A Lagrangian Method for Deriving New Indefinite Integrals of Special Functions

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Abstract

A new method is presented for obtaining indefinite integrals of common special functions. The approach is based on a Lagrangian formulation of the general homogeneous linear ordinary differential equation of second order. A general integral is derived which involves an arbitrary function, and therefore yields an infinite number of indefinite integrals for any special function which obeys such a differential equation. Techniques are presented to obtain the more interesting integrals generated by such an approach, and many integrals, both previously known and completely new are derived using the method. Sample results are given for Bessel functions, Airy functions, Legendre functions and hypergeometric functions. More extensive results are given for the complete elliptic integrals of the first and second kinds. Integrals can be derived which combine common special functions as separate factors.

1 Introduction

In tables of integrals such as [1-4], the bulk of the content consists of definite integrals, with almost token collections of indefinite integrals. However, the relatively few indefinite integrals which have been published tend to be well known and heavily used. This paper presents a new and surprisingly simple method for deriving indefinite integrals of both elementary and special functions, provided the function satisfies an ordinary linear differential equation of the form

$$y''(x) + p(x)\, y'(x) + q(x)\, y(x) = 0. \tag{1}$$

The principal result derived here is the indefinite integral

$$\int f(x)\left(h''(x) + h'(x)\, p(x) + h(x)\, q(x)\right) y(x)\, dx =$$

$$f(x)\left(h'(x)\,y(x) - h(x)\,y'(x)\right) \tag{2}$$

where $y(x)$ is any solution of equation (1), $f(x)$ is given by

$$f(x) = \exp\left(\int p(x)\,dx\right) \tag{3}$$

and $h(x)$ is an arbitrary function. As $h(x)$ is arbitrary, this equation yields an infinite number of integrals for any function $y(x)$ which satisfies equation (1). Many well-known integrals in the literature can be derived very simply from equation (2), together with a large number of interesting new integrals. Techniques for exploiting equation (2) to obtain interesting integrals are presented here, together with sample results for selected special functions. Somewhat more than sample results are presented for the complete elliptic integrals of the first and second kinds, but it is impossible to do more than scratch the surface for any particular function in a single paper. It seems that the total number of indefinite integrals for special functions can be increased by a large factor using equation (2). In particular, it is possible to derive integrals combining products of different special functions, such as Bessel and Legendre functions combined with each other, or with elliptic integrals, or other special functions. Equation (2) was originally derived from an Euler-Lagrange equation, but once known, it can be proved in an elementary manner without variational calculus and both derivations are given here.

Section 2 below presents the Euler-Lagrange formulation of equation (1) and the derivation of equation (2) from it, together with a simpler proof. A second integral is also derived, which will not be considered in detail here. Section 3 presents techniques for exploiting equation (2) and some well-known results are derived using this method. It is shown that functions which are conjugate, in the sense that the differential equations they satisfy have the same $p(x)$ in equation (1), can always be combined as products in the same integral. It is shown that any two equations can be transformed to be mutually conjugate and transformation equations are given to calculate $q(x)$ for arbitrarily specified $p(x)$. It is shown that the reverse process, finding a suitable $p(x)$ to give a desired $q(x)$, is governed by a Riccati equation [5]. Both known and new integrals are derived in this section as examples of the methods presented. Section 4 presents sample results for a selection of special functions. Section 5 gives somewhat more than sample results for the complete elliptic integrals of the first and second kinds, using the method of fragmentary equations presented in Section 3. Table 1 gives the special functions used.

2 Formulation

Equation (1) can be expressed in Lagrangian form as:

$$\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) - \frac{\partial L}{\partial y} = 0 \tag{4}$$

Symbol	Special Function
$\mathrm{Ai}\,(x)$	Airy function of the first kind
$\mathrm{Bi}\,(x)$	Airy function of the second kind
$\mathrm{Ai}'\,(x)\,,\mathrm{Bi}'\,(x)$	Derivatives of the Airy functions
$E(k)$	Complete elliptic integral of the second kind
${}_2F_1\,(a,b;c;x)$	Gauss hypergeometric function
$\mathrm{Gi}\,(x)\,,\mathrm{Hi}\,(x)$	Scorer functions
$I_n\,(x)$	Modified Bessel function of the first kind
$J_\nu(x)$	Bessel function of the first kind
$H_n\,(x)$	Struve function of the first kind
$K(k)$	Complete elliptic integral of the first kind
$K_n\,(x)$	Modified Bessel function of the second kind
$P_\nu\,(x)$	Legendre function of the first kind
$P_\nu^\mu\,(x)$	Associated Legendre function of the first kind
$Q_\nu\,(x)$	Legendre function of the second kind
$Q_\nu^\mu\,(x)$	Associated Legendre function of the second kind
$s_{m,n}\,(x)$	Lommel function
$Y_\nu\,(x)$	Bessel function of the second kind
$\Gamma\,(x)$	Gamma function

Table 1: Special Functions Used

where the Lagrangian L is given by

$$L = f\,(x)\left(y'^2\,(x) - q\,(x)\,y^2\,(x)\right) \tag{5}$$

and the function $f\,(x)$ is a solution of the equation:

$$\frac{f'\,(x)}{f\,(x)} = p\,(x) \tag{6}$$

and hence is given by the integral (3). A principal property of any Lagrangian is:

$$\frac{d}{dx}\left(L - y'\frac{\partial L}{\partial y'}\right) = \frac{\partial L}{\partial x} \tag{7}$$

and from equation (5) we obtain

$$L - y'\frac{\partial L}{\partial y'} = -f\,(x)\left(y'^2\,(x) + q\,(x)\,y^2\,(x)\right) \tag{8}$$

and

$$\frac{\partial L}{\partial x} = f\,(x)'\,y'^2\,(x) - (f\,(x)\,q\,(x))'\,y^2\,(x)\,. \tag{9}$$

Substituting equations (8) and (9) into equation (7) gives immediately the integral

$$\int \left((f\,(x)\,q\,(x))'\,y^2\,(x) - f'\,(x)\,y'^2\,(x)\right)dx = f\,(x)\left(y'^2\,(x) + q\,(x)\,y\,(x)\right) \tag{10}$$

which is analogous to the energy integral in classical mechanics. This integral can be considered to be complementary to equation (2), as it gives different integrals, but these results will be presented separately. Explicitly evaluating both derivatives in equation (7) using equations (8) and (9) gives after cancellation and collection of terms:

$$-\frac{d}{dx}\left(f\left(x\right)y'\left(x\right)\right)=f\left(x\right)q\left(x\right)y\left(x\right) \tag{11}$$

and we obtain a second indefinite integral:

$$\int f\left(x\right)q\left(x\right)y\left(x\right)dx=-f\left(x\right)y'\left(x\right). \tag{12}$$

The differential equation (2) can be transformed by defining $y\left(x\right)=h\left(x\right)z\left(x\right)$ to give:

$$z''+\left(2\frac{h'\left(x\right)}{h\left(x\right)}+p\left(x\right)\right)z'+\left(\frac{h''\left(x\right)}{h\left(\left(x\right)\right)}+\frac{h'\left(x\right)}{h\left(x\right)}p\left(x\right)+q\left(x\right)\right)z=0. \tag{13}$$

This transformed equation can also be put in the Lagrangian form

$$\frac{d}{dx}\left(\frac{\partial\bar{L}}{\partial z'}\right)-\frac{\partial\bar{L}}{\partial z}=0 \tag{14}$$

where

$$\bar{L}=\bar{f}\left(x\right)\left(z'^{2}\left(x\right)-\bar{q}\left(x\right)z\left(x\right)\right) \tag{15}$$

and

$$\bar{q}\left(x\right)=\frac{h''\left(x\right)}{h\left(x\right)}+\frac{h'\left(x\right)}{h\left(x\right)}p\left(x\right)+q\left(x\right). \tag{16}$$

The function $\bar{f}\left(x\right)$, the equivalent of $f\left(x\right)$ in equation (5), is given by

$$\bar{f}\left(x\right)=\exp\left(\int\left(2\frac{h'\left(x\right)}{h\left(x\right)}+p\left(x\right)\right)dx\right)=h^{2}\left(x\right)f\left(x\right) \tag{17}$$

so that the new Lagrangian can be expressed as

$$\bar{L}=h^{2}\left(x\right)f\left(x\right)\left(z'^{2}-\left(\frac{h''\left(x\right)}{h\left(x\right)}+\frac{h'\left(x\right)}{h\left(x\right)}p\left(x\right)+q\left(x\right)\right)z^{2}\left(x\right)\right). \tag{18}$$

The equivalent of the integral (12) for the transformed equation is

$$\int h^{2}\left(x\right)f\left(x\right)\left(\frac{h''\left(x\right)}{h\left(x\right)}+\frac{h'\left(x\right)}{h\left(x\right)}p\left(x\right)+q\left(x\right)\right)z\left(x\right)dx=-h^{2}\left(x\right)f\left(x\right)z'\left(x\right) \tag{19}$$

and since $z\left(x\right)=y\left(x\right)/h\left(x\right)$ and hence

$$z'\left(x\right)=\frac{y'\left(x\right)}{h\left(x\right)}-\frac{h'\left(x\right)y\left(x\right)}{h^{2}\left(x\right)} \tag{20}$$

then equation (19) reduces to equation (2). This relation holds for all homogeneous second-order linear ordinary differential equations for an arbitrary function $h\left(x\right)$. Once known, this result can be proven very simply without reference to variational calculus.

Theorem 1

$$\int f(x)\left(h''(x)+h'(x)\,p(x)+h(x)\,q(x)\right)y(x)\,dx =$$

$$f(x)\left(h'(x)\,y(x)-h(x)\,y'(x)\right) \tag{21}$$

where $f(x)$ and $y(x)$ obey the respective differential equations:

$$f'(x)=p(x)\,f(x) \tag{22}$$

$$y''(x)+p(x)\,y'(x)+q(x)\,y(x)=0 \tag{23}$$

and $h(x)$, $p(x)$ and $q(x)$ are arbitrary functions.

Proof. Arbitrary functions $f(x)$, $h(x)$ and $y(x)$ satisfy the differential identity:

$$\frac{d}{dx}\left(f(x)\left(h'(x)\,y(x)-h(x)\,y'(x)\right)\right)=f'(x)\left(h'(x)\,y(x)-h(x)\,y'(x)\right) \tag{24}$$

$$+f(x)\left(h''(x)\,y(x)-h(x)\,y''(x)\right).$$

Eliminating $f'(x)$ and $y''(x)$ from equation (24) using equations (22) and (23), respectively, gives

$$\frac{d}{dx}\left(f(x)\left(h'(x)\,y(x)-h(x)\,y'(x)\right)\right)=$$

$$f(x)\left(h''(x)+h'(x)\,p(x)+h(x)\,q(x)\right)y(x) \tag{25}$$

for arbitrary $h(x)$ and integration of equation (25) gives (21) and the theorem is proven. Equation (22) can be integrated immediately to give

$$f(x)=\exp\left(\int p(x)\,dx\right). \tag{26}$$

■

In equation (2) we can take $y(x)\equiv y_1(x)$ to be any solution of equation (1) and $h(x)=y_2(x)$ to be another solution, to obtain the Wronskian for the differential equation (1) as:

$$W(x)\equiv\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x)\end{vmatrix}=\frac{A}{f(x)} \tag{27}$$

where A is a constant. From equation (3) this is equivalent to Abel's formula [6,7] for the Wronskian:

$$W(x)=W(x_0)\exp\left(-\int_{x_0}^{x} p(x)\,dx\right). \tag{28}$$

3 Techniques for finding indefinite integrals

Equation (2) yields an infinite number of indefinite integrals for each solution $y(x)$ of equation (2), as any $h(x)$ yields an integral. In equation (2) we can take $h(x) = 1$ or substitute any elementary function such as e^{ax} or e^{-ax^2} and obtain an integral for any special function. Expressions involving elementary functions, such as $h(x) = x^m$ and $h(x) = x^m \left\{ \begin{smallmatrix} \sin x \\ \cos x \end{smallmatrix} \right\}$ often yield integrals which are interesting and new. We can also specify $h(x)$ to be the solution to a differential equation where one or two terms in equation (2) is deleted, such as:

$$h''(x) + p(x)\,h'(x) = 0 \tag{29}$$

$$p(x)\,h'(x) + q(x)\,h(x) = 0 \tag{30}$$

$$h''(x) + q(x)\,h(x) = 0. \tag{31}$$

It is frequently the case that $p(x)$ or $q(x)$ may consist of several terms, and $h(x)$ can be specified as the solution of a differential equation where some of these terms have been deleted. These equations with one or more terms deleted will be referred to as fragmentary equations.

3.1 Integration of the function $y(x)$ itself.

To integrate a solution $y(x)$ of equation (2) it is necessary to specify $h(x)$ to be a solution of the inhomogeneous equation:

$$h''(x) + p(x)\,h'(x) + q(x)\,h(x) = \frac{1}{f(x)}. \tag{32}$$

For the Legendre equation:

$$y''(x) - \frac{2x}{1-x^2}y'(x) + \frac{\nu(\nu+1)}{1-x^2}y(x) = 0 \tag{33}$$

then

$$f(x) \equiv \exp\left(-\int \frac{2x}{1-x^2}dx\right) = 1 - x^2 \tag{34}$$

and equation (32) becomes

$$h''(x) - \frac{2x}{1-x^2}h'(x) + \frac{\nu(\nu+1)}{1-x^2}h(x) = \frac{1}{1-x^2} \tag{35}$$

which has a simple solution

$$h(x) = \frac{1}{\nu(\nu+1)}. \tag{36}$$

Substituting (36) into equation (2) for Legendre's equation gives

$$\int \begin{Bmatrix} P_\nu(x) \\ Q_\nu(x) \end{Bmatrix} dx = \frac{x^2-1}{\nu(\nu+1)} \begin{Bmatrix} P'_\nu(x) \\ Q'_\nu(x) \end{Bmatrix} \tag{37}$$

and the Legendre recurrence relations [1]:

$$\left(x^2-1\right)\begin{Bmatrix} P'_\nu(x) \\ Q'_\nu(x) \end{Bmatrix} = (\nu+1)\begin{Bmatrix} P_{\nu+1}(x) - xP_\nu(x) \\ Q_{\nu+1}(x) - xQ_\nu(x) \end{Bmatrix} \tag{38}$$

$$\left(x^2-1\right)\begin{Bmatrix} P'_\nu(x) \\ Q'_\nu(x) \end{Bmatrix} = \nu\begin{Bmatrix} xP_\nu(x) - P_{\nu-1}(x) \\ xQ_\nu(x) - Q_{\nu-1}(x) \end{Bmatrix} \tag{39}$$

then give the alternative forms

$$\int \begin{Bmatrix} P_\nu(x) \\ Q_\nu(x) \end{Bmatrix} dx = \frac{1}{\nu}\begin{Bmatrix} P_{\nu+1}(x) - xP_\nu(x) \\ Q_{\nu+1}(x) - xQ_\nu(x) \end{Bmatrix} \tag{40}$$

$$\int \begin{Bmatrix} P_\nu(x) \\ Q_\nu(x) \end{Bmatrix} dx = \frac{1}{\nu+1}\begin{Bmatrix} xP_\nu(x) - P_{\nu-1}(x) \\ xQ_\nu(x) - Q_{\nu-1}(x) \end{Bmatrix}. \tag{41}$$

The Legendre equation above is a simple case, and sometimes the solution to equation (32) is given in terms of functions specifically defined to satisfy this inhomogeneous equation . Examples are the Lommel [8,9,] and Struve [9] functions for Bessel's equation and the Scorer functions for Airy's equation [2,10]. The Bessel equation is

$$y''(x) + \frac{1}{x}y'(x) + \left(1 - \frac{n^2}{x^2}\right)y(x) = 0 \tag{42}$$

and has $f(x)$ given by

$$f(x) \equiv \exp\left(\int \frac{dx}{x}\right) = x. \tag{43}$$

The Lommel function $s_{m,n}(x)$ satisfies the equation

$$y''(x) + \frac{1}{x}y'(x) + \left(1 - \frac{n^2}{x^2}\right)y(x) = x^{m-1} \tag{44}$$

with equation (32) being the special case for $m = 0$. Taking $h(x) = s_{m,n}(x)$ in equation (2) gives

$$\int x^m J_n(x)\,dx = x\left(s'_{m,n}(x)\,J_n(x) - s_{m,n}(x)\,J'_n(x)\right) \tag{45}$$

which from the Lommel and Bessel recurrence relations [1]:

$$s'_{m,n}(x) = (m+n-1)\,s_{m-1,n-1}(x)\,J_n(x) - \frac{n}{x}s_{m,n}(x)\,J_n(x) \tag{46}$$

$$\frac{n}{x}J_n(x) + J'_n(x) = J_{n-1}(x) \tag{47}$$

reduces to

$$x\left(s'_{m,n}(x)\,J_n(x) - s_{m,n}(x)\,J'_n(x)\right) \tag{48}$$

and gives

$$\int x^m J_n(x)\,dx = x\left((m+n-1)\,s_{m-1,n-1}(x)\,J_n(x) - s_{m,n}(x)\,J_{n-1}(x)\right). \tag{49}$$

Watson [9] gives a rather different derivation of this result. For the special case where $m = n$, substituting the identities [1]

$$s_{n,n}(x) = \Gamma(n+1/2)\,\sqrt{\pi}2^{n-1}\mathbf{H}_n(x) \tag{50}$$

$$\left(n-\frac{1}{2}\right)\Gamma\left(n-\frac{1}{2}\right) = \Gamma\left(n+\frac{1}{2}\right) \tag{51}$$

into equation (49) gives the integral in terms of the Struve function $\mathbf{H}_n(x)$ as:

$$\int x^n J_n(x)\,dx = \sqrt{\pi}2^{n-1}\Gamma(n+1/2)\,x\left(\mathbf{H}_{n-1}(x)\,J_n(x) - \mathbf{H}_n(x)\,J_{n-1}(x)\right). \tag{52}$$

The Airy functions are a similar case to that of the Bessel functions described above. The Airy functions $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$ obey the differential equation:

$$y''(x) - xy(x) = 0 \tag{53}$$

and the Scorer functions $\mathrm{Gi}(x)$ and $\mathrm{Hi}(x)$ obey the inhomogeneous Airy equations [10]:

$$y''(x) - xy(x) = \mp\frac{1}{\pi} \tag{54}$$

where the negative sign is taken for $\mathrm{Gi}(x)$ and the positive sign for $\mathrm{Hi}(x)$. Taking $h(x)$ in equation (2) to be either $\mathrm{Gi}(x)$ or $\mathrm{Hi}(x)$ gives the alternative integrals, which of course differ only by a constant:

$$\int \begin{Bmatrix} \mathrm{Ai}(x) \\ \mathrm{Bi}(x) \end{Bmatrix} dx = \pi \begin{Bmatrix} \mathrm{Ai}'(x)\,\mathrm{Gi}(x) - \mathrm{Ai}(x)\,\mathrm{Gi}'(x) \\ \mathrm{Bi}'(x)\,\mathrm{Gi}(x) - \mathrm{Bi}(x)\,\mathrm{Gi}'(x) \end{Bmatrix} \tag{55}$$

$$\int \begin{Bmatrix} \mathrm{Ai}(x) \\ \mathrm{Bi}(x) \end{Bmatrix} dx = \pi \begin{Bmatrix} \mathrm{Ai}(x)\,\mathrm{Hi}'(x) - \mathrm{Ai}'(x)\,\mathrm{Hi}(x) \\ \mathrm{Bi}(x)\,\mathrm{Hi}'(x) - \mathrm{Bi}'(x)\,\mathrm{Hi}(x) \end{Bmatrix}. \tag{56}$$

3.2 Conjugate differential equations

Two differential equations of the form (1) are considered conjugate if they have the same $p(x)$, and hence the same $f(x)$, but different $q(x)$. For example, the equation

$$y''(x) + \frac{1}{x}y'(x) + \frac{1}{1-x^2}y(x) = 0 \tag{57}$$

is conjugate to the Bessel equation (42) and has as a general solution [1] the complete elliptic integral expression

$$y(x) = C_1\mathbf{E}(x) + C_2\left(\mathbf{E}(x') - \mathbf{K}(x')\right) \tag{58}$$

where $x' \equiv \sqrt{1-x^2}$ is the complementary modulus. The Bessel equation (42) can be slightly generalized as

$$y''(x) + \frac{1}{x}y'(x) + \left(\alpha^2 - \frac{n^2}{x^2}\right)y(x) = 0 \tag{59}$$

which has the solution $y(x) = C_1 J_n(\alpha x) + C_2 Y_n(\alpha x)$ and two equations of the form (59) are mutually conjugate unless both α and n are the same for both. The modified Bessel equation

$$y''(x) + \frac{1}{x}y'(x) - \left(\alpha^2 + \frac{n^2}{x^2}\right)y(x) = 0 \tag{60}$$

with solution $y(x) = C_1 I_n(\alpha x) + C_2 K_n(\alpha x)$ is always conjugate to equation (59).

In equation (2) we can take $h(x)$ to be any solution of a conjugate equation with $\bar{q}(x)$ instead of $q(x)$. We have

$$h''(x) + p(x)h'(x) = -\bar{q}(x)h \tag{61}$$

and hence

$$h''(x) + p(x)h'(x) + q(x)h(x) = (q(x) - \bar{q}(x))h(x) \tag{62}$$

so that equation (2) gives the integral

$$\int f(x)(q(x) - \bar{q}(x))h(x)y(x)\,dx = f(x)(h'(x)y(x) - h(x)y'(x)) \tag{63}$$

If in this equation we take $y(x) = Z_n(\alpha x)$, where $Z_n(\alpha x) \equiv aJ_n(\alpha x) + bY_n(x)$ is any solution of the Bessel equation (59), and $h(x) = Z_m(\beta x)$ is any solution of the Bessel Equation

$$h''(x) + \frac{1}{x}h'(x) + \left(\beta^2 - \frac{m^2}{x^2}\right)h(x) = 0 \tag{64}$$

then equation (63) gives the well-known Bessel integral [1]:

$$\int \left(\left(\alpha^2 - \beta^2\right)x - \frac{n^2 - m^2}{x}\right) Z_n(\alpha x)\, Z_m(\beta x)\,dx =$$
$$x\left(\beta Z_m'(\beta x)\, Z_n(\alpha x) - Z_m(\beta x)\, Z_n'(\alpha x)\right). \tag{65}$$

Similarly, taking $y(x)$ to be any solution of equation (59) and $h(x) = \bar{Z}_m(\beta x)$, where $\bar{Z}_m(\beta x) \equiv AI_n(\beta x) + BK_n(\beta x)$ is any solution of the modified Bessel equation:

$$h''(x) + \frac{1}{x}h'(x) + \left(\beta^2 - \frac{m^2}{x^2}\right)h(x) = 0 \tag{66}$$

gives the integral

$$\int \left(\left(\alpha^2+\beta^2\right) x - \frac{n^2-m^2}{x} \right) Z_n(\alpha x)\,\bar{Z}_m(\beta x)\,dx =$$

$$x\left(\beta \bar{Z}'_m(\beta x)\,Z_n(\alpha x) - \bar{Z}_m(\beta x)\,Z'_n(\alpha x)\right). \tag{67}$$

In equation (63) we can take $y(x) = Z_n(\alpha x)$ and $h(x) = \mathbf{E}(x)$, a solution of the conjugate equation (57), to obtain

$$\int x \left(\alpha^2 - \frac{n^2}{x^2} - \frac{1}{x'^2} \right) Z_n(\alpha x)\,\mathbf{E}(x)\,dx =$$

$$x\left(\frac{d\mathbf{E}(x)}{dx} Z_n(\alpha x) - \mathbf{E}(x)\,\frac{dZ_n(\alpha x)}{dx} \right) \tag{68}$$

where [1]:

$$\frac{d\mathbf{E}(x)}{dx} = \frac{\mathbf{E}(x)-\mathbf{K}(x)}{x}. \tag{69}$$

A particularly simple special case of equation (69) is

$$\int \frac{x^3}{x'^2} J_0(x)\,\mathbf{E}(x)\,dx = J_0(x)\left(\mathbf{K}(x)-\mathbf{E}(x)\right) - xJ_1(x)\,\mathbf{E}(x). \tag{70}$$

3.3 Transformations of the differential equations

Any differential equation of the form (1) can be made conjugate to any other by a change of dependent variable. This allows integrals to be obtained containing a product of any two specified special functions, provided they both satisfy differential equations of the form (1). The differential equation

$$z''(x) + \bar{p}(x)\,z'(x) + \bar{q}(x)\,z(x) = 0 \tag{71}$$

can be made conjugate to equation (1) by the transformation $y(x) = g(x)\,z(x)$ which yields the equation

$$y''(x) + \left(\frac{2g'(x)}{g(x)} + \bar{p}(x) \right) y'(x) + \left(\frac{g''(x)}{g(x)} + \frac{g'(x)}{g(x)} p(x) + \bar{q}(x) \right) y(x) = 0 \tag{72}$$

and we can choose

$$\frac{2g'(x)}{g(x)} + \bar{p}(x) = p(x) \tag{73}$$

so that

$$g(x) = \exp\left(\frac{1}{2} \int \left(p(x) - \bar{p}(x)\right) dx \right). \tag{74}$$

From equation (74) we obtain the relation:

$$g(x) = \sqrt{\frac{f(x)}{\bar{f}(x)}}. \tag{75}$$

The coefficient of $y(x)$ in equation (72) can be evaluated directly from $g(x)$ but it is usually more convenient to note that

$$\frac{d}{dx}\left(\frac{g'(x)}{g(x)}\right)=\frac{g''(x)}{g(x)}-\left(\frac{g'(x)}{g(x)}\right)^2 \tag{76}$$

and

$$\frac{g'(x)}{g(x)}=\frac{1}{2}\left(p(x)-\bar{p}(x)\right) \tag{77}$$

so that equation (72) becomes

$$y''(x)+p(x)\,y'(x)+\left(\frac{1}{2}\left(p(x)-\bar{p}(x)\right)'+\frac{p^2(x)-\bar{p}^2(x)}{4}+\bar{q}(x)\right)y(x)=0. \tag{78}$$

For example, the equation satisfied by the complete elliptic integral of the first kind $\mathbf{K}(x)$ is

$$z''(x)+\left(\frac{1}{x}-\frac{2x}{1-x^2}\right)z'(x)-\frac{1}{1-x^2}z(x)=0 \tag{79}$$

which has $\bar{f}(x)=x\left(1-x^2\right)$ and general solution [1]:

$$z(x)=C_1\mathbf{K}(x)+C_2\mathbf{K}(x'). \tag{80}$$

The transformed equation conjugate with the Bessel equation is given by equation (78) as

$$y''(x)+\frac{1}{x}y'(x)+\frac{1}{\left(1-x^2\right)^2}y(x)=0 \tag{81}$$

and from the relation (75) the general solution of equation (81) is given by

$$y(x)=C_1x'\mathbf{K}(x)+C_2x'\mathbf{K}(x'). \tag{82}$$

In equation (63) we can take $y(x)$ to be $Z_n(\alpha x)$, the general solution of the Bessel equation (59) and specify $h(x)=x'\mathbf{K}(x)$. This gives the integral

$$\int xx'\left(\alpha^2-\frac{n^2}{x^2}-\frac{1}{x'^4}\right)Z_n(\alpha x)\,\mathbf{K}(x)\,dx=$$

$$x\left(Z_n(\alpha x)\frac{d}{dx}\left(x'\mathbf{K}(x)\right)-x'\mathbf{K}(x)\frac{dZ_n(\alpha x)}{dx}\right) \tag{83}$$

where [1]:

$$\frac{d\mathbf{K}(x)}{dx}=\frac{\mathbf{E}(x)}{xx'^2}-\frac{\mathbf{K}(x)}{x}. \tag{84}$$

A special case of equation (83) is

$$\int\frac{x^3\left(2-x^2\right)}{x'^3}\begin{Bmatrix}J_0(x)\\Y_0(x)\end{Bmatrix}\mathbf{K}(x)\,dx=\begin{Bmatrix}J_0(x)\\Y_0(x)\end{Bmatrix}\frac{\mathbf{K}(x)-\mathbf{E}(x)}{x'}-xx'\begin{Bmatrix}J_1(x)\\Y_1(x)\end{Bmatrix}\mathbf{K}(x). \tag{85}$$

Employing the conjugate equations (57) and (81) in equation (63), with $y(x) = \mathbf{E}(x)$ and $h(x) = x'\mathbf{K}(x)$ gives the integral

$$\int \left(\frac{x}{x'}\right)^3 \mathbf{E}(x)\,\mathbf{K}(x)\,dx = (\mathbf{K}(x) - \mathbf{E}(x))\left(\frac{\mathbf{E}(x)}{x'} - x'\mathbf{K}(x)\right). \tag{86}$$

If the common $p(x)$ of two conjugate differential equations is transformed such that $p(x) \to \hat{p}(x)$ and $f(x) \to \bar{f}(x)$ for both equations, the integral given by equation (63) remains unchanged. This can be shown by applying the transformations $\bar{y}(x) = g(x)\,y(x)$ and $\bar{h}(x) = g(x)\,h(x)$ to the conjugate equations:

$$y''(x) + p(x)\,y'(x) + q(x)\,y(x) = 0 \tag{87}$$

$$h''(x) + p(x)\,h'(x) + \bar{q}(x)\,h(x) = 0 \tag{88}$$

to obtain the still conjugate differential equations:

$$y''(x) + \hat{p}(x)\,y'(x) + q_1(x)\,y(x) = 0 \tag{89}$$

$$\bar{h}''(x) + \hat{p}(x)\,\bar{h}'(x) + \bar{q}_1(x)\,\bar{h}(x) = 0 \tag{90}$$

where from equation (78)

$$q_1(x) = \frac{1}{2}\left(\hat{p}(x) - p(x)\right)' + \frac{\hat{p}^2(x) - p^2(x)}{4} + q(x) \tag{91}$$

$$\bar{q}_1(x) = \frac{1}{2}\left(\hat{p}(x) - p(x)\right)' + \frac{\hat{p}^2(x) - p^2(x)}{4} + \bar{q}(x) \tag{92}$$

and hence

$$q_1(x) - \bar{q}_1(x) = q(x) - \bar{q}(x). \tag{93}$$

From equation (75) the general solutions of the differential equations (89) and (90) are related to the general solutions of equations (87) and (88), respectively, by:

$$\bar{y}(x) = \sqrt{\frac{f(x)}{\bar{f}(x)}}\,y(x) \tag{94}$$

$$\bar{h}(x) = \sqrt{\frac{f(x)}{\bar{f}(x)}}\,h(x) \tag{95}$$

Substituting equations (93)-(95) into equation (63) yields after cancellation

$$\int f(x)\,(q(x) - \bar{q}(x))\,h(x)\,y(x)\,dx = \bar{f}(x)\left(\bar{h}'(x)\,\bar{y}(x) - \bar{h}(x)\,\bar{y}'(x)\right) \tag{96}$$

and hence the transformed equations do not yield a new integral in this manner, though new integrals can be obtained from the transformed equations using fragmentary equations.

3.4 A Riccati Equation

Instead of transforming $p(x)$ in equation (1), we can instead target $q(x)$ for simplification by a dependent variable change. The differential equation

$$z''(x) + \bar{p}(x) z'(x) + \bar{q}(x) z(x) = 0 \tag{97}$$

is transformed by the variable change $y(x) = g(x) z(x)$ to give

$$y''(x) + p(x) y'(x) + q(x) y(x) = 0 \tag{98}$$

and then from equation (78) the function $q(x)$ is given by

$$q(x) = \frac{1}{2}(p(x) - \bar{p}(x))' + \frac{p^2(x) - \bar{p}^2(x)}{4} + \bar{q}(x). \tag{99}$$

If $\bar{p}(x)$ and $\bar{q}(x)$ are considered known and $q(x)$ is to be specified, solving for $p(x)$ as an unknown gives a Riccati equation [5]. Although nonlinear, it is a well-known equation with many useful exact solutions [5]. This can be illustrated with the associated Legendre equation, which has a relatively cumbersome $q(x)$, in the context of applying equation (2):

$$z''(x) - \frac{2x}{1-x^2} z'(x) + \left(\frac{\nu(\nu+1)}{1-x^2} - \frac{\mu^2}{(1-x^2)^2} \right) z(x) = 0. \tag{100}$$

Under a general dependent variable change, the new $q(x)$ is given by (99) as

$$\frac{1}{2}\frac{dp}{dx} + \frac{p^2}{4} + \frac{\nu(\nu+1)}{1-x^2} - \frac{\mu^2 - 1}{(1-x^2)^2} \tag{101}$$

which has a simple solution

$$p(x) = -\frac{2(x-\mu)}{1-x^2} \tag{102}$$

and the transformed associated Legendre equation is therefore

$$y''(x) - \frac{2(x-\mu)}{1-x^2} y'(x) + \frac{\nu(\nu+1)}{1-x^2} y(x) = 0. \tag{103}$$

For this differential equation $f(x)$ and $g(x)$ are given by equations (3) and (74), respectively, as

$$f(x) = \left(1 - x^2\right) \left(\frac{1+x}{1-x} \right)^{\mu} \tag{104}$$

$$g(x) = \left(\frac{1-x}{1+x} \right)^{\mu/2} \tag{105}$$

and hence the general solution of (103) is

$$y(x) = \left(\frac{1-x}{1+x} \right)^{\mu/2} \left(C_1 P_\nu^\mu(x) + C_2 Q_\nu^\mu(x) \right). \tag{106}$$

Simple integrals can be obtained by taking $h(x)$ to be either of the independent solutions of the equation

$$h''(x) - \frac{2(x-\mu)}{1-x^2} h'(x) = 0 \tag{107}$$

which are

$$h(x) = 1 \tag{108}$$

$$h(x) = \left(\frac{1-x}{1+x}\right)^{\mu}. \tag{109}$$

Substituting $h(x) = 1$ into equation (2) gives

$$\int \left(\frac{1+x}{1-x}\right)^{\mu/2} \begin{Bmatrix} P_\nu^\mu(x) \\ Q_\nu^\mu(x) \end{Bmatrix} dx =$$

$$\frac{1}{\nu(\nu+1)} \left(\frac{1+x}{1-x}\right)^{\mu/2} \begin{Bmatrix} \mu P_\nu^\mu(x) + \left(x^2-1\right) P_\nu'^\mu(x) \\ \mu Q_\nu^\mu(x) + \left(x^2-1\right) Q_\nu'^\mu(x) \end{Bmatrix} \tag{110}$$

which from the recurrence relation [1]:

$$\left(1-x^2\right) P_\nu'^\mu(x) = \nu x P_\nu^\mu(x) - (\nu+\mu) P_{\nu-1}(x) \tag{111}$$

can be expressed as

$$\int \left(\frac{1+x}{1-x}\right)^{\mu/2} \begin{Bmatrix} P_\nu^\mu(x) \\ Q_\nu^\mu(x) \end{Bmatrix} dx =$$

$$\frac{1}{\nu(\nu+1)} \left(\frac{1+x}{1-x}\right)^{\mu/2} \begin{Bmatrix} (\nu x+\mu) P_\nu^\mu(x) - (\nu+\mu) P_{\nu-1}^\mu(x) \\ (\nu x+\mu) Q_\nu^\mu(x) - (\nu+\mu) Q_{\nu-1}^\mu(x) \end{Bmatrix}. \tag{112}$$

Substituting equation (109) into equation (2) gives after some reduction:

$$\int \left(\frac{1-x}{1+x}\right)^{\mu/2} \begin{Bmatrix} P_\nu^\mu(x) \\ Q_\nu^\mu(x) \end{Bmatrix} dx =$$

$$\frac{1}{\nu(\nu+1)} \left(\frac{1-x}{1+x}\right)^{\mu/2} \left[\begin{Bmatrix} (\nu x-\mu) P_\nu^\mu(x) - (\nu+\mu) P_{\nu-1}(x) \\ (\nu x-\mu) Q_\nu^\mu(x) - (\nu+\mu) Q_{\nu-1}(x) \end{Bmatrix}\right]. \tag{113}$$

Equations (112) and (113) are previously known integrals which are tabulated in a combined form in [4]. A different integral is obtained by taking $h(x)$ to be the solution of

$$-\frac{2(x-\mu)}{1-x^2} h'(x) + \frac{\nu(\nu+1)}{1-x^2} h(x) = 0 \tag{114}$$

which is:

$$h(x) = (x-\mu)^{\nu(\nu+1)/2}. \tag{115}$$

Substituting equation (115) into equation (2) gives after some reduction

$$\int (x-\mu)^{\nu(\nu+1)/2-2} \frac{(1+x)^{\mu/2+1}}{(1-x)^{\mu/2-1}} \begin{Bmatrix} P_\nu^\mu(x) \\ Q_\nu^\mu(x) \end{Bmatrix} dx = (x-\mu)^{\nu(\nu+1)/2} \left(\frac{1+x}{1-x}\right)^{\mu/2} \times$$

$$\left(\frac{2\left(1-x^2\right)}{(\nu-1)(\nu+2)(x-\mu)} \begin{Bmatrix} P_\nu^\mu(x) \\ Q_\nu^\mu(x) \end{Bmatrix} + \right.$$

$$\left. \frac{4}{(\nu-1)\nu(\nu+1)(\nu+2)} \begin{Bmatrix} (\nu x+\mu) P_\nu^\mu(x) - (\nu+\mu) P_{\nu-1}^\mu(x) \\ (\nu x+\mu) Q_\nu^\mu(x) - (\nu+\mu) Q_{\nu-1}^\mu(x) \end{Bmatrix} \right). \quad (116)$$

This integral seems to be new.

4 Examples for some special functions

There are an unlimited number of cases as $h(x)$ is arbitrary. The art of using equation (2) is to choose $h(x)$ to give an interesting integral.

4.1 Bessel Functions

Equation (2) for Bessel's equation gives:

$$\int x \left(h''(x) + \frac{1}{x} h'(x) + \left(1 - \frac{n^2}{x^2}\right) h(x) \right) J_n(x)\, dx =$$

$$x \left(h'(x) J_n(x) - h(x) J_n'(x) \right) \quad (117)$$

and substituting $h(x) = x^m$ in this equation:

$$\int x^{m+1} \left(1 + \frac{m^2-n^2}{x^2}\right) J_n(x)\, dx = x^{m+1} \left(\frac{m J_n(x)}{x} - J_n'(x) \right) \quad (118)$$

which from the Bessel recurrence

$$J_n'(x) = \frac{1}{2} \left(J_{n-1}(x) - J_{n+1}(x) \right) \quad (119)$$

can be expressed as:

$$\int x^{m+1} \left(1 + \frac{m^2-n^2}{x^2}\right) J_n(x)\, dx =$$

$$x^{m+1} \left(\frac{m J_n(x)}{x} + \frac{1}{2} \left(J_{n+1}(x) - J_{n-1}(x) \right) \right). \quad (120)$$

The second term in equation (120) vanishes for $m = \pm n$, and the Bessel recurrence

$$\frac{m J_n(x)}{x} = \frac{1}{2} \left(J_{n-1}(x) + J_{n+1}(x) \right) \quad (121)$$

then gives the elementary cases

$$\int x^{n+1} J_n(x)\, dx = x^{n+1} J_{n+1}(x)\, dx \quad (122)$$

$$\int x^{-n+1} J_n(x)\, dx = -x^{-n+1} J_{n-1}(x)\, dx. \quad (123)$$

The general integral given by equation (120) appears to be new. For $n = 0$ and $m = 1$ this equation gives the simple expression

$$\int \left(1 + x^2\right) J_0(x)\, dx = x J_0(x) + x^2 J_1(x) \tag{124}$$

which can be readily verified by differentiation.

Substituting $h(x) = x^m \left\{ \begin{smallmatrix} \sin x \\ \cos x \end{smallmatrix} \right\}$ in equation (117) gives the integral

$$\int \left(\left(m^2 - n^2\right) x^{m-1} \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} + (2m+1)\, x^m \begin{Bmatrix} \cos x \\ -\sin x \end{Bmatrix} \right) J_n(x)\, dx$$

$$= x^{m+1} \left[\left(\frac{m J_n(x)}{x} - J_n'(x) \right) \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} + J_n(x) \begin{Bmatrix} \cos x \\ -\sin x \end{Bmatrix} \right]. \tag{125}$$

The integrand of (125) reduces to a single term for $m = n$ or $m = -1/2$. Employing the Bessel recurrence:

$$J_n'(x) = \frac{n J_n(x)}{x} - J_{n+1}(x) \tag{126}$$

gives for $m = n$:

$$\int x^n \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} J_n(x)\, dx = \frac{x^{n+1}}{2n+1} \begin{Bmatrix} J_n(x) \sin x - J_{n+1}(x) \cos x \\ J_n(x) \cos x + J_{n+1}(x) \sin x \end{Bmatrix} \tag{127}$$

and for $m = -1/2$:

$$\int x^{-3/2} J_n(x) \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} dx =$$

$$= \left(\frac{J_n(x)}{(n - 1/2)\sqrt{x}} - \frac{\sqrt{x} J_{n+1}(x)}{n^2 - 1/4} \right) \begin{Bmatrix} \sin x \\ \cos x \end{Bmatrix} - \frac{\sqrt{x} J_n(x)}{n^2 - 1/4} \begin{Bmatrix} \cos x \\ -\sin x \end{Bmatrix}. \tag{128}$$

Equation (127) is tabulated in [3]. Equation (128) is also tabulated in [3] but unfortunately with multiple typographical errors. The same errors are also present in the original Russian edition [11], but it is evident that the authors of [3,11] did originally have equation (128).

Substituting $h(x) = x^n \ln(x)$ in equation (117) gives

$$\int \left(2n x^{n-1} + x^{n+1} \ln(x)\right) J_n(x)\, dx = x^n \left(1 + n \ln x\right) J_n(x) - x^{n+1} \ln(x)\, J_n'(x) \tag{129}$$

which can be simplified with equation (126) to give

$$\int \left(2n x^{n-1} + x^{n+1} \ln(x)\right) J_n(x)\, dx = x^n J_n(x) + \ln(x)\, x^{n+1} J_{n+1}(x)\,. \tag{130}$$

For $n = 0$, equation (130) reduces to

$$\int x \ln(x)\, J_0(x)\, dx = J_0(x) + x \ln(x)\, J_1(x)\,. \tag{131}$$

Despite its simplicity, equation (131) appears to be new, and the Mathematica software [12] is unable to evaluate this integral.

Substituting the power function $h(x) = x^n$ into equation (2) with $y(x)$ given by the Airy equation (53) gives the integral:

$$\int \left(n(n-1)x^{n-2} - x^{n+1}\right) \begin{Bmatrix} \mathrm{Ai}(x) \\ \mathrm{Bi}(x) \end{Bmatrix} dx = nx^{n-1} \begin{Bmatrix} \mathrm{Ai}(x) \\ \mathrm{Bi}(x) \end{Bmatrix} - x^n \begin{Bmatrix} \mathrm{Ai}'(x) \\ \mathrm{Bi}'(x) \end{Bmatrix}. \tag{132}$$

Equation (132) reduces to a single term for $n = 0$ and $n = 1$, to give the well-known integrals:

$$\int x \begin{Bmatrix} \mathrm{Ai}(x) \\ \mathrm{Bi}(x) \end{Bmatrix} dx = \begin{Bmatrix} \mathrm{Ai}'(x) \\ \mathrm{Bi}'(x) \end{Bmatrix} \tag{133}$$

$$\int x^2 \begin{Bmatrix} \mathrm{Ai}(x) \\ \mathrm{Bi}(x) \end{Bmatrix} dx = x \begin{Bmatrix} \mathrm{Ai}'(x) \\ \mathrm{Bi}'(x) \end{Bmatrix} - \begin{Bmatrix} \mathrm{Ai}(x) \\ \mathrm{Bi}(x) \end{Bmatrix}. \tag{134}$$

Defining

$$I_n(x) = \int x^n \begin{Bmatrix} \mathrm{Ai}(x) \\ \mathrm{Bi}(x) \end{Bmatrix} dx \tag{135}$$

then equation (132) can be expressed as a recursion relation for the $I_n(x)$:

$$I_{n+3}(x) = (n+1)(n+2) I_n(x) - (n+2) x^{n+1} \begin{Bmatrix} \mathrm{Ai}(x) \\ \mathrm{Bi}(x) \end{Bmatrix} + x^{n+2} \begin{Bmatrix} \mathrm{Ai}'(x) \\ \mathrm{Bi}'(x) \end{Bmatrix}. \tag{136}$$

Using equations (55)-(56) and (133)-(134) as starting formulas, equation (136) allows closed form solutions to be obtained by upward recursion for all $n \in \mathbb{N}_0$. Downward recursion to negative integer values of n is blocked when (133) and (134) are the starting formulas, but it can be done with either (55) or (56) as a starting formula.

The Airy functions satisfy the differential equation (53) and from this equation it follows that the slightly more general differential equation

$$y''(x) - (x - \alpha) y(x) = 0 \tag{137}$$

has the general solution

$$y(x) = C_1 \mathrm{Ai}(x - \alpha) + C_2 \mathrm{Bi}(x - \alpha). \tag{138}$$

Substituting $h(x) = \sin(x + \phi)$ into equation (2) with $y(x)$ a solution of equation (137) for $\alpha = 1$ gives

$$\int x \sin(x+\phi) \begin{Bmatrix} \mathrm{Ai}(x-1) \\ \mathrm{Bi}(x-1) \end{Bmatrix} dx =$$
$$\sin(x+\phi) \begin{Bmatrix} \mathrm{Ai}'(x-1) \\ \mathrm{Bi}'(x-1) \end{Bmatrix} - \cos(x+\phi) \begin{Bmatrix} \mathrm{Ai}(x-1) \\ \mathrm{Bi}(x-1) \end{Bmatrix}. \tag{139}$$

Similarly, substituting $h(x) = e^{\pm x}$ into equation (2) for $\alpha = -1$ gives

$$\int xe^{\pm x} \begin{Bmatrix} \mathrm{Ai}(x+1) \\ \mathrm{Bi}(x+1) \end{Bmatrix} dx = e^{\pm x} \left(\begin{Bmatrix} \mathrm{Ai}'(x+1) \\ \mathrm{Bi}'(x+1) \end{Bmatrix} \mp \begin{Bmatrix} \mathrm{Ai}(x+1) \\ \mathrm{Bi}(x+1) \end{Bmatrix} \right). \tag{140}$$

Substituting $h(x) = \mathrm{Ai}(x-\beta)$ and $h(x) = \mathrm{Ai}(x-\beta)$ into equation (2) gives, respectively,

$$\int \begin{Bmatrix} \mathrm{Ai}(x-\beta)\,\mathrm{Ai}(x-\alpha) \\ \mathrm{Ai}(x-\beta)\,\mathrm{Bi}(x-\alpha) \end{Bmatrix} dx = \frac{1}{\alpha-\beta} \begin{Bmatrix} \mathrm{Ai}'(x-\beta)\,\mathrm{Ai}(x-\alpha) - \mathrm{Ai}(x-\beta)\,\mathrm{Ai}'(x-\alpha) \\ \mathrm{Ai}'(x-\beta)\,\mathrm{Bi}(x-\alpha) - \mathrm{Ai}(x-\beta)\,\mathrm{Bi}'(x-\alpha) \end{Bmatrix} \tag{141}$$

$$\int \begin{Bmatrix} \mathrm{Bi}(x-\beta)\,\mathrm{Ai}(x-\alpha) \\ \mathrm{Bi}(x-\beta)\,\mathrm{Bi}(x-\alpha) \end{Bmatrix} dx = \frac{1}{\alpha-\beta} \begin{Bmatrix} \mathrm{Bi}'(x-\beta)\,\mathrm{Ai}(x-\alpha) - \mathrm{Ai}(x-\beta)\,\mathrm{Ai}'(x-\alpha) \\ \mathrm{Bi}'(x-\beta)\,\mathrm{Bi}(x-\alpha) - \mathrm{Bi}(x-\beta)\,\mathrm{Bi}'(x-\alpha) \end{Bmatrix}. \tag{142}$$

The integrals given by equations (139)-(142) appear to be new.

4.2 Gauss hypergeometric functions

The Gauss hypergeometric function $y(x) = {}_2F_1(\alpha, \beta; \gamma; x)$ satisfies the differential equation [1]:

$$y''(x) + \frac{\gamma - (\alpha+\beta+1)x}{x(1-x)} y'(x) - \frac{\alpha\beta}{x(1-x)} y(x) = 0 \tag{143}$$

and for this equation $f(x)$ is given by equation (3) as

$$f(x) = x^{\gamma} (1-x)^{\alpha+\beta-\gamma+1}. \tag{144}$$

Substituting $h(x) = 1$ into equation (2) gives

$$\int x^{\gamma-1} (1-x)^{\alpha+\beta-\gamma} {}_2F_1(\alpha, \beta; \gamma; x) = \frac{1}{\gamma} x^{\gamma} (1-x)^{\alpha+\beta-\gamma+1} {}_2F_1(\alpha+1, \beta+1; \gamma+1; x) \tag{145}$$

which is a tabulated integral [4].

We can take $h(x)$ to be a solution of

$$\frac{\gamma - (\alpha+\beta+1)x}{x(1-x)} y'(x) - \frac{\alpha\beta}{x(1-x)} y(x) = 0 \tag{146}$$

which gives

$$h(x) = (\gamma - (\alpha+\beta+1)x)^{-\frac{\alpha\beta}{\alpha+\beta+1}} \tag{147}$$

and substituting this $h(x)$ into equation (2) gives

$$\int (\gamma - (\alpha+\beta+1)x)^{-\frac{\alpha\beta}{\alpha+\beta+1}-2} {}_2F_1(\alpha,\beta;\gamma;x)\,dx =$$

$$\frac{x^\gamma (1-x)^{\alpha+\beta+1-\gamma}}{(\alpha\beta+\alpha+\beta+1)} (\gamma-(\alpha+\beta+1)x)^{-\frac{\alpha\beta}{\alpha+\beta+1}} \times$$
$$\left(\frac{{}_2F_1(\alpha,\beta;\gamma;x)}{\gamma-(\alpha+\beta+1)x} - \frac{{}_2F_1(\alpha+1,\beta+1;\gamma+1;x)}{\gamma}\right) \tag{148}$$

4.2.1 Conjugate hypergeometric functions

In equation (2) we can choose $y(x)$ to be the hypergeometric function

$$y(x) = {}_2F_1(\alpha,\beta;\gamma;x) \tag{149}$$

and $h(x)$ to be a conjugate hypergeometic function:

$$h(x) = {}_2F_1(\alpha+\delta,\beta-\delta;\gamma;x) \tag{150}$$

where $h(x)$ satisfies the equation

$$h''(x) + \frac{\gamma-(\alpha+\beta+1)x}{x(1-x)}h'(x) - \left(\frac{\alpha\beta}{x(1-x)} - \frac{\delta(\alpha-\beta+\delta)}{x(1-x)}\right)h(x) = 0. \tag{151}$$

This gives the integral

$$\gamma\delta(\alpha-\beta+\delta)\int x^{\gamma-1}(1-x)^{\alpha+\beta-\gamma} {}_2F_1(\alpha+\delta,\beta-\delta;\gamma;x)\,{}_2F_1(\alpha,\beta;\gamma;x)\,dx =$$

$$x^\gamma(1-x)^{\alpha+\beta-\gamma+1}\left(\alpha\beta {}_2F_1(\alpha+\delta,\beta-\delta;\gamma;x)\,{}_2F_1(\alpha+1,\beta+1;\gamma+1;x)\right.$$
$$\left. -(\alpha+\delta)(\beta-\delta)\,{}_2F_1(\alpha+\delta+1,\beta-\delta+1;\gamma+1;x)\,{}_2F_1(\alpha,\beta;\gamma;x)\right). \tag{152}$$

5 Complete elliptic integrals of the first and second kinds

In this section the standard notation will be adopted for elliptic integrals, with the modulus denoted by k and the complementary modulus by $k' \equiv \sqrt{1-k^2}$. The complete elliptic integral of the first kind $\mathbf{K}(k)$ is a solution of the differential equation [1]:

$$y''(k) + \left(\frac{1}{k} - \frac{2k}{1-k^2}\right)y'(k) - \frac{1}{1-k^2}y(k) = 0 \tag{153}$$

which has $f(k) = k\left(1-k^2\right)$ and has the general solution

$$y(k) = C_1\mathbf{K}(k) + C_2\mathbf{K}'(k) \tag{154}$$

where $\mathbf{K}'(k) \equiv \mathbf{K}(k')$. The corresponding equation for $\mathbf{E}(k)$, the complete elliptic integral of the second kind, is [1]:

$$z''(k) + \frac{1}{k}z'(k) + \frac{1}{1-k^2}z(k) = 0 \tag{155}$$

which has $f(k) = k$ and the general solution

$$z(k) = C_1\mathbf{E}(k) + C_2(\mathbf{E}'(k) - \mathbf{K}'(k)). \tag{156}$$

Here, only the solutions $\mathbf{K}(k)$ and $\mathbf{E}(k)$ of these equations will be considered, though the other solutions also lead to interesting integrals.

5.1 Transformations of the Equations

These equations can be transformed to give many different forms for $p(k)$, and only a few will be considered here. The dependent variable name $y(k)$ will be retained for transformations of equation (153) and the variable name $z(k)$ will be used for transformations of equation (155). Equation (153) can be transformed to be conjugate with both the Bessel equation (59) and equation (155), which gives:

$$y''(k) + \frac{1}{x}y'(k) + \frac{1}{(1-k^2)^2}y(k) = 0 \tag{157}$$

and this equation has the general solution

$$y(k) = C_1k'\mathbf{K}(k) + C_2k'\mathbf{K}'(k). \tag{158}$$

Transforming equation (153) to be conjugate with the associated Legendre equation (100) gives:

$$y''(k) - \frac{2k}{1-k^2}y'(k) + \frac{1}{4k^2}y(k) = 0. \tag{159}$$

This equation has $f(k) = (1-k^2)$ and the general solution:

$$y(k) = C_1\sqrt{k}\mathbf{K}(k) + C_2\sqrt{k}\mathbf{K}'(k). \tag{160}$$

Equation (155) can be transformed to be conjugate with equation (153), which gives

$$z''(k) + \left(\frac{1}{k} - \frac{2k}{1-k^2}\right)z'(k) - \frac{1}{(1-k^2)^2}z(k) = 0 \tag{161}$$

and this equation has the general solution

$$z(k) = C_1\frac{\mathbf{E}(k)}{k'} + C_2\frac{\mathbf{E}'(k) - \mathbf{K}'(k)}{k'}. \tag{162}$$

Equation (155) can also be transformed to be conjugate with equation (159) and the associated Legendre equation (100). The transformed equation can be expressed in either of the forms:

$$z''(k) - \frac{2k}{1-k^2}z'(k) + \left(\frac{1}{4k^2} + \frac{1}{1-k^2} - \frac{1}{(1-k^2)^2}\right)z(k) = 0 \tag{163}$$

$$z''(k)-\frac{2k}{1-k^2}z'(k)+\left(\frac{1}{4k^2}-\frac{k^2}{\left(1-k^2\right)^2}\right)z(k)=0 \tag{164}$$

and has the general solution

$$z(k)=C_1\frac{\sqrt{k}\mathbf{E}(k)}{k'}+\frac{\sqrt{k}\left(\mathbf{E}'(k)-\mathbf{K}'(k)\right)}{k'}. \tag{165}$$

An unlimited number of similar transformations are possible, each yielding distinct integrals, but only equations (153) and (155) will be considered in detail here. The technique used below is to specify $h(x)$ as a solution to some fragment of the differential equation under consideration. For each equation, this gives a long list of integrals, most new and some very surprising. Since an unlimited number of transformations of the equations are possible, an extremely large number of interesting integrals can be generated. The lists of integrals given below for equations (153) and (155) are not complete, as some fragmentary solutions were rejected as not particularly interesting. However, the reader will be able to generate many additional integrals.

5.2 Integrals from fragments of the equation for the complete elliptic integral of the first kind

In equation (2) with $y(k)$ a solution of equation (153) we can take $h(k)=1$ and employ the recurrence relation:

$$\frac{d\mathbf{K}(k)}{dk}=\frac{\mathbf{E}(k)}{kk'^2}-\frac{\mathbf{K}(k)}{k} \tag{166}$$

to obtain the integral

$$\int k\mathbf{K}(k)\,dk=\mathbf{E}(k)-k'^2\mathbf{K}(k). \tag{167}$$

The six fragmentary equations:

$$h''(k)+\left(\frac{1}{k}-\frac{2k}{1-k^2}\right)h'(k)=0 \tag{168}$$

$$h''(k)+\frac{1}{k}h'(k)=0 \tag{169}$$

$$h''(k)-\frac{2k}{1-k^2}h'(k)=0 \tag{170}$$

$$\left(\frac{1}{k}-\frac{2k}{1-k^2}\right)h'(x)-\frac{1}{1-k^2}h(x)=0 \tag{171}$$

$$\frac{1}{k}h'(x)-\frac{1}{1-k^2}h(x)=0 \tag{172}$$

$$\frac{-2k}{1-k^2}h'(x)-\frac{1}{1-k^2}h(x)=0 \tag{173}$$

have the (non constant) respective solutions:

$$h(k)=\ln(k/k') \tag{174}$$

$$h(k)=\ln(k) \tag{175}$$

$$h(k)=\operatorname{arctanh}(k) \tag{176}$$

$$h(k)=\frac{1}{(3k^2-1)^{1/6}} \tag{177}$$

$$h(k)=\frac{1}{k'} \tag{178}$$

$$h(k)=\frac{1}{\sqrt{k}} \tag{179}$$

and these give the six respective integrals:

$$\int k\ln\left(\frac{k}{k'}\right)\mathbf{K}(k)\,dk=\ln\left(\frac{k}{k'}\right)\left(\mathbf{E}(k)-k'^2\mathbf{K}(k)\right)-\mathbf{K}(k) \tag{180}$$

$$\int k\left(2+\ln(k)\right)\mathbf{K}(k)\,dk=\ln(k)\,\mathbf{E}(k)-k'^2\left(1+\ln(k)\right)\mathbf{K}(k) \tag{181}$$

$$\int\left(1-k\operatorname{arctanh}(k)\right)\mathbf{K}(k)\,dk=k\mathbf{K}(k)-\operatorname{arctanh}(k)\left(\mathbf{E}(k)-k'^2\mathbf{K}(k)\right) \tag{182}$$

$$\int\frac{k\left(1-k^2\right)\left(1+4k^2\right)}{(3k^2-1)^{13/6}}\mathbf{K}(k)\,dx=\frac{\left(2k^2-1\right)\left(1-k^2\right)}{(3k^2-1)^{7/6}}\mathbf{K}(k)-\frac{\mathbf{E}(k)}{(3k^2-1)^{1/6}} \tag{183}$$

$$\int\frac{k\mathbf{K}(k)}{k'^3}dk=\frac{\mathbf{K}(k)-\mathbf{E}(k)}{k'} \tag{184}$$

$$\int\frac{k'^2}{k^{3/2}}\mathbf{K}(x)\,dx=\frac{1}{\sqrt{k}}\left(2k'^2\mathbf{K}(k)-4\mathbf{E}(k)\right). \tag{185}$$

Equations (167) and (184) are given in [4] but equations (180)-(183) and (185) appear to be new. Additional fragmentary equations can be constructed using the identity:

$$\frac{1}{1-k^2}\equiv 1+\frac{k^2}{1-k^2} \tag{186}$$

which gives equation (153) in the alternative form:

$$y''(k)+\left(\frac{1}{k}-\frac{2k}{1-k^2}\right)y'(k)-\left(1+\frac{k^2}{1-k^2}\right)y'(k)=0. \tag{187}$$

Useful fragmentary equations from equation (187) are:

$$h''(k)+\frac{1}{k}h'(k)-h=0 \tag{188}$$

$$h''(k) - h = 0 \tag{189}$$

$$\left(\frac{1}{k} - \frac{2k}{1-k^2}\right) h'(k) - h = 0 \tag{190}$$

$$-\frac{2k}{1-k^2} h'(k) - h = 0 \tag{191}$$

$$\frac{1}{k} h'(k) - h = 0 \tag{192}$$

$$\left(\frac{1}{k} - \frac{2k}{1-k^2}\right) h'(k) - \frac{k^2}{1-k^2} h = 0 \tag{193}$$

$$-\frac{2k}{1-k^2} h'(k) - \frac{k^2}{1-k^2} h = 0 \tag{194}$$

$$\frac{1}{k} h'(k) - \frac{k^2}{1-k^2} h = 0. \tag{195}$$

These equations have the respective solutions

$$h(k) = C_1 K_0(k) + C_2 I_0(k) \tag{196}$$

$$h(k) = e^{\pm k} \tag{197}$$

$$h(k) = \frac{e^{\frac{1}{6}k^2}}{\left(3k^2-1\right)^{\frac{1}{9}}} \tag{198}$$

$$h(k) = \frac{C_1}{\sqrt{k}} e^{\frac{1}{4}k^2} \tag{199}$$

$$h(k) = C_1 e^{\frac{1}{2}k^2} \tag{200}$$

$$h(k) = \frac{e^{-\frac{1}{6}k^2}}{\left(3k^2-1\right)^{\frac{1}{18}}} \tag{201}$$

$$h(k) = e^{-\frac{1}{4}k^2} \tag{202}$$

$$\frac{1}{k'} e^{-\frac{1}{2}k^2} \tag{203}$$

which in turn yield the respective integrals:

$$\int k^2 \begin{Bmatrix} 2K_1(k) - kK_0(k) \\ -2I_1(k) - kI_0(k) \end{Bmatrix} \mathbf{K}(k)\, dk =$$

$$\begin{Bmatrix} k'^2 \left(K_0(k) - kK_1(k)\right) \\ k'^2 \left(I_0(k) + kI_1(k)\right) \end{Bmatrix} \mathbf{K}(k) - \begin{Bmatrix} K_0(k) \\ I_0(k) \end{Bmatrix} \mathbf{E}(k) \tag{204}$$

$$\int \left(1 - 3k^2 \mp k^3\right) e^{\pm k} \mathbf{K}(k)\, dk = \left((k \pm 1) k'^2 \mathbf{K}(k) \mp \mathbf{E}(k)\right) e^{\pm k} \tag{205}$$

$$\int \frac{k\left(1 - k^2 + 6k^4 - 9k^6 - k^8\right)}{\left(3k^2-1\right)^{\frac{19}{9}}} e^{\frac{1}{6}k^2} \mathbf{K}(k)\, dk =$$

$$\left(\frac{\left(1-k^2\right)\left(k^4+2k^2-1\right)}{3k^2-1}\mathbf{K}\left(k\right)-\mathbf{E}\left(k\right)\right)\frac{e^{\frac{1}{6}k^2}}{\left(3k^2-1\right)^{\frac{1}{9}}} \tag{206}$$

$$\int\left(1+k^2-5k^4-k^6\right)\frac{e^{\frac{1}{4}k^2}}{k^{\frac{3}{2}}}\mathbf{K}\left(k\right)dk=\left(2\left(1-k^4\right)\mathbf{K}\left(k\right)-4\mathbf{E}\left(k\right)\right)\frac{e^{\frac{1}{4}k^2}}{k^{1/2}} \tag{207}$$

$$\int k\left(k^4+3k^2-1\right)e^{\frac{1}{2}k^2}\mathbf{K}\left(k\right)dk=e^{\frac{1}{2}k^2}\left(\mathbf{E}\left(k\right)-\left(1-k^4\right)\mathbf{K}\left(k\right)\right) \tag{208}$$

$$\int\frac{k\left(1-k^2\right)\left(1-9k^2+12k^4-k^6\right)}{\left(3k^2-1\right)^{\frac{37}{18}}}e^{-\frac{1}{6}k^2}\mathbf{K}\left(k\right)dk=$$

$$\left(\frac{\left(1-k^2\right)\left(1-3k^2+k^4\right)}{3k^2-1}\mathbf{K}\left(k\right)+\mathbf{E}\left(k\right)\right)\frac{e^{-\frac{1}{6}k^2}}{\left(3k^2-1\right)^{\frac{1}{18}}} \tag{209}$$

$$\int kk'^2\left(\frac{k^2}{4}-2\right)e^{-\frac{1}{4}k^2}\mathbf{K}\left(k\right)dk=k'^2\left[\left(1-\frac{k^2}{2}\right)\mathbf{K}\left(k\right)-\mathbf{E}\left(k\right)\right] \tag{210}$$

$$\int\frac{k\left(5k^2-4k^4+k^6-1\right)}{k'^3}e^{-\frac{1}{2}k^2}\mathbf{K}\left(k\right)dk=$$

$$\frac{e^{-\frac{1}{2}k^2}}{k'}\left(\left(k^4-k^2+1\right)\mathbf{K}\left(k\right)-\mathbf{E}\left(k\right)\right). \tag{211}$$

5.3 Integrals from fragments of the equation for the complete elliptic integral of the second kind

In equation (2) with $y\left(k\right)$ a solution of equation (155) we can take $h\left(k\right)=1$ and employ the recurrence relation:

$$\frac{d\mathbf{E}\left(k\right)}{dk}=\frac{\mathbf{E}\left(k\right)-\mathbf{K}\left(k\right)}{k} \tag{212}$$

to obtain the integral

$$\int\frac{k\mathbf{E}\left(k\right)}{k'^2}dk=\mathbf{K}\left(k\right)-\mathbf{E}\left(k\right). \tag{213}$$

Using the identity (186) equation (155) can be expressed in the alternative form:

$$z''\left(k\right)+\frac{1}{k}z'\left(k\right)+\left(1+\frac{k^2}{1-k^2}\right)z\left(k\right)=0 \tag{214}$$

and suitable fragmentary equations from equations (155) and (214) are:

$$h''\left(k\right)+\frac{1}{k}h'\left(k\right)=0 \tag{215}$$

$$h''\left(k\right)+\frac{1}{1-k^2}h\left(k\right)=0 \tag{216}$$

$$h''(k)+\frac{1}{k}h'(k)+h=0 \tag{217}$$

$$h''(k)+h=0 \tag{218}$$

$$\frac{1}{k}h'(k)+\frac{1}{1-k^2}h(k)=0 \tag{219}$$

$$\frac{1}{k}h'(k)+\frac{k^2}{1-k^2}h(k)=0 \tag{220}$$

$$\frac{1}{k}h'(k)+h=0. \tag{221}$$

Equations (215)-(221) have the respective (non constant) solutions

$$h(k)=\ln(k) \tag{222}$$

$$h(k)=k'\begin{Bmatrix} P^1_{\varphi-1}(k) \\ Q_{\varphi-1}(k) \end{Bmatrix} \tag{223}$$

$$h(k)=\begin{Bmatrix} J_0(k) \\ Y_0(k) \end{Bmatrix} \tag{224}$$

$$h(k)=\begin{Bmatrix} \sin(k) \\ \cos(k) \end{Bmatrix} \tag{225}$$

$$h(k)=k' \tag{226}$$

$$h(k)=k'e^{\frac{1}{2}k^2} \tag{227}$$

$$h(k)=e^{-\frac{1}{2}k^2} \tag{228}$$

where in equation (223), $\varphi \equiv \left(1+\sqrt{5}\right)/2$ is the Golden Ratio. The solutions (222)-(228) give the respective integrals:

$$\int \frac{k\ln(k)\,\mathbf{E}(k)}{k'^2}dk=(1-\ln(k))\,\mathbf{E}(k)+\ln(k)\,\mathbf{K}(k) \tag{229}$$

$$\int kk'\begin{Bmatrix} P^1_{\varphi-1}(k) \\ Q^1_{\varphi-1}(k) \end{Bmatrix}\mathbf{E}(k)\,dk=$$

$$\left(k'\mathbf{K}(k)+\frac{\left(\varphi k^2-1\right)\mathbf{E}(k)}{k'}\right)\begin{Bmatrix} P^1_{\varphi-1}(k) \\ Q^1_{\varphi-1}(k) \end{Bmatrix}-\frac{(\varphi-1)\,k}{k'}\mathbf{E}(k)\begin{Bmatrix} P^1_{\varphi}(k) \\ Q^1_{\varphi}(k) \end{Bmatrix} \tag{230}$$

$$\int \frac{k^3}{k'^2}J_0(k)\,\mathbf{E}(k)\,dk=J_0(k)\,(\mathbf{K}(k)-\mathbf{E}(k))-kJ_1(k)\,\mathbf{E}(k) \tag{231}$$

$$\int\left(\begin{Bmatrix} \cos(k) \\ -\sin(k) \end{Bmatrix}+\frac{k^3}{k'^2}\begin{Bmatrix} \sin(k) \\ \cos(k) \end{Bmatrix}\right)\mathbf{E}(k)\,dk=$$

$$k\begin{Bmatrix} \cos(k) \\ -\sin(k) \end{Bmatrix}\mathbf{E}(k)-\begin{Bmatrix} \sin(k) \\ \cos(k) \end{Bmatrix}(\mathbf{E}(k)-\mathbf{K}(k)) \tag{232}$$

$$\int \frac{k}{k'^3}\mathbf{E}(k)\,dk = \frac{\mathbf{E}(k)}{k'} - k'\mathbf{K}(k) \tag{233}$$

$$\int kk'\left(1+\frac{k^2\left(k^4+k^2-3\right)}{k'^4}\right)e^{\frac{1}{2}k^2}\mathbf{E}(k)\,dk =$$

$$e^{\frac{1}{2}k^2}\left(k'\mathbf{K}(k) - \left(k' + \frac{k^4}{k'}\right)\mathbf{E}(k)\right) \tag{234}$$

$$\int\left(k'^2 - \left(\frac{k}{k'}\right)^2\right)e^{-\frac{1}{2}k^2}\mathbf{E}(k)\,dk = \left[\mathbf{K}(k) - \left(1+k^2\right)\mathbf{E}(k)\right]e^{-\frac{1}{2}k^2}. \tag{235}$$

5.4 Integrals involving the Golden Ratio

Each transformed differential equation such as (157),(159),(161) and (163) yields its own set of fragmentary equations and their corresponding integrals, similar in number to those givenabove for equations (153) and (155), and too numerous to present here. However, of particular interest are integrals related to equation (230) above, which link Legendre/associated Legendre functions with $\mathbf{E}(k)$, $\mathbf{K}(k)$ and the Golden Ratio φ. Equation (163) above has the associated fragmentary equations:

$$h''(k) - \frac{2k}{1-k^2} + \left(\frac{1}{1-k^2} - \frac{1}{\left(1-k^2\right)^2}\right)h(k) = 0 \tag{236}$$

$$h''(k) - \frac{2k}{1-k^2}h'(k) + \frac{1}{1-k^2}h(k) = 0 \tag{237}$$

which have the respective solutions:

$$h(k) = \begin{Bmatrix} P^1_{\varphi-1}(k) \\ Q^1_{\varphi-1}(k) \end{Bmatrix} \tag{238}$$

$$h(k) = \begin{Bmatrix} P_{\varphi-1}(k) \\ Q_{\varphi-1}(k) \end{Bmatrix} \tag{239}$$

and substituting these results in equation (2) gives the respective integrals:

$$\int \frac{k'}{k^{3/2}}\begin{Bmatrix} P^1_{\varphi-1}(k) \\ Q^1_{\varphi-1}(k) \end{Bmatrix}\mathbf{E}(k)\,dk = \frac{4\sqrt{k}}{k'}\times$$

$$\left[\left(\left[\varphi k - \frac{3-k^2}{2k}\right]\mathbf{E}(k) + \frac{k'}{\sqrt{k}}\mathbf{K}(k)\right)\begin{Bmatrix} P^1_{\varphi-1}(k) \\ Q^1_{\varphi-1}(k) \end{Bmatrix} - (\varphi-1)\,\mathbf{E}(k)\begin{Bmatrix} P^1_{\varphi}(k) \\ Q^1_{\varphi}(k) \end{Bmatrix}\right] \tag{240}$$

$$\int \frac{1-6k^2+k^4}{\left(kk'\right)^{3/2}}\begin{Bmatrix} P_{\varphi-1}(k) \\ Q_{\varphi-1}(k) \end{Bmatrix}\mathbf{E}(k)\,dk = \frac{4\sqrt{k}}{k'}\times$$

$$\left[\left[\left(\phi k - \frac{3-k^2}{2k}\right)\mathbf{E}(k) + \frac{k'^2}{k}\mathbf{K}(k)\right]\begin{Bmatrix} P_{\varphi-1}(k) \\ Q_{\varphi-1}(k) \end{Bmatrix} - \varphi\mathbf{E}(k)\begin{Bmatrix} P_{\varphi}(k) \\ Q_{\varphi}(k) \end{Bmatrix}\right] \tag{241}$$

6 Comments and conclusions

A new Lagrangian method has been presented for deriving indefinite integrals of any function which obeys a second order linear ordinary differential equation. The main result can also be proved very simply without variational calculus. Various approaches have been presented to exploit the main formula and transformation methods have been given to multiply the number of interesting integrals obtainable. Sample results have been presented for some special functions, but these only scratch the surface of what is possible. The total number of indefinite integrals given in current tables and handbooks can be multiplied considerably with the method. All of the results presented here have been numerically checked using Mathematica [12].

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