A Refinement of Vietoris' Inequality for Cosine Polynomials

(*To appear in: Analysis and Applications*)

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arXiv:1507.00810v1 [math.CA] 3 Jul 2015

Abstract. Let

$$T_n(x) = \sum_{k=0}^{n} b_k \cos(kx)$$

with

$$b_{2k} = b_{2k+1} = \frac{1}{4^k}\binom{2k}{k} \qquad (k \geq 0).$$

In 1958, Vietoris proved that

$$T_n(x) > 0 \qquad (n \geq 1; x \in (0, \pi)).$$

We offer the following improvement of this result: The inequalities

$$T_n(x) \geq c_0 + c_1 x + c_2 x^2 > 0 \qquad (c_k \in \mathbf{R}, k = 0, 1, 2)$$

hold for all $n \geq 1$ and $x \in (0, \pi)$ if and only if

$$c_0 = \pi^2 c_2, \quad c_1 = -2\pi c_2, \quad 0 < c_2 \leq \alpha,$$

where

$$\alpha = \min_{0 \leq t < \pi} \frac{T_6(t)}{(t-\pi)^2} = 0.12290....$$

2010 Mathematics Subject Classification. 26D05

Keywords. Vietoris theorem, inequalities, cosine polynomials, Sturm's theorem, Jacobi polynomials

[1]The research of this author is supported by the Hong Kong Government GRF Grant PolyU 5003/12P and the Hong Kong Polytechnic University Grants G-UC22 and G-UA10

1. Introduction

In 1958, Vietoris [21] published the following "surprising and quite deep result" [16, p. 1] on inequalities for a class of sine and cosine polynomials.

Proposition 1. *If the real numbers* a_k $(k = 0, 1, ..., n)$ *satisfy*

$$a_0 \geq a_1 \geq \cdots \geq a_n > 0 \quad \textit{and} \quad 2ka_{2k} \leq (2k-1)a_{2k-1} \ \ (k \geq 1),$$

then

$$\sum_{k=1}^{n} a_k \sin(kx) > 0 \quad \textit{and} \quad \sum_{k=0}^{n} a_k \cos(kx) > 0 \quad (0 < x < \pi). \tag{1.1}$$

In order to prove (1.1) it is enough to consider the special case $a_k = b_k$, where

$$b_{2k} = b_{2k+1} = \frac{1}{4^k}\binom{2k}{k} \ \ (k \geq 0)$$

and to apply summation by parts; see Askey and Steinig [8]. In fact, Vietoris proved that

$$S_n(x) > 0 \quad \text{and} \quad T_n(x) > 0 \quad (n \geq 1; 0 < x < \pi), \tag{1.2}$$

where

$$S_n(x) = \sum_{k=1}^{n} b_k \sin(kx) \quad \text{and} \quad T_n(x) = \sum_{k=0}^{n} b_k \cos(kx).$$

In what follows, we maintain these notations.

In 1974, Askey and Steinig [8] offered a simplified proof of (1.2) and showed that these inequalities have remarkable applications in the theory of ultraspherical polynomials and that they can be used to find estimates for the location of zeros of trigonometric polynomials.

In the recent past, Vietoris' inequalities received attention from several authors, who offered new conditions on the coefficients a_k such that (1.1) holds; see Belov [9], Brown [10], Brown and Dai [11], Brown and Hewitt [12], Brown and Yin [13], Koumandos [16], Mondal and Swaminathan [20]. Interesting historical remarks on these inequalities were given by Askey [5].

Is it possible to replace the lower bound 0 in (1.2) by a positive expression? In 2010, this problem was solved for the sine polynomial S_n. Alzer, Koumandos and Lamprecht [2] proved the following

Proposition 2. *The inequalities*

$$S_n(x) > \sum_{k=0}^{4} a_k x^k > 0 \quad (a_k \in \mathbf{R}, k = 0, ..., 4) \tag{1.3}$$

hold for all $n \geq 1$ *and* $x \in (0, \pi)$ *if and only if*

$$a_0 = 0, \quad a_1 = -\pi^2 a_4, \quad a_2 = 3\pi^2 a_4, \quad a_3 = -3\pi a_4, \quad -1/\pi^3 < a_4 < 0.$$

Moreover, in (1.3) *the biquadratic polynomial cannot be replaced by an algebraic polynomial of degree smaller than* 4.

It is natural to ask for a counterpart of Proposition 2 which holds for the cosine polynomial T_n. More precisely, we try to find algebraic polynomials p of smallest degree such that

$$T_n(x) \geq p(x) > 0 \quad (n \geq 1; 0 < x < \pi). \tag{1.4}$$

It is the aim of this paper to determine all quadratic polynomials p satisfying (1.4).

We remark that there is no linear polynomial p such that (1.4) is valid. Otherwise, setting $p(x) = \gamma_0 + \gamma_1 x$ gives

$$T_1(x) = 1 + \cos(x) \geq \gamma_0 + \gamma_1 x > 0.$$

We let x tend to π and obtain $\gamma_0 = -\pi\gamma_1$. Thus,

$$\frac{1+\cos(x)}{\pi - x} \geq -\gamma_1 > 0.$$

But, this contradicts

$$\lim_{x\to\pi} \frac{1+\cos(x)}{\pi - x} = 0.$$

In the next section, we collect twelve lemmas. Our main result is presented in Section 3. We conclude the paper with some remarks which are given in Section 4. Among others, we provide a new inequality for a sum of Jacobi polynomials.

The numerical values in this paper have been calculated via the computer program MAPLE 13. We point out that in four places we apply the classical Sturm theorem to determine the number of distinct zeros of an algebraic polynomial in a given interval. Since the Sturm procedure requires lengthy technical computations we omit the details. However, those details which we do not include in this paper are compiled in the supplementary article [1]. Concerning Sturm's theorem we also refer to van der Waerden [22, p. 248] and Kwong [18].

2. Lemmas

Here, we collect lemmas which play an important role in the proof of our main result.

Lemma 1. *We have*

$$\min_{0\leq x<\pi} \frac{T_6(x)}{(x-\pi)^2} = 0.12290.... \tag{2.1}$$

Proof. We define

$$\eta(x) = 10x^6 + 6x^5 - 12x^4 - \frac{11}{2}x^3 + \frac{29}{8}x^2 + \frac{11}{8}x + \frac{9}{16}$$

and

$$\theta(x) = (\pi - \arccos(x))^2.$$

Let $c = 0.1229$. First, we show that

$$\eta(x) - c\theta(x) > 0 \quad \text{for} \quad x \in (-1, 1]. \tag{2.2}$$

We distinguish six cases.

Case 1. $-1 < x < 0$.
Let

$$\omega(x) = \frac{1}{3}x^4 + \frac{\pi}{6}x^3 + x^2 + \pi x + \frac{\pi^2}{4}.$$

Then we have

$$\omega(x) > 0 \quad \text{and} \quad \omega'(x) > 0. \tag{2.3}$$

Let

$$\phi(x) = \sqrt{\omega(x)} - \sqrt{\theta(x)}.$$

Differentiation gives

$$4(1-x^2)\Big(\frac{\omega'(x)}{2\sqrt{\omega(x)}}+\frac{1}{\sqrt{1-x^2}}\Big)\omega(x)\phi'(x)=-\frac{1}{36}x^4(8x+3\pi)(8x^3+3\pi x^2+16x+9\pi)<0,$$

so that (2.3) leads to

$$\phi'(x)<0 \quad \text{and} \quad \phi(x)>\phi(0)=0.$$

Since

$$\eta(x)-c\omega(x)>0,$$

we obtain

$$\eta(x)-c\theta(x)>c(\omega(x)-\theta(x))>0.$$

Case 2. $0\le x\le 0.3$
Using $\eta'(x)>0$ and $\theta'(x)>0$ gives

$$\eta(x)-c\theta(x)\ge\eta(0)-c\theta(0.3)=0.13....$$

Case 3. $0.3\le x\le 0.5$.
We have $\eta''(x)<0$ and $\theta'(x)>0$. This yields

$$\eta(x)-c\theta(x)\ge\min(\eta(0.3),\eta(0.5))-c\theta(0.5)=\eta(0.5)-c\theta(0.5)=0.52....$$

Case 4. $0.5\le x\le 0.65$.
Since $\eta'(x)<0$ and $\theta'(x)>0$, we get

$$\eta(x)-c\theta(x)\ge\eta(0.65)-c\theta(0.65)=0.14....$$

Case 5. $0.65\le x\le 0.95$.
We have $\eta(x)>0$. Let

$$\lambda(x)=\sqrt{\eta(x)}-\sqrt{c\theta(x)} \quad \text{and} \quad \mu(x)=\eta(x)\eta''(x)-\frac{1}{2}\eta'(x)^2.$$

Differentiation leads to

$$(2.4)\quad 2(1-x^2)^3\Big(\frac{\mu(x)}{\eta(x)^{3/2}}+\frac{2\sqrt{c}x}{(1-x^2)^{3/2}}\Big)\eta(x)^3\lambda''(x)=(1-x^2)^3\mu(x)^2-4cx^2\eta(x)^3=\nu(x), \quad \text{say.}$$

The function μ and ν are polynomials. Applying Sturm's theorem reveals that μ and ν have no zeros on $[0.65,0.95]$. Since $\mu(3/4)>0$ and $\nu(3/4)>0$, we conclude that both functions are positive on $[0.65,0.95]$. From (2.4) we find that $\lambda''(x)>0$. Let $x^*=0.74746$. Then, $\lambda'(x^*)>0$. This implies

$$\lambda(x)\ge\lambda(x^*)+(x-x^*)\lambda'(x^*)\ge\lambda(x^*)+(0.65-x^*)\lambda'(x^*)=0.0000077....$$

Case 6. $0.95\le x\le 1$.
Since $\eta'(x)>0$ and $\theta'(x)>0$, we obtain

$$\eta(x)-c\theta(x)\ge\eta(0.95)-c\theta(1)=1.43....$$

Thus, (2.2) is proved. Let

$$T^*(x)=\frac{T_6(x)}{(\pi-x)^2}.$$

We have $\lim_{x\to\pi}T^*(x)=\infty$ and $T_6(x)=\eta(\cos(x))$. From (2.2) we obtain

$$0.1229<T^*(x)\quad(0\le x<\pi).$$

Since $T^*(0.725)=0.122907...$, we conclude that (2.1) is valid.

Remark. Numerical computation shows that the minimum value is

$$\alpha = 0.12290390650...$$

attained at the unique point $x = 0.72656896349....$

The following lemma is known as l'Hôpital's rule for monotonicity. A slightly weaker version can be found in [14, Proposition 148].

Lemma 2. *Let u and v be real-valued functions which are continuous on $[a,b]$ and differentiable on (a,b). Furthermore, let $v' \neq 0$ on (a,b). If u'/v' is strictly increasing (resp. decreasing) on (a,b), then the functions*

$$x \mapsto \frac{u(x)-u(a)}{v(x)-v(a)} \quad \textit{and} \quad x \mapsto \frac{u(x)-u(b)}{v(x)-v(b)}$$

are strictly increasing (resp. decreasing) on (a,b).

A proof for the next lemma is given by Vietoris in [21].

Lemma 3. *Let $n \geq 2$ and $x \in (0,\pi)$. Then,*

$$1+\cos(x) - \frac{1}{4\sin(x/2)}\Big(1+\sin(3x/2)\Big) \leq T_n(x).$$

Lemma 4. *If $3\pi/8 \leq x < \pi$, then*

$$\frac{123}{1000}(\pi - x)^2 < 1+\cos(x) - \frac{1}{4\sin(x/2)}\Big(1+\sin(3x/2)\Big). \tag{2.5}$$

Proof. Let

$$g(x) = \frac{123}{1000}(\pi-x)^2 \quad \text{and} \quad h(x) = 1+\cos(x) - \frac{1}{4\sin(x/2)}\Big(1+\sin(3x/2)\Big).$$

Then we have

$$g'(x) = \frac{123}{500}(x-\pi) < 0 \quad \text{and} \quad h'(x) = \frac{\cos(x/2)}{8\sin^2(x/2)}\Big(1-8\sin^3(x/2)\Big) < 0.$$

Let $3\pi/8 \leq r \leq x \leq s \leq 2\pi/3$. We obtain

$$h(x) - g(x) \geq h(s) - g(r) = q(r,s), \quad \text{say.}$$

Since

$$q(3\pi/8, 1.27) = 0.0024..., \quad q(1.27, 1.45) = 0.0024..., \quad q(1.45, 1.7) = 0.0008...,$$
$$q(1.70, 1.95) = 0.0072..., \quad q(1.95, 2\pi/3) = 0.0366...,$$

we conclude that (2.5) holds for $x \in [3\pi/8, 2\pi/3]$.

Next, we define

$$f(x) = g(x) - h(x).$$

Let $y \in (0, \pi/3)$. Using

$$\frac{y}{2} < \frac{\sin(y/2)}{\cos(y/2)}$$

yields

$$\begin{aligned} f'(\pi - y) &= \frac{\sin(y/2)}{8\cos^2(y/2)}\Big(8\cos^3(y/2) - 1\Big) - \frac{123y}{500} \\ &> \frac{\sin(y/2)}{8\cos^2(y/2)}\Big(8\cos^3(y/2) - 4\cos(y/2) - 1\Big) \\ &> 0. \end{aligned}$$

Thus, $y \mapsto f(\pi - y)$ is strictly decreasing on $(0, \pi/3)$, so that we get

$$f(\pi - y) < f(\pi) = 0.$$

This proves (2.5) for $x \in (2\pi/3, \pi)$.

Lemma 5. *For $x \in [0, \pi]$,*

$$x^2 < \frac{20000}{99}\Big(1 - \cos\frac{x}{10}\Big). \tag{2.6}$$

Proof. Let

$$u_0(x) = 1 - \cos(x/10) \quad \text{and} \quad v_0(x) = x^2.$$

Since

$$200\,\frac{u_0'(x)}{v_0'(x)} = \frac{\sin(x/10)}{x/10}$$

is decreasing on $[0, \pi]$, we conclude from Lemma 2 that the function

$$w(x) = \frac{1 - \cos(x/10)}{x^2} \quad (0 < x \le \pi), \quad w(0) = \frac{1}{200}$$

is also decreasing on $[0, \pi]$. Thus, for $x \in [0, \pi]$,

$$w(x) \ge w(\pi) = 0.004959... > 0.00495 = \frac{99}{20000}.$$

This settles (2.6).

Lemma 6. *Let a_k, β_k $(k = 1, ..., n)$, and α^* be real numbers such that*

$$\sum_{k=1}^{j} a_k \ge \alpha^* \quad for \quad j = 1, ..., n \quad and \quad \beta_1 \ge \beta_2 \ge \cdots \ge \beta_n \ge 0.$$

Then,

$$\sum_{k=1}^{n} a_k \beta_k \ge \alpha^* \beta_1.$$

Proof. Let

$$A_j = \sum_{k=1}^{j} a_k \quad \text{and} \quad \beta_{n+1} = 0.$$

Summation by parts gives

$$\sum_{k=1}^{n} a_k \beta_k = \sum_{k=1}^{n} A_k(\beta_k - \beta_{k+1}) \ge \sum_{k=1}^{n} \alpha^*(\beta_k - \beta_{k+1}) = \alpha^* \beta_1.$$

Lemma 7. *Let*

$$C_n(x) = \sum_{k=0}^{n} (-1)^k b_k \cos(kx).$$

If $2 \le n \le 21$ $(n \ne 6)$ *and* $x \in (5\pi/8, \pi)$, *then*

$$\frac{820}{33}\Big(1 - \cos\frac{x}{10}\Big) \le C_n(x). \tag{2.7}$$

Proof. We set $y = x/10$ and

$$P_n(y) = C_n(10y) - \frac{820}{33}(1 - \cos(y)).$$

Putting $Y = \cos(y)$ reveals that $P_n(y)$ is an algebraic polynomial in Y. We denote this polynomial by $P_n^*(Y)$, where $Y \in [\cos(\pi/10), \cos(\pi/16)] = [0.951..., 0.980...]$. Applying Sturm's theorem gives that P_n^* has no zero on $[0.951, 0.981]$ and satisfies $P_n^*(0.97) > 0$. It follows that P_n is positive on $[\pi/16, \pi/10]$. This implies that (2.7) holds.

Lemma 8. *Let*

$$\Delta(x) = \sum_{k=0}^{21} (-1)^k (b_k - b_{22}) \cos(kx) - \frac{820}{33}\Big(1 - \cos\frac{x}{10}\Big). \tag{2.8}$$

If $5\pi/8 \le x \le 2.68$, *then* $\Delta(x) > 0.29$;
if $2.68 \le x \le 2.83$, *then* $\Delta(x) > 0.46$;
if $2.83 \le x \le 2.908$, *then* $\Delta(x) > 0.64$;
if $2.908 \le x \le 2.970$, *then* $\Delta(x) > 0.90$;
if $2.970 \le x \le 3.021$, *then* $\Delta(x) > 1.32$;
if $3.021 \le x \le 3.051$, *then* $\Delta(x) > 1.78$.

Proof. Let $5\pi/8 \le x \le 2.68$. We have $\cos(\pi/16) = 0.980...$ and $\cos(0.268) = 0.964....$ The function $\Delta - 0.29$ is an algebraic polynomial in $Y = \cos(x/10)$. An application of Sturm's theorem gives that this function is positive on $[0.964, 0.981]$. This leads to $\Delta(x) > 0.29$ for $x \in [5\pi/8, 2.68]$. Using the same method of proof we obtain that the other estimates for $\Delta(x)$ are also valid.

Lemma 9. *Let* $n \ge 22$,

$$H_n(x) = \sum_{k=0}^{n} (-1)^k \cos(kx) \quad \textit{and} \quad D_n(x) = b_{22} H_{22}(x) + \sum_{k=23}^{n} (-1)^k b_k \cos(kx). \tag{2.9}$$

If $5\pi/8 \le x \le 2.68$, *then* $D_n(x) \ge -0.29$;
if $2.68 \le x \le 2.83$, *then* $D_n(x) \ge -0.46$;
if $2.83 \le x \le 2.908$, *then* $D_n(x) \ge -0.64$;
if $2.908 \le x \le 2.970$, *then* $D_n(x) \ge -0.90$;
if $2.970 \le x \le 3.021$, *then* $D_n(x) \ge -1.32$;
if $3.021 \le x \le 3.051$, *then* $D_n(x) \ge -1.78$.

Proof. We have

$$H_n(x) = \frac{1}{2} + (-1)^n \frac{\cos((n+1/2)x)}{2\cos(x/2)}.$$

Let $5\pi/8 \le x \le 2.68$. Then we obtain

$$(2.10) \quad H_n(x) \ge \frac{1}{2} - \frac{1}{2\cos(x/2)} \ge \frac{1}{2} - \frac{1}{2\cos(1.34)} = -1.68... > -1.72... = -\frac{0.29 \cdot 4^{11}}{\binom{22}{11}} = -\frac{0.29}{b_{22}}.$$

Using (2.10) and

$$b_{22} \ge b_{23} \ge \cdots \ge b_n$$

we conclude from Lemma 6 that $D_n(x) \ge -0.29$. Applying the same method we obtain the other estimates.

As usual, we set

$$(a)_0 = 1, \quad (a)_n = \prod_{k=0}^{n-1}(a+k) = \frac{\Gamma(a+n)}{\Gamma(a)} \quad (n \ge 1).$$

The following result is due to Koumandos [16]; see also Koumandos [17] for background information.

Lemma 10. *Let* $0 < \gamma < 0.6915562$ *be given and*

$$(2.11) \qquad d_{2k} = d_{2k+1} = \frac{(\gamma)_k}{k!} \qquad (k = 0, 1, 2, ...).$$

Then, for $n \ge 1$ *and* $x \in (0, \pi)$,

$$\sum_{k=0}^{n} d_k \cos(kx) > 0.$$

In what follows, we denote by d_k $(k = 0, 1, 2, ...)$ the numbers defined in (2.11) with $\gamma = 0.69$.

Lemma 11. *Let*

$$(2.12) \qquad I(x) = \sum_{k=0}^{21}\Big(b_k - \frac{b_{22}}{d_{22}} d_k\Big)\cos(kx).$$

If $0 < x \le 0.1$, *then* $I(x) > 1.5$.

Proof. Setting $Y = \cos(x)$ gives that $I - 1.5$ is an algebraic polynomial in Y. We have $\cos(0.1) = 0.995....$ Sturm's theorem reveals that this polynomial is positive on $[0.995, 1]$. It follows that $I(x) > 1.5$ for $x \in [0, 0.1]$.

Lemma 12. *Let* $n \ge 22$ *and*

$$(2.13) \qquad J_n(x) = \frac{b_{22}}{d_{22}} \sum_{k=0}^{21} d_k \cos(kx) + \sum_{k=22}^{n} b_k \cos(kx).$$

If $0 < x < \pi$, *then* $J_n(x) \ge 0$.

Proof. Let $x \in (0, \pi)$. We set

$$K_j(x) = \frac{b_{22}}{d_{22}} \sum_{k=0}^{j} d_k \cos(kx) \quad (j = 0, 1, 2, ...)$$

Then, $K_0(x) \equiv b_{22}/d_{22}$ and from Lemma 10 we obtain

$$K_j(x) > 0 \quad \text{for} \quad j \geq 1.$$

Let

$$a_k = \frac{b_{22}}{d_{22}} d_k \cos(kx) \quad (k = 0, 1, ..., n), \quad \beta_0 = \cdots = \beta_{21} = 1, \quad \beta_k = \frac{d_{22} b_k}{b_{22} d_k} \quad (k = 22, ..., n).$$

Since

$$\beta_0 \geq \beta_1 \geq \cdots \geq \beta_n > 0,$$

we conclude from Lemma 6 that

$$J_n(x) = \sum_{k=0}^{n} a_k \beta_k \geq 0.$$

3. Main result

We are now in a position to present positive lower bounds for the cosine polynomial T_n.

Theorem. *The inequalities*

$$T_n(x) \geq c_0 + c_1 x + c_2 x^2 > 0 \quad (c_k \in \mathbf{R}, k = 0, 1, 2) \tag{3.1}$$

hold for all natural numbers n and real numbers $x \in (0, \pi)$ if and only if

$$c_0 = \pi^2 c_2, \quad c_1 = -2\pi c_2, \quad 0 < c_2 \leq \alpha, \tag{3.2}$$

where

$$\alpha = \min_{0 \leq t < \pi} \frac{T_6(t)}{(t - \pi)^2} = 0.12290.... \tag{3.3}$$

Proof. We set

$$Q(x) = c_0 + c_1 x + c_2 x^2.$$

If (3.1) is valid for all $n \geq 1$ and $x \in (0, \pi)$, then we get

$$T_1(x) = 1 + \cos(x) \geq Q(x) > 0.$$

We let x tend to π and obtain $c_0 + c_1 \pi + c_2 \pi^2 = 0$. Thus,

$$\frac{1 + \cos(x)}{x - \pi} \leq \frac{Q(x)}{x - \pi} = c_1 + c_2(x + \pi) < 0.$$

Again, we let x tend to π. This gives

$$c_1 = -2\pi c_2 \quad \text{and} \quad c_0 = -\pi c_1 - \pi^2 c_2 = \pi^2 c_2. \tag{3.4}$$

It follows that

$$Q(x) = c_2 (x - \pi)^2 \quad \text{with} \quad c_2 > 0. \tag{3.5}$$

Moreover, from (3.1) (with $n = 6$) we obtain

$$\frac{T_6(x)}{(x - \pi)^2} \geq \frac{Q(x)}{(x - \pi)^2} = c_2.$$

Using (3.3) leads to

$$\alpha \geq c_2. \tag{3.6}$$

From (3.4) - (3.6) we conclude that (3.2) holds.

Next, we show that (3.2) and (3.3) lead to (3.1). If (3.2) and (3.3) are valid, then

$$0 < c_0 + c_1 x + c_2 x^2 = c_2(x-\pi)^2 \le \alpha(x-\pi)^2.$$

Hence, we have to prove that

$$\alpha(x-\pi)^2 \le T_n(x) \quad (n \ge 1; 0 < x < \pi). \tag{3.7}$$

Applying Lemma 2 we obtain that the function

$$F(x) = \frac{T_1(x)}{(x-\pi)^2} = \frac{1+\cos(x)}{(x-\pi)^2}$$

is strictly increasing on $(0,\pi)$. Thus,

$$F(x) > F(0) = \frac{2}{\pi^2} = 0.202....$$

This settles (3.7) for $n = 1$. From (3.3) we conclude that (3.7) is also valid for $n = 6$. In what follows we prove

$$\frac{123}{1000}(\pi - x)^2 \le T_n(x) \tag{3.8}$$

for $n \ge 2$ $(n \ne 6)$ and $x \in (0,\pi)$. With regard to Lemma 3 and Lemma 4 we may assume that $x \in (0, 3\pi/8)$. We replace in (3.8) x by $\pi - x$. It follows that it is enough to prove

$$\frac{123}{1000}x^2 \le \sum_{k=0}^{n}(-1)^k b_k \cos(kx) = C_n(x) \tag{3.9}$$

for $n \ge 2$ $(n \ne 6)$ and $x \in (5\pi/8, \pi)$. Using Lemma 5 yields that

$$\frac{820}{33}\Big(1 - \cos\frac{x}{10}\Big) \le C_n(x) \tag{3.10}$$

implies (3.9). An application of Lemma 7 reveals that (3.10) is valid if $2 \le n \le 21$ $(n \ne 6)$.

Now, let $n \ge 22$. We have the representation

$$C_n(x) - \frac{820}{33}\Big(1 - \cos\frac{x}{10}\Big) = \Delta(x) + D_n(x), \tag{3.11}$$

with Δ and D_n as defined in (2.8) and (2.9), respectively. Applying Lemma 8 and Lemma 9 reveals that

$$\Delta(x) + D_n(x) \ge 0 \quad \text{for} \quad x \in [5\pi/8, 3.051]. \tag{3.12}$$

From (3.11) and (3.12) we conclude that (3.10) is valid for $x \in [5\pi/8, 3.051]$. This implies that (3.8) holds for $x \in [\pi - 3.051, 3\pi/8]$. Hence, it remains to prove (3.8) for $x \in (0, \pi - 3.051)$. Since

$$\frac{123}{1000}(\pi - x)^2 < 1.22 \quad \text{for} \quad x \in (0, \pi - 3.051)$$

and $\pi - 3.051 = 0.090...$, it suffices to show that

$$1.22 \le T_n(x) \quad \text{for} \quad x \in (0, 0.1]. \tag{3.13}$$

We have

$$T_n(x) = I(x) + J_n(x), \tag{3.14}$$

where I and J_n are defined in (2.12) and (2.13), respectively. Applying Lemma 11 and Lemma 12 we conclude from (3.14) that (3.13) holds. This completes the proof of the Theorem.

4. Concluding remarks

(I) If we set $a_0 = 1$ and $a_k = 1/k$ $(k \geq 1)$ in (1.1), then we find

$$\sum_{k=1}^{n} \frac{\sin(kx)}{k} > 0 \quad \text{and} \quad 1 + \sum_{k=1}^{n} \frac{\cos(kx)}{k} > 0 \quad (n \geq 1; 0 < x < \pi). \tag{4.1}$$

The first inequality is the famous Fejér-Jackson inequality, which was conjectured by Fejér in 1910 and proved one year later by Jackson [15]. Its analogue for the cosine sum was published by Young [23] in 1913. Both inequalities motivated the research of many authors, who presented numerous refinements, extensions, and variants of (4.1). We refer to Askey [3], Askey and Gasper [7], Milovanović, Mitrinović, and Rassias [19, chapter 4] and the references cited therein.

Applying our Theorem we obtain an improvement of Young's inequality:

$$1 + \sum_{k=1}^{n} \frac{\cos(kx)}{k} \geq \alpha(\pi - x)^2 \quad (n \geq 1; 0 < x < \pi),$$

where α is given in (3.3)

(II) Askey and Steinig [8] used Proposition 1 to prove the following interesting result.

Proposition 3. *Let γ_k $(k = 0, 1, ..., n)$ be positive real numbers such that*

$$2k\gamma_k \leq (2k-1)\gamma_{k-1} \quad (k \geq 1). \tag{4.2}$$

Then, for $n \geq 0$ and $t \in (0, 2\pi)$,

$$\sum_{k=0}^{n} \gamma_k \sin\big((k+1/4)t\big) > 0 \quad \textit{and} \quad \sum_{k=0}^{n} \gamma_k \cos\big((k+1/4)t\big) > 0.$$

An application of the Theorem leads to a refinement of the second inequality:

$$\sum_{k=0}^{n} \gamma_k \cos\big((k+1/4)t\big) \geq \frac{\alpha\gamma_0(2\pi - t)^2}{8\cos(t/4)} \quad (n \geq 0; 0 < t < 2\pi). \tag{4.3}$$

In order to prove (4.3) we set

$$\gamma_k^* = \frac{1}{4^k}\binom{2k}{k} = \frac{1}{k!}\Big(\frac{1}{2}\Big)_k \quad (k \geq 0).$$

Then,

$$2\cos(t/4)\sum_{k=0}^{j} \gamma_k^* \cos\big((k+1/4)t\big) = T_{2j+1}(t/2) \geq \alpha(\pi - t/2)^2 \quad \text{for} \quad j = 0, 1, ..., n. \tag{4.4}$$

We define

$$\beta_k = \frac{\gamma_k}{\gamma_k^*} \quad (k \geq 0)$$

and apply (4.2). This yields

$$\beta_0 \geq \beta_1 \geq \cdots \geq \beta_n > 0.$$

Using (4.4) and Lemma 6 leads to

$$\sum_{k=0}^{n} \gamma_k \cos\big((k+1/4)t\big) = \sum_{k=0}^{n} \beta_k \gamma_k^* \cos\big((k+1/4)t\big) \geq \beta_0 \frac{\alpha(\pi - t/2)^2}{2\cos(t/4)}.$$

Since $\beta_0 = \gamma_0$, we get (4.3).

(III) The classical Jacobi polynomials $P_m^{(a,b)}(z)$ are given by

$$P_m^{(a,b)}(z) = \frac{(a+1)_m}{m!} \sum_{k=0}^{m} \frac{(-m)_k (m+a+b+1)_k}{k!(a+1)_k} \Big(\frac{1-z}{2}\Big)^k.$$

A collection of the main properties of these functions can be found, for instance, in [19, chapter 1.2.7]. An application of (4.4) (with $t = 4x, j = n$) and the identity

$$\frac{P_m^{(-1/2,-1/2)}(\cos(x))}{P_m^{(-1/2,-1/2)}(1)} = \cos(mx)$$

(with $m = 4k+1$) yields

$$\sum_{k=0}^{n} \frac{1}{k!} \Big(\frac{1}{2}\Big)_k \frac{P_{4k+1}^{(-1/2,-1/2)}(\cos(x))}{P_{4k+1}^{(-1/2,-1/2)}(1)} \geq \frac{\alpha(\pi-2x)^2}{2\cos(x)} \quad (n \geq 0; 0 < x < \pi/2).$$

For related inequalities we refer to Askey [4], Askey and Gasper [6], [7] and the references therein.

Acknowledgement. We thank the referees for helpful comments.

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