Theorem 4.9.** *Let  $R$  be a Noetherian local ring with maximal ideal  $\mathfrak{m}$ ,  $d = \dim R \geq 2$ . Assume that  $R$  is unmixed, that is  $\dim \hat{R}/\mathfrak{p} = d$  for all  $\mathfrak{p} \in \text{Ass}(\hat{R})$ . The following statements are equivalent.*

- (i)  $R$  is Gorenstein.
- (ii) For all parameter ideals  $\mathfrak{q}$ , we have

$$e_1(I) - e_1(\mathfrak{q}) = 1,$$

where  $I = \mathfrak{q} : \mathfrak{m}$ .

*Proof.* (i)  $\Rightarrow$  (ii). Since  $R$  is Gorenstein,  $R$  is Cohen-Macaulay ring and  $r(R) = 1$ . By Theorem 4.7, we have

$$e_1(I) - e_1(\mathfrak{q}) = r(R) = 1$$

for all parameter ideals  $\mathfrak{q}$ , where  $I = \mathfrak{q} : \mathfrak{m}$ .

(ii)  $\Rightarrow$  (i). Since  $e_1(I) - e_1(\mathfrak{q}) = 1$ , for all parameter ideals  $\mathfrak{q}$ , we have

$$e_1(I) - e_1(\mathfrak{q}) \leq r(R),$$

for all parameter ideals  $\mathfrak{q}$ . By the Proposition 4.6,  $R$  is Cohen-Macaulay. Therefore, we have

$$1 = e_1(I) - e_1(\mathfrak{q}) = r(R),$$

for all parameter ideals  $\mathfrak{q}$ . Hence  $R$  is Gorenstein, as required. □

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