Loops on spheres having a compact-free inner mapping group

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Abstract

We prove that any topological loop homeomorphic to a sphere or to a real projective space and having a compact-free Lie group as the inner mapping group is homeomorphic to the circle. Moreover, we classify the differentiable 1-dimensional compact loops explicitly using the theory of Fourier series.

2000 *Mathematics Subject Classification:* 22A30, 22E99, 20N05, 57S20, 22F30
Key words and phrases: locally compact loops, differentiable loops, multiplications on spheres

Introduction

The only known proper topological compact connected loops such that the groups G topologically generated by their left translations are locally compact and the stabilizers H of their identities in G have no non-trivial compact subgroups are homeomorphic to the 1-sphere. In [8], [9], [7], [10] it is shown that the differentiable 1-dimensional loops can be classified by pairs of real functions which satisfy a differential inequality containing these functions and their first derivatives. A main goal of this paper is to determine the functions satisfying this inequality explicitly in terms of Fourier series.

If L is a topological loop homeomorphic to a sphere or to a real projective space and having a Lie group G as the group topologically generated by the left translations such that the stabilizer of the identity of L is a compact-free Lie subgroup of G, then L is the 1-sphere and G is isomorphic to a finite covering of the group $PSL_2(\mathbb{R})$ (cf. Theorem 4).

To decide which sections $\sigma : G/H \to G$, where G is a Lie group and H is a (closed) subgroup of G containing no normal subgroup $\neq 1$ of G correspond to loops we use systematically a theorem of R. Baer (cf. [3] and [8], Proposition 1.6, p. 18). This statement says that σ corresponds to a loop if and only if the image $\sigma(G/H)$ is also the image for any section $G/H^a \to G$, where $H = a^{-1}Ha$ and $a \in G$. As one of the applications of this we derive in a different way the differential inequality in [8], p. 218, in which the necessary and sufficient conditions for the existence of 1-dimensional differentiable loops are hidden.

Basic facts in loop theory

A set L with a binary operation $(x, y) \mapsto x * y : L \times L \to L$ and an element $e \in L$ such that $e * x = x * e = x$ for all $x \in L$ is called a loop if for any given $a, b \in L$ the equations $a * y = b$ and $x * a = b$ have unique solutions which we denote by $y = a \backslash b$ and $x = b/a$. Every left translation $\lambda_a : y \mapsto a * y : L \to L$, $a \in L$ is a bijection of L and the set $\Lambda = \{\lambda_a,\ a \in L\}$ generates a group G such that Λ forms a system of representatives for the left cosets $\{xH,\ x \in G\}$, where H is the stabilizer of $e \in L$ in G. Moreover, the elements of Λ act on $G/H = \{xH,\ x \in G\}$ such that for any given cosets aH and bH there exists precisely one left translation λ_z with $\lambda_z aH = bH$.

Conversely, let G be a group, H be a subgroup containing no normal subgroup $\neq 1$ of G and let $\sigma : G/H \to G$ be a section with $\sigma(H) = 1 \in G$ such that the set $\sigma(G/H)$ of representatives for the left cosets of H in G generates G and acts sharply transitively on the space G/H (cf. [8], p. 18). Such a section we call a sharply transitive section. Then the multiplication defined by $xH * yH = \sigma(xH)yH$ on the factor space G/H or by $x * y = \sigma(xyH)$ on $\sigma(G/H)$ yields a loop $L(\sigma)$. The group G is isomorphic to the group generated by the left translations of $L(\sigma)$.

We call the group generated by the mappings $\lambda_{x,y} = \lambda_{xy}^{-1}\lambda_x\lambda_y : L \to L$, for all $x, y \in L$, the inner mapping group of the loop L (cf. [8], Definition 1.30, p. 33). According to Lemma 1.31 in [8], p. 33, this group coincides with the stabilizer H of the identity of L in the group generated by the left translations of L.

A locally compact loop L is almost topological if it is a locally compact space and the multiplication $* : L \times L \to L$ is continuous. Moreover, if the maps $(a, b) \mapsto b/a$ and $(a, b) \mapsto a \backslash b$ are continuous then L is a topological loop. An (almost) topological loop L is connected if and only if the group topologically generated by the left translations is connected. We call the loop L strongly almost topological if the group topologically generated by its left translations is locally compact and the corresponding sharply transitive section $\sigma : G/H \to G$, where H is the stabilizer of $e \in L$ in G, is continuous.

If a loop L is a connected differentiable manifold such that the multiplication $* : L \times L \to L$ is continuously differentiable, then L is an almost $\mathcal{C}^1$-differentiable loop (cf. Definition 1.24 in [8], p. 31). Moreover, if the mappings $(a,b) \mapsto b/a$ and $(a,b) \mapsto a\backslash b$ are also continuously differentiable, then the loop L is a $\mathcal{C}^1$-differentiable loop. If an almost $\mathcal{C}^1$-differentiable loop has a Lie group G as the group topologically generated by its left translations, then the sharply transitive section $\sigma : G/H \to G$ is $\mathcal{C}^1$-differentiable. Conversely, any continuous, respectively $\mathcal{C}^1$-differentiable sharply transitive section $\sigma : G/H \to G$ yields an almost topological, respectively an almost $\mathcal{C}^1$-differentiable loop.

It is known that for any (almost) topological loop L homeomorphic to a connected topological manifold there exists a universal covering loop $\tilde{L}$ such that the covering mapping $p : \tilde{L} \to L$ is an epimorphism. The inverse image $p^{-1}(e) = \mathrm{Ker}(p)$ of the identity element e of L is a central discrete subgroup Z of $\tilde{L}$ and it is naturally isomorphic to the fundamental group of L. If Z' is a subgroup of Z, then the factor loop $\tilde{L}/Z'$ is a covering loop of L and any covering loop of L is isomorphic to a factor loop $\tilde{L}/Z'$ with a suitable subgroup Z' (see [5]).

If L' is a covering loop of L, then Lemma 1.34 in [8], p. 33, clarifies the relation between the group topologically generated by the left translations of L' and the group topologically generated by the left translations of L:

Let L be a topological loop homeomorphic to a connected topological manifold. Let the group G topologically generated by the left translations λ_a, $a \in L$, of L be a Lie group. Let $\tilde{L}$ be the universal covering of L and $Z \subseteq \tilde{L}$ be the fundamental group of L. Then the group $\tilde{G}$ topologically generated by the left translations $\tilde{\lambda}_u, u \in \tilde{L}$, of $\tilde{L}$ is the covering group of G such that the kernel of the covering mapping $\varphi : \tilde{G} \to G$ is $Z^* = \{\tilde{\lambda}_z, z \in Z\}$ and Z^* is isomorphic to Z. If we identify $\tilde{L}$ and L with the homogeneous spaces $\tilde{G}/\tilde{H}$ and G/H, where H or $\tilde{H}$ is the stabilizer of the identity of L in G or of $\tilde{L}$ in $\tilde{G}$, respectively, then $\varphi(\tilde{H}) = H$, $\tilde{H} \cap Z^* = \{1\}$, and $\tilde{H}$ is isomorphic to H.

Compact topological loops on the 3-dimensional sphere

Proposition 1. *There is no almost topological proper loop L homeomorphic to the 3-sphere $\mathcal{S}_3$ or to the 3-dimensional real projective space $\mathcal{P}_3$ such that the group G topologically generated by the left translations of L is isomorphic to the group $SL_2(\mathbb{C})$ or to the group $PSL_2(\mathbb{C})$, respectively.*

Proof. We assume that there is an almost topological loop L homeomorphic to $\mathcal{S}_3$ such that the group topologically generated by its left translations is isomorphic to $G = SL_2(\mathbb{C})$. Then there exists a continuous sharply transitive section $\sigma : SL_2(\mathbb{C})/H \to SL_2(\mathbb{C})$, where H is a connected compact-free 3-dimensional subgroup of $SL_2(\mathbb{C})$. According to [2], pp. 273-278, there is a one-parameter family of connected compact-free 3-dimensional subgroups H_r, $r \in \mathbb{R}$ of $SL_2(\mathbb{C})$ such that H_{r_1} is conjugate to H_{r_2} precisely if $r_1 = r_2$. Hence we may assume that the stabilizer H has one of the folowing shapes

$$H_r = \left\{ \begin{pmatrix} \exp[(ri-1)a] & b \\ 0 & \exp[(1-ri)a] \end{pmatrix}; a \in \mathbb{R}, b \in \mathbb{C} \right\}, \quad r \in \mathbb{R},$$

(cf. Theorem 1.11 in [8], p. 21). For each $r \in \mathbb{R}$ the section $\sigma_r : G/H_r \to G$ corresponding to a loop L_r is given by

$$\begin{pmatrix} x & y \\ -\bar{y} & \bar{x} \end{pmatrix} H_r \mapsto \begin{pmatrix} x & y \\ -\bar{y} & \bar{x} \end{pmatrix} \begin{pmatrix} \exp[(ri-1)f(x,y)] & g(x,y) \\ 0 & \exp[(1-ri)f(x,y)] \end{pmatrix},$$

where $x, y \in \mathbb{C}$, $x\bar{x} + y\bar{y} = 1$ such that $f(x,y) : S^3 \to \mathbb{R}$, $g(x,y) : S^3 \to \mathbb{C}$ are continuous functions with $f(1,0) = 0 = g(1,0)$. Since σ_r is a sharply transitive action for each $r \in \mathbb{R}$ the image $\sigma_r(G/H_r)$ forms a system of representatives for all cosets xH_r^{γ}, $\gamma \in G$. This means for all given $c, d \in \mathbb{C}^2$, $c\bar{c} + d\bar{d} = 1$ each coset

$$\begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix} \begin{pmatrix} c & d \\ -\bar{d} & \bar{c} \end{pmatrix} H_r \begin{pmatrix} \bar{c} & -d \\ \bar{d} & c \end{pmatrix},$$

where $u, v \in \mathbb{C}$, $u\bar{u} + v\bar{v} = 1$, contains precisely one element of $\sigma_r(G/H_r)$. This is the case if and only if for all given $c, d, u, v \in \mathbb{C}$ with $u\bar{u} + v\bar{v} = 1 = c\bar{c} + d\bar{d}$ there exists a unique triple $(x, y, q) \in \mathbb{C}^3$ with $x\bar{x} + y\bar{y} = 1$ and a real number m such that the following matrix equation holds:

$$\begin{pmatrix} \bar{u}\bar{c} - \bar{v}d & -ud - v\bar{c} \\ \bar{v}c + \bar{u}\bar{d} & uc - v\bar{d} \end{pmatrix} \begin{pmatrix} x & y \\ -\bar{y} & \bar{x} \end{pmatrix} \begin{pmatrix} \exp[(ri-1)f(x,y)] & g(x,y) \\ 0 & \exp[(1-ri)f(x,y)] \end{pmatrix}$$

$$= \begin{pmatrix} \exp[(ri-1)m] & q \\ 0 & \exp[(1-ri)m] \end{pmatrix} \begin{pmatrix} \bar{c} & -d \\ \bar{d} & c \end{pmatrix}. \tag{1}$$

The (1,1)- and (2,1)-entry of the matrix equation (1) give the following system A of equations:

$$[(\bar{u}x + v\bar{y})\bar{c} + (u\bar{y} - \bar{v}x)d] \exp[(ri-1)f(x,y)] = \exp[(ri-1)m]\bar{c} + q\bar{d} \tag{2}$$

$$[(\bar{v}x - u\bar{y})c + (\bar{u}x + v\bar{y})\bar{d}] \exp[(ri-1)f(x,y)] = \exp[(1-ri)m]\bar{d}. \tag{3}$$

If we take c and d as independent variables the system A yields the following system B of equations:

$$(\bar{u}x + v\bar{y}) \exp[irf(x,y)] \exp[-f(x,y)] = \exp(irm)\exp(-m) \tag{4}$$

$$(u\bar{y} - \bar{v}x)\exp[(ri-1)f(x,y)]d = \bar{d}q \tag{5}$$

$$(\bar{u}x + v\bar{y})\exp[irf(x,y)]\exp[-f(x,y)] = \exp(m)\exp(-irm). \tag{6}$$

Since equation (5) must be satisfied for all $d \in \mathbb{C}$ we obtain $q = 0$. From equation (4) it follows

$$\bar{u}x + v\bar{y} = \exp(irm)\exp(-m)\exp[-irf(x,y)]\exp[f(x,y)]. \tag{7}$$

Putting (7) into (6) one obtains

$$\exp(irm)\exp(-m) = \exp(m)\exp(-irm) \tag{8}$$

which is equivalent to

$$\exp[2(ir-1)m] = 1. \tag{9}$$

The equation (9) is satisfied if and only if $m = 0$. Hence the matrix equation (1) reduces to the matrix equation

$$\begin{pmatrix} x & y \\ -\bar{y} & \bar{x} \end{pmatrix} \begin{pmatrix} \exp[(ri-1)f(x,y)] & g(x,y) \\ 0 & \exp[(1-ri)f(x,y)] \end{pmatrix} = \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix}.$$

and therefore the matrix

$$M = \begin{pmatrix} \exp[(ri-1)f(x,y)] & g(x,y) \\ 0 & \exp[(1-ri)f(x,y)] \end{pmatrix}$$

is an element of $SU_2(\mathbb{C})$. This is the case if and only if $f(x,y) = 0 = g(x,y)$ for all $(x,y) \in \mathbb{C}^2$ with $x\bar{x} + y\bar{y} = 1$. Since for each $r \in \mathbb{R}$ the loop L_r is isomorphic to the loop $L_r(\sigma_r)$, hence to the group $SU_2(\mathbb{C})$, there is no connected almost topological proper loop L homeomorphic to $\mathcal{S}_3$ such that the group topologically generated by its left translations is isomorphic to the group $SL_2(\mathbb{C})$.

The universal covering of an almost topological proper loop L homeomorphic to the real projective space $\mathcal{P}_3$ is an almost topological proper loop $\tilde{L}$ homeomorphic to $\mathcal{S}_3$. If the group topologically generated by the left translations of L is isomorphic to $PSL_2(\mathbb{C})$ then the group topologically generated by the left translations of $\tilde{L}$ is isomorphic to $SL_2(\mathbb{C})$. Since no proper loop $\tilde{L}$ exists the Proposition is proved. □

Proposition 2. *There is no almost topological proper loop L homeomorphic to the 3-dimensional real projective space $\mathcal{P}_3$ or to the 3-sphere $\mathcal{S}_3$ such that the group G topologically generated by the left translations of L is isomorphic to the group $SL_3(\mathbb{R})$ or to the universal covering group $\widetilde{SL_3(\mathbb{R})}$, respectively.*

Proof. First we assume that there exists an almost topological loop L homeomorphic to $\mathcal{P}_3$ such that the group topologically generated by its left translations is isomorphic to $G = SL_3(\mathbb{R})$. Then there is a continuous sharply transitive section $\sigma : SL_3(\mathbb{R})/H \to SL_3(\mathbb{R})$, where H is a connected compact-free 5-dimensional subgroup of $SL_3(\mathbb{R})$. According to Theorem 2.7, p. 187, in [4] and to Theorem 1.11, p. 21, in [8] we may assume that

$$H = \left\{ \begin{pmatrix} a & k & v \\ 0 & b & l \\ 0 & 0 & (ab)^{-1} \end{pmatrix}; a > 0, b > 0, k, l, v \in \mathbb{R} \right\}. \tag{10}$$

Using Euler angles every element of $SO_3(\mathbb{R})$ can be represented by the following matrix

$$g(t,u,z) := \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos z & \sin z \\ 0 & -\sin z & \cos z \end{pmatrix} \begin{pmatrix} \cos u & \sin u & 0 \\ -\sin u & \cos u & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \cos t\ \cos u - \sin t\ \cos z\ \sin u & \cos t\ \sin u + \sin t\ \cos z\ \cos u & \sin t\ \sin z \\ -\sin t\ \cos u - \cos t\ \cos z\ \sin u & -\sin t\ \sin u + \cos t\ \cos z\ \cos u & \cos t\ \sin z \\ \sin z\ \sin u & -\sin z\ \cos u & \cos z \end{pmatrix},$$

where $t, u \in [0, 2\pi]$ and $z \in [0, \pi]$.

The section $\sigma : SL_3(\mathbb{R})/H \to SL_3(\mathbb{R})$ is given by

$$g(t,u,z)H \mapsto g(t,u,z) \begin{pmatrix} f_1(t,u,z) & f_2(t,u,z) & f_3(t,u,z) \\ 0 & f_4(t,u,z) & f_5(t,u,z) \\ 0 & 0 & f_1^{-1}(t,u,z) f_4^{-1}(t,u,z) \end{pmatrix}, \tag{11}$$

where $t, u \in [0, 2\pi]$, $z \in [0, \pi]$ and $f_i(t,u,z) : [0, 2\pi] \times [0, 2\pi] \times [0, \pi] \to \mathbb{R}$ are continuous functions such that for $i \in \{1, 4\}$ the functions f_i are positive with $f_i(0,0,0) = 1$ and for $j = \{2, 3, 5\}$ the functions $f_j(t,u,z)$ satisfy that $f_j(0,0,0) = 0$. As σ is sharply transitive the image $\sigma(SL_3(\mathbb{R})/H)$ forms a system of representatives for all cosets xH^δ, $\delta \in SL_3(\mathbb{R})$. Since the elements x and δ can be chosen in the group $SO_3(\mathbb{R})$ we may take x as the matrix

$$\begin{pmatrix} \cos q\ \cos r - \sin q\ \sin r\ \cos p & \cos q\ \sin r + \sin q\ \cos r\ \cos p & \sin q\ \sin p \\ -\sin q\ \cos r - \cos q\ \sin r\ \cos p & -\sin q\ \sin r + \cos q\ \cos r\ \cos p & \cos q\ \sin p \\ \sin p\ \sin r & -\sin p\ \cos r & \cos p \end{pmatrix}$$

and δ as the matrix

$$\begin{pmatrix} \cos\alpha\ \cos\beta - \sin\alpha\ \sin\beta\ \cos\gamma & \cos\alpha\ \sin\beta + \sin\alpha\ \cos\beta\ \cos\gamma & \sin\alpha\ \sin\gamma \\ -\sin\alpha\ \cos\beta - \cos\alpha\ \sin\beta\ \cos\gamma & -\sin\alpha\ \sin\beta + \cos\alpha\ \cos\beta\ \cos\gamma & \cos\alpha\ \sin\gamma \\ \sin\gamma\ \sin\beta & -\sin\gamma\ \cos\beta & \cos\gamma \end{pmatrix},$$

where $q, r, \alpha, \beta \in [0, 2\pi]$ and $p, \gamma \in [0, \pi]$. The image $\sigma(SL_3(\mathbb{R})/H)$ forms for all given $\delta \in SO_3(\mathbb{R})$ and $x \in SO_3(\mathbb{R})$ a system of representatives for the cosets xH^δ if and only if there exists unique angles $t,u\in[0,2\pi]$ and $z\in[0,\pi]$ and unique positive real numbers a,b as well as unique real numbers k,l,v such that the following equation holds

$$\delta x^{-1}g(t,u,z)f=h\delta, \tag{12}$$

where the matrices δ,x have the form as above,

$$f=\begin{pmatrix} f_1(t,u,z) & f_2(t,u,z) & f_3(t,u,z) \\ 0 & f_4(t,u,z) & f_5(t,u,z) \\ 0 & 0 & f_1^{-1}(t,u,z)f_4^{-1}(t,u,z) \end{pmatrix}$$

and

$$h=\begin{pmatrix} a & k & v \\ 0 & b & l \\ 0 & 0 & (ab)^{-1} \end{pmatrix}.$$

Comparing the first column of the left and the right side of the equation (12) we obtain the following three equations:

$f_1(t,u,z)\{[(\cos\alpha\ \cos\beta-\sin\alpha\ \sin\beta\ \cos\gamma)(\cos r\ \cos q-\sin r\ \sin q\ \cos p)+$
$(\cos\alpha\ \sin\beta+\sin\alpha\ \cos\beta\ \cos\gamma)(\sin r\ \cos q+\cos r\ \sin q\ \cos p)+$
$\sin\alpha\sin\gamma\sin p\sin q](\cos t\ \cos u-\sin t\ \sin u\ \cos z)-$
$[-(\cos\alpha\ \cos\beta-\sin\alpha\ \sin\beta\ \cos\gamma)(\cos r\ \sin q+\sin r\ \cos q\ \cos p)+$
$(\cos\alpha\ \sin\beta+\sin\alpha\ \cos\beta\ \cos\gamma)(-\sin r\ \sin q+\cos r\ \cos q\ \cos p)+$
$\sin\alpha\sin\gamma\sin p\cos q](\sin t\ \cos u+\cos t\ \sin u\ \cos z)+$
$[(\cos\alpha\ \cos\beta-\sin\alpha\ \sin\beta\ \cos\gamma)\sin r\ \sin p-$
$(\cos\alpha\ \sin\beta+\sin\alpha\ \cos\beta\ \cos\gamma)\cos r\sin p+\sin\alpha\ \sin\gamma\ \cos p]\sin z\ \sin u\}=$
$a(\cos\alpha\ \cos\beta-\sin\alpha\ \sin\beta\ \cos\gamma)-k(\sin\alpha\ \cos\beta+\cos\alpha\ \sin\beta\ \cos\gamma)+$
$v\sin\gamma\ \sin\beta,$

$f_1(t,u,z)\{[-(\sin\alpha\ \cos\beta+\cos\alpha\ \sin\beta\ \cos\gamma)(\cos r\ \cos q-\sin r\ \sin q\ \cos p)-$
$(-\sin\alpha\ \sin\beta+\cos\alpha\ \cos\beta\ \cos\gamma)(\sin r\ \cos q+\cos r\ \sin q\ \cos p)+$
$\cos\alpha\sin\gamma\sin p\ \sin q](\cos t\ \cos u-\sin t\ \sin u\ \cos z)-$
$[(\sin\alpha\ \cos\beta+\cos\alpha\ \sin\beta\ \cos\gamma)(\cos r\ \sin q+\sin r\ \cos q\ \cos p)+$
$(-\sin\alpha\ \sin\beta+\cos\alpha\ \cos\beta\ \cos\gamma)(-\sin r\ \sin q+\cos r\ \cos q\ \cos p)+$
$\cos\alpha\sin\gamma\sin p\cos q](\sin t\ \cos u+\cos t\ \sin u\ \cos z)+$
$[-(\sin\alpha\ \cos\beta+\cos\alpha\ \sin\beta\ \cos\gamma)\sin r\ \sin p-(\cos\alpha\ \cos\beta\ \cos\gamma-\sin\alpha\ \sin\beta)$
$\cos r\sin p+\cos\alpha\ \sin\gamma\ \cos p]\sin z\ \sin u\}=$
$-b(\sin\alpha\ \cos\beta+\cos\alpha\ \sin\beta\ \cos\gamma)+l\sin\gamma\ \sin\beta,$

$f_1(t,u,z)\{[(\cos r\ \cos q-\sin r\ \sin q\ \cos p)\sin\gamma\ \sin\beta-$
$(\sin r\ \cos q+\cos r\ \sin q\ \cos p)\sin\gamma\ \cos\beta+\cos\gamma\sin p\sin q]$
$(\cos t\ \cos u-\sin t\ \sin u\ \cos z)+[(\cos r\ \sin q+\sin r\ \cos q\ \cos p)\sin\gamma\ \sin\beta+$
$(-\sin r\ \sin q+\cos r\ \cos q\ \cos p)\sin\gamma\ \cos\beta+\cos\gamma\sin p\cos q]$

$(\sin t \ \cos u + \cos t \ \sin u \ \cos z)+$
$[(\sin\gamma \ \sin\beta \ \sin r \ \sin p + \sin\gamma \ \cos\beta \ \cos r \ \sin p) + \cos\gamma \cos p] \sin z \ \sin u\} =$
$(ab)^{-1} \sin\gamma \ \sin\beta.$

If we take $\sin\gamma \ \sin\beta$ and $\cos\gamma$ as independent variables the third equation turns to the following equations

$$\begin{aligned} 0 &= f_1(t,u,z)[\sin p \ \sin q(\cos t \ \cos u - \sin t \ \sin u \ \cos z) - \\ & \sin p \ \cos q(\sin t \ \cos u + \cos t \ \sin u \ \cos z) + \cos p \ \sin z \ \sin u] \qquad (13) \\ (ab)^{-1} &= \{[(\cos r \ \cos q - \sin r \ \sin q \ \cos p)(\cos t \ \cos u - \sin t \ \sin u \ \cos z) + \\ & (\cos r \ \sin q + \sin r \ \cos q \ \cos p)(\sin t \ \cos u + \cos t \ \sin u \ \cos z) + \\ & \sin r \ \sin p \ \sin z \ \sin u] - \\ & \frac{\cos\beta}{\sin\beta}[(\sin r \ \cos q + \cos r \ \sin q \ \cos p)(\cos t \ \cos u - \sin t \ \sin u \ \cos z) - \\ & (-\sin r \ \sin q + \cos r \ \cos q \ \cos p)(\sin t \ \cos u + \cos t \ \sin u \ \cos z) - \\ & \cos r \ \sin p \ \sin z \ \sin u]\} f_1(t,u,z). \qquad (14) \end{aligned}$$

If we take $\cos\alpha \ \sin\beta \ \cos\gamma$, $\sin\beta \ \sin\gamma$ as independent variables from the second equation it follows

$$\begin{aligned} l &= \frac{\cos\alpha}{\sin\beta} f_1(t,u,z)[\sin p \ \sin q(\cos t \ \cos u - \sin t \ \sin u \ \cos z) - \\ & \sin p \ \cos q(\sin t \ \cos u + \cos t \ \sin u \ \cos z) + \cos p \ \sin z \ \sin u] \qquad (15) \\ -b &= \{[-(\cos r \ \cos q - \sin r \ \sin q \ \cos p)(\cos t \ \cos u - \sin t \ \sin u \ \cos z) - \\ & (\cos r \ \sin q + \sin r \ \cos q \ \cos p)(\sin t \ \cos u + \cos t \ \sin u \ \cos z) - \\ & \sin r \ \sin p \ \sin z \ \sin u] - \\ & \frac{\cos\beta}{\sin\beta}[(\sin r \ \cos q + \cos r \ \sin q \ \cos p)(\cos t \ \cos u - \sin t \ \sin u \ \cos z) - \\ & (-\sin r \ \sin q + \cos r \ \cos q \ \cos p)(\sin t \ \cos u + \cos t \ \sin u \ \cos z) - \\ & \cos r \ \sin p \ \sin z \ \sin u]\} f_1(t,u,z). \qquad (16) \end{aligned}$$

If we choose $\sin\alpha \ \sin\beta \ \cos\gamma$, $\sin\beta \ \sin\gamma$ as independent variables the first equation yields

$$\begin{aligned} v &= \frac{\sin\alpha}{\sin\beta} f_1(t,u,z)[\sin p \ \sin q(\cos t \ \cos u - \sin t \ \sin u \ \cos z) - \\ & \sin p \ \cos q(\sin t \ \cos u + \cos t \ \sin u \ \cos z) + \cos p \ \sin z \ \sin u] \qquad (17) \end{aligned}$$

$$\begin{aligned} a + k\frac{\cos\alpha}{\sin\alpha} &= \{[(\cos r \ \cos q - \sin r \ \sin q \ \cos p)(\cos t \ \cos u - \sin t \ \sin u \ \cos z) - \\ & (\cos r \ \sin q + \sin r \ \cos q \ \cos p)(\sin t \ \cos u + \cos t \ \sin u \ \cos z) + \\ & \sin r \ \sin p \ \sin z \ \sin u] - \\ & \frac{\cos\beta}{\sin\beta}[(\sin r \ \cos q + \cos r \ \sin q \ \cos p)(\cos t \ \cos u - \sin t \ \sin u \ \cos z) - \\ & (-\sin r \ \sin q + \cos r \ \cos q \ \cos p)(\sin t \ \cos u + \cos t \ \sin u \ \cos z) - \\ & \cos r \ \sin p \ \sin z \ \sin u]\} f_1(t,u,z). \qquad (18) \end{aligned}$$

Since $f_1(t,u,z) > 0$ from equation (13) it follows that

$$\begin{aligned} 0 &= \sin p\ \sin q(\cos t\ \cos u - \sin t\ \sin u\ \cos z) + \\ &\quad \sin p\ \cos q(\sin t\ \cos u + \cos t\ \sin u\ \cos z) + \cos p\ \sin z\ \sin u. \quad (19) \end{aligned}$$

Using this it follows from (15) that $l = 0$ holds and from equation (17) that $v = 0$. Since the equation (14) must be satisfied for all $\beta \in [0, 2\pi]$ we have

$$\begin{aligned} (ab)^{-1} &= [(\cos r\ \cos q - \sin r\ \sin q\ \cos p)(\cos t\ \cos u - \sin t\ \sin u\ \cos z) + \\ &\quad (\cos r\ \sin q + \sin r\ \cos q\ \cos p)(\sin t\ \cos u + \cos t\ \sin u\ \cos z) + \\ &\quad \sin r\ \sin p\ \sin z\ \sin u] f_1(t,u,z) \quad (20) \\ 0 &= [(\sin r\ \cos q + \cos r\ \sin q\ \cos p)(\cos t\ \cos u - \sin t\ \sin u\ \cos z) - \\ &\quad (-\sin r\ \sin q + \cos r\ \cos q\ \cos p)(\sin t\ \cos u + \cos t\ \sin u\ \cos z) - \\ &\quad \cos r\ \sin p\ \sin z\ \sin u]. \quad (21) \end{aligned}$$

Using equation (21) and comparing the equations (20) and (16) we obtain that $(ab)^{-1} = b$. With equation (21) the equation (18) turns to

$$\begin{aligned} a + k\frac{\cos\alpha}{\sin\alpha} &= [(\cos r\ \cos q - \sin r\ \sin q\ \cos p)(\cos t\ \cos u - \sin t\ \sin u\ \cos z) - \\ &\quad (\cos r\ \sin q + \sin r\ \cos q\ \cos p)(\sin t\ \cos u + \cos t\ \sin u\ \cos z) + \\ &\quad \sin r\ \sin p\ \sin z\ \sin u] f_1(t,u,z). \quad (22) \end{aligned}$$

Since the equation (22) must be satisfied for all $\alpha \in [0, 2\pi]$ we obtain $k = 0$. Using this, the equations (22) and (20) yield $(ab)^{-1} = a$. Since $1 = ab(ab)^{-1} = a^3$ it follows that $a = 1$ and hence the matrix h is the identity. But then the matrix equation (12) turns to the matrix equation

$$g(t,u,z)f = x.$$

As x and $g(t,u,z)$ are elements of $SO_3(\mathbb{R})$ one has $f = xg^{-1}(t,u,z) \in SO_3(\mathbb{R})$. But then f is the identity, which means that

$$f_1(t,u,z) = 1 = f_4(t,u,z), \quad f_2(t,u,z) = f_3(t,u,z) = f_5(t,u,z) = 0,$$

for all $t, u \in [0, 2\pi]$ and $z \in [0, \pi]$. Since the loop L is isomorphic to the loop $L(\sigma)$ and $L(\sigma) \cong SO_3(\mathbb{R})$ there is no connected almost topological proper loop L homeomorphic to $\mathcal{P}_3$ such that the group topologically generated by its left translations is isomorphic to $SL_3(\mathbb{R})$.

Now we assume that there is an almost topological loop L homeomorphic to $\mathcal{S}_3$ such that the group G topologically generated by its left translations is isomorphic to the universal covering group $\widetilde{SL_3(\mathbb{R})}$. Then the stabilizer H of the identity of L may be chosen as the group (10). Then there exists a local section $\sigma : U/H \to G$, where U is a suitable neighbourhood of H in G/H which has the shape (11) with sufficiently small $t, u \in [0, 2\pi]$, $z \in [0, \pi]$ and continuous functions $f_i(t,u,z)$:

$[0, 2\pi] \times [0, 2\pi] \times [0, \pi] \to \mathbb{R}$ satisfying the same conditions as there. The image $\sigma(U/H)$ is a local section for the space of the left cosets $\{xH^\delta;\ x \in G, \delta \in G\}$ precisely if for all suitable matrices $x := g(q, r, p)$ with sufficiently small $(q, r, p) \in [0, 2\pi] \times [0, 2\pi] \times [0, \pi]$ there exist a unique element $g(t, u, z) \in Spin_3(\mathbb{R})$ with sufficiently small $(t, u, z) \in [0, 2\pi] \times [0, 2\pi] \times [0, \pi]$ and unique positive real numbers a, b as well as unique real numbers k, l, v such that the matrix equation (12) holds. Then we see as in the case of the group $SL_3(\mathbb{R})$ that for small x and $g(t, u, z)$ the matrix f is the identity. Therefore any subloop T of L which is homeomorphic to $\mathcal{S}_1$ is locally commutative. Then according to [8], Corollary 18.19, p. 248, each subloop T is isomorphic to a 1-dimensional torus group. It follows that the restriction of the matrix f to T is the identity. Since L is covered by such 1-dimensional tori the matrix f is the identity for all elements of $\mathcal{S}_3$. Hence there is no proper loop L homeomorphic to $\mathcal{S}_3$ such that the group G topologically generated by its left translations is isomorphic to the universal covering group $\widetilde{SL_3(\mathbb{R})}$. □

Compact loops with compact-free inner mapping groups

Proposition 3. *Let L be an almost topological loop homeomorphic to a compact connected Lie group K. Then the group G topologically generated by the left translations of L cannot be isomorphic to a split extension of a solvable group R homeomorphic to $\mathbb{R}^n$ $(n \geq 1)$ by the group K.*

Proof. Denote by H the stabilizer of the identity of L in G. If G has the structure as in the assertion then the elements of G can be represented by the pairs (k, r) with $k \in K$ and $r \in R$. Since L is homeomorphic to K the loop L is isomorphic to the loop $L(\sigma)$ given by a sharply transitive section $\sigma : G/H \to G$ the image of which is the set $\mathfrak{S} = \{(k, f(k));\ k \in K\}$, where f is a continuous function from K into R with $f(1) = 1 \in R$. The multiplication of $(L(\sigma), *)$ on $\mathfrak{S}$ is given by $(x, f(x)) * (y, f(y)) = \sigma((xy, f(x)f(y))H)$.

Let T be a 1-dimensional torus of K. Then the set $\{(t, f(t));\ t \in T\}$ topologically generates a compact subloop $\tilde{T}$ of $L(\sigma)$ such that the group topologically generated by its left translations has the shape TU with $T \cap U = 1$, where U is a normal solvable subgroup of TU homeomorphic to $\mathbb{R}^n$ for some $n \geq 1$. The multiplication $*$ in the subloop $\tilde{T}$ is given by

$$(x, f(x)) * (y, f(y)) = \sigma((xy, f(x)f(y))H) = (xy, f(xy)),$$

where $x, y \in T$. Hence $\tilde{T}$ is a subloop homeomorphic to a 1-sphere which has a solvable Lie group S as the group topologically generated by the left translations. It follows that $\tilde{T}$ is a 1-dimensional torus group since otherwise the group S would be not solvable (cf. [8], Proposition 18.2, p. 235). As $f : \tilde{T} \to U$ is a homomorphism and U is homeomorphic to $\mathbb{R}^n$ it follows that the restriction of f to $\tilde{T}$ is the constant function $f(\tilde{T}) = 1$. Since the exponential map of a compact group is surjective any element of K is contained in a one-parameter subgroup of K. It follows $f(K) = 1$ and L is the group K which is a contradiction. □

Theorem 4. *Let L be an almost topological proper loop homeomorphic to a sphere or to a real projective space. If the group G topologically generated by the left translations of L is a Lie group and the stabilizer H of the identity of L in G is a compact-free subgroup of G, then L is homeomorphic to the 1-sphere and G is a finite covering of the group $PSL_2(\mathbb{R})$.*

Proof. If dim $L = 1$ then according to Brouwer's theorem (cf. [11], 96.30, p. 639) the transitive group G on S_1 is a finite covering of $PSL_2(\mathbb{R})$.

Now let dim $L > 1$. Since the universal covering of the n-dimensional real projective space is the n-sphere $\mathcal{S}_n$ we may assume that L is homeomorphic to $\mathcal{S}_n$, $n \geq 2$. Since L is a multiplication with identity e on S_n one has $n \in \{3, 7\}$ (cf. [1]).

Any maximal compact subgroup K of G acts transitively on L (cf. [11], 96.19, p. 636). As $H \cap K = \{1\}$ the group K operates sharply transitively on L. Since there is no compact group acting sharply transitively on the 7-sphere (cf. [11], 96.21, p. 637), the loop L is homeomorphic to the 3-sphere. The only compact group homeomorphic to the 3-sphere is the unitary group $SU_2(\mathbb{C})$. If the group G were not simple, then G would be a semidirect product of the at most 3-dimensional solvable radical R with the group $SU_2(\mathbb{C})$ (cf. [4], p. 187 and Theorem 2.1, p. 180). But according to Proposition 3 such a group cannot be the group topologically generated by the left translations of L. Hence G is a non-compact Lie group the Lie algebra of which is simple. But then G is isomorphic either to the group $SL_2(\mathbb{C})$ or to the universal covering of the group $SL_3(\mathbb{R})$. It follows from Proposition 1 and 2 that no of these groups can be the group topologically generated by the left translations of an almost topological proper loop L. □

The classification of 1-dimensional compact connected $\mathcal{C}^1$-loops

If L is a connected strongly almost topological 1-dimensional compact loop, then L is homeomorphic to the 1-sphere and the group topologically generated by its left translations is a finite covering of the group $PSL_2(\mathbb{R})$

(cf. Proposition 18.2 in [8], p. 235). We want to classify explicitly all 1-dimensional $\mathcal{C}^1$-differentiable compact connected loops which have either the group $PSL_2(\mathbb{R})$ or $SL_2(\mathbb{R})$ as the group topologically generated by the left translations.

First we classify the 1-dimensional compact connected loops having $G = SL_2(\mathbb{R})$ as the group topologically generated by their left translations. Since the stabilizer H is compact free and may be chosen as the group of upper triangular matrices (see Theorem 1.11, in [8], p. 21) this is equivalent to the classification of all loops $L(\sigma)$ belonging to the sharply transitive $\mathcal{C}^1$-differentiable sections

$$\begin{aligned}\sigma \quad : \quad & \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}; a > 0, b \in \mathbb{R} \right\} \to \\ & \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} f(t) & g(t) \\ 0 & f^{-1}(t) \end{pmatrix} \text{with} \quad t \in \mathbb{R}. \end{aligned} \tag{23}$$

Definition 1. *Let $\mathcal{F}$ be the set of series*

$$a_0 + \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt), \quad t \in \mathbb{R},$$

such that

$$1 - a_0 = \sum_{k=1}^{\infty} \frac{a_k + k b_k}{1 + k^2},$$

$$a_0 > \sum_{k=1}^{\infty} \frac{k a_k - b_k}{1 + k^2} \sin kt - \frac{a_k + k b_k}{1 + k^2} \cos kt \quad \textit{for all} \quad t \in [0, 2\pi],$$

$$2a_0 \geq \sum_{k=1}^{\infty} (a_k^2 + b_k^2) \frac{k^2 - 1}{k^2 + 1}.$$

Lemma 5. *The set $\mathcal{F}$ consists of Fourier series of continuous functions.*

Proof. Since $\sum\limits_{k=2}^{\infty} a_k^2 + b_k^2 < \frac{10}{3} a_0$ it follows from [14], p. 4, that any series in $\mathcal{F}$ converges uniformly to a continuous function f and hence it is the Fourier series of f (cf. [14], Theorem 6.3, p. 12). □

Let σ be a sharply transitive section of the shape (23). Then $f(t)$, $g(t)$ are periodic continuously differentiable functions $\mathbb{R} \to \mathbb{R}$, such that $f(t)$ is strictly positive with $f(2k\pi) = 1$ and $g(2k\pi) = 0$ for all $k \in \mathbb{Z}$.

As σ is sharply transitive the image $\sigma(G/H)$ forms a system of representatives for the cosets xH^{ρ} for all $\rho \in G$ (cf. [3]). All conjugate groups H^{ρ} can be already obtained if ρ is an element of $K = \left\{ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, t \in \mathbb{R} \right\}$. Since $K^{\kappa}H^{\kappa} = KH^{\kappa}$ for any $\kappa \in K$ the group K forms a system of representatives for the left cosets xH^{κ}.

We want to determine the left coset $x(t)H^{\kappa}$ containing the element

$$\varphi(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} f(t) & g(t) \\ 0 & f^{-1}(t) \end{pmatrix},$$

where $\kappa = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix}$ and $x(t) = \begin{pmatrix} \cos\ \eta(t) & \sin\ \eta(t) \\ -\sin\ \eta(t) & \cos\ \eta(t) \end{pmatrix}$. The element $\varphi(t)$ lies in the left coset $x(t)H^{\kappa}$ if and only if $\varphi(t)^{\kappa^{-1}} \in x(t)^{\kappa^{-1}}H = x(t)H$. Hence we have to solve the following matrix equation

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \left[\kappa \begin{pmatrix} f(t) & g(t) \\ 0 & f^{-1}(t) \end{pmatrix} \kappa^{-1} \right] = \begin{pmatrix} \cos\ \eta(t) & \sin\ \eta(t) \\ -\sin\ \eta(t) & \cos\ \eta(t) \end{pmatrix} \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \tag{24}$$

for suitable $a > 0, b \in \mathbb{R}$. Comparing both sides of the matrix equation (24) we have

$$f(t)\cos\beta(\sin t\cos\beta - \cos t\sin\beta) - g(t)\sin\beta(\sin t\cos\beta - \cos t\sin\beta) + f(t)^{-1}\sin\beta(\sin t\sin\beta + \cos t\cos\beta) = \sin\eta(t)a$$

and

$$f(t)\cos\beta(\cos t\cos\beta + \sin t\sin\beta) - g(t)\sin\beta(\cos t\cos\beta + \sin t\sin\beta) + f(t)^{-1}\sin\beta(\cos t\sin\beta - \sin t\cos\beta) = \cos\eta(t)a.$$

From this it follows

$$\tan\eta_{\beta}(t) = \frac{(f(t) - g(t)\tan\beta)(\tan t - \tan\beta) + f^{-1}(t)\tan\beta(1 + \tan t\tan\beta)}{(f(t) - g(t)\tan\beta)(1 + \tan t\tan\beta) + f^{-1}(t)\tan\beta(\tan\beta - \tan t)}.$$

Since β can be chosen in the intervall $0 \le \beta < \frac{\pi}{2}$ and $\frac{\pi}{2} < \beta < \pi$ we may replace the parameter $\tan\beta$ by any $w \in \mathbb{R}$.

A $\mathcal{C}^1$-differentiable loop L corresponding to σ exists if and only if the function $t \mapsto \eta_w(t)$ is strictly increasing, i.e. if $\eta_w'(t) > 0$ (cf. Proposition 18.3, p. 238, in [8]). The function $a_w(t) : t \mapsto \tan\eta_w(t) : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ is strictly increasing if and only if $\eta_w'(t) > 0$ since

$$\frac{d}{dt}\tan(\eta_w(t)) = \frac{1}{\cos^2(\eta_w(t))}\eta_w'(t).$$

A straightforward calculation shows that

$$\begin{aligned}\frac{d}{dt}\tan(\eta_w(t)) &= \frac{w^2+1}{\cos^2(t)}[w^2(g'(t)f(t)+g(t)f'(t)+g^2(t)f^2(t)+1)+ \\ & \quad w(-2f(t)f'(t)-2g(t)f^3(t))+f^4(t)]. \qquad (25)\end{aligned}$$

Hence the loop $L(\sigma)$ exists if and only if for all $w \in \mathbb{R}$ the inequality

$$\begin{aligned}0 &< w^2(g'(t)f(t)+g(t)f'(t)+g^2(t)f^2(t)+1)+ \\ & \quad w(-2f(t)f'(t)-2g(t)f^3(t))+f^4(t) \qquad (26)\end{aligned}$$

holds. For $w=0$ the expression (26) equals to $f^4(t)>0$. Therefore the inequality (26) satisfies for all $w \in \mathbb{R}$ if and only if one has

$$f'^2(t)+g(t)f^2(t)f'(t)-g'(t)f^3(t)-f^2(t)<0 \quad \text{and} \quad g'(0)>f'^2(0)-1 \qquad (27)$$

for all $t \in \mathbb{R}$. Putting $f(t)=\hat{f}^{-1}(t)$ and $g(t)=-\hat{g}(t)$ these conditions are equivalent to the conditions

$$\hat{f}'^2(t)+\hat{g}(t)\hat{f}'(t)+\hat{g}'(t)\hat{f}(t)-\hat{f}^2(t)<0 \quad \text{and} \quad \hat{g}'(0)<1-\hat{f}'^2(0) \qquad (28)$$

(cf. [8], Section 18, (C), p. 238).
Now we treat the differential inequality (28). The solution $h(t)$ of the linear differential equation

$$h'(t)+h(t)\frac{\hat{f}'(t)}{\hat{f}(t)}+\frac{\hat{f}'^2(t)}{\hat{f}(t)}-\hat{f}(t)=0 \qquad (29)$$

with the initial conditions $h(0)=0$ and $h'(0)=1-\hat{f}'^2(0)$ is given by

$$h(t)=\hat{f}(t)^{-1}\int_0^t(\hat{f}^2(t)-\hat{f}'^2(t))dt.$$

Since $\hat{g}(0)=h(0)=0$ and $\hat{g}'(0)<h'(0)$ it follows from VI in [13] (p. 66) that $\hat{g}(t)$ is a subfunction of the differential equation (29), i.e. that $\hat{g}(t)$ satisfies the differential inequality (28). Moreover, according to Theorem V in [13] (p. 65) one has $\hat{g}(t)<h(t)$ for all $t \in (0,2\pi)$. Since the functions $\hat{g}(t)$ and $h(t)$ are continuous $0=\hat{g}(2\pi)\le h(2\pi)$. This yields the following integral inequality

$$\int_0^{2\pi}(\hat{f}^2(t)-\hat{f}'^2(t))dt \ge 0. \qquad (30)$$

We consider the real function $R(t)$ defined by $R(t) = \hat{f}(t) - \hat{f}'(t)$. Since $\hat{f}(0) = \hat{f}(2\pi) = 1$ and $\hat{f}'(0) = \hat{f}'(2\pi)$ we have $R(0) = 1 - \hat{f}'(0) = 1 - \hat{f}'(2\pi) = R(2\pi)$.

The linear differential equation

$$y'(t) - y(t) + R(t) = 0 \quad \text{with} \quad y(0) = 1 \tag{31}$$

has the solution

$$y(t) = e^t(1 - \int_0^t R(u)e^{-u}du). \tag{32}$$

This solution is unique (cf. [6], p. 2) and hence it is the function $\hat{f}(t)$. The condition $\hat{f}(2\pi) = 1$ is satisfied if and only if $\int_0^{2\pi} R(u)e^{-u}du = 1 - \frac{1}{e^{2\pi}}$. Since $R(t)$ has periode 2π its Fourier series is given by

$$a_0 + \sum_{k=1}^{\infty}(a_k \cos kt + b_k \sin kt), \tag{33}$$

where $a_0 = \frac{1}{\pi}\int_0^{2\pi} R(t)\, dt$, $a_k = \frac{1}{\pi}\int_0^{2\pi} R(t)\cos kt\, dt$, and $b_k = \frac{1}{\pi}\int_0^{2\pi} R(t)\sin kt\, dt$. Partial integration yields

$$\int_0^t \sin ku\ e^{-u}du = \frac{k - k\cos kt\ e^{-t} - \sin kt\ e^{-t}}{1+k^2} \tag{34}$$

$$\int_0^t \cos ku\ e^{-u}du = \frac{1 + k\sin kt\ e^{-t} - \cos kt\ e^{-t}}{1+k^2}. \tag{35}$$

Using (34) and (35), we obtain by partial integration

$$\int_0^t R(u)e^{-u}\ du = a_0 - a_0e^{-t} + \sum_{k=1}^{\infty}[\int_0^t a_k \cos ku\ e^{-u}du + \int_0^t b_k \sin ku\ e^{-u}du] =$$

$$a_0 - a_0e^{-t} + \sum_{k=1}^{\infty} \frac{a_k(1 + k\sin kt\ e^{-t} - \cos kt\ e^{-t})}{1+k^2} + \frac{b_k(k - k\cos kt\ e^{-t} - \sin kt\ e^{-t})}{1+k^2}. \tag{36}$$

Now for the real coefficients a_0, a_k, b_k $(k \geq 1)$ it follows

$$1 - \frac{1}{e^{2\pi}} = \int_0^{2\pi} R(u)e^{-u}du = (a_0 + \sum_{k=1}^{\infty} \frac{a_k + kb_k}{1+k^2})(1 - \frac{1}{e^{2\pi}}).$$

Hence one has

$$a_0 + \sum_{k=1}^{\infty} \frac{a_k + kb_k}{1+k^2} = 1. \tag{37}$$

The function $\hat{f}(t)$ is positive if and only if

$$1 > \int_0^t R(u)e^{-u}du \quad \text{for all} \quad t \in [0, 2\pi]. \tag{38}$$

Applying (34) and (35) again we see that the inequality (38) is equivalent to

$$a_0 > \sum_{k=1}^{\infty} [\frac{a_k k - b_k}{1+k^2} \sin kt - \frac{a_k + b_k k}{1+k^2} \cos kt]. \tag{39}$$

Since $\hat{f}'(t) + \hat{f}(t) = 2e^t(1 - \int_0^t R(u)e^{-u}du) - R(t)$ the function $\hat{f}(t)$ satisfies the integral inequality (30) if and only if

$$\int_0^{2\pi} R(t)[2e^t(1 - \int_0^t R(u)e^{-u}du) - R(t)]dt \geq 0. \tag{40}$$

The left side of (40) can be written as

$$2\int_0^{2\pi} R(t)e^t dt - 2\int_0^{2\pi} R(t)e^t(\int_0^t R(u)e^{-u}du)dt - \int_0^{2\pi} R^2(t)dt. \tag{41}$$

Using partial integration and representing $R(u)$ by a Fourier series (33) we have

$$\int_0^{2\pi} R(t)e^t dt = (a_0 + \sum_{k=1}^{\infty} \frac{a_k - b_k k}{1+k^2})(e^{2\pi} - 1). \tag{42}$$

From (36) it follows

$$\int_0^{2\pi} R(t)e^t \left(\int_0^t R(u)e^{-u}du \right) dt =$$

$$a_0 \int_0^{2\pi} R(t)e^t dt - a_0 \int_0^{2\pi} R(t)dt + \sum_{k=1}^{\infty} \int_0^{2\pi} \left(\frac{a_k + kb_k}{1+k^2} \right) R(t)e^t dt +$$

$$\sum_{k=1}^{\infty}\int_0^{2\pi}\left(\frac{ka_k-b_k}{1+k^2}\right)R(t)\sin kt\ dt-\sum_{k=1}^{\infty}\int_0^{2\pi}\left(\frac{a_k+kb_k}{1+k^2}\right)R(t)\cos kt\ dt. \tag{43}$$

Substituting for $R(t)$ its Fourier series and applying the relation (a) in [12] (p. 10) we have

$$\int_0^{2\pi} R(t)dt = 2\pi a_0.$$

Futhermore, one has

$$\sum_{k=1}^{\infty}\int_0^{2\pi}\left(\frac{ka_k-b_k}{1+k^2}\right)R(t)\sin kt\ dt=$$

$$\sum_{k=1}^{\infty}\int_0^{2\pi}\left(\frac{ka_k-b_k}{1+k^2}\right)[a_0+\sum_{l=1}^{\infty}(a_l\cos lt+b_l\sin lt)]\sin kt\ dt=$$

$$a_0\sum_{k=1}^{\infty}\int_0^{2\pi}\left(\frac{ka_k-b_k}{1+k^2}\right)\sin kt\ dt+\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\int_0^{2\pi}\left(\frac{ka_k-b_k}{1+k^2}\right)a_l\cos lt\ \sin kt\ dt+$$

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\int_0^{2\pi}\left(\frac{ka_k-b_k}{1+k^2}\right)b_l\sin lt\ \sin kt\ dt.$$

The relations (a), (b), (c), (d) in [12], p. 10, yield

$$\sum_{k=1}^{\infty}\int_0^{2\pi}\left(\frac{ka_k-b_k}{1+k^2}\right)R(t)\sin kt\ dt=\sum_{k=1}^{\infty}\int_0^{2\pi}\left(\frac{ka_k-b_k}{1+k^2}\right)b_k\sin^2 kt\ dt=\sum_{k=1}^{\infty}(\frac{ka_k-b_k}{1+k^2})b_k\pi.$$

Analogously we obtain that

$$\sum_{k=1}^{\infty}\int_0^{2\pi}\left(\frac{a_k+kb_k}{1+k^2}\right)R(t)\cos kt\ dt=\sum_{k=1}^{\infty}\int_0^{2\pi}\left(\frac{ka_k+b_k}{1+k^2}\right)b_k\cos^2 kt\ dt=\sum_{k=1}^{\infty}(\frac{a_k+kb_k}{1+k^2})a_k\pi.$$

Using the equality (37) one has

$$\int_0^{2\pi}R(t)e^t\left(\int_0^t R(u)e^{-u}du\right)dt=$$

$$[a_0+\sum_{k=1}^{\infty}\frac{a_k-kb_k}{1+k^2}](e^{2\pi}-1)-\pi\sum_{k=1}^{\infty}\frac{b_k^2+a_k^2}{1+k^2}-2\pi a_0^2. \tag{44}$$

Substituting for $R(t)$ its Fourier series we have

$$\int_0^{2\pi} R^2(t)\ dt = \int_0^{2\pi} a_0^2\ dt + 2a_0 \sum_{k=1}^{\infty} \int_0^{2\pi} (a_k \cos kt\ + b_k \sin kt)\ dt -$$

$$\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \int_0^{2\pi} (a_k a_l \cos kt \cos lt + a_k b_l \cos kt \sin lt +$$

$$b_k a_l \sin kt \cos lt + b_k b_l \sin kt \sin lt)\ dt.$$

Applying the relations (a), (b), (c), (d) in [12] (p. 10) we obtain

$$\int_0^{2\pi} R^2(t)\ dt = 2\pi a_0^2 + \pi \sum_{k=1}^{\infty} (a_k^2 + b_k^2).$$

Hence the integral inequality (30) holds if and only if

$$2a_0 \geq \sum_{k=1}^{\infty} (a_k^2 + b_k^2) \frac{k^2 - 1}{k^2 + 1}.$$

Since the Fourier series of $R(t)$ lies in the set $\mathcal{F}$ of series the Fourier series of R converges uniformly to R (Lemma 5).

Summarizing our discussion we obtain the main part of the following

Theorem 6. *Let L be a 1-dimensional connected $\mathcal{C}^1$-differentiable loop such that the group topologically generated by its left translations is isomorphic to the group $SL_2(\mathbb{R})$. Then L is compact and belongs to a $\mathcal{C}^1$-differentiable sharply transitive section σ of the form*

$$\sigma\ :\ \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}; a > 0, b \in \mathbb{R} \right\} \to$$
$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} f(t) & g(t) \\ 0 & f^{-1}(t) \end{pmatrix} \quad \textit{with} \quad t \in \mathbb{R} \tag{45}$$

such that the inverse function f^{-1} has the shape

$$f^{-1}(t) = e^t (1 - \int_0^t R(u) e^{-u}\ du) =$$

$$a_0 + \sum_{k=1}^{\infty} \frac{(ka_k - b_k) \sin kt + (a_k + kb_k) \cos kt}{1 + k^2}, \tag{46}$$

where $R(u)$ is a continuous function the Fourier series of which is contained in the set $\mathcal{F}$ and converges uniformly to R, and g is a periodic $\mathcal{C}^1$-differentiable function with $g(0) = g(2\pi) = 0$ such that

$$g(t) > -f(t) \int_0^t \frac{(f^2(u) - f'^2(u))}{f^4(u)} \, du \quad for \quad all \quad t \in (0, 2\pi). \tag{47}$$

Conversely, if $R(u)$ is a continuous function the Fourier series of which is contained in $\mathcal{F}$, then the section σ of the form (45) belongs to a loop if f is defined by (46) *and g is a $\mathcal{C}^1$-differentiable periodic function with $g(0) = g(2\pi) = 0$ satisfying* (47).

The isomorphism classes of loops defined by σ are in one-to-one correspondence to the 2*-sets $\{(f(t), g(t)), (f(-t), -g(-t))\}$.*

Proof. The only part of the assertion which has to be discussed is the isomorphism question. It follows from [7], Theorem 3, p. 3, that any isomorphism class of the loops L contains precisely two pairs (f_1, g_1) and (f_2, g_2). If $(f_1, g_1) \neq (f_2, g_2)$ and if (f_1, g_1) satisfy the inequality (27), then we have

$$f_2'^2(-t) + g_2(-t)f_2^2(-t)f_2'(-t) - g_2'(-t)f_2^3(-t) - f_2^2(-t) < 0.$$

since from $f_1(t) = f_2(-t)$ and $g_1(t) = -g_2(-t)$ we have $f_1'(t) = -f_2'(-t)$ and $g_1'(t) = g_2'(-t)$. □

Remark. A loop $\tilde{L}$ belonging to a section σ of shape (45) is a 2-covering of a $\mathcal{C}^1$-differentiable loop L having the group $PSL_2(\mathbb{R})$ as the group topologically generated by the left translations if and only if for the functions f and g one has $f(\pi) = 1$ and $g(\pi) = 0$ (cf. [9], p. 5106). Moreover, L is the factor loop $\tilde{L}/\left\{\begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}; \epsilon = \pm 1\right\}$. Any n-covering of L is a non-split central extension $\hat{L}$ of the cyclic group of order n by L. The loop $\hat{L}$ has the n-covering of $PSL_2(\mathbb{R})$ as the group topologically generated by its left translations.

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