Spherically symmetric Einstein-aether perfect fluid models

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Abstract. We investigate spherically symmetric cosmological models in Einstein-aether theory with a tilted (non-comoving) perfect fluid source. We use a 1+3 frame formalism and adopt the comoving aether gauge to derive the evolution equations, which form a well-posed system of first order partial differential equations in two variables. We then introduce normalized variables. The formalism is particularly well-suited for numerical computations and the study of the qualitative properties of the models, which are also solutions of Horava gravity. We study the local stability of the equilibrium points of the resulting dynamical system corresponding to physically realistic inhomogeneous cosmological models and astrophysical objects with values for the parameters which are consistent with current constraints. In particular, we consider dust models in $(\beta-)$ normalized variables and derive a reduced (closed) evolution system and we obtain the general evolution equations for the spatially homogeneous Kantowski-Sachs models using appropriate bounded normalized variables. We then analyse these models, with special emphasis on the future asymptotic behaviour for different values of the parameters. Finally, we investigate static models for a mixture of a (necessarily non-tilted) perfect fluid with a barotropic equations of state and a scalar field.

Keywords: Spherical symmetry, Einstein-aether theory, perfect fluid

Contents

1 Introduction

Since the vacuum in quantum gravity may determine a preferred rest frame at the microscopic level, gravitational Lorentz violation has been studied within the framework of general relativity (GR), where the background tensor field(s) breaking the symmetry must be dynamical [1]. Einstein-aether theory [2, 3] consists of GR coupled, at second derivative order, to a dynamical timelike unit vector field, the aether. In this effective field theory approach, the aether vector field and the metric tensor together determine the local spacetime structure.

The aether spontaneously breaks Lorentz invariance by picking out a preferred frame at each point in spacetime while maintaining local rotational symmetry (breaking only the boost sector of the Lorentz symmetry). Since the aether is a unit vector, it is everywhere non-zero in any solution, including flat spacetime. A systematic construction of an Einstein-aether gravity theory with a Lorentz violating dynamical field that preserves locality and covariance in the presence of an additional 'aether' vector field has been presented [2–6].

In the infra-red limit of (extended) Horava gravity [7] [a candidate ultra-violet completion in the consistent non-projectable extension of Horava-Lifschitz gravity], the aether vector is assumed to be hypersurface-orthogonal; hence every hypersurface-orthogonal Einstein-aether solution is a Horava solution (most of the solutions studied). The relationship between Einstein-aether theory and Horava gravity is further clarified in [8], where it is shown how Horava gravity can formally be obtained from Einstein-aether theory in the limit that the twist coupling constant goes to infinity.

Cosmological models in aether theories of gravity are currently of interest. The impact of Lorentz violation on the inflationary scenario has been explored [4–6] (also see the review [9]). [1] In particular, the primordial spectra of perturbations generated by inflation in the presence of a timelike Lorentz-violating vector field has been computed, and the amplitude of perturbation spectra were found to be modified which, in general, leads to a violation of the inflationary consistency relationship [10].

In particular, it is of importance to study inhomogeneous cosmologies, in both GR and alternative gravitational theories, partially motivated by current cosmological observations. Measurements of anisotropies of the cosmic microwave background (CMB) from experiments including the WMAP [11] and Planck [12] satellites, have provided strong support for the standard model of cosmology with dark energy (and specifically a cosmological constant, Λ). However the latest measurements are in tension with local measurements of the Hubble expansion rate from supernovae Ia [13] and other cosmological observables which point towards a lower growth rate of large-scale structure (LSS) (which may be evidence for deviations from the standard ΛCDM cosmological model). The possible observation by the BICEP2 experiment [14] of B-mode polarisation in the CMB in excess of the signal due to lensing would indicate the first detection of gravitational waves, perhaps generated during an inflationary era. In particular, there is a growing body of work on the imprints of gravitational waves on large-scale structure.

In this paper we will study spherically symmetric Einstein-aether models. We shall study perfect fluid matter models in general, and various subcases in particular. In a companion paper [15] we study spherically symmetric Einstein-aether scalar field models with an

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[1]We note that in scalar field models in which the dimensionless parameters of the models are not constant (e.g., depend on the scalar field), it was found that inflationary solutions are possible even in the absence of a scalar field potential [5].

exponential self-interaction potential. Einstein-aether models with an exponential potential were recently studied in [16–18].

We shall use the 1+3 frame formalism [19–21] to write down the evolution equations for non-comoving perfect fluid spherically symmetric models and show they form a well-posed system of first order partial differential equations (PDEs) in two variables. We adopt the so-called comoving aether gauge (which implies a preferred foliation, the only remaining freedom is coordinate time and space reparameterization). We introduce normalized variables. The formalism is particularly well-suited for numerical and qualitative analysis [22].

In particular, we derive the governing equations for an aether and a tilted perfect fluid assuming that the acceleration $\dot{u}$ is non-zero and introduce (so-called β-) normalized variables (some of the technical details are relegated to the Appendix B). The evolution equations are presented in various different forms. We also rigorously derive the evolution equations when $\dot{u} = 0$. We also consider the special subset $\dot{U} = v = 0$ (where $\dot{U}$ is the normalized acceleration, v is the tilt, and we also assume the model parameters $c_\theta = 0$ and $c_\sigma \neq 0$) and derive the final reduced phase space equations in normalized variables. We briefly review the Friedmann-Lemaître-Robertson-Walker (FLRW) models in which the source must be of the form of a comoving perfect fluid (or vacuum) and the aether must be comoving. We study in detail a number of special cases of particular physical interest.

We first consider dust models, which are of particular interest at late times. We investigate a special dust model with $\dot{U} = 0$ and $v = 0$ in normalized variables (assuming $c_\sigma \neq 0$) and derive a reduced (closed) evolution system. The FLRW models in this special dust model correspond to an equilibrium point. We are particularly interested in the future asymptotic behaviour of the models for different values of the parameters. We then consider the spatially homogeneous Kantowski-Sachs models using appropriate normalized variables (non-β–normalized variables which are bounded), and obtain the general evolution equations. A full global dynamical analysis of these models is possible. We then consider a special case and analyse the qualitative behaviour for physically reasonable values of the parameters at both early and late times. Finally, we consider static models for a mixture of a (necessarily non-tilted) perfect fluid with a barotropic equations of state and a scalar field, which are also of physical importance (although perhaps more from the astrophysical point of view than from the cosmological one). A brief discussion of the physical conclusions is presented at the end.

1.1 The models

The evolution equations follow from the field equations (FE) derived from the Einstein-aether action [2, 3]. In an Einstein-aether model there will be additional terms in the FE which include (see the technical details in the Section 2.2):

- The effects on the geometry from the anisotropy and inhomogeneities (e.g., the curvature) of the spherically symmetric models under consideration.

- The Einstein FE are generalised by the contribution of an additional stress tensor, $T_{ab}^{æ}$, for the aether field which depends on the dimensionless parameters of the aether model (e.g., "the c_i"). In GR, all of the $c_i = 0$. To study the effects of matter, we could perhaps assume the corresponding GR values (or close to them) in the first instance.

- When the phenomenology of theories with a preferred frame is studied, it is generally assumed that this frame coincides, at least roughly, with the cosmological rest frame

defined by the Hubble expansion of the universe. In particular, in an isotropic and spatially homogeneous Friedmann universe the aether field will be aligned with the (natural preferred CMB rest frame) cosmic frame and is thus related to the expansion rate of the universe. In principle, the preferred frame determined by the aether can be different from (i.e., tilted with respect to) the CMB rest frame in spherically symmetric models. This adds additional terms to the aether stress tensor $T_{ab}^{æ}$, which can be characterized by a hyperbolic tilt angle, $v(t)$, measuring the boost of the aether relative to the (perfect fluid) CMB rest frame [4, 5]. The tilt is expected to decay to the future in anisotropic but spatially homogeneous models [23].

1.2 Spherical symmetry

All spherically symmetric aether fields are hypersurface orthogonal and, hence, all spherically symmetric solutions of aether theory will also be solutions of the IR limit of Horava gravity. The converse is not true in general, but it does hold in spherical symmetry for solutions with a regular center [8].

The c_i are dimensionless constants in the model. When spherical symmetry is imposed the aether is hypersurface orthogonal, and so it has vanishing twist. Thus it is possible to set c_4 to zero without loss of generality [1]. After the parameter redefinition to eliminate c_4, one is left with a 3- dimensional parameter space. The c_i contribute to the effective Newtonian gravitational constant G; so a renormalization of the parameters in the model can be then used to set $8\pi G = 1$ (i.e., another condition on the c_i can effectively be specified). The remaining parameters in the model can be characterized by two non-trivial constant parameters. The other constraints imposed on the c_i have been summarized in [1] (e.g., see equations 43-46 in [24]; also see Appendix A). In GR $c_i = 0$. We shall study the qualitative properties of models with values for the non-GR parameters which are consistent with current constraints.

Some of the models studied in this paper involve a static metric coupled to a stationary aether. This situation will be referred to here as "stationary spherical symmetry". This case will be treated separately later. An important special case occurs when the aether is parallel to the Killing vector. We refer to this special case as a "static aether". A spherically symmetric static vacuum solution is known explicitly [25].

1.3 Stars and black holes

Spherically symmetric static and stationary solutions are physical important. Unlike GR, Einstein-aether theory has a spherically symmetric mode, corresponding to radial tilting of the aether. The time-independent spherically symmetric solutions and black holes were studied in [25] and [26], respectively, and surveyed in [1], and recently revisited for a more viable coupling parameter c_1 in [25] and [24]. In general, within this same parameter space, the dynamics of the cosmological scale factor and perturbations differ little from GR, and non-rotating neutron star and black hole solutions are quite close to those of GR. A thorough examination of the fully nonlinear solutions has not been carried out to date. A fully nonlinear energy positivity has, however, been established for spherically symmetric solutions at a moment of time symmetry [27].

Let us discuss this in more detail. There is a three-parameter family of spherically symmetric static vacuum solutions [25]. In the Einstein-aether theory the aether vector and its derivative provide two additional degrees of freedom at each point. If asymptotic flatness is imposed and the mass is fixed, there remains a one-parameter family (i.e., imposing

asymptotic flatness reduces this to a two parameter family [28]), whereas GR has the unique Schwarzschild solution (Birkhoff's theorem). In GR asymptotic flatness is a consequence of the vacuum field equations without any tuning of initial data, so the one-parameter family of local (Schwarzschild) solutions is automatically asymptotically flat. The radial tilt of the aether provides another local degree of freedom in aether theory, so spherical solutions need not be time-independent (even when restricting to stationary spherically symmetric aether theory). Not only are spherical solutions not generally static, but even if we restrict to static, spherical solutions, they are not necessarily asymptotically flat. It was shown in [3] that the Reissner-Nordstrom metric in a spherically symmetric static gauge with fixed norm is a solution, although this is not the only solution in that special case [25].

Requiring that the aether be aligned with the timelike Killing field restricts the static aether solution to one parameter (the single parameter c_{14}, essentially the total mass [25]). Thus the solution outside a static star is the unique vacuum solution for a given mass in the static aether case [25], and is asymptotically flat. In [29] it was found that this static "wormhole" aether solution is generally stable to linear perturbations under the same conditions as for flat spacetime. In the pure GR limit ($c_1 = 0$), we have just the Schwarzschild solution. For small values of r, the solutions can behave quite differently from the Schwarzschild solution. More recently, an analytic static spherically symmetric vacuum solution in the Einstein-aether theory was presented (demonstrated numerically) by use of the Euler-Lagrange equations [30].

Unlike the singular wormhole, the static solutions have a regular origin [25]. It is known that pure aether stars do not exist; i.e., there are no asymptotically flat self-gravitating aether solutions with a regular origin [25]. It has been shown that in the presence of a perfect fluid, regular asymptotically flat star solutions exist and are parameterized (for a given equation of state) by the central pressure (see also [31]).

For black holes the aether cannot be aligned with the Killing vector, since the latter is not timelike on and inside the horizon. Instead, the aether is at rest at spatial infinity and flows inward at finite radii. The condition of regularity (at the spin-0 horizon) selects a unique solution from the one-parameter family of spherical stationary solutions for a given mass [25, 26]. Such black holes are rather close to Schwarzschild outside the horizon for a wide range of couplings. Inside the horizon the solutions differ more (but typically no more than a few percent), and like the Schwarzschild solution they contain a spacelike singularity.

More recently, static spherically symmetric, asymptotically flat, regular (non-rotating) black-hole solutions in Einstein-aether theory have been studied (numerically) [24], generalizing previous results. It has been found that spherical black-hole solutions formed by gravitational collapse exist for all viable parameter values of the theory and a notion of black hole thus persists. Indeed, static spherically symmetric solutions in Lorentz-violating theories, in which the causal structure of gravity is greatly modified, still possess a special hypersurface, called a "Universal horizon", that acts as a genuine absolute causal boundary because it traps all excitations, even those which could be traveling at arbitrarily high propagation speeds [24]. The Universal horizon satisfies a first law of black-hole mechanics [32], and evidence has been found that Hawking radiation is associated with the Universal horizon [33, 34].

Finally, it would be of interest to determine the structure of rotating solutions; rapidly rotating black holes, unlike the non-rotating ones, might turn out to be very different from the Kerr metrics of GR.

2 Spherically symmetric Einstein-aether Models

We shall use the 1+3 frame formalism [19, 20] to write down the evolution equations for spherically symmetric models as a well-posed system of first order PDEs in two variables. The formalism is particularly well-suited for studying perfect fluid spherically symmetric models [21], and especially for numerical and qualitative analysis [22]. We follow a similar approach to that in the resource paper [35] (wherein all relevant quantitites are explicitly defined).

2.1 Restrictions on the kinematic and auxiliary variables:

The metric is:

$$ds^2 = -N^2 dt^2 + (e_1{}^1)^{-2} dx^2 + (e_2{}^2)^{-2}(d\vartheta^2 + \sin^2\vartheta d\varphi^2). \tag{2.1}$$

The Killing vector fields (KVF) are given by [36]:

$$\partial_\varphi, \quad \cos\varphi \; \partial_\vartheta - \sin\varphi \cot\vartheta \; \partial_\varphi, \quad \sin\varphi \; \partial_\vartheta + \cos\varphi \cot\vartheta \; \partial_\varphi. \tag{2.2}$$

The frame vectors in coordinate form are:

$$\mathbf{e}_0 = N^{-1}\partial_t, \quad \mathbf{e}_1 = e_1{}^1\partial_x, \quad \mathbf{e}_2 = e_2{}^2\partial_\vartheta, \quad \mathbf{e}_3 = e_3{}^3\partial_\varphi, \tag{2.3}$$

where $e_3{}^3 = e_2{}^2/\sin\vartheta$. N, $e_1{}^1$ and $e_2{}^2$ are functions of t and x.

This leads to the following restrictions on the kinematic variables:

$$\sigma_{\alpha\beta} = \mathrm{diag}(-2\sigma_+, \sigma_+, \sigma_+), \quad \omega_{\alpha\beta} = 0, \quad \dot{u}_\alpha = (\dot{u}_1, 0, 0), \tag{2.4}$$

where

$$\dot{u}_\alpha = u^\beta \nabla_\beta u_\alpha; \tag{2.5}$$

$$\dot{u}_1 = \mathbf{e}_1 \ln N; \tag{2.6}$$

on the spatial commutation functions:

$$a_\alpha = (a_1, a_2, 0), \quad n_{\alpha\beta} = \begin{pmatrix} 0 & 0 & n_{13} \\ 0 & 0 & 0 \\ n_{13} & 0 & 0 \end{pmatrix}, \tag{2.7}$$

where

$$a_1 = \mathbf{e}_1 \ln e_2{}^2, \quad a_2 = n_{13} = -\frac{1}{2}e_2{}^2 \cot\vartheta; \tag{2.8}$$

and on the matter components:

$$q_\alpha = (q_1, 0, 0), \quad \pi_{\alpha\beta} = \mathrm{diag}(-2\pi_+, \pi_+, \pi_+). \tag{2.9}$$

The frame rotation $\Omega_{\alpha\beta}$ is also zero.

Furthermore, n_{13} only appears in the equations together with $\mathbf{e}_2 n_{13}$ in the form of the Gauss curvature of the spheres

$$^2K := 2(\mathbf{e}_2 - 2n_{13})n_{13}, \tag{2.10}$$

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which simplifies to

$$^2K = (e_2{}^2)^2. \tag{2.11}$$

Thus the dependence on ϑ is hidden in the equations. We will also use ^2K in place of $e_2{}^2$.

To simplify notation, we will write

$$^2K, \ \dot{u}_1, \ a_1$$

as

$$K, \ \dot{u}, \ a.$$

To summarize, the essential variables are

$$N, e_1{}^1, \ K, \ \theta, \ \sigma_+, \ a, \ \dot{u}, \ \mu, \ q_1, \ p, \ \pi_+, \tag{2.12}$$

where N is the lapse function, $e_1{}^1$ is the non null component of the frame vector $\mathbf{e}_1$, K is the Gauss curvature of the spheres, θ is the (volume) rate of expansion scalar, σ_+ is related to the magnitude of the rate of shear tensor (a measure of the anisotropies present in the model), a is the radial component of the object (spatial commutation function) a_α, $\dot{u}$ is the acceleration, μ denotes the total energy density scalar, q_1 is a component of the total energy current density vector, p is the total isotropic pressure scalar and π_+ is related to the magnitude of the total anisotropic pressure tensor [19, 20].

In the case of spherical symmetry in Einstein-aether theory one must be careful in choosing the gauge (an additional gauge condition). Normally, in GR, spherically symmetric coordinates are chosen so that the metric is simplified (e.g., a choice for N) or so that the fluid is comoving. Here we chose the aether vector field to be aligned with the timelike frame vector $\mathbf{e}_0$ (the comoving aether gauge, and hence in general $N(t,x)$ cannot be simplified any further). This may make comparisons with GR difficult in some special cases. Our formulation is perhaps better suited for fluids/matter and cosmology, although the static case is not necessarily aligned (see later).

We note that the tilt is defined relative to matter; one important question is to investigate whether this tilt decays to the future.

2.2 Einstein-aether theory

The action for Einstein-aether theory is the most general generally covariant functional of the spacetime metric g_{ab} and aether field u^a involving no more than two derivatives (not including total derivatives) [1, 27]. The action is [1, 37]:

$$S = \int d^4x \sqrt{-g} \left[\frac{1}{2}R - K^{ab}{}_{cd}\nabla_a u^c \nabla_b u^d + \lambda \left(u^c u_c + 1 \right) + \mathcal{L}_m \right], \tag{2.13}$$

where

$$K^{ab}{}_{cd} \equiv c_1 g^{ab} g_{cd} + c_2 \delta^a_c \delta^b_d + c_3 \delta^a_d \delta^b_c + c_4 u^a u^b g_{cd}. \tag{2.14}$$

The action (2.13) contains an Einstein-Hilbert term for the metric, a kinetic term for the aether with four dimensionless coefficients c_i, and λ is a Lagrange multiplier enforcing the time-like constraint on the aether. ^{2} The convention used in this paper for the metric

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^{2}We set the vector norm to unity in order to obtain a unit time-like aether. Comparing with [27, 39], the tensor K^{ab}_{mn} was rescaled by a factor of 2, c_4 was taken with the opposite sign, and J_{ab} was redefined taking the opposite sign (i.e., the constant c's here and λ have been rescaled by a factor of 2).

signature is $(-+++)$ and the units are chosen so that the speed of light defined by the metric g_{ab} is unity and $\kappa^2 \equiv 8\pi G = 1$. The field equations from varying (2.13) with respect to g^{ab}, u^a, and λ are given, respectively, by [39]:

$$G_{ab} = T_{ab}^{TOT} \tag{2.15}$$

$$\lambda u_b = \nabla_a J^a{}_b + c_4 \dot{u}_a \nabla_b u^a \tag{2.16}$$

$$u^a u_a = -1. \tag{2.17}$$

Here G_{ab} is the Einstein tensor of the metric g_{ab}. T_{ab}^{TOT} is the *total* energy momentum tensor, $T_{ab}^{TOT} = T_{ab}^{æ} + T_{ab}^{mat}$, where T_{ab}^{mat} is the total contribution from all matter sources. We shall omit T_{ab}^{mat} for the moment (and add in later for perfect fluid and scalar field sources), and so we begin with the *vacuum* case ($\mathcal{L}_m = 0$) first with a non-trivial aether stress-energy $T_{ab}^{æ}$ (which we will refer to as "pure" Einstein-aether theory which is a theory of the spacetime metric g_{ab} and a vector field (the "aether") u^a).

The quantities $J^a{}_b$, $\dot{u}_a$ and the aether stress-energy $T_{ab}^{æ}$ are given by

$$J^a{}_m = -K^{ab}{}_{mn}\nabla_b u^n \tag{2.18a}$$

$$\dot{u}_a = u^b \nabla_b u_a \tag{2.18b}$$

$$\begin{aligned}
T_{ab}^{æ} &= 2c_1(\nabla_a u^c \nabla_b u_c - \nabla^c u_a \nabla_c u_b) \\
&\quad - 2[\nabla_c(u_{(a}J^c{}_{b)}) + \nabla_c(u^c J_{(ab)}) - \nabla_c(u_{(a}J_{b)}{}^c)] - 2c_4\dot{u}_a\dot{u}_b + \\
&\quad + 2\lambda u_a u_b + g_{ab}\mathcal{L}_u
\end{aligned} \tag{2.18c}$$

where

$$\mathcal{L}_u \equiv -K^{ab}{}_{cd}\nabla_a u^c \nabla_b u^d, \tag{2.19}$$

is the Einstein-aether Lagrangian [40].

Taking the contraction of (2.16) with u^b and with the induced metric $h^{bc} := g^{bc} + u^b u^c$ we obtain the equations

$$\lambda = -u^b \nabla_a J^a_b - c_4 \dot{u}_a \dot{u}^a, \tag{2.20a}$$

$$0 = h^{bc}\nabla_a J^a_b + c_4 h^{bc}\dot{u}_a \nabla_b u^a. \tag{2.20b}$$

We shall use the equation (2.20a) as a definition for the Lagrange multiplier, whereas the second equation (2.20b) leads to a set of restrictions that the aether vector must satisfy.

The Einstein FE, Jacobi identities and contracted Bianchi identities gives a system of partial differential equations on the frame and commutator functions, while (2.18c) defines the components of the energy momentum tensor and (2.20b) gives one extra equation for the aether. We choose a gauge in which the aether is aligned with $\mathbf{e}_0$, the comoving aether

temporal gauge (all that then remains is the time and space reparameterization freedom): [3]

$$\mathbf{e}_0(e_1{}^1) = -\tfrac{1}{3}(\theta - 6\sigma_+)e_1{}^1, \tag{2.21a}$$

$$\mathbf{e}_0(K) = -\tfrac{2}{3}(\theta + 3\sigma_+)K, \tag{2.21b}$$

$$\mathbf{e}_0(\theta) - \mathbf{e}_1(\dot{u}) = -\tfrac{1}{3}\theta^2 - 6\sigma_+^2 + (\dot{u} - 2a)\dot{u} - \frac{1}{2}(\mu + 3p), \tag{2.21c}$$

$$\mathbf{e}_0(\sigma_+) - \frac{1}{3}\mathbf{e}_1(\dot{u} - a) = -\theta\sigma_+ - \frac{1}{3}(a + \dot{u})\dot{u} - \frac{1}{3}K + \pi_+, \tag{2.21d}$$

$$\mathbf{e}_0(a) = -\tfrac{1}{3}(\theta + 3\sigma_+)(a + \dot{u}) - \tfrac{1}{2}q_1, \tag{2.21e}$$

$$\mathbf{e}_0(\mu) + \mathbf{e}_1(q_1) = -\theta(\mu + p) + 2(a - \dot{u})q_1 - 6\sigma_+\pi_+, \tag{2.21f}$$

$$\mathbf{e}_0(q_1) + \mathbf{e}_1(p) = -\tfrac{2}{3}(2\theta - 3\sigma_+)q_1 - 2(3a - \dot{u} - \mathbf{e}_1)\pi_+ - \dot{u}(\mu + p), \tag{2.21g}$$

Constraints:

$$\mathbf{e}_1(\ln N) = \dot{u}, \tag{2.22a}$$

$$\mathbf{e}_1(\ln K) = 2a, \tag{2.22b}$$

$$\mu = 3H^2 - 3\sigma_+^2 + K - 3a^2 + 2\mathbf{e}_1(a), \tag{2.22c}$$

$$q_1 = -6a\sigma_+ + \tfrac{2}{3}\mathbf{e}_1(\theta + 3\sigma_+), \tag{2.22d}$$

where [4] $(\mu, \ p, \ q_1, \ \pi_+) = (\mu_{\text{æ}}, \ p_{\text{æ}}, \ q_{\text{æ}}, \ \pi_{\text{æ}})$ can be computed from (2.18c):

$$\mu_{\text{æ}} = (c_1 - c_4)(2\mathbf{e}_1 - 4a + \dot{u})\dot{u} - \tfrac{1}{3}(c_1 + 3c_2 + c_3)\theta^2 - 6(c_1 + c_3)\sigma_+^2, \tag{2.23a}$$

$$p_{\text{æ}} = \tfrac{1}{3}(c_1 + 3c_2 + c_3)(2\mathbf{e}_0 + \theta)\theta - 6(c_1 + c_3)\sigma_+^2 + \tfrac{1}{3}(c_1 - c_4)\dot{u}^2, \tag{2.23b}$$

$$q_{\text{æ}} = -\tfrac{4}{3}(c_1 - c_4)(\theta + 3\sigma_+)\dot{u} - 2(c_1 - c_4)\mathbf{e}_0(\dot{u}), \tag{2.23c}$$

$$\pi_{\text{æ}} = \tfrac{2}{3}(c_1 - c_4)\dot{u}^2 + 2(c_1 + c_3)(\mathbf{e}_0 + \theta)\sigma_+. \tag{2.23d}$$

The aether equation (2.20b) becomes (and is true regardless of whether $\dot{u}$ is zero or not)

$$\begin{aligned} (c_1 - c_4)\mathbf{e}_0(\dot{u}) = &-\tfrac{2}{3}(c_1 - c_4)(\theta + 3\sigma_+)\dot{u} + 6(c_1 + c_3)a\sigma_+ \\ &+ \tfrac{1}{3}(c_1 + 3c_2 + c_3)\mathbf{e}_1(\theta) - 2(c_1 + c_3)\mathbf{e}_1(\sigma_+). \end{aligned} \tag{2.24}$$

To simplify these expressions it is convenient to make a reparameterization of the aether parameters, analogous to the one given in [8]:

$$c_\theta = c_2 + (c_1 + c_3)/3, \ c_\sigma = c_1 + c_3, \ c_\omega = c_1 - c_3, \ c_a = c_4 - c_1,$$

where the new parameters correspond to terms in the Lagrangian relating to expansion, shear, acceleration and twist of the aether. Since the spherically symmetric models are hypersurface orthogonal the aether field has vanishing twist and is therefore independent of the twist parameter c_ω (the coupling $c_1 - c_3$ does not occur in the field equations (only $c_1 + c_3$ does) [8]; this is equivalent to being able to set $c_4 = 0$ [1]).

A second condition on the c_i can effectively be specified by a renormalization of the Newtonian gravitational constant G. The remaining parameters in the model can therefore

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be characterized by two non-trivial constant parameters. The other constraints imposed on the c_i have been summarized in [1].

It may be useful later to define $c^2 \equiv 1 - 2c_\sigma \leq 1$. In particular, some special cases of interest are (see the Appendix A): **case A:** $c_\sigma = \frac{1}{2}(1 - c^2) \geq 0, c_a = -\frac{d}{(1+d)}c_\sigma \leq 0, c_\theta = 0$: **case B(ii):** $c_\sigma = \frac{1}{2}(1 - c^2) \geq 0, c_a = -\frac{1}{2}(1 - c^2), c_\theta = 0$: **case C:** $c_\sigma = \frac{1}{2}(1 - c^2) \geq 0, c_\theta = -\frac{1}{3}(1 - c^2) \leq 0, [c_a = 0]$

The Lagrangian (2.19) becomes

$$\mathcal{L}_u = -\left(c_a \dot{u}^2 + c_\theta \theta^2 + 6c_\sigma \sigma_+^2 \right) \tag{2.25}$$

and the aether energy components $(\mu, \ p, \ q_1, \ \pi_+) = (\mu_æ, \ p_æ, \ q_æ, \ \pi_æ)$ become

$$\mu_æ = -c_\theta \theta^2 - 6c_\sigma \sigma_+^2 - c_a(\dot{u} + 2\mathbf{e}_1 - 4a)\dot{u}, \tag{2.26a}$$

$$p_æ = -\tfrac{1}{3}c_a \dot{u}^2 - 6c_\sigma \sigma_+^2 + c_\theta(2\mathbf{e}_0 + \theta)\theta, \tag{2.26b}$$

$$q_æ = \tfrac{4}{3}c_a(\theta + 3\sigma_+)\dot{u} + 2c_a \mathbf{e}_0(\dot{u}), \tag{2.26c}$$

$$\pi_æ = -\tfrac{2}{3}c_a \dot{u}^2 + 2c_\sigma(\mathbf{e}_0 + \theta)\sigma_+. \tag{2.26d}$$

and the aether equation (2.20b) reads

$$c_a \mathbf{e}_0(\dot{u}) = -\tfrac{2}{3}c_a(\theta + 3\sigma_+)\dot{u} - 6c_\sigma a\sigma_+ - \mathbf{e}_1(c_\theta \theta - 2c_\sigma \sigma_+). \tag{2.27}$$

Combining all of the above equations, and assuming $\dot{u} \neq 0$ (the special case $\dot{u} = 0$ will be dealt with later), we obtain

$$\mathbf{e}_0(e_1{}^1) = -\tfrac{1}{3}(\theta - 6\sigma_+)e_1{}^1, \tag{2.28a}$$

$$\mathbf{e}_0(K) = -\tfrac{2}{3}(\theta + 3\sigma_+)K, \tag{2.28b}$$

$$\mathbf{e}_0(\dot{u}) - \frac{(3c_\theta + 2c_\sigma)\,\mathbf{e}_1(\theta)}{3c_a\,(2c_\sigma - 1)} = -\tfrac{2}{3}\dot{u}(\theta + 3\sigma_+), \tag{2.28c}$$

$$\mathbf{e}_0(\theta) - \frac{(c_a + 1)\,\mathbf{e}_1(\dot{u})}{(3c_\theta + 1)} = -\tfrac{1}{3}\theta^2 + \frac{(c_a + 1)\,\dot{u}^2}{(3c_\theta + 1)} - \frac{2\,(c_a + 1)\,a\dot{u}}{(3c_\theta + 1)} + \frac{6\,(2c_\sigma - 1)\,\sigma_+^2}{(3c_\theta + 1)}, \tag{2.28d}$$

$$\mathbf{e}_0(\sigma_+) - \frac{(c_a + 1)\,\mathbf{e}_1(\dot{u})}{3\,(2c_\sigma - 1)} = -\theta\sigma_+ + \tfrac{1}{2}\sigma_+^2 + \frac{(3c_\theta + 1)\,\theta^2}{18\,(2c_\sigma - 1)} + \frac{(1 - 2c_a)\,a\dot{u}}{3\,(2c_\sigma - 1)} +$$
$$+ \frac{(5c_a + 2)\,\dot{u}^2}{6\,(2c_\sigma - 1)} - \frac{a^2}{2\,(2c_\sigma - 1)} + \frac{K}{2\,(2c_\sigma - 1)}, \tag{2.28e}$$

$$\mathbf{e}_0(a) + \frac{(3c_\theta + 2c_\sigma)\,\mathbf{e}_1(\theta)}{6c_\sigma - 3} = -\tfrac{1}{3}(a + \dot{u})(\theta + 3\sigma_+). \tag{2.28f}$$

Constraints:

$$\mathbf{e}_1(\ln N) = \dot{u}, \tag{2.29a}$$

$$\mathbf{e}_1(\ln K) = 2a, \tag{2.29b}$$

$$\mathbf{e}_1(a) + c_a \mathbf{e}_1(\dot{u}) = -\tfrac{1}{6}\,(3c_\theta + 1)\,\theta^2 - \frac{3}{2}\,(2c_\sigma - 1)\,\sigma_+^2 - \frac{K}{2} + \frac{3a^2}{2} + 2c_a a\dot{u} - \frac{c_a \dot{u}^2}{2}, \tag{2.29c}$$

$$\mathbf{e}_1(\sigma_+) - \frac{(3c_\theta + 1)\,\mathbf{e}_1(\theta)}{6c_\sigma - 3} = 3a\sigma_+. \tag{2.29d}$$

Commutator:

$$[\mathbf{e}_0, \mathbf{e}_1] = \dot{u}\mathbf{e}_0 - \tfrac{1}{3}(\theta - 6\sigma_+)\mathbf{e}_1. \tag{2.30}$$

Integrability conditions. In the Einstein-aether analysis, one of the "field equations" is the spatial projection (with the induced metric h^{bc}) of the equation obtained by the contraction of the velocity variation of the action. In many cases this equation does not involve the appropriate time derivatives, and hence this equation is not an evolution equation, but rather it is a constraint [41]. In the spherically symmetric case here it can be shown that the constraint is conserved and is compatible with all of the other (evolution) equations.

3   Aether and a tilted perfect fluid

The energy momentum-tensor for the matter field is

$$T_{ab}^m \equiv -2\frac{\delta\mathcal{L}_m}{\delta g^{ab}} + \mathcal{L}_m g_{ab} = \hat{\mu}u_a u_b + \hat{p}(g_{ab} + u_a u_b). \tag{3.1}$$

with $\hat{p}$ to be specified. In general, the 4-velocity vector $\mathbf{u}$ of the perfect fluid is not aligned with the vector $\mathbf{e}_0$ of a chosen temporal gauge. In spherically symmetric models, $\mathbf{u}$ is allowed to be of the form
$$\mathbf{u} = \Gamma(\mathbf{e}_0 + v\mathbf{e}_1), \quad \Gamma = (1 - v^2)^{-\frac{1}{2}}, \tag{3.2}$$
where v is the tilt parameter. We choose a linear equation of state for the perfect fluid:

$$\hat{p} = (\gamma - 1)\hat{\mu}, \tag{3.3}$$

where γ is a constant satisfying $1 \leq \gamma < 2$. Then we obtain for the tilted fluid:

$$\mu = \frac{G_+}{1 - v^2}\hat{\mu} \tag{3.4a}$$

$$p = \frac{(\gamma - 1)(1 - v^2) + \frac{1}{3}\gamma v^2}{1 - v^2}\hat{\mu} \tag{3.4b}$$

$$q_1 = \frac{\gamma\hat{\mu}}{1 - v^2}v \tag{3.4c}$$

$$\pi_+ = -\frac{1}{3}\frac{\gamma\hat{\mu}}{1 - v^2}v^2, \tag{3.4d}$$

where $G_\pm = 1 \pm (\gamma - 1)v^2$. Thus (the total) μ, p, q_1 and π_+ are given in terms of $\hat{\mu}$ and v. These are then substituted into the evolution and constraint equations.

Assuming $\dot{u} \neq 0$ (and under the general conditions that $2c_\sigma - 1 \neq 0, 3c_\theta - 1 \neq 0$) we

obtain:

$$\mathbf{e}_0(e_1{}^1) = -\tfrac{1}{3}(\theta - 6\sigma_+)e_1{}^1, \tag{3.5a}$$

$$\mathbf{e}_0(K) = -\tfrac{2}{3}(\theta + 3\sigma_+)K, \tag{3.5b}$$

$$\mathbf{e}_0(\dot{u}) - \frac{(3c_\theta + 2c_\sigma)\,\mathbf{e}_1(\theta)}{3c_a\,(2c_\sigma - 1)} = -\tfrac{2}{3}\dot{u}(\theta + 3\sigma_+) + \frac{\gamma c_\sigma \hat{\mu} v}{c_a\,(2c_\sigma - 1)\,(1 - v^2)}, \tag{3.5c}$$

$$\mathbf{e}_0(\theta) - \frac{(c_a + 1)\,\mathbf{e}_1(\dot{u})}{(3c_\theta + 1)} = -\tfrac{1}{3}\theta^2 + \frac{(c_a + 1)\,\dot{u}^2}{(3c_\theta + 1)} - \frac{2\,(c_a + 1)\,a\dot{u}}{(3c_\theta + 1)} + \frac{6\,(2c_\sigma - 1)\,\sigma_+^2}{(3c_\theta + 1)} +$$

$$+ \frac{\hat{\mu}\left((\gamma - 2)v^2 - 3\gamma + 2\right)}{2\,(3c_\theta + 1)\,(1 - v^2)}, \tag{3.5d}$$

$$\mathbf{e}_0(\sigma_+) - \frac{(c_a + 1)\,\mathbf{e}_1(\dot{u})}{3\,(2c_\sigma - 1)} = -\theta\sigma_+ + \frac{1}{2}\sigma_+^2 + \frac{(3c_\theta + 1)\,\theta^2}{18\,(2c_\sigma - 1)} + \frac{(1 - 2c_a)\,a\dot{u}}{3\,(2c_\sigma - 1)} +$$

$$+ \frac{(5c_a + 2)\,\dot{u}^2}{6\,(2c_\sigma - 1)} - \frac{a^2}{2\,(2c_\sigma - 1)} + \frac{K}{2\,(2c_\sigma - 1)} + \frac{\hat{\mu}\left((\gamma + 1)v^2 - 1\right)}{6\,(2c_\sigma - 1)\,(1 - v^2)}, \tag{3.5e}$$

$$\mathbf{e}_0(a) + \frac{(3c_\theta + 2c_\sigma)\,\mathbf{e}_1(\theta)}{3(2c_\sigma - 1)} = -\tfrac{1}{3}(a + \dot{u})(\theta + 3\sigma_+) - \frac{\gamma \hat{\mu} v}{2\,(2c_\sigma - 1)\,(1 - v^2)}, \tag{3.5f}$$

$$\mathbf{e}_0(\hat{\mu}) - \frac{\mathbf{e}_1\,(\hat{\mu})\,v\,(2 - \gamma)}{G_-} - \frac{\gamma \hat{\mu} \mathbf{e}_1\,(v)}{G_-} = -\frac{2\gamma \hat{\mu} v^2 \sigma_+}{G_-} + \frac{\gamma\,(v^2 - 3)\,\hat{\mu}\theta}{3G_-} - 2\frac{\gamma \hat{\mu} v a}{G_-}, \tag{3.5g}$$

$$\mathbf{e}_0(v) - \frac{\mathbf{e}_1\,(\hat{\mu})\,(1 - v^2)^2\,(\gamma - 1)}{\gamma \hat{\mu} G_-} - \frac{v\,(2 - \gamma)\,\mathbf{e}_1\,(v)}{G_-} = \frac{2v\,(1 - v^2)\,\sigma_+}{G_-} +$$

$$\frac{v\,(1 - v^2)\,(3\gamma - 4)\,\theta}{3G_-} + \frac{2v^2\,(1 - v^2)\,(\gamma - 1)\,a}{G_-} + (1 - v^2)\dot{u}. \tag{3.5h}$$

Constraints:

$$\mathbf{e}_1(\ln N) = \dot{u}, \tag{3.6a}$$

$$\mathbf{e}_1(\ln K) = 2a, \tag{3.6b}$$

$$\mathbf{e}_1(a) + c_a \mathbf{e}_1(\dot{u}) = \frac{G_+ \hat{\mu}}{2(1 - v^2)} - \frac{1}{6}\,(3c_\theta + 1)\,\theta^2 - \frac{3}{2}\,(2c_\sigma - 1)\,\sigma_+^2 +$$

$$- \frac{K}{2} + \frac{3a^2}{2} + 2c_a a\dot{u} - \frac{c_a \dot{u}^2}{2}, \tag{3.6c}$$

$$\mathbf{e}_1(\sigma_+) - \frac{(3c_\theta + 1)\,\mathbf{e}_1(\theta)}{3(2c_\sigma - 1)} = 3a\sigma_+ + \frac{\gamma \hat{\mu} v}{2\,(2c_\sigma - 1)\,(1 - v^2)}. \tag{3.6d}$$

3.1   Well-posedness

We now show that the system of evolution equations plus restrictions for the state vector

$$\left[e_1{}^1, K, \dot{u}, \theta, \sigma_+, a, \hat{\mu}, v\right]^T$$

is well-posed for $\gamma \geq 1$. The coefficient matrix for the spatial derivative terms (for $[\dot{u}, \theta, \sigma_+, a, \hat{\mu}, v]^T$) is: [5]

$$
\begin{pmatrix}
0 & -\frac{(3c_\theta + 2c_\sigma)}{3c_a(2c_\sigma - 1)} & 0 & 0 & 0 & 0 \\
-\frac{c_a + 1}{(3c_\theta + 1)} & 0 & 0 & 0 & 0 & 0 \\
-\frac{c_a + 1}{3(2c_\sigma - 1)} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{3c_\theta + 2c_\sigma}{3(2c_\sigma - 1)} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{v(2 - \gamma)}{(\gamma - 1)v^2 - 1} & \frac{\gamma\hat{\mu}}{(\gamma - 1)v^2 - 1} \\
0 & 0 & 0 & 0 & \frac{(1 - v^2)^2(\gamma - 1)}{((\gamma - 1)v^2 - 1)\gamma\hat{\mu}} & \frac{v(2 - \gamma)}{(\gamma - 1)v^2 - 1}
\end{pmatrix}.
\tag{3.7}
$$

Its eigenvalues are

$$
0, 0, \pm\sqrt{\frac{(c_a + 1)(3c_\theta + 2c_\sigma)}{3c_a(3c_\theta + 1)(2c_\sigma - 1)}}, \quad -\frac{(2 - \gamma)v \pm \sqrt{\gamma - 1}(1 - v^2)}{G_-},
\tag{3.8}
$$

with corresponding eigenvectors (for example)

$$
\begin{pmatrix}
-\frac{1}{c_a} \\
-\sqrt{\frac{(c_a+1)(6c_\sigma - 3)}{c_a(3c_\theta+1)(3c_\theta+2c_\sigma)}} \\
-\sqrt{\frac{(c_a+1)(3c_\theta+1)}{c_a(3c_\theta+2c_\sigma)(6c_\sigma - 3)}} \\
1 \\
0 \\
0
\end{pmatrix},
\begin{pmatrix}
-\frac{1}{c_a} \\
\sqrt{\frac{(c_a+1)(6c_\sigma - 3)}{c_a(3c_\theta+1)(3c_\theta+2c_\sigma)}} \\
\sqrt{\frac{(c_a+1)(3c_\theta+1)}{c_a(3c_\theta+2c_\sigma)(6c_\sigma - 3)}} \\
1 \\
0 \\
0
\end{pmatrix},
\begin{pmatrix}
0 \\
0 \\
1 \\
0 \\
0 \\
0
\end{pmatrix},
$$

$$
\begin{pmatrix}
0 \\
0 \\
0 \\
1 \\
0 \\
0
\end{pmatrix},
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
\frac{\gamma\hat{\mu}}{(1-v^2)\sqrt{\gamma - 1}} \\
1
\end{pmatrix},
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
-\frac{\gamma\hat{\mu}}{(1-v^2)\sqrt{\gamma - 1}} \\
1
\end{pmatrix}.
\tag{3.9}
$$

The matrix is diagonalizable (all its eigenvalues are real) for $\gamma > 1$ and $\frac{(c_a+1)(3c_\theta+2c_\sigma)}{c_a(3c_\theta+1)(6c_\sigma - 3)} \geq 0$, with $c_s = \sqrt{\gamma - 1}$ being the speed of sound in the perfect fluid. The system (3.5a)-(3.5h) is thus well-posed if the above conditions are fulfilled, otherwise the system is elliptic and not well-posed.

3.2 Normalized variables

We introduce the normalized variables (for $\dot{u} \neq 0$ using the β-normalization):

$$
\left\{ \mathcal{N}^{-1}, E_1^1, \mathcal{Q}, \Sigma, \mathcal{A}, \dot{U} \right\} = \left\{ N^{-1}, e_1^1, \tfrac{\theta}{3}, \sigma_+, a, \dot{u} \right\} / \beta
$$

$$
\left\{ \Omega, \Omega_k, \mathcal{K}, \mathcal{S}_+ \right\} = \left\{ \hat{\mu}, -\tfrac{1}{2}{}^3R, K, {}^3S_+ \right\} / (3\beta^2),
$$

where $\beta = \frac{1}{3}(\theta + 3\sigma_+)$ and

$$
{}^3R = 4\mathbf{e}_1 a - 6a^2 + 2K, \quad {}^3S_+ = -\tfrac{1}{3}\mathbf{e}_1 a + \tfrac{1}{3}K.
$$

---

[5] Strictly speaking, we should also include the factor $e_1{}^1$ in the matrix, but the result on well-posedness is the same.

By definition $\mathcal{Q} + \Sigma = 1$. In the above we assume that $\beta \neq 0$. In general the variables are unbounded. However, physically $\Omega \geq 0$, and if the expansion and the shear are both positive, then $0 \leq \mathcal{Q} \leq 1$.

We also introduce the normalized differential operators,

$$\boldsymbol{\partial}_\alpha := \frac{\mathbf{e}_\alpha}{\beta} \quad ; \text{ where } \quad \alpha = 0, 1.$$

Moreover, we define q and r analogous to the usual volume deceleration parameter and "Hubble spatial gradient" as follows

$$\boldsymbol{\partial}_0 \beta := -(1+q)\beta, \quad \boldsymbol{\partial}_1 \beta := -r\beta. \tag{3.10}$$

so that

$$\boldsymbol{\partial}_0 E_1{}^1 \equiv E_1{}^1 (1+q) + \frac{1}{\beta^2}\mathbf{e}_0(e_1{}^1), \quad \boldsymbol{\partial}_0 \mathcal{Q} \equiv \mathcal{Q}(1+q) + \frac{1}{3\beta^2}\mathbf{e}_0(\theta),$$

$$\boldsymbol{\partial}_0 \Sigma \equiv \Sigma(1+q) + \frac{1}{\beta^2}\mathbf{e}_0(\sigma_+), \quad \boldsymbol{\partial}_0 \dot{U} \equiv \dot{U}(1+q) + \frac{1}{\beta^2}\mathbf{e}_0(\dot{u}), \quad \boldsymbol{\partial}_0 \mathcal{A} \equiv \mathcal{A}(1+q) + \frac{1}{\beta^2}\mathbf{e}_0(a),$$

$$\boldsymbol{\partial}_0 \mathcal{K} \equiv 2\mathcal{K}(1+q) + \frac{1}{3\beta^3}\mathbf{e}_0(K), \quad \boldsymbol{\partial}_0 \Omega \equiv 2\Omega(1+q) + \frac{1}{3\beta^3}\mathbf{e}_0(\hat{\mu}), \quad \boldsymbol{\partial}_0 v \equiv \frac{1}{\beta}\mathbf{e}_0 v,$$

and

$$\boldsymbol{\partial}_1 \mathcal{Q} \equiv r\mathcal{Q} + \frac{1}{3\beta^2}\mathbf{e}_1(\theta), \quad \boldsymbol{\partial}_1 \Sigma \equiv r\Sigma + \frac{1}{\beta^2}\mathbf{e}_1(\sigma_+), \quad \boldsymbol{\partial}_1 \dot{U} \equiv r\dot{U} + \frac{1}{\beta^2}\mathbf{e}_1(\dot{u}), \quad \boldsymbol{\partial}_1 \mathcal{A} \equiv r\mathcal{A} + \frac{1}{\beta^2}\mathbf{e}_1(a),$$

$$\boldsymbol{\partial}_1 \mathcal{K} \equiv 2r\mathcal{K} + \frac{1}{3\beta^3}\mathbf{e}_1(K), \quad \boldsymbol{\partial}_1 \Omega \equiv 2r\Omega + \frac{1}{3\beta^3}\mathbf{e}_1(\hat{\mu}), \quad \boldsymbol{\partial}_1 v \equiv \frac{1}{\beta}\mathbf{e}_1 v.$$

Thus, the field equations reduce to

$$\boldsymbol{\partial}_0 E_1^1 = (q + 3\Sigma) E_1^1, \tag{3.13a}$$

$$\boldsymbol{\partial}_0 \mathcal{K} = 2q\mathcal{K}, \tag{3.13b}$$

$$c_a \boldsymbol{\partial}_0 \dot{U} - \frac{(3c_\theta + 2c_\sigma)\boldsymbol{\partial}_1 \mathcal{Q}}{(2c_\sigma - 1)} = c_a \dot{U}(q-1) - \frac{r(3c_\theta + 2c_\sigma)\mathcal{Q}}{(2c_\sigma - 1)} + \frac{3\gamma c_\sigma v\Omega}{(2c_\sigma - 1)(1 - v^2)}, \tag{3.13c}$$

$$\boldsymbol{\partial}_0 \mathcal{Q} - \frac{(c_a + 1)\boldsymbol{\partial}_1 \dot{U}}{3(3c_\theta + 1)} = \mathcal{Q}(1 + q - \mathcal{Q}) - \frac{r(c_a + 1)\dot{U}}{3(3c_\theta + 1)} - \frac{2(c_a + 1)\mathcal{A}\dot{U}}{3(3c_\theta + 1)} +$$

$$+ \frac{(c_a + 1)\dot{U}^2}{3(3c_\theta + 1)} + \frac{2(2c_\sigma - 1)\Sigma^2}{3c_\theta + 1} + \frac{\Omega\left((\gamma - 2)v^2 - 3\gamma + 2\right)}{2(3c_\theta + 1)(1 - v^2)}, \tag{3.13d}$$

$$\boldsymbol{\partial}_0 \Sigma - \frac{(c_a + 1)\boldsymbol{\partial}_1 \dot{U}}{3(2c_\sigma - 1)} = \Sigma(1 + q - 3\mathcal{Q}) - \frac{r(c_a + 1)\dot{U}}{3(2c_\sigma - 1)} + \frac{(1 - 2c_a)\mathcal{A}\dot{U}}{3(2c_\sigma - 1)} +$$

$$+ \frac{3\mathcal{K}}{2(2c_\sigma - 1)} + \frac{(5c_a + 2)\dot{U}^2}{6(2c_\sigma - 1)} - \frac{\mathcal{A}^2}{2(2c_\sigma - 1)} + \frac{(3c_\theta + 1)\mathcal{Q}^2}{2(2c_\sigma - 1)} + \frac{\Omega\left((\gamma + 1)v^2 - 1\right)}{2(2c_\sigma - 1)(1 - v^2)} + \frac{1}{2}\Sigma^2, \tag{3.13e}$$

$$\boldsymbol{\partial}_0 \mathcal{A} + \frac{(3c_\theta + 2c_\sigma)\boldsymbol{\partial}_1 \mathcal{Q}}{2c_\sigma - 1} = q\mathcal{A} + \frac{r(3c_\theta + 2c_\sigma)\mathcal{Q}}{2c_\sigma - 1} - \frac{3\gamma v\Omega}{2(2c_\sigma - 1)(1 - v^2)} - \dot{U}, \tag{3.13f}$$

$$\boldsymbol{\partial}_0 \Omega + \frac{(\gamma - 2)v\boldsymbol{\partial}_1 \Omega}{G_-} - \frac{\gamma\Omega\boldsymbol{\partial}_1 v}{G_-} = -\frac{2\gamma\mathcal{A}v\Omega}{G_-} + \frac{2q\Omega\left(1 - (\gamma - 1)v^2\right)}{G_-} +$$

$$- \frac{3\gamma\mathcal{Q}\left(1 - v^2\right)\Omega}{G_-} + \frac{2(\gamma - 2)rv\Omega}{G_-} + \frac{2\Omega\left((1 - 2\gamma)v^2 + 1\right)}{G_-}, \tag{3.13g}$$

$$\boldsymbol{\partial}_0 v - \frac{(\gamma - 1)\left(1 - v^2\right)^2 \boldsymbol{\partial}_1 \Omega}{\gamma\Omega G_-} + \frac{(\gamma - 2)v\boldsymbol{\partial}_1 v}{G_-} = \frac{2(\gamma - 1)\mathcal{A}\left(1 - v^2\right)v^2}{G_-} +$$

$$- \frac{3(\gamma - 2)\mathcal{Q}\left(1 - v^2\right)v}{G_-} - \frac{2(\gamma - 1)r\left(1 - v^2\right)^2}{\gamma G_-} + \frac{2\left(1 - v^2\right)v}{G_-} + \dot{U}\left(1 - v^2\right), \tag{3.13h}$$

and

$$\boldsymbol{\partial}_1 \mathcal{N}^{-1} = (r - \dot{U})\mathcal{N}^{-1}, \tag{3.14a}$$

$$\boldsymbol{\partial}_1 \mathcal{K} = 2(r + \mathcal{A})\mathcal{K}, \tag{3.14b}$$

$$\boldsymbol{\partial}_1 \mathcal{A} + c_a \boldsymbol{\partial}_1 \dot{U} = -\frac{3}{2}\mathcal{K} + r\left(c_a \dot{U} + \mathcal{A}\right) + 2c_a \mathcal{A}\dot{U} - \frac{1}{2}c_a \dot{U}^2 + \frac{3}{2}\mathcal{A}^2 +$$

$$- \frac{3}{2}(3c_\theta + 2c_\sigma)\mathcal{Q}^2 + 3(2c_\sigma - 1)\mathcal{Q} + \frac{3\Omega\left((\gamma - 1)v^2 + 1\right)}{2(1 - v^2)} - 3c_\sigma + \frac{3}{2}, \tag{3.14c}$$

$$\boldsymbol{\partial}_1 \Sigma - \frac{(3c_\theta + 1)\boldsymbol{\partial}_1 \mathcal{Q}}{2c_\sigma - 1} = r\Sigma + 3\mathcal{A}\Sigma - \frac{r(3c_\theta + 1)\mathcal{Q}}{2c_\sigma - 1} + \frac{3\gamma v\Omega}{2(2c_\sigma - 1)(1 - v^2)}, \tag{3.14d}$$

$$\boldsymbol{\partial}_0 \mathcal{A} = (q + 3\Sigma)\mathcal{A} - \dot{U} + r. \tag{3.14e}$$

Note that $\mathcal{A}^2, \mathcal{K}$, only appear in the equations (3.13d), (3.13e), (3.13f) and (3.14c) via the combination $\mathcal{D} \equiv \mathcal{A}^2 - 3\mathcal{K}$ (and in the other equations also via terms of the form $\mathcal{A}\dot{U}, \mathcal{A}v$, the definition of q (B.1), etc.). We further develop the governing equations; however, since this is rather technical we continue this development in Appendix B.

4 Dust models

Let us consider dust models with $\gamma = 1$ ($p = 0$). The special case of dust, in which the governing equations simplify considerably, is of particular interest at late times. In GR we immediately obtain the simple Lemaître-Tolman-Bondi (LTB) model with $\dot{u} = 0$ (and since $N = N(t)$ is a function of t, we can set $N = 1$ by a time rescaling), where the fluid is "comoving" ($v = 0$); i.e., $\dot{u} = 0$ and $v = 0$ simultaneously (see Appendix C). This is not possible in the models here; in general v cannot be zero in the dust case and there is no GR-like "LTB" model.

If $v = 0$ ($\gamma = 1$), from equations (3.5c)-(3.5h) we immediately find that $\dot{u} = 0$, which is a contradiction (for equations (3.5c)-(3.5h)). Therefore, in our formalism, the perfect fluid must be tilting ($v \neq 0$). In general, we thus need to investigate dust with $v \neq 0$ and $\dot{u} \neq 0$ (i.e., non-"LTB"). Let us next consider the case $\dot{u} = 0$ (see equations (B.21a)-(B.21g)). From equations (B.22a)-(B.22d), if $v = 0$ we then find that (either) $3c_\theta + 2c_\sigma = 0$ ($c_\theta = c_\sigma = 0$ in GR) (or $\mathbf{e}_1(\theta) = 0$, which is valid in the spatially homogeneous models); we could investigate this special model further.

Let us also consider the subcase $\dot{U} = 0$ and $v = 0$ in normalized variables.

4.1 Normalized equations

Let us study the special subset $\dot{U} = v = 0$, with $\gamma = 1$ (see Appendix B). We also assume that $c_\theta = 0$ and $c_\sigma \neq 0$:

$$\boldsymbol{\partial}_0 \mathcal{K} = 2q\mathcal{K}, \tag{4.1a}$$

$$\boldsymbol{\partial}_0 \mathcal{Q} = \mathcal{Q}(1 + q - \mathcal{Q}) + 2(2c_\sigma - 1)(1 - \mathcal{Q})^2 - \frac{1}{2}\Omega, \tag{4.1b}$$

$$\boldsymbol{\partial}_0 \mathcal{A} = q\mathcal{A}, \tag{4.1c}$$

$$\boldsymbol{\partial}_0 \Omega = (2q - 3\mathcal{Q} + 2)\Omega, \tag{4.1d}$$

subject to the restrictions:

$$\boldsymbol{\partial}_1 \mathcal{K} = 2(r + \mathcal{A})\mathcal{K}, \tag{4.2a}$$

$$\boldsymbol{\partial}_1 \mathcal{A} = -\frac{3}{2}\mathcal{K} + r\mathcal{A} + \frac{3}{2}\mathcal{A}^2 - 3c_\sigma \mathcal{Q}^2 + \frac{3}{2}(2c_\sigma - 1)(2\mathcal{Q} - 1) + \frac{3}{2}\Omega, \tag{4.2b}$$

$$\boldsymbol{\partial}_1 \mathcal{Q} = r\mathcal{Q}, \tag{4.2c}$$

where q and r are defined by:

$$q = \frac{1}{2(2c_\sigma - 1)}\Big\{ -3\mathcal{K} + \mathcal{A}^2 - 2c_\sigma(8c_\sigma - 3)\mathcal{Q}^2 + 16c_\sigma(2c_\sigma - 1)\mathcal{Q}$$
$$+ 2c_\sigma\Omega + (1 - 8c_\sigma)(2c_\sigma - 1)\Big\}. \tag{4.3}$$

$$r = -3\mathcal{A}(1 - \mathcal{Q}). \tag{4.4}$$

We also have that $\boldsymbol{\partial}_0 := \mathcal{N}^{-1}\partial_t$, $\boldsymbol{\partial}_1 := E_1^1 \partial_x$. [The only remaining freedom is the coordinate rescalings $t \to f(t)$ and $x \to g(x)$].

4.2 Special dust model

The terms $\mathcal{K}, \mathcal{A}^2$, only appear in the evolution equations for $\boldsymbol{\partial}_0 \Omega, \boldsymbol{\partial}_0 \mathcal{Q}$ via (through q) the combination $\mathcal{D} \equiv \mathcal{A}^2 - 3\mathcal{K}$. Hence (assuming $c_\sigma \neq 0$) we have the reduced (closed) evolution system:

$$\boldsymbol{\partial}_0 \mathcal{D} = 2q\mathcal{D}, \tag{4.5a}$$

$$\boldsymbol{\partial}_0 \mathcal{Q} = \mathcal{Q}(1 + q - \mathcal{Q}) + 2\left(2c_\sigma - 1\right)\left(1 - \mathcal{Q}\right)^2 - \frac{1}{2}\Omega, \tag{4.5b}$$

$$\boldsymbol{\partial}_0 \Omega = (2q - 3\mathcal{Q} + 2)\Omega, \tag{4.5c}$$

where

$$q = \frac{1}{2(2c_\sigma - 1)}\left\{\mathcal{D} - 2c_\sigma\left(8c_\sigma - 3\right)\mathcal{Q}^2 + 16c_\sigma(2c_\sigma - 1)\mathcal{Q}\right.$$
$$\left. + 2c_\sigma\Omega + (1 - 8c_\sigma)(2c_\sigma - 1)\right\}. \tag{4.6}$$

In the decoupled evolution equations above (which are only valid strictly speaking for $c_\sigma \neq 0$) we have not yet applied any constraints. The constraint eqns for "LTB"-like models (i.e., dust models in Einstein-aether theory with $\dot{U} = 0$ and $v = 0$) imply either $3c_\theta + 2c_\sigma = 0$ or $\mathbf{e}_1(\theta) = 0$. The problems regarding "LTB" come from the constraints for $\mathbf{e}_1(\theta)$ and $\mathbf{e}_1(\sigma_+)$; when normalizing with $3\beta = \theta + 3\sigma_+$, these constraints get hidden in the normalized variables (because θ decouples in the normalized eqns and Σ is related to $\sigma_+/(\theta + 3\sigma_+)$), and so there are no problems per se with normalized equations But they do not represent any "LTB" model because the constraints are not satisfied.

The FLRW models in this special dust model (with $c_\sigma \neq 0$) have $\mathcal{D} = 0, \Omega = 1, \mathcal{Q} = 1$ ($q = \frac{1}{2}$) and correspond to an equilibrium point (the point P_4 below). In the Kantowski-Sachs models, $\mathcal{A} = 0, r = 0$, and there are no spatial derivatives, and the constraints can be used to eliminate the (non-zero) $\mathcal{D}$ and the resulting system becomes 2-dimensional (the Kantowski-Sachs models will be studied later using a different normalization).

Assuming $\mathcal{A} \neq 0$, we can define the new spatial derivative $\boldsymbol{\partial}_\eta \equiv \mathcal{A}^{-1}\boldsymbol{\partial}_1$, whence the spatial derivatives become:

$$\boldsymbol{\partial}_\eta \mathcal{D} = 3\left(-2c_\sigma(1 - \mathcal{Q})^2 + (\mathcal{D} - 1)(-1 + 2\mathcal{Q}) + \Omega\right), \tag{4.7a}$$
$$\boldsymbol{\partial}_\eta \mathcal{Q} = -3(1 - \mathcal{Q})\mathcal{Q}. \tag{4.7b}$$

The commutator equation is given by

$$[\boldsymbol{\partial}_\eta, \boldsymbol{\partial}_\tau] = \mathcal{A}^{-1}\left(q\boldsymbol{\partial}_0 + [\boldsymbol{\partial}_1, \boldsymbol{\partial}_0]\right). \tag{4.8}$$

There is no spatial restriction for Ω; thus, it is freely specified at the initial spatial hypersurface.

Let us summarize the equilibrium points of the system (4.5) and their eigenvalues (see Table 1), and discuss their stability:

1. Point P_1 exists (i.e., with $-1 \leq \mathcal{Q} \leq 1$) for $c_\sigma < 0$ or $0 < c_\sigma \leq \frac{1}{2}$. It is a source for $c_\sigma < 0$ or $0 < c_\sigma < \frac{3}{8}$; a saddle for $\frac{3}{8} < c_\sigma < \frac{1}{2}$. (The equilibrium points are non-hyperbolic for other values of the parameter c_σ).

Label	$(\mathcal{D}, \mathcal{Q}, \Omega)$	Eigenvalues
P_1	$\left(0, 1 + \frac{\sqrt{1-2c_\sigma}}{2c_\sigma} - \frac{1}{2c_\sigma}, 0\right)$	$\frac{4c_\sigma + 3\sqrt{1-2c_\sigma} - 3}{c_\sigma}, \frac{8c_\sigma + 3\sqrt{1-2c_\sigma} - 3}{2c_\sigma}, \frac{3\left(2c_\sigma + \sqrt{1-2c_\sigma} - 1\right)}{2c_\sigma}$
P_2	$\left(0, 1 - \frac{\sqrt{1-2c_\sigma}}{2c_\sigma} - \frac{1}{2c_\sigma}, 0\right)$	$\frac{4c_\sigma - 3\sqrt{1-2c_\sigma} - 3}{c_\sigma}, -\frac{-8c_\sigma + 3\sqrt{1-2c_\sigma} + 3}{2c_\sigma}, -\frac{3\left(-2c_\sigma + \sqrt{1-2c_\sigma} + 1\right)}{2c_\sigma}$
P_3	$\left(0, \frac{4(2c_\sigma - 1)}{8c_\sigma - 3}, 0\right)$	$-\frac{3}{8c_\sigma - 3}, -\frac{32c_\sigma - 15}{2(8c_\sigma - 3)}, -1$
P_4	$(0, 1, 1)$	$1, 1, -\frac{3}{2}$
P_5	$\left(1 - 2c_\sigma, \frac{2}{3}, \frac{8c_\sigma}{9}\right)$	$1, \frac{-3 + \sqrt{9 - 48c_\sigma}}{6}, \frac{-3 - \sqrt{9 - 48c_\sigma}}{6}$
P_6	$(1, 1, 0)$	$-1, -1 - \sqrt{\frac{2c_\sigma}{2c_\sigma - 1}}, -1 + \sqrt{\frac{2c_\sigma}{2c_\sigma - 1}}$
P_7	$\left(\frac{3(2c_\sigma - 1)(8c_\sigma - 3)}{(4c_\sigma - 3)^2}, \frac{2(2c_\sigma - 1)}{4c_\sigma - 3}, 0\right)$	$-\frac{4c_\sigma}{4c_\sigma - 3}, -\frac{3}{4c_\sigma - 3}, -\frac{8c_\sigma - 3}{4c_\sigma - 3}$

Table 1. Equilibrium points and eigenvalues of the system (4.5).

2. Point P_2 exists for $\frac{3}{8} \le c_\sigma \le \frac{1}{2}$. It is a sink for $\frac{3}{8} \le c_\sigma < \frac{15}{32}$; a saddle for $\frac{15}{32} < c_\sigma < \frac{1}{2}$.

3. Point P_3 exists for $c_\sigma \ge \frac{7}{16}$. It is a sink for $c_\sigma > \frac{15}{32}$; a saddle for $\frac{7}{16} \le c_\sigma < \frac{15}{32}$.

4. The [FLRW] point P_4 always exists and it is a saddle.

5. The point P_5 always exists and it is a saddle for $c_\sigma \ne 0$.

6. The point P_6 always exists. It is sink for $c_\sigma < \frac{1}{2}$ [two complex conjugate eigenvalues with negative real part for $0 < c_\sigma < \frac{1}{2}$]. It is a saddle for $c_\sigma > \frac{1}{2}$. For $c_\sigma = \frac{1}{2}$ it is a saddle too.

7. The point P_7 exists for $c_\sigma \le \frac{5}{8}$. It is a source for $\frac{3}{8} < c_\sigma \le \frac{5}{8}$. Non-hyperbolic for $c_\sigma \in \{0, \frac{3}{8}\}$. Saddle otherwise.

Discussion: Let us define

$$D_\sigma = (1 - 8c_\sigma)(1 - 2c_\sigma) + 16c_\sigma(1 - 2c_\sigma)\mathcal{Q} - 2c_\sigma(3 - 8c_\sigma)\mathcal{Q}^2,$$

and consider the equilibrium points at finite values:
(a) $\mathcal{D} = 0, \Omega = 0, \mathcal{Q} = Q_a$ (where there are constraints on the parameter c_σ in order for Q_a to be physical) [the points P_1, P_2, P_3 in Table 1]; generically a saddle.
(b) $\mathcal{D} = 0, \Omega = 1, \mathcal{Q} = 1$ ($q = \frac{1}{2}$). Eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 1$ and $\lambda_3 = -\frac{3}{2}$ The FLRW equilibium point [point P_4 in Table 1] is always a saddle.
(c) $\mathcal{D} = 1 - 2c_\sigma, \Omega = \frac{8}{9}c_\sigma, \mathcal{Q} = \frac{2}{3}$ ($q = 0$) [point P_5 in Table 1.]
(d) $\mathcal{D} = D_\sigma, \Omega = 0$ ($q = \frac{1}{2}$), and either (i) $\mathcal{Q} = 1$ ($\mathcal{D} = 1$; no shear) or (ii) $\mathcal{Q} = Q_d = 2(1 - 2c_\sigma)/(3 - 4c_\sigma)$ ($\mathcal{D} = 3(1 - 2c_\sigma)(3 - 8c_\sigma)/(3 - 4c_\sigma)^2$). Eigenvalues: (i) ($\mathcal{Q} = 1$ [point P_6 in Table 1]) $\lambda_1 = -1$, $\lambda_{2,3} = -1 \pm \sqrt{\frac{-2c_\sigma}{(1-2c_\sigma)}}$ (negative real part for $c_\sigma < \frac{1}{2}$) – corresponding
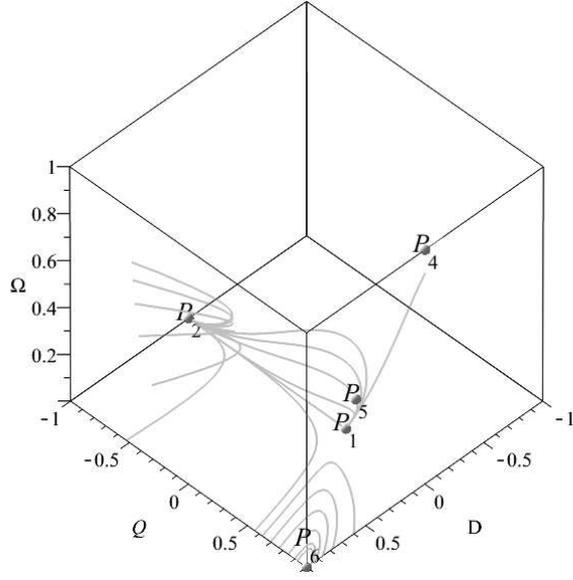
Figure 1. Phase space of the system (4.5) for the choice $c_\sigma = \frac{3}{8}$. The sinks are P_2 and P_6. P_7 and P_1 coincide; they are non-hyperbolic and behave as the sources.
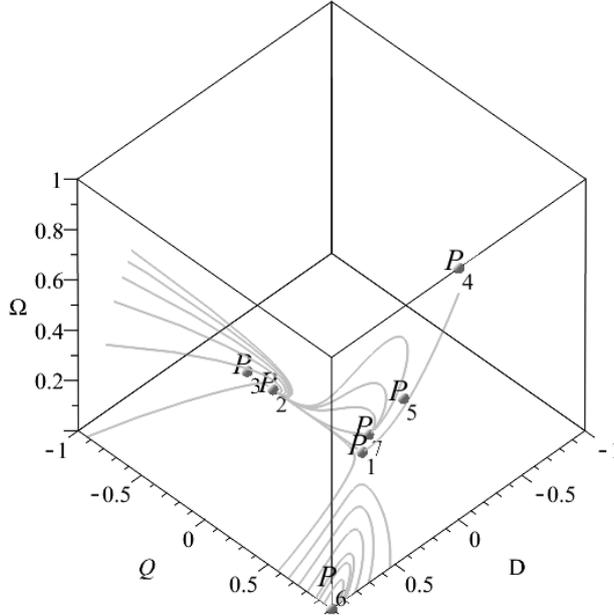
Figure 2. Phase space of the system (4.5) for the choice $c_\sigma = 0.45 \in \left(\frac{3}{8}, \frac{15}{32}\right)$. The sinks are P_2 and P_6, and P_7 is the source.

to a sink, (ii) [point P_7 in Table 1] $\lambda_1 = \frac{4c_\sigma}{(3-4c_\sigma)}$, $\lambda_2 = -1 + O(c_\sigma)$, $\lambda_3 = 1 + O(c_\sigma)$ – which is a saddle for small c_σ.

Summary of sinks: P_6 for $c_\sigma < \frac{1}{2}$, P_2 for $\frac{3}{8} \leq c_\sigma < \frac{15}{32}$, P_3 for $c_\sigma > \frac{15}{32}$. In all cases $\Omega \to 0$ to the future. For P_2 and P_3, $\mathcal{D} \to 0$, but for P_6, $\mathcal{D} \to 1$ ($\mathcal{Q} \to 1$) and the shear goes to zero at late times (for small c_σ). There is a range of values of the parameter c_σ for which

Figure 3. Phase space of the system (4.5) for the choice $c_\sigma = 0.7$. The sink is P_3.

the sinks P_2 and P_3 represent inflationary solutions.

For illustration, we present some phase portraits of the system (4.5) for the x-constant surfaces in figures 1 - 3. In figure 1 we present some orbits of the phase space for the parameter $c_\sigma = \frac{3}{8}$. The sinks are P_2 and P_6. P_7 and P_1 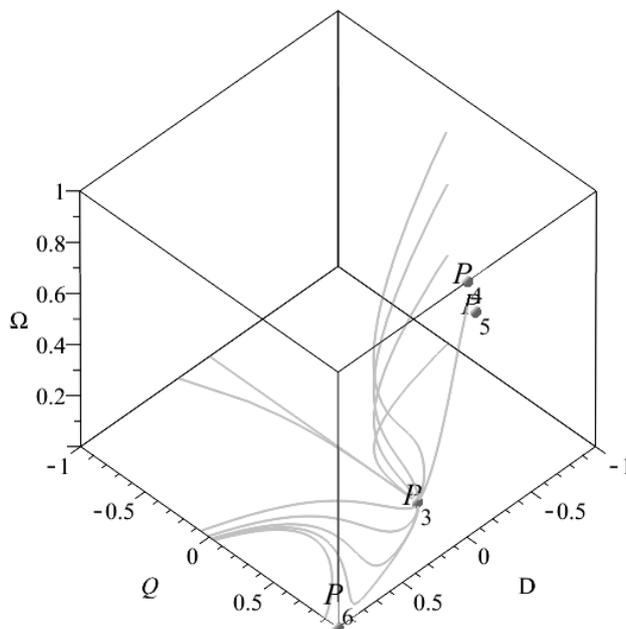coincide; they are non-hyperbolic and behave as the sources. In figure 2 we present the evolution of the system (4.5) for $c_\sigma = 0.45 \in \left(\frac{3}{8}, \frac{15}{32}\right)$. The sinks are P_2 and P_6. P_7 is the source. Finally, in figure 3 we present the phase portrait for the choice $c_\sigma = 0.7$. The sink is P_3.

5 Special cases with extra Killing vectors

Spherically symmetric models with more than 3 KVF are either spatially homogeneous or static. Spatially homogeneous spherically symmetric models are either Kantowski-Sachs models, or the Friedmann-Lemaître-Robertson-Walker (FLRW) models (with or without cosmological constant Λ) [or locally rotationally symmetric (LRS) Bianchi I and Bianchi III models]. For the FLRW and Kantowski-Sachs models we use the equations in the case that $\dot{u} = 0$, which follows immediately from the condition that $N = N(t)$. Static and self-similar spherically symmetric models have been studied in [21, 46]. [6]

5.1 The FLRW models

For the FLRW models the source must be of the form of a comoving perfect fluid (or vacuum) and the aether must be comoving. The metric has the form

$$ds^2 = -N(t)^2 dt^2 + \ell^2(t)dx^2 + \ell^2(t)f^2(x)(d\vartheta^2 + \sin^2\vartheta d\varphi^2), \tag{5.1}$$

with

$$f(x) = \sin x, \ x, \ \sinh x, \tag{5.2}$$

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[6]Recall that we have chosen a gauge so that the aether is aligned with $\mathbf{e}_0$.

for closed, flat, and open FLRW models, respectively. The frame coefficients are given by $e_1{}^1 = \ell^{-1}(t)$ and $e_2{}^2 = \ell^{-1}(t)f^{-1}(x)$. Then $\sigma_+ = \frac{1}{3}\mathbf{e}_0 \ln(e_1{}^1/e_2{}^2)$ vanishes. Furthermore, $a = -\frac{\partial_x f(x)}{f(x)\ell(t)}$. $N = N(t)$ implies that $\dot{u} = 0$; i.e., the temporal gauge is synchronous, and we can set N to any positive function of t (we usually choose $N = 1$). The Hubble scalar $H = \mathbf{e}_0 \ln \ell(t)$ is also a function of t. [7]

For the spatial curvatures, ${}^3S_+$ does vanish because (5.2) implies $\mathbf{e}_1 a = K$,[8] while ${}^3R$ simplifies to

$$^3R = \frac{6\kappa}{\ell^2}, \quad \kappa = 1, 0, -1, \tag{5.3}$$

for closed, flat, and open FLRW, respectively. The evolution equation for σ_+ and the Codazzi constraint then imply that $\pi_+ = 0 = q_1$.

FLRW cosmological models with aether and a comoving perfect fluid comoving have been studied previously [1, 3, 6, 37]. It was found that there is no essential affects on standard cosmology in the minimal aether theory. FLRW cosmological models with a scalar field were studied in [2, 16, 17]. The decay of tilt has also been studied in (anisotropic and non-comoving) models with Λ [4, 5].

5.1.1 The FLRW models in normalized coordinates

The FLRW models in normalized coordinates are characterized by $\Sigma = 0$ ($Q = 1$), $\dot{U} = 0$, $v = 0$, and we can use the remaining coordinate freedom to set $N = 1$ (where $\beta = \frac{\dot{\ell}}{N\ell}$). We recall that $\boldsymbol{\partial}_1 := E_1^1 \partial_x$ and $\boldsymbol{\partial}_0 E_1^1 = qE_1^1$. We use the remaining spatial freedom to simplify $f(x)$ as in equation (5.2) for the FLRW metric as above, where $(\partial_x f(x))^2 \equiv 1 - \kappa f^2$, $\partial_x \partial_x f(x) \equiv -\kappa f$, and so we obtain: [9]

$$\mathcal{K} = \frac{N^2}{3\dot{\ell}^2 f^2} \tag{5.4}$$

$$\mathcal{A} = -\frac{N\partial_x f}{\dot{\ell} f}, \tag{5.5}$$

and hence

$$\mathcal{A}^2 - 3\mathcal{K} = -\frac{\kappa N^2}{\dot{\ell}^2}, \qquad\qquad \boldsymbol{\partial}_1 \mathcal{A} = \frac{3}{N}\mathcal{K}. \tag{5.6}$$

5.1.2 The subset $\dot{U} = v = 0$ with $Q = 1$

We take the equations in the case $\dot{U} = v = 0$ presented earlier, and set $Q = 1$. We again assume that $c_\theta = 0$ and $c_\sigma \neq 0$ (and, in principle, $\gamma \neq 1$). Since $Q = 1$, $\Sigma = 0$ (i.e., the shear is zero), which is not in general an invariant set. We immediately have that $r = 0$, whence $\boldsymbol{\partial}_1 \mathcal{N}^{-1} = 0$, and we can rescale time so that $N = 1$ and $\boldsymbol{\partial}_0 := \partial_\tau$, where τ is essentially logarithmic time. We also have that Ω is independent of space, and that

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[7] We shall not list the KVFs as they are complicated in spherically symmetric coordinates and not needed here.

[8] That $\mathbf{e}_1 a$ does not vanish is consistent with the frame vector $\mathbf{e}_1$ not being group-invariant.

[9] Flat FLRW power law models: In the flat case $\kappa = 0$, $f(x) = x$ (and $\tau = \ln(\ell)$). At the equilibrium points we have that $\ell = t^p$.

$$\boldsymbol{\partial}_\tau E_1^1 = qE_1^1, \tag{5.7a}$$

$$\boldsymbol{\partial}_\tau \mathcal{K} = 2q\mathcal{K}, \tag{5.7b}$$

$$\boldsymbol{\partial}_\tau \mathcal{A} = q\mathcal{A}, \tag{5.7c}$$

$$\boldsymbol{\partial}_\tau \Omega = (2q - 3\gamma + 2)\Omega, \tag{5.7d}$$

subject to the restrictions:

$$\boldsymbol{\partial}_1 \mathcal{K} = 2\mathcal{A}\mathcal{K}, \tag{5.8a}$$

$$\boldsymbol{\partial}_1 \mathcal{A} = -\frac{3}{2}\mathcal{K} + \frac{3}{2}\mathcal{A}^2 + \frac{3}{2}(\Omega - 1), \tag{5.8b}$$

$$\tag{5.8c}$$

where q is defined by:

$$q = \frac{1}{2}\Omega(3\gamma - 2), \tag{5.9}$$

and where

$$0 = -3\mathcal{K} + \mathcal{A}^2 + \Omega - 1. \tag{5.10}$$

Note that if we differentiate this constraint and use the constraint and the definition of q, we obtain zero; hence the constraint is conserved along the evolution. We can use this constraint to eliminate $\mathcal{K}$ from the above equations. We note, as expected, that all dependence on c_σ has dropped out. We also note that the above system has the equilibrium points $\Omega = 0, q = 0$, corresponding to late time vacuum, and $\Omega = 1, q = \frac{1}{2}(3\gamma - 2)$, the early time flat solution.

Finally, using equations ([5.4](#), [5.5](#) [5.6](#)), we obtain

$$\boldsymbol{\partial}_\tau \Omega = (2q - 3\gamma + 2)\Omega, \tag{5.11}$$

where $q = \frac{1}{2}\Omega(3\gamma - 2)$ and $\Omega = 1 + \frac{\kappa}{\ell^2}$, as expected.

Note that in the general solution (i.e., not FLRW) we can define $\mathcal{D} = \mathcal{A}^2 - 3\mathcal{K}$, so that $\boldsymbol{\partial}_\tau \mathcal{D} = 2q\mathcal{D}$, subject to the restrictions: $\boldsymbol{\partial}_\eta \mathcal{D} = 3(\mathcal{D} + \Omega - 1)$, where we have introduced the new spatial coordinate $\boldsymbol{\partial}_\eta \equiv \boldsymbol{\partial}_1/\mathcal{A}$. Since $0 = \mathcal{D} + \Omega - 1$ we then obtain $\boldsymbol{\partial}_\eta \mathcal{D} = 0$, which implies $\boldsymbol{\partial}_\eta \Omega = 0$. We then obtain

$$\mathcal{D} = \frac{1}{e^{-3\gamma\tau - c_1 + 2\tau} + 1}, \quad \Omega = \frac{e^{2\tau}}{e^{3\gamma\tau + c_1} + e^{2\tau}}, \quad q = \frac{(3\gamma - 2)e^{2\tau}}{2(e^{3\gamma\tau + c_1} + e^{2\tau})},$$

$$\mathcal{A} = \frac{c_2(\eta)e^{\frac{3\gamma\tau}{2}}}{\sqrt{e^{3\gamma\tau + c_1} + e^{2\tau}}}, \quad \mathcal{K} = \frac{e^{3\gamma\tau}(c_2(\eta)^2 - e^{c_1})}{3(e^{3\gamma\tau + c_1} + e^{2\tau})}.$$

The equations

$$\boldsymbol{\partial}_\eta \mathcal{K} = 2\mathcal{K}, \tag{5.12a}$$

$$\mathcal{A}\boldsymbol{\partial}_\eta \mathcal{A} = -\frac{3}{2}\mathcal{K} + \frac{3}{2}\mathcal{A}^2 + \frac{3}{2}(\Omega - 1), \tag{5.12b}$$

are identically satisfied if

$$e^{c_1} + c_2(\eta) \left(c_2'(\eta) - c_2(\eta)\right) = 0, c_2(\eta) \neq 0. \tag{5.13}$$

The above equations admit the solutions

$$c_2(\eta) = \pm\sqrt{e^{c_1} - e^{2c_2+2\eta}}, \ E_1^1 = \frac{c_3(\eta)e^{\frac{3\gamma\tau}{2}}}{\sqrt{e^{3\gamma\tau+c_1} + e^{2\tau}}},$$

$$\mathcal{K} = -\frac{e^{3\gamma\tau+2c_2+2\eta}}{3\left(e^{3\gamma\tau+c_1} + e^{2\tau}\right)}, \ \mathcal{A} = \pm\frac{\sqrt{e^{c_1} - e^{2(c_2+\eta)}}e^{\frac{3\gamma\tau}{2}}}{\sqrt{e^{3\gamma\tau+c_1} + e^{2\tau}}}.$$

5.2 The Kantowski-Sachs models

We now investigate the spatially homogeneous subcase, in which a full global analysis is possible. It is of particular interest whether general solutions can asymptote towards spatially homogeneous solutions at late or early times. The spatially homogeneous spherically symmetric models (that has 4 Killing vectors, the fourth being ∂_x) are the so-called Kantowski-Sachs models [36]. We shall consider the special comoving aether case. The metric (2.1) simplifies to

$$ds^2 = -N(t)^2 dt^2 + (e_1{}^1(t))^{-2} dx^2 + (e_2{}^2(t))^{-2}(d\vartheta^2 + \sin^2\vartheta d\varphi^2); \tag{5.14}$$

i.e., N, $e_1{}^1$ and $e_2{}^2$ are now independent of x. The spatial derivative terms $\mathbf{e}_1(\ )$ vanish and as a result $a = 0 = \dot{u}$. Since $\dot{u} = 0$, N is a positive function of t which under a time rescaling can be set to one. This metric choice forces the fluid to be non-tilted ($v = 0$) [assuming $\mu > 0, \gamma > 0$].

The evolution equations for the Kantowski-Sachs metric for an Einstein-aether spherically symmetric cosmology, in the presence of a perfect fluid, are:

$$\mathbf{e}_0(e_1{}^1) = -\tfrac{1}{3}(\theta - 6\sigma_+)e_1{}^1 \tag{5.15a}$$

$$\mathbf{e}_0(K) = -\tfrac{2}{3}(\theta + 3\sigma_+)K \tag{5.15b}$$

$$\mathbf{e}_0(\theta) = -\frac{\theta^2}{3} + \frac{6(2c_\sigma - 1)\sigma_+^2}{3c_\theta + 1} + \frac{(2 - 3\gamma)\hat{\mu}}{2(3c_\theta + 1)}, \tag{5.15c}$$

$$\mathbf{e}_0(\sigma_+) = -\frac{(3c_\theta + 1)\theta^2}{9(2c_\sigma - 1)} - \theta\sigma_+ - \sigma_+^2 + \frac{\hat{\mu}}{3(2c_\sigma - 1)}, \tag{5.15d}$$

$$\mathbf{e}_0(\hat{\mu}) = -\gamma\theta\hat{\mu} \tag{5.15e}$$

with the constraint

$$K + \frac{(3c_\theta + 1)\theta^2}{3} = \hat{\mu} - 3(2c_\sigma - 1)\sigma_+^2. \tag{5.16}$$

We choose the following normalized variables (which are bounded for $1 - 2c_\sigma \geq 0$; note that we do not use the $\beta-$normalization for convenience here):

$$x = \frac{\sqrt{\hat{\mu}}}{D}, y = \frac{\sqrt{3}\sigma_+}{D}, z = \frac{\sqrt{K}}{D}, Q = \frac{\theta}{\sqrt{3}D} \tag{5.17}$$

where

$$D = \sqrt{K + \frac{\theta^2}{3}}, \tag{5.18}$$

and the new time variable $f' \equiv \frac{1}{D}\mathbf{e}_0(f)$.

We then obtain the full 4 dimensional (4D) system:

$$x' = x\left(\frac{Qy^2\left(2 - 4c_\sigma\right)}{\sqrt{3}\left(3c_\theta + 1\right)} + \frac{Q^3}{\sqrt{3}} + \frac{Q\left(2z^2 - 3\gamma\right)}{2\sqrt{3}} + \frac{yz^2}{\sqrt{3}}\right) + \frac{(3\gamma - 2)Qx^3}{2\sqrt{3}\left(3c_\theta + 1\right)} \tag{5.19a}$$

$$y' = \frac{\sqrt{3}Q^2\left(-3c_\theta - 2y^2c_\sigma + y^2 - 1\right)}{6c_\sigma - 3} + x^2\left(\frac{(3\gamma - 2)Qy}{2\sqrt{3}\left(3c_\theta + 1\right)} + \frac{1}{\sqrt{3}\left(2c_\sigma - 1\right)}\right) +$$
$$- \frac{2\sqrt{3}Qy\left(3c_\theta + 2y^2c_\sigma - y^2 + 1\right)}{9c_\theta + 3} \tag{5.19b}$$

$$z' = -z\left(\frac{2\sqrt{3}Qy^2\left(1 - 2c_\sigma\right)}{9c_\theta + 3} - \frac{Q^2y}{\sqrt{3}}\right) + \frac{(3\gamma - 2)Qx^2z}{2\sqrt{3}\left(3c_\theta + 1\right)} \tag{5.19c}$$

$$Q' = z^2\left(\frac{2\sqrt{3}y^2\left(2c_\sigma - 1\right)}{9c_\theta + 3} + \frac{Qy}{\sqrt{3}}\right) + \frac{(2 - 3\gamma)x^2z^2}{2\sqrt{3}\left(3c_\theta + 1\right)}, \tag{5.19d}$$

The variables (5.17) are related through the constraints

$$-3c_\theta Q^2 + x^2 - (2c_\sigma - 1)y^2 = 1, \tag{5.20a}$$

$$Q^2 + z^2 = 1, \tag{5.20b}$$

which are preserved by the 4D system. From the equations (5.20) it follows that Q and z are bounded in the intervals $Q \in [-1, 1]$, $z \in [0, 1]$ (for expanding universes $Q \geq 0$). However, since $1 - 2c_\sigma$ is not necessarily non-negative it follows that x and y are unbounded, unless $1 - 2c_\sigma \geq 0$.

The restrictions (5.20) allow the elimination of two variables, say x and z. This leads to the following 2-dimensional dynamical system:

$$y' = \frac{\sqrt{3}Q^2}{3 - 6c_\sigma} + \frac{\sqrt{3}Qy\left(c_\theta\left((3\gamma - 2)Q^2 - 4\right) + \gamma - 2\right)}{6c_\theta + 2} + \frac{\sqrt{3}(\gamma - 2)Qy^3\left(2c_\sigma - 1\right)}{6c_\theta + 2} +$$
$$+ \frac{1}{\sqrt{3}\left(2c_\sigma - 1\right)} - \frac{\left(Q^2 - 1\right)y^2}{\sqrt{3}} \tag{5.21a}$$

$$Q' = \frac{\sqrt{3}(2 - 3\gamma)}{18c_\theta + 6} + \frac{\sqrt{3}(3\gamma - 2)Q^4c_\theta}{6c_\theta + 2} - \frac{\sqrt{3}(3\gamma - 2)Q^2\left(3c_\theta - 1\right)}{18c_\theta + 6} +$$
$$+ \frac{\sqrt{3}(\gamma - 2)\left(Q^2 - 1\right)y^2\left(2c_\sigma - 1\right)}{6c_\theta + 2} + \frac{(1 - Q^2)Qy}{\sqrt{3}} \tag{5.21b}$$

We shall study the general case in future work (using the β−normalization). Let us consider the following special case here.

5.2.1 Special case.

Let us assume

$$3c_\theta \equiv c_1 + 3c_2 + c_3 = 0 \tag{5.22}$$

(see Appendix A and the references [16–18]), and define $c^2 \equiv 1 - 2c_\sigma = 1 - 2(c_1 + c_3) \geq 0$. This choice leads to a compact phase space.

With these special values of the c's, the evolution equations for Kantowski-Sachs models simplify and the constraint becomes

$$K + \frac{\theta^2}{3} = \hat{\mu} + 3c^2\sigma_+^2. \tag{5.23}$$

The following normalized variable

$$y_1 = \frac{\sqrt{3}c\sigma_+}{D} \tag{5.24}$$

is chosen for convenience, whence the variables are related through the constraints

$$x^2 + y_1^2 = 1, \tag{5.25a}$$
$$Q^2 + z^2 = 1. \tag{5.25b}$$

Thus, the phase space is compact with $x \in [-1,1]$, $y_1 \in [-1,1]$ and $Q \in [-1,1]$, $z \in [0,1]$ (for expanding universes $Q \geq 0$).

The system for (y_1, Q) reduces to

$$y_1' = -\frac{\left(y_1^2 - 1\right)\left(3c(\gamma - 2)Qy_1 + 2Q^2 - 2\right)}{2\sqrt{3}c}, \tag{5.26a}$$
$$Q' = -\frac{\left(Q^2 - 1\right)\left(c\left(-3\gamma + 3(\gamma - 2)y_1^2 + 2\right) + 2Qy_1\right)}{2\sqrt{3}c} \tag{5.26b}$$

Since the evolution equations are invariant under the transformation $y_1 \to -y_1$ and $c \to -c$, without loss of generality we can assume $c > 0$. Scaling the time derivative by the positive factor $2\sqrt{3}c$, we then obtain:

$$y_1' = -\left(y_1^2 - 1\right)\left(3c(\gamma - 2)Qy_1 + 2Q^2 - 2\right), \tag{5.27a}$$
$$Q' = -\left(Q^2 - 1\right)\left(c\left(-3\gamma + 3(\gamma - 2)y_1^2 + 2\right) + 2Qy_1\right) \tag{5.27b}$$

In tables 2 and 3 we present the equilibrium points of the system (5.27) and discuss their stability. We have that $c > 0, -1 \leq Q \leq 1, -1 \leq y_1 \leq 1$. Some of the equilibrium points do not exist for certain values of "c". We have not analyzed the non-hyperbolic "stiff fluid" case, $\gamma = 2$, in which there are zero eigenvalues. Clearly, the case $c = 0$ is not included here (the GR case), since the equations are not valid in that case.

Let us enumerate the stability conditions for the hyperbolic equilibrium points:

1. The equilibrium point P_1 is a source for $c > 0, 0 \leq \gamma < \frac{2}{3}$, and a saddle for $\frac{2}{3} < \gamma \leq 2$. Non-hyperbolic for $\gamma = \frac{2}{3}$ or $\gamma = 2$.

2. The equilibrium point P_2 is a sink for $c > 0, 0 \leq \gamma < \frac{2}{3}$, and a saddle for $\frac{2}{3} < \gamma < 2$. Non-hyperbolic for $\gamma = \frac{2}{3}$ or $\gamma = 2$.

3. The equilibrium point P_3 is a sink for $0 \leq \gamma < 2, c > \frac{1}{2}$, and non-hyperbolic for $c = \frac{1}{2}$ or $\gamma = 2$. Saddle otherwise.

4. The equilibrium point P_4 is a source for $c > 0, 0 \leq \gamma < 2$. Non-hyperbolic for $\gamma = 2$.

5. The equilibrium point P_5 is a sink for $c > 0, 0 \leq \gamma < 2$. Non-hyperbolic for $\gamma = 2$.

Label	Coordinates: (y_1, Q)	Eigenvalues
P_1	$(0, -1)$	$3c(2-\gamma), 2c(2-3\gamma)$
P_2	$(0, 1)$	$-3c(2-\gamma), -2c(2-3\gamma)$
P_3	$(-1, -1)$	$-6c(2-\gamma), 4(1-2c)$
P_4	$(-1, 1)$	$6c(2-\gamma), 4(1+2c)$
P_5	$(1, -1)$	$-6c(2-\gamma), -4(1+2c)$
P_6	$(1, 1)$	$6c(2-\gamma), -4(1-2c)$
P_7	$(-1, -2c)$	$2(4c^2-1), 4\left[c^2(3\gamma-2)-1\right]$
P_8	$(1, 2c)$	$-2(4c^2-1), -4\left[c^2(3\gamma-2)-1\right]$
P_9	$\left(\frac{c(2-3\gamma)}{d}, -\frac{2}{d}\right)$	$\frac{c(-e-3\gamma+6)}{d}, \frac{c(e-3\gamma+6)}{d}$
P_{10}	$\left(-\frac{c(2-3\gamma)}{d}, \frac{2}{d}\right)$	$\frac{c(-e+3\gamma-6)}{d}, \frac{c(e+3\gamma-6)}{d}$

Table 2. Equilibrium points of the system (5.27) and their eigenvalues. We use the notation $d = \sqrt{3(\gamma-2)(3\gamma-2)c^2+4}$ and $e \equiv \sqrt{3}\sqrt{2-\gamma}\sqrt{8c^2(2-3\gamma)^2 - 27\gamma + 22}$.

6. The equilibrium point P_6 is a source for $0 \le \gamma < 2, c > \frac{1}{2}$. A saddle for $0 \le \gamma < 2, 0 < c < \frac{1}{2}$. Non-hyperbolic for $\gamma = 2$ or $c = \frac{1}{2}$.

7. The equilibrium point P_7 exist for $0 \le \gamma \le 2, 0 < c \le \frac{1}{2}$. It is a sink for $0 \le \gamma \le 2, 0 < c < \frac{1}{2}$. Non-hyperbolic otherwise.

8. The equilibrium point P_8 exists for $0 \le \gamma \le 2, 0 < c \le \frac{1}{2}$. It is a source for $0 \le \gamma \le 2, 0 < c < \frac{1}{2}$. Non-hyperbolic otherwise.

9. The equilibrium point P_9 exists for $0 < c \le \frac{1}{2}, 0 \le \gamma \le \frac{2}{3}$, or $0 < c \le \frac{1}{2}, \gamma = 2$, or $c > \frac{1}{2}, 0 \le \gamma \le \frac{2}{3}$. It is a saddle for $0 \le \gamma < \frac{2}{3}, c > 0$. Non-hyperbolic for $\gamma = \frac{2}{3}$ or $\gamma = 2$.

10. The equilibrium point P_{10} exists for $0 < c \le \frac{1}{2}, 0 \le \gamma \le \frac{2}{3}$, or $0 < c \le \frac{1}{2}, \gamma = 2$, or $c > \frac{1}{2}, 0 \le \gamma \le \frac{2}{3}$. It is a saddle for $0 \le \gamma < \frac{2}{3}, c > 0$. Non-hyperbolic for $\gamma = \frac{2}{3}$ or $\gamma = 2$.

Discussion. In the case $c_\sigma < \frac{1}{2}$ (i.e., $c > 0$), when $\gamma < \frac{2}{3}$, P_2 is the unique shear-free, zero curvature (FLRW) inflationary future attractor, and for $\frac{3}{8} < c_\sigma < \frac{1}{2}$ (i.e., $0 < c < \frac{1}{2}$) and $0 \le \gamma < 2$ the sources and sinks are, respectively, P_4 & P_8 and P_5 & P_7. All of these sources and sinks have maximal shearing and all, except P_7, have zero curvature; the sink P_7 does not have zero curvature. For $c_\sigma < \frac{3}{8}$ (i.e., $c > \frac{1}{2}$) the points P_7 & P_8 do not exist, and the sources and sinks with maximal shearing are P_4 and P_5, respectively.

In figures 4 – 7 we present some orbits in the phase plane of the system (5.27) for different choices of the parameters. In figure 4, $\gamma = 0$ and $c_\sigma = 0.3$. The sinks are P_2, P_5

Eq. Pt.	γ value	$0 < c < \frac{1}{2}$	$c = \frac{1}{2}$	$\frac{1}{2} < c$
P_1	$0 \leq \gamma < \frac{2}{3}$	$++$ source	$++$ source	$++$ source
	$\gamma = \frac{2}{3}$	$+0$	$+0$	$+0$
	$\frac{2}{3} < \gamma < 2$	$+-$ saddle	$+-$ saddle	$+-$ saddle
	$\gamma = 2$	$0-$	$0-$	$0-$
P_2	$0 \leq \gamma < \frac{2}{3}$	$--$sink	$--$sink	$--$sink
	$\gamma = \frac{2}{3}$	-0	-0	-0
	$\frac{2}{3} < \gamma < 2$	$-+$ saddle	$-+$ saddle	$-+$ saddle
	$\gamma = 2$	$0+$	$0+$	$0+$
P_3	$0 \leq \gamma < 2$	$-+$ saddle	-0	$--$ sink
	$\gamma = 2$	$0+$	00	$0-$
P_4	$0 \leq \gamma < 2$	$++$ source	$++$ source	$++$ source
	$\gamma = 2$	$0+$	$0+$	$0+$
P_5	$0 \leq \gamma < 2$	$--$ sink	$--$ sink	$--$ sink
	$\gamma = 2$	$0-$	$0-$	$0+$
P_6	$0 \leq \gamma < 2$	$-+$saddle	$+0$	$++$ source
	$\gamma = 2$	$0-$	00	$0+$
P_7	$0 \leq \gamma < 2$	$--$ sink	-0	DNE
	$\gamma = 2$	$--$ sink	00	DNE
P_8	$0 \leq \gamma < 2$	$++$ source	$+0$	DNE
	$\gamma = 2$	$++$ source	00	DNE
P_9	$0 \leq \gamma < \frac{2}{3}$	$-+$ saddle	$-+$saddle	$-+$saddle
	$\gamma = \frac{2}{3}$	$+0$	$+0$	$+0$
	$\gamma = 2$	00	00	00
P_{10}	$0 \leq \gamma < \frac{2}{3}$	$-+$ saddle	$-+$ saddle	$-+$ saddle
	$\gamma = \frac{2}{3}$	-0	-0	-0
	$\gamma = 2$	00	00	00

Table 3. Stability of the equilibrium points of the system (5.27).

and P_7. The sources are P_1, P_4 and P_8. P_3, P_6, P_9 and P_{10} are saddles. In figure 5, $\gamma = 0$ and $c_\sigma = 0.6$. The sinks are P_2, P_3 and P_5. The sources are P_1, P_4 and P_6. P_7 and P_8 do not exist. The saddles are P_9 and P_{10}. In figure 6 we present the phase plane of the system (5.27) for the choice of parameters $\gamma = 1$ and $c_\sigma = 0.2$. The sinks are P_5 and P_7. The sources are P_4 and P_8. The saddles are P_1, P_2, P_3 and P_6. P_9 and P_{10} do not exist. Finally, in figure
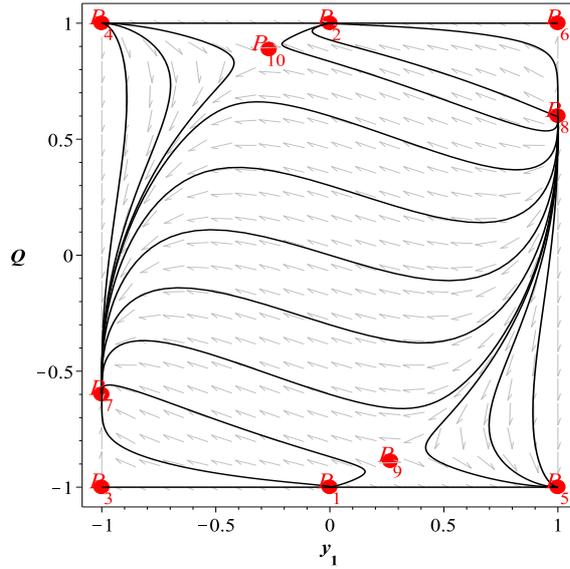
Figure 4. Phase plane of the system (5.27) for the choice of parameters $\gamma = 0$ and $c_\sigma = 0.3$. The sinks are P_2, P_5 and P_7. The sources are P_1, P_4 and P_8. P_3, P_6, P_9 and P_{10} are saddles.
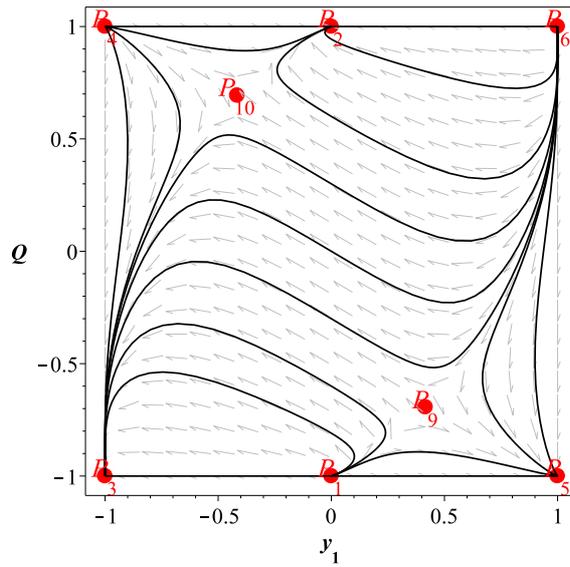

Figure 5. Phase plane of the system (5.27) for the choice of parameters $\gamma = 0$ and $c_\sigma = 0.6$. The sinks are P_2, P_3 and P_5. The sources are P_1, P_4 and P_6. P_7 and P_8 do not exist. The saddles are P_9 and P_{10}.

7, $\gamma = 1$ and $c_\sigma = 0.6$. The sinks are P_3 and P_5. The sources are P_4 and P_6. P_1 and P_2 are saddles. The points P_7-P_{10} do not exist.

Figure 6. Phase plane of the system (5.27) for the choice of parameters $\gamma = 1$ and $c_\sigma = 0.2$. The sinks are P_5 and P_7. The sources are P_4 and P_8. The saddles are P_1, P_2, P_3 and P_6. P_9 and P_{10} do not exist.
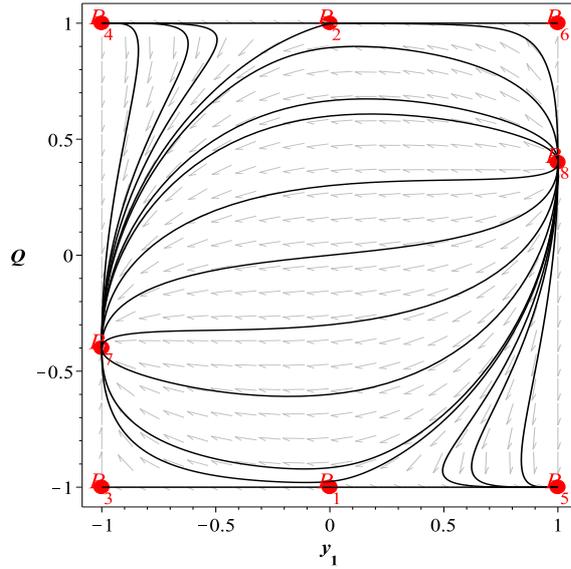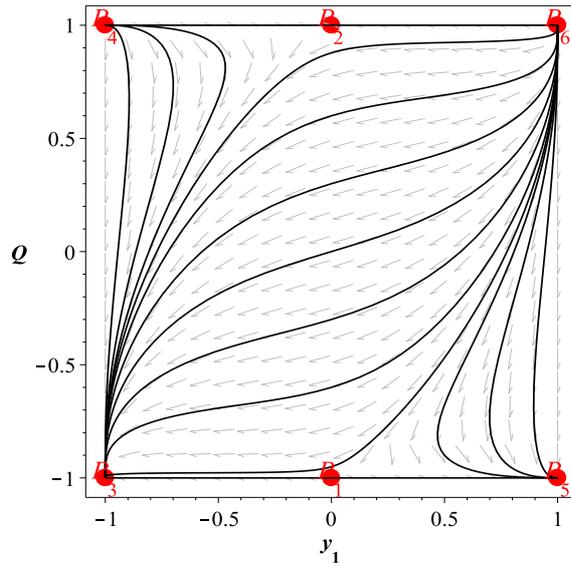

Figure 7. Phase plane of the system (5.27) for the choice of parameters $\gamma = 1$ and $c_\sigma = 0.6$. The sinks are P_3 and P_5. The sources are P_4 and P_6. P_1 and P_2 are saddles. The points P_7-P_{10} do not exist.

6 Static models

Models that are not evolving with time are also of physical importance, although perhaps more from the astrophysical point of view than from the cosmological one. In particular, much physical information can be obtained from a qualitative analysis of the models. Let us consider the static case $\mathbf{e}_0(\cdot) = 0$, for a mixture of a perfect fluid and a scalar field. [In this case

$\dot{u} \equiv d \ln N/dx \neq 0$ and the perfect fluid is forced to be non-tilted $(v = 0)$.] We will consider barotropic equations of state $\hat{\mu} = \hat{\mu}(\hat{p})$. Since $\theta = \sigma_+ = 0$ in the static subcase, V depends only on the scalar field $\phi = \phi(x)$. Furthermore, the irreducible components of the scalar field energy-momentum tensor are given by $\mu^\phi = \frac{1}{2}\mathbf{e}_1(\phi)^2 + V, p^\phi = -\frac{1}{6}\mathbf{e}_1(\phi)^2 - V, q^\phi = 0$ and $\pi_+^\phi = -\frac{1}{3}\mathbf{e}_1(\phi)^2$ [15].

The equations for the variables $a, \dot{u}, \hat{p}, \phi, K, N$ are:

$$\mathbf{e}_1(a) = \frac{\hat{\mu} + 3\hat{p}}{2(c_a + 1)} + \mathbf{e}_1(\phi)^2 - \frac{V}{c_a + 1} + 2c_a\dot{u}^2 + 3a\dot{u} + K, \tag{6.1a}$$

$$\mathbf{e}_1(\dot{u}) = \frac{\hat{\mu} + 3\hat{p}}{2(c_a + 1)} - \frac{V}{c_a + 1} + 2a\dot{u} - \dot{u}^2, \tag{6.1b}$$

$$\mathbf{e}_1(\hat{p}) = -\dot{u}(\hat{\mu} + \hat{p}) \tag{6.1c}$$

$$\mathbf{e}_1(\mathbf{e}_1(\phi)) = -(\dot{u} - 2a)\,\mathbf{e}_1(\phi) + V_\phi, \tag{6.1d}$$

$$\mathbf{e}_1(K) = 2aK, \tag{6.1e}$$

where V_ϕ denotes differentiation with respect to ϕ. The system satisfies the restriction

$$a^2 = c_a\dot{u}^2 + 2a\dot{u} + \hat{p} + \frac{1}{2}\mathbf{e}_1(\phi)^2 - V + K. \tag{6.2}$$

Taking the differential operator $\mathbf{e}_1(...)$ of both sides of (6.2), using the equations (6.1) to substitute for the spatial derivatives, and again using the restriction (6.2) solved for K, we obtain an identity. Thus, the Gauss constraint is a first integral of the system. The aether constraint is identically zero.

Let us now show how equations (6.1) can be used to obtain exact solutions and, additionally, use the dynamical systems approach to investigate the structure of the whole solution space. The first thing to do is to select a suitable radial coordinate. We may choose a new radial coordinate λ such that the equation (6.1e) has a trivial solution. A reasonable way to do this is to select an r-coordinate such that $\mathbf{e}_1(f) \equiv -ar\partial_r(f)$ (as in [47]). This implies $K \propto r^{-2}$ and $\int \mathbf{e}_1{}^1(x)^{-1}dx = -\int a^{-1}d\ln r$. As we will see later, for the dynamical systems investigation it is better to use the new time variable $\tau = \ln r$, which takes values over the whole real line.

Here we shall study the two special cases: (i) perfect fluid (previous work has assumed a comoving aether and a comoving fluid). (ii) vacuum (stationary with a scalar field and harmonic potential). In particular, for the stationary aether case, it is also of interest to choose a frame in which the aether is non-comoving. For a non-comoving stationary aether and a tilted fluid, it follows [in an analogous way to the static case] that the perfect fluid must be non-tilted $(v = 0)$. Additionally, since $\theta = \sigma_+ = 0$ in the stationary subcase, V depends only on the scalar field $\phi = \phi(x)$. We shall present a more comprehensive analysis in [46]; in particular, the "evolution" equations for the tilt α, a and $\dot{u}$ are given in the Appendix therein.

6.1 Static case with perfect fluid with linear equation of state and no scalar field.

To investigate this model we will use the approach of [48]. First, let us consider no scalar field in (6.1) and use the linear equation of state

$$\hat{\mu} = \mu_0 + (\eta - 1)\hat{p}, \tag{6.3}$$

where the constants μ_0 and η satisfy $\mu_0 \geq 0, \eta \geq 1$. The case $\eta = 1$ corresponds to an incompressible fluid with constant energy density, while the case $\mu_0 = 0$ describes a scale-invariant equation of state.

Introducing the new dimensionless variables

$$x_1 = \frac{\mu_0}{a^2}, \quad x_2 = \frac{\dot{u}}{a}, \quad x_3 = \frac{\hat{p}}{a^2}, \quad x_4 = \frac{K}{a^2}, \tag{6.4}$$

we obtain the dynamical system

$$\frac{dx_1}{d\tau} = x_1 \left(\frac{x_1 + x_3(\eta + 2)}{c_a + 1} + 4x_2^2 c_a + 2x_4 + 6x_2 \right), \tag{6.5a}$$

$$\frac{dx_2}{d\tau} = \frac{(x_2 - 1)(x_1 + x_3(\eta + 2))}{2(c_a + 1)} + 2x_2^3 c_a + x_2(x_4 + 4x_2 - 2), \tag{6.5b}$$

$$\frac{dx_3}{d\tau} = \frac{x_3(x_1 + x_3(\eta + 2))}{c_a + 1} + 4x_3 x_2^2 c_a + x_2(x_1 + x_3(\eta + 6)) + 2x_4 x_3, \tag{6.5c}$$

$$\frac{dx_4}{d\tau} = x_4 \left(\frac{x_1 + x_3(\eta + 2)}{c_a + 1} + 4x_2^2 c_a + 2x_4 + 6x_2 - 2 \right), \tag{6.5d}$$

subject to the constraint

$$1 = x_2^2 c_a + 2x_2 + x_4 + x_3. \tag{6.6}$$

The constraint (6.6) is preserved by the dynamical system (6.5). Solving the constraint (6.6) for x_4 and substituting back into the system (6.5) we obtain the reduced system:

$$\frac{dx_1}{d\tau} = \frac{x_1 x_3 (\eta - 2c_a)}{c_a + 1} + 2x_1 x_2^2 c_a + x_1 \left(\frac{x_1}{c_a + 1} + 2 \right) + 2x_1 x_2, \tag{6.7a}$$

$$\frac{dx_2}{d\tau} = x_2 \left(\frac{x_1}{2c_a + 2} + x_3 \left(\frac{\eta + 2}{2c_a + 2} - 1 \right) - 1 \right) - \frac{x_1}{2c_a + 2} - \frac{x_3(\eta + 2)}{2c_a + 2} + x_2^3 c_a + 2x_2^2, \tag{6.7b}$$

$$\frac{dx_3}{d\tau} = x_3 \left(\frac{x_1}{c_a + 1} + 2 \right) + x_3^2 \left(\frac{\eta + 2}{c_a + 1} - 2 \right) + 2x_3 x_2^2 c_a + x_2(x_1 + x_3(\eta + 2)), \tag{6.7c}$$

defined on the phase space

$$\Psi = \left\{ (x_1, x_2, x_3) : x_1 \geq 0, x_2^2 c_a + 2x_2 + x_3 \leq 1 \right\}. \tag{6.8}$$

The equilibrium points of the system (6.7) are given in table 4. Let us discuss their stability.

1. The equilibrium point P_1 is always a saddle. It satisfies $K = a^2$ asymptotically, which implies $e_1{}^1 \sim e^{-\tau} = \frac{1}{r}, e_2{}^2 \sim e^{-\tau} = \frac{1}{r}$. Since $\dot{u} \ll a$ as $r \to \infty$, it follows that $N \ll r^{-1}$.

2. Although the equilibrium point P_2 can be an attractor for $\eta \geq 1, c_a \leq -\eta - 3$ or $\eta \geq 1, -\eta - 3 < c_a < -1$, since it can never belong to the phase space (denoted $\notin \Psi_2$ in table), we do not discuss it further.

3. The equilibrium point P_3 is always a saddle.

4. The equilibrium point P_4 is a source for $\eta > 2, \frac{\eta - 2}{4} < c_a < \frac{1}{16}(\eta - 2)(\eta + 6)$. Otherwise it is a saddle.

Label	x_1	x_2	x_3	Existence
P_1	0	0	0	always
P_2	$-2-\eta$	0	1	$\notin \Psi$
P_3	0	$-\frac{2}{\eta}$	$\frac{4(c_a+1)}{\eta^2}$	$c_a \leq \Delta_1$
P_4	0	$\frac{\eta+2}{-4c_a+\eta-2}$	$-\frac{(c_a+1)((\eta-2)(\eta+6)-16c_a)}{(-4c_a+\eta-2)^2}$	$1 \leq \eta < 2, \Delta_2 \leq c_a < \frac{\eta-2}{4}$ or $\eta > 2, \frac{\eta-2}{4} < c_a \leq \Delta_2$
P_5	$-\frac{\eta(c_a+1)(4c_a+3)}{(2c_a+1)^2}$	$\frac{1}{-2c_a-1}$	$\frac{(c_a+1)(4c_a+3)}{(2c_a+1)^2}$	$\eta \geq 1, -1 \leq c_a \leq -\frac{3}{4}$
$P_{6,7}$	0	$\frac{1}{1\pm\sqrt{1+c_a}}$	0	always

Table 4. Equilibrium points of the system (6.7). We use the notation $\Delta_1 = \frac{1}{8}\left(\eta^2 + 4\eta - 4\right)$ and $\Delta_2 = \frac{1}{16}\left(\eta^2 + 4\eta - 12\right)$.

Label	Eigenvalues
P_1	$-1, 2, 2$
P_2	$-2, -1 + \frac{c_a+\eta+3}{\sqrt{(c_a+1)(c_a+\eta+3)}}, -1 - \frac{c_a+\eta+3}{\sqrt{(c_a+1)(c_a+\eta+3)}}$
P_3	$2, -\frac{\eta+2+\sqrt{64c_a-7\eta(\eta+4)+36}}{2\eta}, -\frac{\eta+2-\sqrt{64c_a-7\eta(\eta+4)+36}}{2\eta}$
P_4	$\frac{\eta^2-4}{2(4c_a-\eta+2)} - 2, \frac{\eta(\eta+2)}{4c_a-\eta+2} - 2, \frac{\eta(\eta+2)}{4c_a-\eta+2}$
P_5	$-\frac{\eta}{2c_a+1}, -2, \frac{1}{-2c_a-1} - 2$
$P_{6,7}$	$\frac{2\left(\left(c_a\mp\sqrt{c_a+1}\right)+1\right)}{c_a}, \frac{(\eta+6)\left(1\pm\sqrt{c_a+1}\right)+4c_a}{\left(\sqrt{c_a+1}-1\right)^2}, \frac{2\left(\left(2c_a\mp\sqrt{c_a+1}\right)+1\right)}{c_a}$

Table 5. Eigenvalues of the equilibrium points of the system (6.7).

5. The equilibrium point P_5 is a saddle for $\eta \geq 1, -1 \leq c_a < -\frac{3}{4}$. It is non-hyperbolic for $c_a = -\frac{3}{4}$ [but it behaves as a saddle].

6. P_6 is a source for $\eta \geq 1, -1 < c_a < 0$, or $\eta \geq 1, c_a > 0$.

7. P_7 is a sink for $1 \leq \eta < 2, \frac{1}{16}(\eta-2)(\eta+6) < c_a < 0$. It is a source for $1 \leq \eta \leq 2, c_a > 0$ or $\eta > 2, c_a > \frac{1}{16}(\eta-2)(\eta+6)$. It is a saddle otherwise.

6.2 Static vacuum aether with a scalar field with harmonic potential.

Let us investigate a static vacuum aether with a scalar field with harmonic potential $V(\phi) = \frac{m^2\phi^2}{2}$ (also see [15]). Introducing the new dimensionless variables

$$x_2 = \frac{\dot{u}}{a}, \quad x_4 = \frac{K}{a^2}, \quad x_5 = \frac{\sqrt{2}m}{a}, \quad x_6 = \frac{\mathbf{e}_1(\phi)}{a}, \quad x_7 = \frac{m\phi}{\sqrt{2}a}, \tag{6.9}$$

we obtain the dynamical system

$$\frac{dx_2}{d\tau} = 2x_2^3 c_a - \frac{(x_2 - 1)x_7^2}{c_a + 1} + x_2 \left(x_4 + 4x_2 + x_6^2 - 2 \right), \tag{6.10a}$$

$$\frac{dx_4}{d\tau} = 2x_4 \left(2x_2^2 c_a - \frac{x_7^2}{c_a + 1} + x_4 + 3x_2 + x_6^2 - 1 \right), \tag{6.10b}$$

$$\frac{dx_5}{d\tau} = x_5 \left(2x_2^2 c_a - \frac{x_7^2}{c_a + 1} + x_4 + 3x_2 + x_6^2 \right), \tag{6.10c}$$

$$\frac{dx_6}{d\tau} = x_6 \left(-\frac{x_7^2}{c_a + 1} + x_4 + 4x_2 + x_6^2 - 2 \right) + 2x_2^2 x_6 c_a - x_5 x_7, \tag{6.10d}$$

$$\frac{dx_7}{d\tau} = x_7 \left(-\frac{x_7^2}{c_a + 1} + x_4 + 3x_2 + x_6^2 \right) + 2x_2^2 x_7 c_a - \frac{x_5 x_6}{2}, \tag{6.10e}$$

subject to the constraint

$$1 = c_a x_2{}^2 + 2x_2 + x_4 + \frac{1}{2}x_6^2 - x_7^2. \tag{6.11}$$

The constraint (6.11) is preserved by the dynamical system (6.10). Solving the constraint (6.11) for x_4 and substituting back into the system (6.10) we obtain the reduced system:

$$\frac{dx_2}{d\tau} = x_2^3 c_a - \frac{(x_2 - 1)x_7^2}{c_a + 1} + \frac{1}{2}x_2 \left(4x_2 + 2x_7^2 + x_6^2 - 2 \right), \tag{6.12a}$$

$$\frac{dx_5}{d\tau} = \frac{1}{2}x_5 \left(2c_a \left(\frac{x_7^2}{c_a + 1} + x_2^2 \right) + 2x_2 + x_6^2 + 2 \right), \tag{6.12b}$$

$$\frac{dx_6}{d\tau} = x_6 c_a \left(\frac{x_7^2}{c_a + 1} + x_2^2 \right) + \frac{1}{2} \left(x_6 \left(4x_2 + x_6^2 - 2 \right) - 2x_5 x_7 \right), \tag{6.12c}$$

$$\frac{dx_7}{d\tau} = x_7 c_a \left(\frac{x_7^2}{c_a + 1} + x_2^2 \right) - \frac{x_5 x_6}{2} + x_7 \left(x_2 + \frac{x_6^2}{2} + 1 \right), \tag{6.12d}$$

defined in the phase space

$$\Psi = \left\{ (x_2, x_5, x_6, x_7) : c_a x_2{}^2 + 2x_2 + x_4 + \frac{1}{2}x_6^2 - x_7^2 \leq 1 \right\}. \tag{6.13}$$

The equilibrium points of the system (6.12) are described in tables 6 and 7. Let us discuss their stability.

1. Q_1 is always a saddle.

2. The line of equilibrium points $Q_{2,3}$ is normally hyperbolic and is stable when $x_2^\star > 2$.

3. The equilibrium points $Q_{4,5}$ are non-hyperbolic. They have a 3D stable manifold and a 1D center manifold for $-1 \leq c_a < -\frac{3}{4}$ and a 1D stable manifold and a 3D center manifold for $c_a = -\frac{3}{4}$.

4. $Q_{6,7}$ are non-hyperbolic. They have a 3D stable manifold and a 1D center manifold for $-\frac{3}{4} < c_a < -\frac{1}{2}$. They have a 3D unstable manifold and a 1D center manifold for $c_a < -1$ or $c_a > -\frac{1}{2}$ [the non zero eigenvalues are always of the same sign].

Label	x_2	x_5	x_6	x_7	Existence
Q_1	0	0	0	0	always
$Q_{2,3}$	$x_2^\star$	0	$\pm\sqrt{2}\sqrt{1 - 2x_2^\star - c_a x_2^{\star 2}}$	0	$1 - 2x_2^\star - c_a x_2^{\star 2} \geq 0$
$Q_{4,5}$	$-\frac{1}{2c_a+1}$	0	0	$\pm\frac{\sqrt{(-c_a-1)(4c_a+3)}}{2c_a+1}$	$(c_a + 1)(4c_a + 3) \leq 0$
$Q_{6,7}$	$-\frac{1}{2c_a+1}$	0	$\pm\frac{\sqrt{2}\sqrt{(c_a+1)(4c_a+3)}}{2c_a+1}$	0	$(c_a + 1)(4c_a + 3) \geq 0$
$Q_{8,9}$	$\frac{1}{1\pm\sqrt{c_a+1}}$	0	0	0	always

Table 6. Equilibrium points of the system (6.12). $x_2^\star$ is a parameter and hence the curves $Q_{2,3}$ represent lines of equilibrium points ($x_2^\star = 0, x_2^\star = 2$ are special points on these curves).

Label	Eigenvalues
Q_1	$-1, -1, 1, 1$
$Q_{2,3}$	$2 - x_2^\star, 2 - x_2^\star, 2(1 - x_2^\star), 0$
$Q_{4,5}$	$-\frac{4c_a+3}{2c_a+1}, -\frac{4c_a+3}{2c_a+1}, -2, 0$
$Q_{6,7}$	$\frac{4(c_a+1)}{2c_a+1}, \frac{4c_a+3}{2c_a+1}, \frac{4c_a+3}{2c_a+1}, 0$
$Q_{8,9}$	$\frac{(2c_a \mp \sqrt{c_a+1})+1}{c_a}, \frac{(2c_a \mp \sqrt{c_a+1})+1}{c_a}, \frac{2((c_a \mp \sqrt{c_a+1})+1)}{c_a}, 0$

Table 7. Eigenvalues of the equilibrium points of the system (6.12) given in the previous table.

5. Q_8 is non-hyperbolic. It has a 3D unstable manifold and a 1D center manifold for $-1 < c_a < 0$ or $c_a > 0$. Otherwise, its center manifold has dimension greater than 1.

6. Q_9 is non-hyperbolic. It has a 3D stable manifold and a 1D center manifold for $-\frac{3}{4} < c_a < 0$. It has a 3D unstable manifold and a 1D center manifold for $c_a > 0$. Finally, Q_9 has a 2D unstable manifold, a 1D center manifold and a 1D stable manifold for $-1 < c_a < -\frac{3}{4}$.

Although static models are of particular physical importance, in this paper we have primarily focused on the mathematical properties of the solution space. It can be observed that the phase spaces (6.8) and (6.13) are in general non-compact; thus a more detailed analysis requires the introduction of compact variables. In addition, since the equilibrium points in (6.13) are generically non-hyperbolic, the use of the center manifold theorem is required, which is beyond the linear analysis provided here. A more detailed stability analysis for the equilibrium points of both the dynamical systems (6.7) and (6.12), and the study of the tilted aether static model, is left for the companion paper [46].

7 Discussion

In this paper we have studied spherically symmetric Einstein-aether models with tilting perfect fluid matter, which are also solutions of the IR limit of Horava gravity [8]. We used the 1+3 frame formalism [19–21] to write down the evolution equations for non-comoving perfect fluid spherically symmetric models and showed they form a well-posed system of first order PDEs in two variables. We adopted the so-called comoving aether gauge (which implies a preferred foliation, the only remaining freedom is the coordinate time and space reparameterization freedom). We also introduced (β-) normalized variables. The formalism is particularly well-suited for numerical and qualitative analysis. In particular, we considered the special subset $\dot{U} = v = 0$ (where we also assumed $c_\theta = 0$ and $c_\sigma \neq 0$) and derived the final reduced phase space equations in normalized variables.

The formalism adopted here is appropriate for the study of the qualitative properties of astrophysical and cosmological models with values for the non-GR parameters which are consistent with current constraints. In particular, motivated by current cosmological observations, we have studied inhomogeneous cosmologies in Einstein-aether theories of gravity.

We first considered dust models. We investigated a special dust model with $\dot{U} = 0$ and $v = 0$ in normalized variables (assuming $c_\sigma \neq 0$) and derived a reduced (closed) evolution system. The FLRW models in this special dust model correspond to an equilibrium point. In these models we are particularly interested physically in their late time evolution. Therefore, we paid particular attention to the sinks for different values of the parameter c_σ (which were summarized earlier). In all cases $\Omega \to 0$ to the future. For all solutions with small $c_\sigma < 3/8$, $\mathcal{D} \to 1$ ($\mathcal{Q} \to 1$) and the shear goes to zero at late times. Consequently, the models close to GR isotropize to the future.

We briefly reviewed the FLRW models in which the source must be of the form of a comoving perfect fluid (or vacuum) and the aether must be comoving. We then considered the spatially homogeneous Kantowski-Sachs models [36] using appropriate normalized variables (which are bounded; note that we did not use the $\beta-$normalization here), and obtained the general evolution equations. We then considered a special case with $3c_\theta \equiv c_1+3c_2+c_3 = 0$ and analysed the qualitative behaviour. In this case a full global dynamical analysis is possible, and we determined both the early and late time behaviour of the models and their physical properties.

In the case $c_\sigma < \frac{1}{2}$ (i.e., $c > 0$), when $\gamma < \frac{2}{3}$, there is the unique shear-free, zero curvature (FLRW) inflationary future attractor (P_2), and for $\frac{3}{8} < c_\sigma < \frac{1}{2}$ (i.e., $0 < c < \frac{1}{2}$) and $0 \leq \gamma < 2$ all of the sources and sinks (respectively, P_4 & P_8 and P_5 & P_7) have maximal shearing and all except one sink (P_7) have zero curvature. For $c_\sigma < \frac{3}{8}$ (i.e., $c > \frac{1}{2}$), the points P_7 & P_8 do not exist, and the sources and sinks with maximal shearing are P_4 and P_5, respectively.

Finally, we considered static models for a mixture of a (necessarily non-tilted with $v = 0$) perfect fluid with a barotropic equations of state and a scalar field (with a self-interaction potential V that depends only on the scalar field). In particular, we studied the special cases of a tilted perfect fluid and no scalar field (previous work had assumed a comoving aether and a comoving fluid) and a stationary vacuum with a scalar field (with a harmonic potential). The equilibrium points in the resulting dynamical systems in these two cases were determined and their stability was investigated. Although models that are not evolving with time are of physical importance, and physical information can be obtained from their qualitative analysis, we have primarily focussed on the mathematical properties of the solution space in

this paper. The physical interpretation of this analysis will be comprehensively discussed in [46].

We also examined the conditions for the existence of McVittie-like solutions in the context of Einstein-aether theory. We found that they only exist for the choice of parameters $c_a = 0, \gamma = 0$, and for an aligned aether ($v = 0$). Since $\gamma = 0$, the matter fluid corresponds to a cosmological constant (and θ is always a constant). Irrespective of the sign of the initial expansion, the physical variables tend to zero as $t \to +\infty$.

In future work we shall investigate the general Kantowski-Sachs models and the static models more comprehensively. In particular, it would be of interest to determine the structure of stationary rotating solutions; rapidly rotating black holes, unlike the non-rotating ones, might turn out to be very different from the Kerr metrics of GR.

We note that the tilt is defined relative to matter; one important question is to investigate whether this tilt decays to the future in general. We shall also study spherically symmetric, self-similar spacetimes which also admit, in addition to the three Killing vectors, a homothetic vector [21].

A   Models and the parameters c_i

We can study different models with different dimensionless parameters c_i. From earlier:

$$c_\theta = c_2 + (c_1 + c_3)/3, \ c_\sigma = c_1 + c_3, \ c_\omega = c_1 - c_3, \ c_a = c_4 - c_1.$$

Since the spherically symmetric models are hypersurface orthogonal the aether field has vanishing twist and the field equations are therefore independent of the twist parameter c_ω [8] (this is equivalent to being able to set $c_4 = 0$ without loss of generality [1]).

A second condition on the c_i can effectively be specified by a renormalization the Newtonian gravitational constant G. From [1] we have that $G_N = G\left(1 - \frac{1}{2}(c_1 + c_4)\right)^{-1}$. So long as $(c_1 + c_4) < 2$, so that the gravitational constant is positive, we can effectively renormalize and specify $(c_1 + c_4)$. If not, and we reduce the theory to a one parameter model, the theory might be pure GR in disguise [in GR $c_i = 0$].

The remaining two non-trivial constant parameters in the model must satisfy additional constraints (it will be useful here to define $c^2 \equiv 1 - 2c_\sigma \leq 1$):

Observations: The models (i.e., the values of the c_i) must be consistent with all observations. In general, if the magnitudes of all of the c_i are (non-zero and) small (e.g., less than 10^{-2}), then the models will be physical [1, 24].

Self-consistency: There are also a number of self-consistency requirements [1, 24]:

$$0 \leq c_1 + c_3 \leq 1, \tag{A.1}$$

$$0 \leq c_1 - c_3 \leq \frac{(c_1 + c_3)}{3[1 - (c_1 + c_3)]}, \tag{A.2}$$

which can be written in terms of c_θ, c_σ, c_a. Note that this imples that $0 \leq c_1, c_3 \leq c_1 \leq 1 - c_3$.

A.1   Case A:

All of the c_i are small (and not all zero). We set $c_4 = 0$. We renormalize and chose c_1 so that $c_1 + 3c_2 + c_3 = 0$ (i.e., $c_\theta = 0$). The self-consistency relations (A.1, A.2) then imply that $0 \leq c_3 \leq c_1 \leq 2c_3$. We thus have a two parameter model with small $c_\sigma \geq 0$, $c_a \leq 0$. Using $c_1 \equiv dc_3$ ($1 \leq d \leq 2$), we have that $c_\sigma = \frac{1}{2}(1 - c^2)$ and $c_a = -\frac{d}{(1+d)}c_\sigma$. Note that it is not possible for $c_a = -c_\sigma$ in this case.

Summary case A: $c_\sigma = \frac{1}{2}(1 - c^2) \geq 0, c_a = -\frac{d}{(1+d)}c_\sigma \leq 0, c_\theta = 0$

A.2   Case B:

We set

$$c_4 = -\frac{c_3{}^2}{c_1}, \tag{A.3}$$

$$c_2 = -\frac{(2c_1{}^2 + c_1 c_3 - c_3{}^2)}{3c_1} \tag{A.4}$$

(before the field redefinition of c_4), so that the parameterized post-Newtonian (PPN) parameters $\alpha_1 = \alpha_2 = 0$ (and hence all solar system tests are trivially satified [1]). A two parameter family of models ($c_1 \neq 0, c_3$) satisfying the consistency conditions (A.1, A.2) results. Here (**Summary**):

$$c_a = -\frac{(c_1{}^2 + c_3{}^2)}{c_1} \le 0, \tag{A.5}$$

$$0 \le c_\sigma = c_1 + c_3 \le 1, \tag{A.6}$$

$$c_\theta = -\frac{(c_1{}^2 - c_3{}^2)}{3c_1} \le 0, \tag{A.7}$$

and the c_i satisfy

$$(c_a{}^2 - 9c_\theta{}^2) = (c_a + 3c_\theta + 2c_\sigma)^2. \tag{A.8}$$

A.2.1   Case B(ii):

If we also renormalise the Newtonian gravitational potential by setting $c_1 + c_4 = 0$, then from (A.3) we obtain $c_1{}^2 = c_3{}^2$, and hence $c_1 = c_3$ (since $c_1 = -c_3$ cannot satisfy (A.1, A.2)). In this case $c_\theta = 0$. Hence, $c_a = -c_\sigma$, and we have that $c_\sigma = \frac{1}{2}(1 - c^2) \ge 0$ and $c_a = -\frac{1}{2}(1 - c^2) \le 0$, where $c^2 \le 1$ but need not be small (and the self-consistency relations (A.1, A.2) are satisfied).

Summary case B(ii): $c_\sigma = \frac{1}{2}(1 - c^2) \ge 0, c_a = -\frac{1}{2}(1 - c^2), c_\theta = 0$

A.3   Case C:

In principle we can study the physics of the models for different parameter ranges of the c_i. If we study the models in the early universe (where the constants c_i can be replaced with evolving parameters [5]), then the observational constraints above need not apply.

In one particularly interesting theoretical case (see section 4), we could consider $3c_\theta + 2c_\sigma = 0$ [we could also use the renormalization of the Newtonian gravitational constant and consider the case $c_a = 0$]. Note that $3c_\theta + 2c_\sigma = c_1 + c_2 + c_3 = 0$. This implies that the PPN parameter α_1 diverges [1] and conditions (A.1, A.2) can only be satisfied when $c_1 = c_2 = c_3 = 0$, the GR case. However, for theoretical reasons it may be of interest to study this case in early universe cosmological models.

Summary case C: $c_\sigma = \frac{1}{2}(1 - c^2) \ge 0, c_\theta = -\frac{1}{3}(1 - c^2) \le 0, [c_a = 0]$

B Further development of the governing equations

Let us further develop the governing equations presented in Section 3. Combining equations (3.13d) and (3.13e) and using the identity $\Sigma = 1 - \mathcal{Q}$, we obtain

$$
q = -\frac{3\mathcal{K}}{2(2c_\sigma - 1)} + \frac{1}{3}\mathcal{A}\dot{U}\left(\frac{2\left(c_a + 1\right)}{3c_\theta + 1} + \frac{1 - 2c_a}{1 - 2c_\sigma}\right) - \frac{\left(c_a + 1\right)\left(3c_\theta + 2c_\sigma\right)\boldsymbol{\partial}_1\dot{U}}{3\left(3c_\theta + 1\right)\left(2c_\sigma - 1\right)} +
$$
$$
+ \frac{1}{3}r\left(c_a + 1\right)\left(\frac{1}{3c_\theta + 1} + \frac{1}{2c_\sigma - 1}\right)\dot{U} + \frac{1}{6}\dot{U}^2\left(\frac{5c_a + 2}{1 - 2c_\sigma} - \frac{2\left(c_a + 1\right)}{3c_\theta + 1}\right) +
$$
$$
+ \frac{\mathcal{A}^2}{2(2c_\sigma - 1)} + \frac{4\left(3c_\theta + 2c_\sigma\right)\mathcal{Q}}{3c_\theta + 1} + \left(\frac{6 - 3c_\theta - 14c_\sigma}{4c_\sigma - 2} + \frac{2 - 4c_\sigma}{3c_\theta + 1} + 1\right)\mathcal{Q}^2 +
$$
$$
- \frac{\Omega\left(v^2\left(3(\gamma + 1)c_\theta + 2(\gamma - 2)c_\sigma + 3\right) - 6\gamma c_\sigma - 3c_\theta + 4c_\sigma + 3\gamma - 3\right)}{2\left(3c_\theta + 1\right)\left(2c_\sigma - 1\right)\left(1 - v^2\right)} +
$$
$$
+ \frac{1 - 9c_\theta - 8c_\sigma}{6c_\theta + 2}. \tag{B.1}
$$

Combining equations (3.13c), (3.13f) and (3.14d) and (3.14e) and using the identity $\Sigma = 1 - \mathcal{Q}$, we obtain

$$
r = (q - 1)c_a\dot{U} - c_a\boldsymbol{\partial}_0\dot{U} - 3\mathcal{A}(1 - \mathcal{Q}) + \frac{3\gamma v\Omega}{2(1 - v^2)}, \tag{B.2a}
$$

$$
\boldsymbol{\partial}_0\mathcal{A} = (q + 3(1 - \mathcal{Q}))\mathcal{A} - \dot{U} + r, \tag{B.2b}
$$

$$
\boldsymbol{\partial}_1\mathcal{Q} = -\frac{3\left(2c_\sigma - 1\right)\mathcal{A}(1 - \mathcal{Q})}{3c_\theta + 2c_\sigma} + r\left(\mathcal{Q} + \frac{1 - 2c_\sigma}{3c_\theta + 2c_\sigma}\right) - \frac{3\gamma v\Omega}{2\left(3c_\theta + 2c_\sigma\right)\left(1 - v^2\right)},. \tag{B.2c}
$$

Finally, solving the equations (B.1) and (B.2a) for q and r we obtain:

$$
q = -\frac{1}{3\left(2 - \frac{2c_a(c_a + 1)(3c_\theta + 2c_\sigma)\dot{U}^2}{3(3c_\theta + 1)(2c_\sigma - 1)}\right)}\left\{\frac{3\mathcal{A}^2}{1 - 2c_\sigma} - \frac{4\left(c_a + 1\right)\dot{U}\mathcal{A}}{3c_\theta + 1} +\right.
$$
$$
+ \frac{2\left(1 - 2c_a\right)\dot{U}\mathcal{A}}{2c_\sigma - 1} - 6\mathcal{Q}^2 + \frac{2\left(c_a + 1\right)\dot{U}^2}{3c_\theta + 1} + \frac{\left(5c_a + 2\right)\dot{U}^2}{2c_\sigma - 1} +
$$
$$
+ \frac{12(\mathcal{Q} - 1)^2\left(2c_\sigma - 1\right)}{3c_\theta + 1} + \frac{3\mathcal{Q}\left(-16c_\sigma + \mathcal{Q}\left(3c_\theta + 14c_\sigma - 6\right) + 8\right)}{2c_\sigma - 1} +
$$
$$
- \frac{3\left((\gamma - 2)v^2 - 3\gamma + 2\right)\Omega}{\left(3c_\theta + 1\right)\left(v^2 - 1\right)} - \frac{3\left((\gamma + 1)v^2 - 1\right)\Omega}{\left(2c_\sigma - 1\right)\left(v^2 - 1\right)} +
$$
$$
+ \frac{\left(c_a + 1\right)\left(3c_\theta + 2c_\sigma\right)\dot{U}\left(-6\mathcal{A}(\mathcal{Q} - 1) + 2c_a(\boldsymbol{\partial}_0\dot{U} + \dot{U}) + \frac{3\gamma v\Omega}{v^2 - 1}\right)}{\left(3c_\theta + 1\right)\left(2c_\sigma - 1\right)} +
$$
$$
\left. + \frac{2\boldsymbol{\partial}_1\dot{U}\left(c_a + 1\right)}{3c_\theta + 1} + \frac{3}{1 - 2c_\sigma} + \frac{9\mathcal{K}}{2c_\sigma - 1} + \frac{2\boldsymbol{\partial}_1\dot{U}\left(c_a + 1\right)}{2c_\sigma - 1} + \frac{3}{2c_\sigma - 1} + 9\right\}, \tag{B.3a}
$$

$$r = 3\mathcal{Q}\mathcal{A} - 3\mathcal{A} - c_a\boldsymbol{\partial}_0\dot{U} - c_a\dot{U} + \frac{3\gamma v\Omega}{2(1-v^2)} +$$

$$- \frac{c_a\dot{U}}{3\left(2 - \frac{2c_a(c_a+1)(3c_\theta+2c_\sigma)\dot{U}^2}{3(3c_\theta+1)(2c_\sigma-1)}\right)}\left\{\frac{3\mathcal{A}^2}{1-2c_\sigma} - \frac{4\,(c_a+1)\,\dot{U}\mathcal{A}}{3c_\theta+1} +\right.$$

$$+ \frac{2\,(1-2c_a)\,\dot{U}\mathcal{A}}{2c_\sigma - 1} - 6\mathcal{Q}^2 + \frac{2\,(c_a+1)\,\dot{U}^2}{3c_\theta+1} + \frac{(5c_a+2)\,\dot{U}^2}{2c_\sigma-1} +$$

$$+ \frac{12(\mathcal{Q}-1)^2\,(2c_\sigma-1)}{3c_\theta+1} + \frac{3\mathcal{Q}\,(-16c_\sigma + \mathcal{Q}\,(3c_\theta + 14c_\sigma - 6) + 8)}{2c_\sigma - 1} +$$

$$- \frac{3\left((\gamma-2)v^2 - 3\gamma + 2\right)\Omega}{(3c_\theta+1)\,(v^2-1)} - \frac{3\left((\gamma+1)v^2 - 1\right)\Omega}{(2c_\sigma-1)\,(v^2-1)} +$$

$$+ \frac{(c_a+1)\,(3c_\theta+2c_\sigma)\,\dot{U}\left(-6\mathcal{A}(\mathcal{Q}-1) + 2c_a(\boldsymbol{\partial}_0\dot{U} + \dot{U}) + \frac{3\gamma v\Omega}{v^2-1}\right)}{(3c_\theta+1)\,(2c_\sigma-1)} +$$

$$\left.+ \frac{2\boldsymbol{\partial}_1\dot{U}\,(c_a+1)}{3c_\theta+1} + \frac{3}{1-2c_\sigma} + \frac{9\mathcal{K}}{2c_\sigma-1} + \frac{2\boldsymbol{\partial}_1\dot{U}\,(c_a+1)}{2c_\sigma-1} + \frac{3}{2c_\sigma-1} + 9\right\}. \qquad \text{(B.3b)}$$

These equations give the expressions for q and r in terms of the normalized variables and the derivatives $\boldsymbol{\partial}_0\dot{U}, \boldsymbol{\partial}_1\dot{U}$. The expression (B.2c), is used to eliminate the spatial derivative $\boldsymbol{\partial}_1\mathcal{Q}$ from the equations.

The final equations for the reduced phase space $\left(\mathcal{N}, \dot{U}, E_1^1, \mathcal{K}, \mathcal{Q}, \Omega, v\right)^T$ are equations (3.13a,3.13b,3.13d,3.13g,3.13h) and (3.14e) for $\boldsymbol{\partial}_0\mathcal{A}$, subject to the constraints (3.14a-3.14d) and a constraint for $\boldsymbol{\partial}_1\mathcal{Q}$ (rather than $\boldsymbol{\partial}_1\Sigma$), with q and r defined as in (B.3). From the equations we have either the evolution equation for $\dot{U}$ given by (3.13c) or the definition of r given by (B.2a) (or by (B.3)(b), after q-elimination in (B.2a)).

The commutator equation (2.30) can be expressed in terms of the normalized variables by

$$[\boldsymbol{\partial}_0, \boldsymbol{\partial}_1] = \frac{1}{\beta^2}[\mathbf{e}_0, \mathbf{e}_1] + (1+q)\boldsymbol{\partial}_1 - r\boldsymbol{\partial}_0$$
$$= (\dot{U} - r)\boldsymbol{\partial}_0 + (q + 3 - 3\mathcal{Q})\boldsymbol{\partial}_1. \qquad \text{(B.4)}$$

On the other hand

$$\boldsymbol{\partial}_0 r = \boldsymbol{\partial}_0\boldsymbol{\partial}_1(-\ln\beta) = -[\boldsymbol{\partial}_0, \boldsymbol{\partial}_1]\ln\beta - \boldsymbol{\partial}_1\boldsymbol{\partial}_0\ln\beta$$
$$= \dot{U}(1+q) + r(2 - 3\mathcal{Q}) + \boldsymbol{\partial}_1 q, \qquad \text{(B.5)}$$

where we have used $\boldsymbol{\partial}_0\ln\beta = -(1+q)$ and q is given by (B.1). [10]

---

[10] Note that the evolution equation for r will contain the second order derivative term $\boldsymbol{\partial}_1\boldsymbol{\partial}_1\dot{U}$ (via the term $\boldsymbol{\partial}_1 q$), as occurs in the GR setting [42].

The final equations for the reduced phase space $\left(\mathcal{N}, \dot{U}, E_1^1, \mathcal{K}, \mathcal{Q}, \Omega, v, r\right)^T$ are then

$$\boldsymbol{\partial}_0 E_1^1 = (q + 3\Sigma)E_1^1, \tag{B.6a}$$

$$\boldsymbol{\partial}_0 \mathcal{K} = 2q\mathcal{K}, \tag{B.6b}$$

$$c_a \boldsymbol{\partial}_0 \dot{U} = (q - 1)c_a \dot{U} - r - 3\mathcal{A}(1 - \mathcal{Q}) + \frac{3\gamma v \Omega}{2(1 - v^2)}, \tag{B.6c}$$

$$\boldsymbol{\partial}_0 \mathcal{Q} - \frac{(c_a + 1)\boldsymbol{\partial}_1 \dot{U}}{3(3c_\theta + 1)} = \mathcal{Q}(1 + q - \mathcal{Q}) - \frac{r(c_a + 1)\dot{U}}{3(3c_\theta + 1)} - \frac{2(c_a + 1)\mathcal{A}\dot{U}}{3(3c_\theta + 1)} +$$
$$+ \frac{(c_a + 1)\dot{U}^2}{3(3c_\theta + 1)} + \frac{2(2c_\sigma - 1)(1 - \mathcal{Q})^2}{3c_\theta + 1} + \frac{\Omega\left((\gamma - 2)v^2 - 3\gamma + 2\right)}{2(3c_\theta + 1)(1 - v^2)}, \tag{B.6d}$$

$$\boldsymbol{\partial}_0 \mathcal{A} = (q + 3\Sigma)\mathcal{A} - \dot{U} + r, \tag{B.6e}$$

$$\boldsymbol{\partial}_0 \Omega + \frac{(\gamma - 2)v \boldsymbol{\partial}_1 \Omega}{G_-} - \frac{\gamma \Omega \boldsymbol{\partial}_1 v}{G_-} = -\frac{2\gamma \mathcal{A} v \Omega}{G_-} + \frac{2q\Omega\left(1 - (\gamma - 1)v^2\right)}{G_-} +$$
$$- \frac{3\gamma \mathcal{Q}\left(1 - v^2\right)\Omega}{G_-} + \frac{2(\gamma - 2)rv\Omega}{G_-} + \frac{2\Omega\left((1 - 2\gamma)v^2 + 1\right)}{G_-}, \tag{B.6f}$$

$$\boldsymbol{\partial}_0 v - \frac{(\gamma - 1)\left(1 - v^2\right)^2 \boldsymbol{\partial}_1 \Omega}{\gamma \Omega G_-} + \frac{(\gamma - 2)v \boldsymbol{\partial}_1 v}{G_-} = \frac{2(\gamma - 1)\mathcal{A}\left(1 - v^2\right)v^2}{G_-} +$$
$$- \frac{3(\gamma - 2)\mathcal{Q}\left(1 - v^2\right)v}{G_-} - \frac{2(\gamma - 1)r\left(1 - v^2\right)^2}{\gamma G_-} + \frac{2\left(1 - v^2\right)v}{G_-} + \dot{U}\left(1 - v^2\right), \tag{B.6g}$$

$$\boldsymbol{\partial}_0 r = \dot{U}(1 + q) + r(2 - 3\mathcal{Q}) + \boldsymbol{\partial}_1 q, \tag{B.6h}$$

where

$$q = -\frac{3\mathcal{K}}{2(2c_\sigma - 1)} + \frac{1}{3}\mathcal{A}\dot{U}\left(\frac{2(c_a + 1)}{3c_\theta + 1} + \frac{1 - 2c_a}{1 - 2c_\sigma}\right) - \frac{(c_a + 1)(3c_\theta + 2c_\sigma)\boldsymbol{\partial}_1 \dot{U}}{3(3c_\theta + 1)(2c_\sigma - 1)} +$$
$$+ \frac{1}{3}r(c_a + 1)\left(\frac{1}{3c_\theta + 1} + \frac{1}{2c_\sigma - 1}\right)\dot{U} + \frac{1}{6}\dot{U}^2\left(\frac{5c_a + 2}{1 - 2c_\sigma} - \frac{2(c_a + 1)}{3c_\theta + 1}\right) +$$
$$+ \frac{\mathcal{A}^2}{2(2c_\sigma - 1)} + \frac{4(3c_\theta + 2c_\sigma)\mathcal{Q}}{3c_\theta + 1} + \left(\frac{6 - 3c_\theta - 14c_\sigma}{4c_\sigma - 2} + \frac{2 - 4c_\sigma}{3c_\theta + 1} + 1\right)\mathcal{Q}^2 +$$
$$- \frac{\Omega\left(v^2\left(3(\gamma + 1)c_\theta + 2(\gamma - 2)c_\sigma + 3\right) - 6\gamma c_\sigma - 3c_\theta + 4c_\sigma + 3\gamma - 3\right)}{2(3c_\theta + 1)(2c_\sigma - 1)(1 - v^2)} +$$
$$+ \frac{1 - 9c_\theta - 8c_\sigma}{6c_\theta + 2}, \tag{B.7}$$

subject to the constraints

$$\boldsymbol{\partial}_1 \mathcal{N}^{-1} = (r - \dot{U})\mathcal{N}^{-1}, \tag{B.8a}$$

$$\boldsymbol{\partial}_1 \mathcal{K} = 2(r + \mathcal{A})\mathcal{K}, \tag{B.8b}$$

$$\boldsymbol{\partial}_1 \mathcal{A} + c_a \boldsymbol{\partial}_1 \dot{U} = -\frac{3}{2}\mathcal{K} + r\left(c_a \dot{U} + \mathcal{A}\right) + 2c_a \mathcal{A}\dot{U} - \frac{1}{2}c_a \dot{U}^2 + \frac{3}{2}\mathcal{A}^2 +$$
$$- \frac{3}{2}(3c_\theta + 2c_\sigma)\mathcal{Q}^2 + 3(2c_\sigma - 1)\mathcal{Q} + \frac{3\Omega\left((\gamma - 1)v^2 + 1\right)}{2(1 - v^2)} - 3c_\sigma + \frac{3}{2}, \tag{B.8c}$$

$$\boldsymbol{\partial}_1 \mathcal{Q} = -\frac{3(2c_\sigma - 1)\mathcal{A}(1 - \mathcal{Q})}{3c_\theta + 2c_\sigma} + r\left(\mathcal{Q} + \frac{1 - 2c_\sigma}{3c_\theta + 2c_\sigma}\right) - \frac{3\gamma v \Omega}{2(3c_\theta + 2c_\sigma)(1 - v^2)}. \tag{B.8d}$$

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We can choose to study the system (3.13), (3.14), (B.3) or the system (B.6), (B.7), (B.8), depending on the particular application.

In the comoving aether temporal gauge, which implies a preferred foliation, the only remaining freedom is the coordinate rescalings $t \rightarrow f(t)$ and $x \rightarrow g(x)$ (time and space reparameterization freedom), consistent with

$$\boldsymbol{\partial}_0 E_1^1 = (q + 3\Sigma)E_1^1, \tag{B.9}$$

$$\boldsymbol{\partial}_1 \mathcal{N}^{-1} = (r - \dot{U})\mathcal{N}^{-1}, \tag{B.10}$$

where we recall that $\boldsymbol{\partial}_0 := \mathcal{N}^{-1}\partial_t, \quad \boldsymbol{\partial}_1 := E_1^1 \partial_x$.

How do we best treat $\mathcal{N}^{-1}$ in the evolutions eqns? There is no evolution eqn for $\mathcal{N}^{-1}$ but, in principle, we can integrate the spatial constraint and use the time reparameterization to determine $\mathcal{N}^{-1}$. [This is what happens in GR in the separable gauge, where neither an algebraic equation or an evolution equation for $\mathcal{N}^{-1}$ is available; the conditions are sufficient to integrate the spatial derivative constraint equation, whence a time redefinition can be employed to set $\mathcal{N}^{-1} = 1$.]

Depending on the application, we could do one of the following: (i) Since $\mathcal{N}^{-1}$ is positive-definite, we can determine the qualitative behaviour of the system by simply studying the right-hand-sides of the evolution equations. (ii) In many special cases of interest we can integrate for $\mathcal{N}^{-1}$ and replace the left-hand-sides by partial time derivatives (as in GR in the separable gauge). (iii) Numerically we could solve for $\mathcal{N}^{-1}$ in the integration (although this looks messy). (iv) Analytically, we could, for example, use the commutators to obtain evolution and constraint equations for r, and then $\mathcal{N}^{-1} = \exp(\int(\dot{U} - r)dx)$ and define new variables (e.g., use $\dot{U} - r$ as a variable). (v) While the comoving aether gauge choice is motivated physically, it is not ideal for doing analysis and numerics; we could change to a separable gauge.

In practice, it is often useful to choose a gauge in order to compare with the FLRW model as easily as possible (e.g., so we can choose integration functions for $\mathcal{N}, E_1^1$ in the above to be trivial).

As an example, let us consider McVittie-like models [43]. The line element is given by

$$-\left(\frac{1 - \frac{M}{2Ax}}{1 + \frac{M}{2Ax}}\right)^2 dt^2 + A^2\left(1 + \frac{M}{2Ax}\right)^4 \left[dx^2 + x^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2)\right], \tag{B.11}$$

where $A = A(t)$ and M is a constant. We recall that as $x \rightarrow +\infty$ the metric approaches the flat FLRW solution and for constant A we obtain the Schwarzschild solution. Several aspects of the McVittie solution, of geometrical and physical relevance have been investigated by many authors [44]. In GR the McVittie solution is unique under the following assumptions: (i) The metric is spherically symmetric with a singularity at the centre. (ii) The matter distribution is a perfect fluid. (iii) The metric must asymptotically tend to an isotropic cosmological form. (iv) The fluid flow is shear-free. McVittie-like models were investigated, e.g., in the references [45].

For this metric we obtain in our scenario:

$$\dot{u} = -\frac{16Mx^2A^2}{(M-2xA)(M+2xA)^3}, \tag{B.12a}$$

$$a = \frac{4xA(M-2xA)}{(M+2xA)^3}, \tag{B.12b}$$

$$K = \frac{16x^2A^2}{(2xA+M)^4}, \tag{B.12c}$$

$${}^3R = 0, \tag{B.12d}$$

$${}^3S_+ = \frac{64Mx^3A^3}{(2xA+M)^6}, \tag{B.12e}$$

$$\sigma_+ = 0, \tag{B.12f}$$

$$\theta = \frac{3\dot{A}}{A}, \tag{B.12g}$$

which implies that

$$\mathcal{Q} = 1, \Sigma = 0, r = 0,$$

$$q = \frac{A\ddot{A}(2xA+M)}{\dot{A}^2(M-2xA)} - \frac{2M}{M-2xA}$$

For this metric, the equations (B.6c) and (B.8d) lead to a contradiction, unless $c_a = 0$ and either $v = 0$ or $\Omega = 0$, or both. Assuming $c_a = v = \Omega = 0$ and substituting into equation (B.6d) we obtain

$$\frac{A\ddot{A}(2xA+M) - 2M\dot{A}^2}{\dot{A}(M-2xA)} = 0 \tag{B.13}$$

which, in general, is not satisfied as we vary t and x. Thus, assuming $c_a = 0$ and $v = 0, \Omega \neq 0$, we obtain

$$\Omega = \frac{2\left(3c_\theta + 1\right)\left(A\ddot{A}(2xA+M) - 2M\dot{A}^2\right)}{(3\gamma - 2)\dot{A}^2(M-2xA)}. \tag{B.14}$$

The equations (B.6) and the constraints (B.8), with the exception of (B.6f), (B.6g) and (B.8c), are identically satisfied. The equations (B.6f) and (B.6g) reduce to

$$\boldsymbol{\partial}_0\Omega = (2q + 2 - 3\gamma)\Omega, \tag{B.15a}$$

$$(1-\gamma)\boldsymbol{\partial}_1\Omega = \gamma\Omega\dot{U}. \tag{B.15b}$$

which, after the substitution of (B.14), lead to

$$\ddot{A} = \frac{\dot{A}^2((\gamma+1)M - 2(\gamma-1)xA)}{A(2xA+M)}, \tag{B.16a}$$

$$\dddot{A} = \frac{\dot{A}^3\left(4xA\left((\gamma-1)(3\gamma-1)xA - 3\gamma^2M + M\right) + (\gamma(3\gamma+2)+1)M^2\right)}{A^2(2xA+M)^2}. \tag{B.16b}$$

and the restriction (B.8c) becomes

$$\frac{3\left(3c_\theta + 1\right)\left(\dot{A}^2((4-6\gamma)xA + (3\gamma+2)M) - 2A\ddot{A}(2xA+M)\right)}{2(3\gamma-2)\dot{A}^2(M-2xA)} = 0. \tag{B.17}$$

The last three equations, in general, are not satisfied simultaneously for all t and x unless we set $\gamma = 0$, which implies

$$A(t) = c_2 e^{c_1 t}, \quad \Omega = 1 + 3c_\theta. \tag{B.18}$$

Summarizing: the McVittie-like models only exist for the choice of parameters $c_a = 0, \gamma = 0$, and for aligned aether ($v = 0$). Since $\gamma = 0$, the matter fluid corresponds to a cosmological constant. The solution is characterized by

$$\dot{u} = -\frac{16c_2^2 M x^2 e^{2c_1 t}}{\left(M - 2c_2 x e^{c_1 t}\right)\left(2c_2 x e^{c_1 t} + M\right)^3}, \tag{B.19a}$$

$$a = \frac{4c_2 x e^{c_1 t}\left(M - 2c_2 x e^{c_1 t}\right)}{\left(2c_2 x e^{c_1 t} + M\right)^3}, \tag{B.19b}$$

$$K = \frac{16c_2^2 x^2 e^{2c_1 t}}{\left(2c_2 x e^{c_1 t} + M\right)^4}, \tag{B.19c}$$

$${}^3R = 0, \tag{B.19d}$$

$${}^3S_+ = \frac{64c_2^3 M x^3 e^{3c_1 t}}{\left(2c_2 x e^{c_1 t} + M\right)^6}, \tag{B.19e}$$

$$\sigma_+ = 0, \tag{B.19f}$$

$$\theta = 3c_1. \tag{B.19g}$$

and

$$\mathcal{Q} = 1, \Sigma = 0, r = 0, v = 0, \Omega = 1 + 3c_\theta, q = -1. \tag{B.20}$$

Irrespective of the sign of the expansion ($\theta > 0$ or $\theta < 0$), the quantities $\dot{u}, a, K, {}^3S_+$ tend to zero as $t \to +\infty$.

B.1 Special case: $\dot{u} = 0$

The evolution equations above were derived under the assumption that $\dot{u} \neq 0$. We now consider the special case of $\dot{u} = 0$ and display the appropriate eqns:

$$\mathbf{e}_0(e_1{}^1) = -\tfrac{1}{3}(\theta - 6\sigma_+)e_1{}^1, \tag{B.21a}$$

$$\mathbf{e}_0(K) = -\tfrac{2}{3}(\theta + 3\sigma_+)K, \tag{B.21b}$$

$$\mathbf{e}_0(\theta) = -\frac{1}{3}\theta^2 + \frac{6(2c_\sigma - 1)\sigma_+^2}{3c_\theta + 1} + \frac{\left(2 - 3\gamma + (\gamma - 2)v^2\right)\hat{\mu}}{2(3c_\theta + 1)(1 - v^2)}, \tag{B.21c}$$

$$\mathbf{e}_0(\sigma_+) = \frac{1}{2}\sigma_+^2 - \theta\sigma_+ + \frac{(3c_\theta + 1)\theta^2}{18(2c_\sigma - 1)} - \frac{a^2}{2(2c_\sigma - 1)} + \frac{K}{2(2c_\sigma - 1)} +$$

$$+ \frac{\left((1 + \gamma)v^2 - 1\right)\hat{\mu}}{6(2c_\sigma - 1)(1 - v^2)}, \tag{B.21d}$$

$$\mathbf{e}_0(a) = -\frac{1}{3}a(\theta + 3\sigma_+) + \frac{\gamma v\hat{\mu}}{2(1 - v^2)}, \tag{B.21e}$$

$$\mathbf{e}_0(\hat{\mu}) - \frac{\mathbf{e}_1(\hat{\mu})\,v\,(2 - \gamma)}{G_-} - \frac{\gamma\hat{\mu}\mathbf{e}_1(v)}{G_-} = -\frac{2\gamma\hat{\mu}v^2\sigma_+}{G_-} + \frac{\gamma\left(v^2 - 3\right)\hat{\mu}\theta}{3G_-} - 2\frac{\gamma\hat{\mu}va}{G_-}, \tag{B.21f}$$

$$\mathbf{e}_0(v) - \frac{\mathbf{e}_1(\hat{\mu})\left(1 - v^2\right)^2(\gamma - 1)}{\gamma\hat{\mu}G_-} - \frac{v\,(2 - \gamma)\,\mathbf{e}_1(v)}{G_-} = \frac{2v\left(1 - v^2\right)\sigma_+}{G_-} +$$

$$\frac{v\left(1 - v^2\right)(3\gamma - 4)\,\theta}{3G_-} + \frac{2v^2\left(1 - v^2\right)(\gamma - 1)\,a}{G_-}. \tag{B.21g}$$

Constraints ($3c_\theta + 2c_\sigma \neq 0$):

$$\mathbf{e}_1(\ln K) = 2a, \tag{B.22a}$$

$$\mathbf{e}_1(a) = \frac{G_+\hat{\mu}}{2(1 - v^2)} - \frac{1}{6}(3c_\theta + 1)\,\theta^2 - \frac{3}{2}(2c_\sigma - 1)\,\sigma_+^2 - \frac{K}{2} + \frac{3a^2}{2} \tag{B.22b}$$

$$\mathbf{e}_1(\theta) = -\frac{3\gamma c_\sigma v\hat{\mu}}{(3c_\theta + 2c_\sigma)(1 - v^2)}, \tag{B.22c}$$

$$\mathbf{e}_1(\sigma_+) = 3a\sigma_+ - \frac{3\gamma c_\theta v\hat{\mu}}{2(3c_\theta + 2c_\sigma)(1 - v^2)}. \tag{B.22d}$$

B.1.1 Normalized variables

The β-normalized equations are:

$$\boldsymbol{\partial}_0 E_1^1 = (q + 3\Sigma)E_1^1, \tag{B.23a}$$

$$\boldsymbol{\partial}_0 \mathcal{K} = 2q\mathcal{K}, \tag{B.23b}$$

$$\boldsymbol{\partial}_0 \mathcal{Q} = \mathcal{Q}(1 + q - \mathcal{Q}) + \frac{2\left(2c_\sigma - 1\right)\Sigma^2}{3c_\theta + 1} + \frac{\Omega\left((\gamma - 2)v^2 - 3\gamma + 2\right)}{2\left(3c_\theta + 1\right)\left(1 - v^2\right)}, \tag{B.23c}$$

$$\boldsymbol{\partial}_0 \Sigma = \Sigma(1 + q - 3\mathcal{Q}) + \frac{1}{2}\Sigma^2 + \frac{3\mathcal{K}}{2(2c_\sigma - 1)} - \frac{\mathcal{A}^2}{2(2c_\sigma - 1)} + \frac{\left(3c_\theta + 1\right)\mathcal{Q}^2}{2(2c_\sigma - 1)} +$$
$$+ \frac{\Omega\left((\gamma + 1)v^2 - 1\right)}{2\left(2c_\sigma - 1\right)\left(1 - v^2\right)}, \tag{B.23d}$$

$$\boldsymbol{\partial}_0 \mathcal{A} = q\mathcal{A} + \frac{3\gamma v\Omega}{2\left(1 - v^2\right)}, \tag{B.23e}$$

$$\boldsymbol{\partial}_0 \Omega + \frac{(\gamma - 2)v\boldsymbol{\partial}_1\Omega}{G_-} - \frac{\gamma\Omega\boldsymbol{\partial}_1 v}{G_-} = -\frac{2\gamma\mathcal{A}v\Omega}{G_-} + \frac{2q\Omega\left(1 - (\gamma - 1)v^2\right)}{G_-} +$$
$$- \frac{3\gamma\mathcal{Q}\left(1 - v^2\right)\Omega}{G_-} + \frac{2(\gamma - 2)rv\Omega}{G_-} + \frac{2\Omega\left((1 - 2\gamma)v^2 + 1\right)}{G_-}, \tag{B.23f}$$

$$\boldsymbol{\partial}_0 v - \frac{(\gamma - 1)\left(1 - v^2\right)^2\boldsymbol{\partial}_1\Omega}{\gamma\Omega G_-} + \frac{(\gamma - 2)v\boldsymbol{\partial}_1 v}{G_-} = \frac{2(\gamma - 1)\mathcal{A}\left(1 - v^2\right)v^2}{G_-} +$$
$$- \frac{3(\gamma - 2)\mathcal{Q}\left(1 - v^2\right)v}{G_-} - \frac{2(\gamma - 1)r\left(1 - v^2\right)^2}{\gamma G_-} + \frac{2\left(1 - v^2\right)v}{G_-}, \tag{B.23g}$$

subject to the restrictions:

$$\boldsymbol{\partial}_1 \mathcal{N}^{-1} = r\mathcal{N}^{-1}, \tag{B.24a}$$

$$\boldsymbol{\partial}_1 \mathcal{K} = 2(r + \mathcal{A})\mathcal{K}, \tag{B.24b}$$

$$\boldsymbol{\partial}_1 \mathcal{A} = -\frac{3}{2}\mathcal{K} + r\mathcal{A} + \frac{3}{2}\mathcal{A}^2 - \frac{3}{2}\left(3c_\theta + 2c_\sigma\right)\mathcal{Q}^2 + 3\left(2c_\sigma - 1\right)\mathcal{Q}$$
$$+ \frac{3\Omega\left((\gamma - 1)v^2 + 1\right)}{2\left(1 - v^2\right)} - 3c_\sigma + \frac{3}{2}, \tag{B.24c}$$

$$\boldsymbol{\partial}_1 \mathcal{Q} = r\mathcal{Q} - \frac{3\gamma c_\sigma v\Omega}{\left(3c_\theta + 2c_\sigma\right)\left(1 - v^2\right)}, \tag{B.24d}$$

$$\boldsymbol{\partial}_1 \Sigma = r\Sigma + 3\mathcal{A}\Sigma - \frac{9\gamma c_\theta v\Omega}{2\left(3c_\theta + 2c_\sigma\right)\left(1 - v^2\right)}. \tag{B.24e}$$

Using the identity $\Sigma = 1 - \mathcal{Q}$, and combining equations (B.23c) and (B.23d) and equations (B.24d) and (B.24e), respectively, we obtain:

$$q = -\frac{3\mathcal{K}}{2(2c_\sigma - 1)} + \frac{\mathcal{A}^2}{2(2c_\sigma - 1)} - \frac{\left(3c_\theta + 2c_\sigma\right)\left(3c_\theta + 8c_\sigma - 3\right)\mathcal{Q}^2}{2\left(3c_\theta + 1\right)\left(2c_\sigma - 1\right)} + \frac{4\left(3c_\theta + 2c_\sigma\right)\mathcal{Q}}{3c_\theta + 1} +$$
$$- \frac{\Omega\left(v^2\left(3(\gamma + 1)c_\theta + 2(\gamma - 2)c_\sigma + 3\right) - 6\gamma c_\sigma - 3c_\theta + 4c_\sigma + 3\gamma - 3\right)}{2\left(3c_\theta + 1\right)\left(2c_\sigma - 1\right)\left(1 - v^2\right)} + \frac{1 - 9c_\theta - 8c_\sigma}{2(3c_\theta + 1)}, \tag{B.25}$$

$$r = -3\mathcal{A}(1 - \mathcal{Q}) + \frac{3\gamma v \Omega}{2(1 - v^2)}. \tag{B.26}$$

The final equations for the reduced phase space $\left(\mathcal{N}, E_1^1, \mathcal{K}, \mathcal{Q}, \Omega, v\right)^T$ are the evolution equations and restrictions displayed above (less the equations (B.23d),(B.24e) for the frame derivatives of Σ), where q and r are defined by (B.25) and (B.26), respectively. (It will be useful to define $\mathcal{D} \equiv \mathcal{A}^2 - 3\mathcal{K}$ in some computations). Again we note that formally the same equations can be obtained from the previous case ($\dot{u} \neq 0$) by setting $\dot{u} = 0$; but here we have derived the equations properly.

B.2 The subset $\dot{U} = v = 0$

Let us consider the special subset $\dot{U} = v = 0$. We also assume $c_\theta = 0$ and $c_\sigma \neq 0$. We first assume that $\gamma \neq 1$. The final equations for the reduced phase space are then:

$$\boldsymbol{\partial}_0 E_1^1 = (q + 3(1 - \mathcal{Q}))E_1^1, \tag{B.27a}$$

$$\boldsymbol{\partial}_0 \mathcal{K} = 2q\mathcal{K}, \tag{B.27b}$$

$$\boldsymbol{\partial}_0 \mathcal{Q} = \mathcal{Q}(1 + q - \mathcal{Q}) + 2\left(2c_\sigma - 1\right)\left(1 - \mathcal{Q}\right)^2 + \frac{1}{2}\Omega\left(2 - 3\gamma\right), \tag{B.27c}$$

$$\boldsymbol{\partial}_0 \mathcal{A} = q\mathcal{A}, \tag{B.27d}$$

$$\boldsymbol{\partial}_0 \Omega = (2q - 3\gamma\mathcal{Q} + 2)\Omega, \tag{B.27e}$$

subject to the restrictions:

$$\boldsymbol{\partial}_1 \mathcal{N}^{-1} = r\mathcal{N}^{-1}, \tag{B.28a}$$

$$\boldsymbol{\partial}_1 \mathcal{K} = 2(r + \mathcal{A})\mathcal{K}, \tag{B.28b}$$

$$\boldsymbol{\partial}_1 \mathcal{A} = -\frac{3}{2}\mathcal{K} + r\mathcal{A} + \frac{3}{2}\mathcal{A}^2 - 3c_\sigma \mathcal{Q}^2 + \frac{3}{2}\left(2c_\sigma - 1\right)\left(2\mathcal{Q} - 1\right) + \frac{3}{2}\Omega, \tag{B.28c}$$

$$\boldsymbol{\partial}_1 \mathcal{Q} = r\mathcal{Q}, \tag{B.28d}$$

$$\boldsymbol{\partial}_1 \Omega = 2r\Omega, \tag{B.28e}$$

where q and r are defined by:

$$q = \frac{1}{2(2c_\sigma - 1)}\bigg\{ -3\mathcal{K} + \mathcal{A}^2 - 2c_\sigma\left(8c_\sigma - 3\right)\mathcal{Q}^2 + 16c_\sigma(2c_\sigma - 1)\mathcal{Q}$$
$$- \Omega\left(-2c_\sigma(3\gamma - 2) + 3(\gamma - 1)\right) + (1 - 8c_\sigma)(2c_\sigma - 1)\bigg\}. \tag{B.29}$$

$$r = -3\mathcal{A}(1 - \mathcal{Q}). \tag{B.30}$$

We recall that $\boldsymbol{\partial}_0 E_1^1 = (q + 3(1 - \mathcal{Q}))E_1^1$, $\boldsymbol{\partial}_1 \mathcal{N}^{-1} = r\mathcal{N}^{-1}$, where $\boldsymbol{\partial}_0 := \mathcal{N}^{-1}\partial_t$, $\boldsymbol{\partial}_1 := E_1^1\partial_x$. [The only remaining freedom is the coordinate rescalings $t \to f(t)$ and $x \to g(x)$]. [12]

B.2.1 The special case $\gamma = 1$

In the special case of dust, we lose the equation $\boldsymbol{\partial}_1 \Omega = 2r\Omega$, but all of the remaining equations are valid with $\gamma = 1$.

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[12] $\mathcal{A}^2$ and $\mathcal{K}$ only appear in the evolution equations via the combination $\mathcal{D} \equiv \mathcal{A}^2 - 3\mathcal{K}$ (but also appear in the constraints; e.g., via r).

C Lemaître-Tolman-Bondi model

The Lemaître-Tolman-Bondi (LTB) model in GR [49] is the spherically symmetric dust solution of the Einstein equations which can be regarded as a generalization of the FLRW universe. LTB metrics with dust source and a comoving and geodesic 4-velocity constitute a well known class of exact solutions of Einstein's field equations [50].

The line element for a spherically symmetric comoving dust is:

$$ds^2 = -dt^2 + \frac{[R'(t,x)]^2}{1+2E}dx^2 + R(t,x)^2[d\theta^2 + \sin^2\theta d\phi^2] \tag{C.1}$$

(where an overdot denotes a t-derivative and a prime denotes an x-derivative). The geometric variables, $\{H, \sigma_+, {}^3R, {}^3S_+\}$, are defined by

$$H + \sigma_+ = \frac{\dot{R}}{R}; \ H - 2\sigma_+ = \frac{\dot{R}'}{R'}; \ {}^3R = -\frac{4(ER)'}{R^2 R'}; \ {}^3S_+ = -\frac{1}{12}{}^3R - \frac{E}{R^2}. \tag{C.2}$$

where

$$ {}^3R = 4\mathbf{e}_1 a - 6a^2 + 2K, \ {}^3S_+ = -\frac{1}{3}\mathbf{e}_1 a + \frac{1}{3}K, \tag{C.3}$$

$$ a = -\frac{\sqrt{1+E}}{R(t,x)}, \ K = \frac{1}{R(t,x)^2}. \tag{C.4}$$

The matter variable is defined by $\rho = \frac{2M'}{R^2 R'}$. The 3 free functions $R(t,r)$, E, M are subject to the Gauss constraint. We can introduce Hubble-normalized variables [51]:

$$\Sigma_+ = \frac{\sigma_+}{H}, \quad \Omega = \frac{\rho}{3H^2}, \quad \Omega_k = -\frac{{}^3R}{6H^2}, \quad \mathcal{S}_+ = \frac{{}^3S_+}{3H^2}, \tag{C.5}$$

where (the deceleration parameter) $q = 2\Sigma_+^2 + \frac{1}{2}\Omega$, and $D = \Omega_k - 6\mathcal{S}_+$. By introducing a new time variable $\tau(t,r)$ defined by $\partial t/\partial \tau = 1/H$, where $H > 0$, H decouples from the evolution equations. Using these variables, expressed in the coordinates above, the evolution equations for GR are (the dynamical system) (see also [52]): [13]

$$\partial_\tau \Omega = (\Omega + 4\Sigma_+^2 - 1)\Omega, \tag{C.6a}$$

$$\partial_\tau \Sigma_+ = (\frac{1}{2}\Omega + 2\Sigma_+^2 - 2)\Sigma_+ - \frac{1}{2}(1 - \Omega - \Sigma_+^2 - D), \tag{C.6b}$$

$$\partial_\tau D = (\Omega + 4\Sigma_+^2 - 2\Sigma_+)D. \tag{C.6c}$$

The flat FLRW equilibrium point is given by $(\Omega, \Sigma_+, D) = (1, 0, 0)$, where $(\Omega_k, \mathcal{S}_+) = (0, 0)$. It is a saddle with eigenvalues $(1, -\frac{3}{2}, 1)$ [51].

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[13]We note that the exact GR LTB is not a solution to the Einstein-aether equations (see section 4).

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