Topological Representation of Precontact Algebras and a Connected Version of the Stone Duality Theorem – I

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Abstract

The notions of a *2-precontact space* and a *2-contact space* are introduced. Using them, new representation theorems for precontact and contact algebras are proved. They incorporate and strengthen both the discrete and topological representation theorems from [8, 5, 6, 9, 24]. It is shown that there are bijective correspondences between such kinds of algebras and such kinds of spaces. As applications of the obtained results, we get new connected versions of the Stone Duality Theorems [22, 19] for Boolean algebras and for complete Boolean algebras, as well as a Smirnov-type theorem (in the sense of [20]) for a kind of compact T_0-extensions of compact Hausdorff extremally disconnected spaces. We also introduce the notion of a *Stone adjacency space* and using it, we prove another representation theorem for precontact algebras. We even obtain a bijective correspondence between the class of all, up to isomorphism, precontact algebras and the class of all, up to isomorphism, Stone adjacency spaces.

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1 Introduction

In this paper we give the proofs of the results announced in the first seven sections of our paper [7] (the results of the remaining sections will be proved in the second part of the paper) and obtain many new additional results and some new applications. In it we present a common approach both to the discrete and to the non-discrete region-based theory of space. The paper is a continuation of the line of investigations started in [24] and continued in [5, 6, 9].

Standard models of non-discrete theories of space are the contact algebras of regular closed subsets of some topological spaces ([21, 18, 24, 5, 6, 9]). In a sense these topological models reflect the continuous nature of the space. However, in the "real-world" applications, where digital methods of modeling are used, the continuous models of space are not enough. This motivates a search for good "discrete" versions of the theory of space. One kind of discrete models are the so called *adjacency spaces*, introduced by Galton [12] and generalized by Düntsch and Vakarelov in [8]. Based on the Galton's approach, Li and Ying [16] presented a "discrete" generalization of the Region Connection Calculus (RCC). The latter, introduced in [17], is one of the main systems in the non-discrete region-based theory of space. A natural class of Boolean algebras related to adjacency spaces are the *precontact algebras*, introduced in [8] under the name of *proximity algebras*. The notion of precontact algebra is a generalization of the notion of contact algebra. Each adjacency space generates canonically a precontact algebra. It is proved in [8] (using another terminology) that each precontact algebra can be embedded in the precontact algebra of an adjacency space. In [5] we proved that each contact algebra can be embedded in the standard contact algebra of a compact semiregular T_0-space, answering the question of Düntsch and Winter, posed in [9], whether the contact algebras have a topological representation. This shows that contact algebras possess both a discrete and a non-discrete (topological) representation. In this paper we extend the representation techniques developed in [5, 6] to precontact algebras, proving that each precontact algebra can be embedded in a special topological object, called a *2-precontact space*. We also establish a bijective correspondence between precontact algebras and 2-precontact spaces. This result is new even in the special case of contact algebras: introducing the notion of *2-contact space* as a specialization of the notion of a 2-precontact space, we show that there is a bijective correspondence between contact algebras and 2-contact spaces. Also, we introduce the notion of a *Stone adjacency space* and using it, we prove another representation theorem for precontact algebras. We even obtain a bijective correspondence between the class of all, up to isomorphism, precontact algebras and the class of all, up to isomorphism, Stone adjacency spaces.

The developed theory permits us to obtain as corollaries the celebrated Stone Representation Theorem [22] and a new connected version of it. They correspond, respectively, to the extremal contact relations on Boolean algebras: the smallest one and the largest one. We show as well that the new connected version of the Stone Representation Theorem can be extended to a new connected version of the Stone Duality Theorem. Let us explain what we mean by a "connected version". The celebrated Stone Duality Theorem [22] states that the category **Bool** of all Boolean algebras and Boolean homomorphisms is dually equivalent to the category **Stone** of compact Hausdorff totally disconnected spaces (i.e., *Stone spaces*) and continuous maps. The restriction of the Stone duality to the category **CBool** of complete Boolean algebras and Boolean homomorphisms is a duality between the category **CBool** and the category of compact Hausdorff extremally disconnected spaces and continuous maps. We introduce the notion of a *Stone 2-space* and the category **2Stone** of Stone 2-spaces and suitable morphisms between them, and we show that the category **2Stone** is dually equivalent to the category **Bool**. The Stone 2-spaces are pairs (X, X_0) of a compact *connected* T_0-space X and a dense subspace X_0 of X, satisfying some mild conditions. We introduce as well the notion of an *extremally connected space* and show that the category **ECS** of extremally connected spaces and continuous maps between them satisfying a natural condition, is dually equivalent to the category **CBool**. The extremally connected spaces are compact *connected* T_0-spaces satisfying an additional condition, and the open maps are part of the morphisms of the category **ECS**.

As another application of the obtained results, we prove a Smirnov-type theorem (in the sense of [20]). In his celebrated Compactification Theorem, Ju. M. Smirnov [20] proved that there exists an isomorphism between the ordered set of all Efremovič proximities on a Tychonoff space X and all, up to equivalence, compact Hausdorff extensions of X. The notion of a contact relation on a Boolean algebra is a generalization of the notion of a proximity. We show that there exist an isomorphism between the ordered by inclusion set of all contact relations on a complete Boolean algebra B and the ordered (by the injective order) set of all, up to isomorphism, *C-semiregular* extensions of its Stone space $S(B)$. In this way we describe C-semiregular extensions of extremally disconnected compact Hausdorff spaces. The notion of a *C-semiregular space* was introduced in [5]. It appears naturally in the theory of contact algebras. The class of C-semiregular spaces is a subclass of the class of compact T_0-spaces. As a corollary, we obtain that every extremally disconnected compact Hausdorff space X has a largest C-semiregular extension $(\gamma X, \gamma_X)$; moreover, γX is an extremally connected space and this characterizes it between all C-semiregular extensions of X. We show that every continuous map $f : X \longrightarrow Y$ between two extremally disconnected compact Hausdorff spaces X and Y has a continuous extension $\gamma f : \gamma X \longrightarrow \gamma Y$. We obtain, as well, some other similar results about continuous extensions of continuous maps.

The paper is organized as follows. In Section 2 we introduce the notions of precontact and contact algebra and give the two main examples of them: the precontact algebras on adjacency spaces, and the contact algebras on topological spaces. In Section 3 we introduce three kinds of points in precontact algebras: ultrafilters, grills and clans. Also, the notions of a *topological adjacency space* and a *Stone adjacency space* are introduced and our first representation theorem for precontact algebras is proved there. In Section 4 we introduce the notions of *2-precontact space* and *canonical precontact algebra of a 2-precontact space*. In Section 5 we associate with each precontact algebra $\underline{B}$ a 2-precontact space, called the *canonical 2-precontact space of* $\underline{B}$. In Section 6 we present the main theorem of the paper: the second representation theorem for precontact algebras. In Section 7 we introduce the notion of a *2-contact space* and we prove that there exists a bijective correspondence between the class of all (up to isomorphism) contact algebras and the class of all, up to isomorphism, 2-contact spaces. This is a generalization of the similar result about complete contact algebras obtained in [5]. In Section 8, the results of which are completely new and were not announced in our paper [7], we demonstrate that the Stone Representation Theorem [22] and the new connected version of it, which we now obtain, follow from our representation theorem for contact algebras presented in Section 7. Here we obtain also the new connected versions of the Stone Duality Theorems for Boolean algebras and for complete Boolean algebras, about which we already mentioned above. In the last Section 9, we collect our results about C-semiregular extensions of extremally disconnected compact Hausdorff spaces about which we also mentioned above. These results were not presented in the paper [7] and are new.

We now fix the notations.

All lattices are with top (= unit) and bottom (= zero) elements, denoted respectively by 1 and 0. We do not require the elements 0 and 1 to be distinct.

If (X, τ) is a topological space and M is a subset of X, we denote by $\mathrm{cl}_{(X,\tau)}(M)$ (or simply by $\mathrm{cl}(M)$ or $\mathrm{cl}_X(M)$) the closure of M in (X, τ) and by $\mathrm{int}_{(X,\tau)}(M)$ (or briefly by $\mathrm{int}(M)$ or $\mathrm{int}_X(M)$) the interior of M in (X, τ). The open maps between topological spaces are supposed to be continuous. The extremally disconnected spaces and compact spaces are not assumed to be Hausdorff (as it is adopted in [10]).

If X is a topological space, we denote by $CO(X)$ the set of all clopen subsets of X. Obviously, $(CO(X), \cup, \cap, \setminus, \emptyset, X)$ is a Boolean algebra.

If X is a set, we denote by 2^X the power set of X.

If $\mathcal{C}$ denotes a category, we write $X \in |\mathcal{C}|$ if X is an object of $\mathcal{C}$, and $f \in \mathcal{C}(X, Y)$ if f is a morphism of $\mathcal{C}$ with domain X and codomain Y.

The main reference books for all notions which are not defined here are [10, 15, 1].

2 Precontact algebras

Precontact and contact algebras.

Definition 2.1. An algebraic system $\underline{B} = (B, C)$ is called a *precontact algebra* ([8]) (abbreviated as PCA) if the following holds:

- $B = (B, 0, 1, +, ., *)$ is a Boolean algebra (where the complement is denoted by "$*$");
- C is a binary relation on B (called a *precontact relation*) satisfying the following axioms:

$(C0)$ If aCb then $a \neq 0$ and $b \neq 0$;

$(C+)$ $aC(b + c)$ iff aCb or aCc; $(a + b)Cc$ iff aCc or bCc.

A precontact algebra (B, C) is said to be *complete* if the Boolean algebra B is complete. Two precontact algebras $\underline{B} = (B, C)$ and $\underline{B_1} = (B_1, C_1)$ are said to be *PCA-isomorphic* (or, simply, *isomorphic*) if there exists a *PCA-isomorphism* between them, i.e., a Boolean isomorphism $\varphi : B \longrightarrow B_1$ such that, for every $a, b \in B$, aCb iff $\varphi(a)C_1\varphi(b)$.

The negation of the relation C is denoted by $(-C)$.

For any PCA (B, C), we define a binary relation " $\ll_C$" on B (called *non-tangential inclusion*) by

$$a \ll_C b \leftrightarrow a(-C)b^*.$$

Sometimes we will write simply " $\ll$" instead of " $\ll_C$".

We will also consider precontact algebras satisfying some additional axioms:

$(Cref)$ If $a \neq 0$ then aCa (reflexivity axiom);

$(Csym)$ If aCb then bCa (symmetry axiom);

(Ctr) If $a \ll_C c$ then $(\exists b)(a \ll_C b \ll_C c)$ (transitivity axiom);

$(Ccon)$ If $a \neq 0, 1$ then aCa^* or a^*Ca (connectedness axiom).

A precontact algebra (B, C) is called a *contact algebra* ([5]) (and C is called a *contact relation*) if it satisfies the axioms $(Cref)$ and $(Csym)$. We say that two contact algebras are *CA-isomorphic* if they are PCA-isomorphic; also, a PCA-isomorphism between two contact algebras will be called a *CA-isomorphism*.

A precontact algebra (B, C) is called *connected* if it satisfies the axiom $(Ccon)$.

The following lemma says that in every precontact algebra we can define a contact relation.

Lemma 2.2. *Let (B, C) be a precontact algebra. Define*

$$aC^{\#}b \iff ((aCb) \vee (bCa) \vee (a.b \neq 0)).$$

Then $C^{\#}$ is a contact relation on B and hence $(B, C^{\#})$ is a contact algebra.

Proof. If $a \neq 0$ then $a.a = a \neq 0$ and thus $aC^{\#}a$. So, $C^{\#}$ satisfies the axiom $(Cref)$. Further, let $aC^{\#}b$. Then there are three possibilities: (1) if aCb then $bC^{\#}a$; (2) if bCa then $bC^{\#}a$; (3) if $a.b \neq 0$ then $b.a \neq 0$ and thus $bC^{\#}a$. Therefore, $C^{\#}$ satisfies the axiom $(Csym)$. □

Remark 2.3. We will also consider precontact algebras satisfying the following variant of the transitivity axiom (Ctr):

$(Ctr\#)$ If $a \ll_{C\#} c$ then $(\exists b)(a \ll_{C\#} b \ll_{C\#} c)$.

The axiom $(Ctr\#)$ is known as the "Interpolation axiom".

A contact algebra (B, C) is called a *normal contact algebra* ([4, 11]) if it satisfies the axiom $(Ctr\#)$ and the following one:

(C6) If $a \neq 1$ then there exists $b \neq 0$ such that $b(-C)a$.

The notion of a normal contact algebra was introduced by Fedorchuk [11] (under the name of "*Boolean δ-algebra*") as an equivalent expression of the notion of a *compingent Boolean algebra* of de Vries [4] (see its definition below). We call such algebras "normal contact algebras" because they form a subclass of the class of contact algebras and naturally arise in normal Hausdorff spaces.

The relations C and $\ll$ are inter-definable. For example, normal contact algebras could be equivalently defined (and exactly in this way they were introduced (under the name of *compingent Boolean algebras*) by de Vries in [4]) as a pair of a Boolean algebra $B = (B, 0, 1, +, ., ^*)$ and a binary relation $\ll$ on B subject to the following axioms:

($\ll$1) $a \ll b$ implies $a \leq b$;
($\ll$2) $0 \ll 0$;
($\ll$3) $a \leq b \ll c \leq t$ implies $a \ll t$;
($\ll$4) ($a \ll b$ and $a \ll c$) implies $a \ll b.c$;
($\ll$5) If $a \ll c$ then $a \ll b \ll c$ for some $b \in B$;
($\ll$6) If $a \neq 0$ then there exists $b \neq 0$ such that $b \ll a$;
($\ll$7) $a \ll b$ implies $b^* \ll a^*$.

Note that if $0 \neq 1$ then the axiom ($\ll$2) follows from the axioms ($\ll$3), ($\ll$4), ($\ll$6) and ($\ll$7).

Obviously, contact algebras could be equivalently defined as a pair of a Boolean algebra B and a binary relation $\ll$ on B subject to the axioms ($\ll$1)-($\ll$4) and ($\ll$7); then, clearly, the relation $\ll$ satisfies also the axioms

($\ll$2') $1 \ll 1$;
($\ll$4') ($a \ll c$ and $b \ll c$) implies $(a + b) \ll c$.

It is not difficult to see that precontact algebras could be equivalently defined as a pair of a Boolean algebra B and a binary relation $\ll$ on B subject to the axioms ($\ll$2), ($\ll$2'), ($\ll$3), ($\ll$4) and ($\ll$4').

It is easy to see that axiom (C6) can be stated equivalently in the form of ($\ll$6).

Examples of precontact and contact algebras

1. Extremal contact relations

Example 2.4. Let B be a Boolean algebra. Then there exist a largest and a smallest contact relations on B; the largest one, ρ_l (sometimes we will write ρ_l^B), is defined by

$$a\rho_l b \iff (a \neq 0 \text{ and } b \neq 0),$$

and the smallest one, ρ_s (sometimes we will write ρ_s^B), by

$$a\rho_s b \iff a \wedge b \neq 0.$$

Note that, for $a, b \in B$,

$$a \ll_{\rho_s} b \iff a \leq b;$$

hence $a \ll_{\rho_s} a$, for any $a \in B$. Thus (B, ρ_s) is a normal contact algebra.

2. Precontact algebras on adjacency spaces. (Galton [12], Düntsch and Vakarelov [8])

By an *adjacency space* we mean a relational system (W, R), where W is a non-empty set whose elements are called *cells*, and R is a binary relation on W called the *adjacency relation*; the subsets of W are called *regions*.

The reflexive and symmetric closure $R^\flat$ of R is defined as follows:

(1) $xR^\flat y \iff ((xRy) \vee (yRx) \vee (x = y))$.

A precontact relation C_R between the regions of an adjacency space (W, R) is defined as follows: for every $a, b \subseteq W$,

(2) $aC_R b$ iff $(\exists x \in a)(\exists y \in b)(xRy)$.

Proposition 2.5. ([8]) *Let (W, R) be an adjacency space and let 2^W be the Boolean algebra of all subsets of W. Then:*

(a) $(2^W, C_R)$ *is a precontact algebra;*

(b) $(2^W, C_R)$ *is a contact algebra iff R is a reflexive and symmetric relation on W. If R is a reflexive and symmetric relation on W then C_R coincides with $(C_R)^\#$ and $C_{R^\flat}$;*

(c) C_R *satisfies the axiom (Ctr) iff R is a transitive relation on W;*

(d) C_R *satisfies the axiom $(Ccon)$ iff R is a connected relation on W (which means that if $x, y \in W$ and $x \neq y$ then there is an R-path from x to y or from y to x).*

Theorem 2.6. ([8]) *Each precontact algebra (B, C) can be isomorphically embedded into the precontact algebra $(2^W, C_R)$ of some adjacency space (W, R). Moreover, if (B, C) satisfies the axiom (Cref) (respectively, (Csym); (Ctr)) then the relation R is reflexive (respectively, symmetric; transitive).*

3. Contact algebras on topological spaces.

2.7. Let X be a topological space and let $RC(X)$ be the set of all regular closed subsets of X (recall that a subset F of X is said to be *regular closed* if $F = \mathrm{cl}(\mathrm{int}(F))$). Let us equip $RC(X)$ with the following Boolean operations and contact relation C_X:

- $a + b = a \cup b$;
- $a^* = \mathrm{cl}(X \setminus a)$;
- $a.b = \mathrm{cl}(\mathrm{int}(a \cap b))(= (a^* \cup b^*)^*)$;
- $0 = \emptyset$, $1 = X$;
- $aC_X b$ iff $a \cap b \neq \emptyset$.

The following lemma is a well-known fact.

Lemma 2.8. *Let X be a topological space. Then*

$$(RC(X), C_X) = (RC(X), 0, 1, +, ., *, C_X)$$

is a contact algebra.

The contact algebras of the type $(RC(X), C_X)$, where X is a topological space, are called *standard contact algebras*.

Recall that a space X is said to be *semiregular* if $RC(X)$ is a closed base for X. Recall as well the following definition: if $(A, \leq)$ is a poset and $B \subseteq A$ then B is said to be a *dense subset of* A if for any $a \in A \setminus \{0\}$ there exists $b \in B \setminus \{0\}$ such that $b \leq a$; when $(B, \leq_1)$ is a poset and $f : A \longrightarrow B$ is a map, then we will say that f is a *dense map* if $f(A)$ is a dense subset of $(B, \leq_1)$.

The following theorem answers the question, posed by Düntsch and Winter in [9], whether contact algebras have a topological representation:

Theorem 2.9. ([5]) *For each contact algebra $\underline{B} = (B, C)$ there exists a dense embedding g_B of B into a standard contact algebra $(RC(X, \tau), C_X)$, where (X, τ) is a compact semiregular T_0-space. The algebra B is connected iff the space X is connected. When B is complete then the embedding g_B becomes an isomorphism between contact algebras (B, C) and $(RC(X), C_X)$.*

The aim of this work is to generalize Theorem 2.6 and Theorem 2.9 in several ways: to find a topological representation of precontact algebras which incorporates both the "discrete" and the "continuous" nature of the space; to find representation theorems in the style of the Stone representation of Boolean algebras instead of embedding theorems; to establish, again as in the Stone theory, a bijective correspondence between precontact algebras and the corresponding topological objects; to find some new applications of the obtained results.

3 Points in precontact algebras

In this section we introduce three kinds of abstract points in precontact algebras: ultrafilters, grills and clans. This is done by analogy with the case of contact algebras (see, e.g., [5, 24]). We assume that the notions of a filter and ultrafilter of a Boolean algebra are familiar. Clans were introduced by Thron [23] in proximity theory. Our definition is a lattice-theoretic generalization of Thron's definition.

The set of all ultrafilters of a Boolean algebra B is denoted by $Ult(B)$.

Definition 3.1. Let $\underline{B} = (B, C)$ be a precontact algebra. A non-empty subset Γ of B is called a *clan* if it satisfies the following conditions:

$(Clan1)$ $0 \notin \Gamma$;

$(Clan2)$ If $a \in \Gamma$ and $a \leq b$ then $b \in \Gamma$;

$(Clan3)$ If $a + b \in \Gamma$ then $a \in \Gamma$ or $b \in \Gamma$;

$(Clan4)$ If $a, b \in \Gamma$ then $aC^{\#}b$.

The set of all clans of a precontact algebra $\underline{B}$ is denoted by $Clans(\underline{B})$.

The following lemma is obvious:

Lemma 3.2. *Let $\underline{B} = (B, C)$ be a precontact algebra. Each ultrafilter of B is a clan of $\underline{B}$ and hence $Ult(B) \subseteq Clans(\underline{B})$.*

Now, for any precontact algebra $\underline{B} = (B, C)$, we define a binary relation $R_{\underline{B}}$ between the ultrafilters of B making the set $Ult(B)$ an adjacency space.

Definition 3.3. Let $\underline{B} = (B, C)$ be a precontact algebra and let U_1, U_2 be ultrafilters of B. We set

(3) $U_1 R_{\underline{B}} U_2$ iff $(\forall a \in U_1)(\forall b \in U_2)(aCb)$ (i.e., iff $U_1 \times U_2 \subseteq C$).

The relational system $(Ult(B), R_{\underline{B}})$ is called the *canonical adjacency space of* $\underline{B}$.

We say that U_1, U_2 are *connected* iff $U_1 (R_{\underline{B}})^{\flat} U_2$ (see (1) for the notation $R^{\flat}$).

The next lemma is obvious.

Lemma 3.4. *Let $\underline{B} = (B, C)$ be a precontact algebra and let I be a set of connected ultrafilters. Then the union $\Gamma = \bigcup\{U \mid U \in I\}$ is a clan.*

Lemma 3.5. ([5, 8]) (Ultrafilter and clan characterizations of precontact and contact relations.) *Let $\underline{B} = (B, C)$ be a precontact algebra and $(Ult(B), R_{\underline{B}})$ be the canonical adjacency space of $\underline{B}$. Then the following is true for any $a, b \in B$:*

(a) *aCb iff $(\exists U_1, U_2 \in Ult(B))((a \in U_1) \wedge (b \in U_2) \wedge (U_1 R_{\underline{B}} U_2))$;*

(b) *$aC^{\#}b$ iff $(\exists U_1, U_2 \in Ult(B))((a \in U_1) \wedge (b \in U_2) \wedge (U_1 R^{\flat}_{(B,C)} U_2))$;*

(c) *$aC^{\#}b$ iff $(\exists \Gamma \in Clans(\underline{B}))(a, b \in \Gamma)$;*

(d) *$R_{\underline{B}}$ is a reflexive relation iff $\underline{B}$ satisfies the axiom $(Cref)$;*

(e) *$R_{\underline{B}}$ is a symmetric relation iff $\underline{B}$ satisfies the axiom $(Csym)$;*

(f) *$R_{\underline{B}}$ is a transitive relation iff $\underline{B}$ satisfies the axiom (Ctr).*

Recall that a non-empty subset of a Boolean algebra B is called a *grill* if it satisfies the axioms (Clan1)-(Clan3). The set of all grills of B will be denoted by $Grills(B)$. The next lemma is well known (see, e.g., [23]):

Lemma 3.6. (Grill Lemma.) *If F is a filter of a Boolean algebra B and G is a grill of B such that $F \subseteq G$ then there exists an ultrafilter U of B with $F \subseteq U \subseteq G$.*

Stone adjacency spaces and representation of precontact algebras

Definition 3.7. Let X be a non-empty topological space and R be a binary relation on X. Then the pair $(CO(X), C_R)$ (see (2) for C_R) is a precontact algebra (by Proposition 2.5(a)), called the *canonical precontact algebra of the relational system* (X, R).

Definition 3.8. An adjacency space (X, R) is called a *topological adjacency space* (abbreviated as TAS) if X is a topological space and R is a closed relation on X. When X is a compact Hausdorff zero-dimensional space (i.e., when X is a *Stone space*), we say that the topological adjacency space (X, R) is a *Stone adjacency space.*

Two topological adjacency spaces (X, R) and (X_1, R_1) are said to be *TAS-isomorphic* if there exists a homeomorphism $f : X \longrightarrow X_1$ such that, for every $x, y \in X$, xRy iff $f(x)R_1f(y)$.

Recall that the Stone space $S(A)$ of a Boolean algebra A is the set $X = Ult(A)$ endowed with a topology $\mathfrak{T}$ having as a closed base the family $\{s_A(a) \mid a \in A\}$, where

(4) $\quad s_A(a) = \{u \in X \mid a \in u\}$,

for every $a \in A$; then

$$S(A) = (X, \mathfrak{T})$$

is a compact Hausdorff zero-dimensional space, $s_A(A) = CO(X)$ and *the Stone map*

(5) $\quad s_A : A \longrightarrow CO(X), \quad a \mapsto s_A(a)$,

is a Boolean isomorphism; also, the family $\{s_A(a) \mid a \in A\}$ is an open base of $(X, \mathfrak{T})$. Further, for every Stone space X and for every $x \in X$, we set

(6) $\quad u_x = \{P \in CO(X) \mid x \in P\}$

(sometimes we will write also u_x^X instead of u_x). Then $u_x \in Ult(CO(X))$ and the map

$$f : X \longrightarrow S(CO(X)), \quad x \mapsto u_x,$$

is a homeomorphism.

When $\underline{B} = (B, C)$ is a precontact algebra, the pair $(S(B), R_{\underline{B}})$ is said to be *the canonical Stone adjacency space of* $\underline{B}$.

Now we can obtain the following strengthening of Theorem 2.6:

Theorem 3.9. (a) *Each precontact algebra* $\underline{B} = (B, C)$ *is isomorphic to the canonical precontact algebra* $(CO(X, \mathfrak{T}), C_{R_{\underline{B}}})$ *of the Stone adjacency space* $((X, \mathfrak{T}), R_{\underline{B}})$*, where* $(X, \mathfrak{T}) = S(B)$ *and for every* $u, v \in X$, $uR_{\underline{B}}v \iff u \times v \subseteq C$*; the isomorphism between them is just the Stone map* $s_B : B \longrightarrow CO(X, \mathfrak{T})$*. Moreover, the relation* C *satisfies the axiom (Cref) (resp., (Csym); (Ctr)) iff the relation* $R_{\underline{B}}$ *is reflexive (resp., symmetric; transitive).*

(b) *There exists a bijective correspondence between the class of all, up to PCA-isomorphism, precontact algebras and the class of all, up to TAS-isomorphism, Stone* *adjacency spaces (X,R); namely, for each precontact algebra $\underline{B}=(B,C)$, the PCA-isomorphism class $[\underline{B}]$ of $\underline{B}$ corresponds to the TAS-isomorphism class of the canonical Stone adjacency space $(S(B),R_{\underline{B}})$ of $\underline{B}$, and for each Stone adjacency space (X,R), the TAS-isomorphism class $[(X,R)]$ of (X,R) corresponds to the PCA-isomorphism class of the canonical precontact algebra $(CO(X),C_R)$ of (X,R) (see (2) for C_R).*

Proof. (a) Let $\underline{B}=(B,C)$ be a precontact algebra, $(X,\mathcal{T})=S(B)$ and for every $u,v\in X$, $uR_{\underline{B}}v \iff u\times v\subseteq C$. We will show that $R_{\underline{B}}$ is a closed relation. Let $(u,v)\notin R_{\underline{B}}$. Then there exist $a\in u$ and $b\in v$ such that $a(-C)b$. Hence $u\in s_B(a)$ and $v\in s_B(b)$. Also, $(s_B(a)\times s_B(b))\cap R_{\underline{B}}=\emptyset$. Indeed, let $u'\in s_B(a)$ and $v'\in s_B(b)$; then $a\in u'$ and $b\in v'$; since $a(-C)b$, we get that $u'(-R_{\underline{B}})v'$, i.e. $(u',v')\notin R_{\underline{B}}$. Therefore, $R_{\underline{B}}$ is a closed relation. Thus $((X,\mathcal{T}),R_{\underline{B}})$ is a Stone adjacency space. We have, by the Stone Representation Theorem, that $CO(X,\mathcal{T})=s_B(B)$. Further, we have that for every $a,b\in B$, $s_B(a)C_{R_{\underline{B}}}s_B(b) \iff (\exists u\in s_B(a))(\exists v\in s_B(b))(uR_{\underline{B}}v)$. It is easy to see that $(CO(X,\mathcal{T}),C_{R_{\underline{B}}})$ is a precontact algebra. We will show that $s_B:(B,C)\longrightarrow(CO(X,\mathcal{T}),C_{R_{\underline{B}}})$ is a PCA-isomorphism. We know, by the Stone Representation Theorem, that $s_B:B\longrightarrow CO(X,\mathcal{T})$ is a Boolean isomorphism. Let $a,b\in B$. Then, using Lemma 3.5(a), we obtain that $s_B(a)C_{R_{\underline{B}}}s_B(b) \iff (\exists u\in s_B(a))(\exists v\in s_B(b))(uR_{\underline{B}}v) \iff (\exists u,v\in X)((a\in u)\wedge(b\in v)\wedge(uR_{\underline{B}}v)) \iff aCb$. Therefore, $s_B:(B,C)\longrightarrow(CO(X,\mathcal{T}),C_{R_{\underline{B}}})$ is a PCA-isomorphism. The rest follows from Lemma 3.5(d,e,f).

(b) Let us set, for every precontact algebra $\underline{B}=(B,C)$,

$$\Phi(\underline{B})=(S(B),R_{\underline{B}}).$$

Then, by (a), $\Phi(\underline{B})$ is a Stone adjacency space. Further, for every Stone adjacency space (X,R), we set

$$\Psi(X,R)=(CO(X),C_R),$$

where, for every $F,G\in CO(X)$, $FC_RG \iff (\exists x\in F)(\exists y\in G)(xRy)$. Clearly, $\Psi(X,R)$ is a precontact algebra.

Let $\underline{B}=(B,C)$ be a precontact algebra. Then $\underline{B}$ is PCA-isomorphic to the precontact algebra $\Psi(\Phi(\underline{B}))$. Indeed, we have that $\Psi(\Phi(\underline{B}))=\Psi(S(B),R_{\underline{B}})=(CO(S(B)),C_{R_{\underline{B}}})$. Then, by (a), $s_B:(B,C)\longrightarrow\Psi(\Phi(\underline{B}))$ is a PCA-isomorphism.

Let (X,R) be a Stone adjacency space. Then (X,R) is TAS-isomorphic to the Stone adjacency space $\Phi(\Psi(X,R))$. Indeed, let $B=CO(X)$ and $\underline{B}=(B,C_R)$. Then $\Phi(\Psi(X,R))=\Phi(\underline{B})=(S(B),R_{\underline{B}})$. By the Stone Representation Theorem, we have that the map

(7) $\quad f:X\longrightarrow S(B),\ x\mapsto u_x,\ $ is a homeomorphism.

Let $x,y\in X$ and xRy. Since, for every $F\in u_x$ and every $G\in u_y$, we have that $x\in F$ and $y\in G$, we get that $u_xR_{\underline{B}}u_y$, i.e. $f(x)R_{\underline{B}}f(y)$. Conversely, let $x,y\in X$ and $f(x)R_{\underline{B}}f(y)$, i.e. $u_xR_{\underline{B}}u_y$. Suppose that $x(-R)y$. Then $(x,y)\notin R$. Since R is a closed relation, there exist $F,G\in CO(X)$ such that $x\in F$, $y\in G$ and $(F\times G)\cap R=\emptyset$. Then $F\in u_x$, $G\in u_y$ and if $x'\in F$, $y'\in G$ then $(x',y')\notin R$, i.e. $x'(-R)y'$. This implies that $u_x(-R_{\underline{B}})u_y$, a contradiction. Therefore, xRy. Hence, $f : (X, R) \longrightarrow \Phi(\Psi(X, R))$ is a TAS-isomorphism. □

Notation 3.10. Let B be a Boolean algebra and $(X, \mathfrak{T}) = S(B)$. Then we denote by $\mathrm{PCRel}(B)$ (resp., $\mathrm{CRel}(B)$) the set of all precontact (resp., contact) relations on B and by $\mathrm{CloRel}(X, \mathfrak{T})$ (resp., $\mathrm{CloRSRel}(X, \mathfrak{T})$) the set of all closed relations (resp., all reflexive and symmetric closed relations) on $(X, \mathfrak{T})$.

Corollary 3.11. *Let B be a Boolean algebra and $(X, \mathfrak{T}) = S(B)$. Then there exists an isomorphism between the ordered sets* $(\mathrm{PCRel}(B), \subseteq)$ *(resp.,* $(\mathrm{CRel}(B), \subseteq)$*) and* $(\mathrm{CloRel}(X, \mathfrak{T}), \subseteq)$ *(resp.,* $(\mathrm{CloRSRel}(X, \mathfrak{T}), \subseteq)$*).*

Proof. By Theorem 3.9(a), for every precontact relation C on B, the relation $R_{\underline{B}}$ (defined there), where $\underline{B} = (B, C)$, is a closed relation on $(X, \mathfrak{T})$. Now we define the correspondences

$$\Phi' : \mathrm{PCRel}(B) \longrightarrow \mathrm{CloRel}(X, \mathfrak{T}), \quad C \mapsto R_{\underline{B}}, \quad \text{and}$$

$$\Psi' : \mathrm{CloRel}(X, \mathfrak{T}) \longrightarrow \mathrm{PCRel}(B), \quad R \mapsto C^R,$$

where, for any $a, b \in B$,

$$aC^R b \iff s_B(a)\, C_R\, s_B(b).$$

Then, using Theorem 3.9(a), we get that $\Psi' \circ \Phi' = id_{\mathrm{PCRel}(B)}$. Further, arguing as in the proof of Theorem 3.9(b), we will show that $\Phi' \circ \Psi' = id_{\mathrm{CloRel}(X,\mathfrak{T})}$. Let $R \in \mathrm{CloRel}(X, \mathfrak{T})$. Then $\Phi'(\Psi'(R)) = R_{(B,C^R)}$. Obviously, for every $u, v \in X$, we have that $uR_{(B,C^R)}v \iff u \times v \subseteq C^R \iff (\forall a \in u)(\forall b \in v)(aC^R b) \iff (\forall a \in u)(\forall b \in v)(s_B(a)\, C_R\, s_B(b))$. Since $s_B(a)\, C_R\, s_B(b) \iff (\exists u' \in s_B(a))(\exists v' \in s_B(b))(u'Rv')$, we get immediately that $R \subseteq R_{(B,C^R)}$. Let now $u, v \in X$ and $uR_{(B,C^R)}v$. Suppose that $u(-R)v$. Since R is a closed relation, there exist $a, b \in B$ such that $(u, v) \in s_B(a) \times s_B(b) \subseteq X^2 \setminus R$. Then $a \in u$, $b \in v$ and $(s_B(a) \times s_B(b)) \cap R = \emptyset$. Thus, for every $u' \in s_B(a)$ and for every $v' \in s_B(b)$, we get that $u'(-R)v'$. Hence $s_B(a)(-C_R)s_B(b)$, a contradiction. So, uRv. Therefore, $\Phi' \circ \Psi' = id_{\mathrm{CloRel}(X,\mathfrak{T})}$. Hence Φ' and Ψ' are bijections. Note that, by Theorem 3.9(a), C is a contact relation on B iff $R_{(B,C)}$ is, in addition, a reflexive and symmetric relation on X. Finally, it is clear from the corresponding definitions that for any $C_1, C_2 \in \mathrm{PCRel}(B)$, $C_1 \subseteq C_2$ iff $R_{(B,C_1)} \subseteq R_{(B,C_2)}$ iff $\Phi'(C_1) \subseteq \Phi'(C_2)$. □

As it is shown in [8], there is no bijective correspondence between the classes of all, up to corresponding isomorphisms, precontact algebras and adjacency spaces. Hence, the role of the topology in Theorem 3.9 is essential. However, Theorem 3.9 is not completely satisfactory because the representation of the precontact algebras (B, C) obtained here does not give a topological representation of the contact algebras $(B, C^{\#})$ generated by (B, C); we would like to have an isomorphism f such that, for every $a, b \in B$, aCb iff $f(a)C_R f(b)$, and $aC^{\#}b$ iff $f(a) \cap f(b) \neq \emptyset$ (see (2) for C_R). The isomorphism s_B in Theorem 3.9 is not of this type. Indeed, there are many examples of contact algebras (B, C) where $a.b = 0$ (and hence $s_B(a) \cap s_B(b) = \emptyset$)

but aCb (note that C and $C^{\#}$ coincide for contact algebras). We now construct some natural topological objects which correspond bijectively to the precontact algebras and satisfy the above requirement. In the case when (B, C) is a contact algebra, we will show that these topological objects are just topological pairs satisfying some natural conditions. In such a way we will obtain new representation theorems for the contact algebras, completely different from those given in [24, 5, 6, 9].

4 2-Precontact spaces

Let us start with recalling the following well known statement (see, e.g., [3], p.271).

Lemma 4.1. *Let X be a dense subspace of a topological space Y. Then the functions*

$$r : RC(Y) \longrightarrow RC(X),\ F \mapsto F \cap X,$$

and

$$e : RC(X) \longrightarrow RC(Y),\ G \mapsto \mathrm{cl}_Y(G),$$

are Boolean isomorphisms between Boolean algebras $RC(X)$ and $RC(Y)$, and $e \circ r = id_{RC(Y)}$, $r \circ e = id_{RC(X)}$. (We will sometimes write $r_{X,Y}$ (resp., $e_{X,Y}$) instead of r (resp., e).)

Definition 4.2. (a) Let X be a topological space and X_0 be a dense subspace of X. Then the pair (X, X_0) is called a *topological pair*.

(b) Let (X, X_0) be a topological pair. Then we set

(8) $\quad RC(X, X_0) = \{cl_X(A) \mid A \in CO(X_0)\}$.

Lemma 4.3. *Let (X, X_0) be a topological pair. Then $RC(X, X_0) \subseteq RC(X)$; the set $RC(X, X_0)$ with the standard Boolean operations on the regular closed subsets of X is a Boolean subalgebra of $RC(X)$; $RC(X, X_0)$ is isomorphic to the Boolean algebra $CO(X_0)$; the sets $RC(X)$ and $RC(X, X_0)$ coincide iff X_0 is an extremally disconnected space. If*

$$C_{(X,X_0)}$$

is the restriction of the contact relation C_X (see Lemma 2.8) to $RC(X, X_0)$, then $(RC(X, X_0), C_{(X,X_0)})$ is a contact subalgebra of $(RC(X), C_X)$.

Proof. Since $CO(X_0)$ is a Boolean subalgebra of the Boolean algebra $RC(X_0)$ and (in the notation of Lemma 4.1) $e(CO(X_0)) = RC(X, X_0)$ and e is a Boolean isomorphism, we get that $RC(X, X_0) \subseteq RC(X)$ and the set $RC(X, X_0)$ with the standard Boolean operations on the regular closed subsets of X is a Boolean subalgebra of $RC(X)$. Clearly, the restriction $e_0 = e|_{CO(X_0)} : CO(X_0) \longrightarrow RC(X, X_0)$ is a Boolean isomorphism. Using the above arguments, we get that $RC(X) = RC(X, X_0)$ iff $RC(X_0) = CO(X_0)$. As it is well known, the later equality is true iff X_0 is an extremally disconnected space; hence $RC(X) = RC(X, X_0)$ iff X_0 is an extremally disconnected space. Finally, it is obvious that $(RC(X, X_0), C_{(X,X_0)})$ is a contact subalgebra of $(RC(X), C_X)$. □

Notation 4.4. Let (X, τ) be a topological space, X_0 be a subspace of X, $x \in X$ and B be a subalgebra of the Boolean algebra $(RC(X), +, ., *, \emptyset, X)$ defined in 2.7. We put

(9) $\sigma_x^B = \{F \in B \mid x \in F\}$; $\Gamma_{x,X_0} = \{F \in CO(X_0) \mid x \in cl_X(F)\}$.

We set also

(10) $\nu_x^B = \{F \in B \mid x \in \mathrm{int}_X(F)\}$.

When $B = RC(X)$, we will often write simply σ_x and ν_x instead of, respectively, σ_x^B and ν_x^B; in this case we will sometimes use the notation σ_x^X and ν_x^X as well.

Definition 4.5. (2-Precontact spaces.)
(a) A triple $\underline{X} = (X, X_0, R)$ is called a *2-precontact space* (abbreviated as PCS) if the following conditions are satisfied:

$(PCS1)$ (X, X_0) is a topological pair and X is a T_0-space;

$(PCS2)$ (X_0, R) is a Stone adjacency space;

$(PCS3)$ $RC(X, X_0)$ is a closed base for X;

$(PCS4)$ For every $F, G \in CO(X_0)$, $\mathrm{cl}_X(F) \cap \mathrm{cl}_X(G) \neq \emptyset$ implies that $F(C_R)^{\#}G$ (see (2) for C_R);

$(PCS5)$ If $\Gamma \in Clans(CO(X_0), C_R)$ then there exists a point $x \in X$ such that $\Gamma = \Gamma_{x,X_0}$ (see (9) for Γ_{x,X_0}).

(b) Let $\underline{X} = (X, X_0, R)$ be a 2-precontact space. Define, for every $F, G \in RC(X, X_0)$,

$$F \; C_{\underline{X}} \; G \iff ((\exists x \in F \cap X_0)(\exists y \in G \cap X_0)(xRy)).$$

Then the precontact algebra

$$\underline{B}(\underline{X}) = (RC(X, X_0), C_{\underline{X}})$$

is said to be the *canonical precontact algebra of* $\underline{X}$.
(c) A 2-precontact space $\underline{X} = (X, X_0, R)$ is called *reflexive* (resp., *symmetric*; *transitive*) if the relation R is reflexive (resp., symmetric; transitive); $\underline{X}$ is called *connected* if the space X is connected.
(d) Let $\underline{X} = (X, X_0, R)$ and $\underline{\widehat{X}} = (\widehat{X}, \widehat{X}_0, \widehat{R})$ be two 2-precontact spaces. We say that $\underline{X}$ and $\underline{\widehat{X}}$ are *PCS-isomorphic* (or, simply, *isomorphic*) if there exists a homeomorphism $f : X \longrightarrow \widehat{X}$ such that:

(ISO1) $f(X_0) = \widehat{X}_0$; and

(ISO2) $(\forall x, y \in X_0)(xRy \leftrightarrow f(x)\widehat{R}f(y))$.

Remark 4.6. It is very easy to see that the canonical precontact algebra of a 2-precontact space, defined in Definition 4.5(b), is indeed a precontact algebra.

Proposition 4.7. (a) *Let* (X, X_0, R) *be a 2-precontact space. Then* X *is a semiregular space and, for every* $F, G \in CO(X_0)$,

(11) $cl_X(F) \cap cl_X(G) \neq \emptyset$ *iff* $F(C_R)^{\#}G$.

(b) Let $\underline{X} = (X, X_0, R)$ *and* $\widehat{\underline{X}} = (\widehat{X}, \widehat{X}_0, \widehat{R})$ *be two isomorphic 2-precontact spaces. Then the corresponding canonical precontact algebras* $\underline{B}(\underline{X})$ *and* $\underline{B}(\widehat{\underline{X}})$ *are PCA-isomorphic.*

Proof. (a) By the axiom (PCS3), the family $RC(X, X_0)$ is a closed base for the space X. Since $RC(X, X_0) \subseteq RC(X)$ (see Lemma 4.3(a)), we get that X is a semiregular space.

Let $F, G \in CO(X_0)$ and $F(C_R)^{\#}G$. The pair $(CO(X_0), C_R)$ is a precontact algebra (see Proposition 2.5(a)). Hence, by Proposition 3.5(c), there exists a clan Γ in $(CO(X_0), C_R)$ such that $F, G \in \Gamma$. The axiom (PCS5) implies that there exists $x \in X$ such that $\Gamma = \Gamma_{x,X_0}$. Thus $x \in cl_X(F) \cap cl_X(G)$. Therefore, $cl_X(F) \cap cl_X(G) \neq \emptyset$. The converse implication follows from the axiom (PCS4).

(b) This is obvious. □

Lemma 4.8. (Correspondence Lemma.) *Let* $\underline{X} = (X, X_0, R)$ *be a 2-precontact space and let* $\underline{B}(\underline{X}) = (RC(X, X_0), C_{\underline{X}})$ *be the canonical precontact algebra of* $\underline{X}$. *Then the following equivalences hold:*

(a) *The space* $\underline{X}$ *is reflexive iff the algebra* $\underline{B}(\underline{X})$ *satisfies the axiom* $(Cref)$;

(b) *The space* $\underline{X}$ *is symmetric iff* $\underline{B}(\underline{X})$ *satisfies the axiom* $(Csym)$.;

(c) *The space* $\underline{X}$ *is transitive iff* $\underline{B}(\underline{X})$ *satisfies the axiom* (Ctr);

(d) *The space* $\underline{X}$ *is connected iff* $\underline{B}(\underline{X})$ *is connected.*

Proof. By (the proof of) Lemma 4.3(a), the map

$$\varphi : CO(X_0) \longrightarrow RC(X, X_0), \quad A \mapsto \mathrm{cl}_X(A),$$

is a Boolean isomorphism. From the definitions of the relations $C_{\underline{X}}$ and C_R (see, respectively, Definition 4.5(b) and (2)) it follows immediately that the map

(12) $\varphi : (CO(X_0), C_R) \longrightarrow (RC(X, X_0), C_{\underline{X}})$ is a PCA-isomorphism.

Now the assertions (a), (b) and (c) follow from Definition 4.5(c), Theorem 3.9 and Lemma 3.5(d,e,f). Indeed, by the axiom (PCS2), (X_0, R) is a Stone adjacency space; hence, by Theorem 3.9(b) and in the notation of its proof, $\Psi(X_0, R) = (CO(X_0), C_R)$ and $\Phi(\Psi(X_0, R)) = (S(CO(X_0)), R_{(CO(X_0),C_R)})$ is TAS-isomorphic to the Stone adjacency space (X_0, R); now we can apply Lemma 3.5(d,e,f) to the precontact algebra $(CO(X_0), C_R)$ and its canonical adjacency space $(Ult(CO(X_0)), R_{(CO(X_0),C_R)}) =$

$(S(CO(X_0)), R_{(CO(X_0),C_R)})$. So, the assertions (a), (b) and (c) are proved. Let us prove the assertion (d). Let X be a connected space. Suppose that the precontact algebra $(CO(X_0), C_R)$ is not connected. Then there exists an $F \in CO(X_0)$ such that $F \neq \emptyset, X_0$, $F(-C_R)F^*$ and $F^*(-C_R)F$. Since $F.F^* = \emptyset = 0$, we get that $F(-(C_R)^{\#})F^*$. Thus, by (11), $\mathrm{cl}_X(F) \cap \mathrm{cl}_X(F^*) = \emptyset$. Since $F^* = X_0 \setminus F$ and X_0 is dense in X, we obtain that $\mathrm{cl}_X(F) \cup \mathrm{cl}_X(F^*) = \mathrm{cl}_X(F) \cup \mathrm{cl}_X(X_0 \setminus F) = \mathrm{cl}_X(X_0) = X$. Hence $\mathrm{cl}_X(F)$ is a clopen subset of X. Since $\mathrm{cl}_X(F) \neq \emptyset, X$, we get a contradiction. Thus, the precontact algebra $(CO(X_0), C_R)$ is connected. Conversely, let the precontact algebra $(CO(X_0), C_R)$ be connected. Suppose that X is not connected. Then there exists a clopen in X subset G of X such that $G \neq \emptyset, X$. Let $F = X_0 \cap G$. Then $F \in CO(X_0)$ and $F \neq \emptyset, X_0$. Thus, FC_RF^* or F^*C_RF. Hence $F(C_R)^{\#}F^*$. Using (11), we get that $\mathrm{cl}_X(F) \cap \mathrm{cl}_X(F^*) \neq \emptyset$. Since $\mathrm{cl}_X(F) = \mathrm{cl}_X(G \cap X_0) = \mathrm{cl}_X(G) = G$ and, analogously, $\mathrm{cl}_X(F^*) = \mathrm{cl}_X(X_0 \setminus F) = \mathrm{cl}_X((X \setminus G) \cap X_0) = X \setminus G$, we get a contradiction. Hence, X is connected. □

5 The canonical 2-precontact space of a precontact algebra

In this section we will associate with each precontact algebra a 2-precontact space.

Definition 5.1. Let $\underline{B} = (B, C)$ be a precontact algebra. We associate with $\underline{B}$ a 2-precontact space

$$\underline{X}(\underline{B}) = (X, X_0, R),$$

called the *canonical 2-precontact space of* $\underline{B}$, as follows:

- $X = Clans(\underline{B})$ and $X_0 = Ult(B)$;
- The topology τ on the set X is defined in the following way: the family

  $$\{g_B(a) \mid a \in B\},$$

  where, for any $a \in B$,

  (13) $g_B(a) = \{\Gamma \in X \mid a \in \Gamma\}$,

  is a closed base of τ. The topology on X_0 is the subspace topology induced by (X, τ).
- $R = R_{\underline{B}}$ (see (3) for the notation $R_{\underline{B}}$).

Remark 5.2. Note that, in the notation of Definition 5.1, setting, for every $a \in B$,

$$g_0^B(a) = g_B(a) \cap X_0,$$

we get that the family $\{g_0^B(a) \mid a \in B\}$ is a closed base of X_0 and $g_0^B(a) = s_B(a)$, where $s_B : B \longrightarrow CO(X_0)$ is the Stone map.

Proposition 5.3. *Let $\underline{B} = (B, C)$ be a precontact algebra. Then the canonical 2-precontact space $\underline{X}(\underline{B}) = (X, X_0, R)$ of $\underline{B}$ defined above is indeed a 2-precontact space.*

Proof. Since $Clans(B, C) \equiv Clans(B, C^{\#})$ and $(B, C^{\#})$ is a contact algebra, we can use [5, Lemma 5.1(i), Lemma 5.7(i) and Lemma 5.3(ii)] which imply that the family $\{g_B(a) \mid a \in B\}$ can be taken as a closed base for a topology on the set X, that (X, τ) is a semiregular compact T_0-space and that, for every $a \in B$, $g_B(a) \in RC(X)$.

Let us set, for every $a \in B$, $h_B(a) = X \setminus g_B(a)$. Then

$$h_B(a) = \{\Gamma \in X \mid a \notin \Gamma\}$$

and $\{h_B(a) \mid a \in B\}$ is an open base of (X, τ). Let us show that X_0 is dense in (X, τ). Let $a \in B$ and $a \neq 1$ (i.e., $h_B(a) \neq \emptyset$). Then $a^* \neq 0$ and thus there exists an ultrafilter u in B such that $a^* \in u$. Thus $a \notin u$ and hence $u \in h_B(a)$. Therefore, $h_B(a) \cap X_0 \neq \emptyset$. Hence, X_0 is dense in (X, τ). So, the axiom (PCS1) is satisfied. Since $g_0^B(a) = s_B(a)$, for every $a \in B$ (see Remark 5.2), Theorem 3.9 implies that $((X_0, \tau|_{X_0}), R)$ is a Stone adjacency space. Thus, the axiom (PCS2) is also satisfied. Further, we have that for every $a \in B$, $g_B(a) \in RC(X)$; thus, using Lemma 4.1, we get that

(14) $g_B(a) = \mathrm{cl}_X(g_0^B(a))$.

Since $CO(X_0) = \{g_0^B(a) \mid a \in B\}$, we obtain that $RC(X, X_0) = \{g_B(a) \mid a \in B\}$. Hence, the axiom (PCS3) is satisfied.

Let $F, G \in CO(X_0)$. Then there exist $a, b \in B$ such that $F = s_B(a)(= g_0^B(a))$ and $G = s_B(b)(= g_0^B(b))$. Let $\mathrm{cl}_X(F) \cap \mathrm{cl}_X(G) \neq \emptyset$. Then, by (14), $g_B(a) \cap g_B(b) \neq \emptyset$. Hence, there exists $\Gamma \in g_B(a) \cap g_B(b)$. Then $a, b \in \Gamma$ and thus, $aC^{\#}b$. There are three possibilities:

1) aCb: then, by Lemma 3.5(a), there exist $u, v \in X_0$ such that $a \in u$, $b \in v$ and uRv; since $u \in F$ and $v \in G$, we get that FC_RG and therefore, $F(C_R)^{\#}G$;

2) bCa: then, by Lemma 3.5(a), there exist $u, v \in X_0$ such that $b \in u$, $a \in v$ and uRv; since $v \in F$ and $u \in G$, we get that GC_RF and therefore, $F(C_R)^{\#}G$;

3) $a.b \neq 0$: then, as it is well known (see, e.g., [15, Corollary 2.17]), there exists $u \in X_0$ such that $a.b \in u$; thus $a, b \in u$, i.e., $u \in F \cap G$ and therefore, $F(C_R)^{\#}G$.

So, the axiom (PCS4) is satisfied.

Let $\Gamma \in Clans(CO(X_0), C_R)$. We have, by Theorem 3.9(a), that

(15) $s_{(B,C)} : (B, C) \longrightarrow (CO(X_0), C_R), \quad b \mapsto s_B(b),$ is a PCA-isomorphism.

Set $x = s^{-1}_{(B,C)}(\Gamma)$. Then x is a clan of (B, C), i.e. $x \in X$. We will show that $\Gamma = \Gamma_{x,X_0}$ (see (9) for the notation Γ_{x,X_0}). We have to prove that $(\forall F \in CO(X_0))((x \in \mathrm{cl}_X(F)) \iff (F \in \Gamma))$. Let $F \in CO(X_0)$. Then there exists $a \in B$ such that $F = s_{(B,C)}(a)$. Let $F \in \Gamma$. Then $s_{(B,C)}(a) \in \Gamma$ and thus $s^{-1}_{(B,C)}(s_{(B,C)}(a)) \in s^{-1}_{(B,C)}(\Gamma)$, i.e. $a \in x$ and hence $x \in g_B(a) = \mathrm{cl}_X(s_B(a)) = \mathrm{cl}_X(F)$. Conversely, let $x \in \mathrm{cl}_X(F)$. Since $g_B(a) = \mathrm{cl}_X(s_B(a))$, we get that $a \in x$. Then $s_B(a) \in s_{(B,C)}(x) = \Gamma$. Therefore, $F \in \Gamma$. Hence there exists $x \in X$ such that $\Gamma = \Gamma_{x,X_0}$. Thus the axiom (PCS5) is satisfied. □

Proposition 5.4. *Let $\underline{B_1}$ and $\underline{B_2}$ be two isomorphic precontact algebras. Then the corresponding canonical 2-precontact spaces $\underline{X}(\underline{B_1})$ and $\underline{X}(\underline{B_2})$ are isomorphic.*

Proof. It is obvious. □

Proposition 5.5. *Let $\underline{B}$ be a precontact algebra, let $\underline{X}(\underline{B})$ be the canonical 2-precontact space of $\underline{B}$ and let $\underline{B'}$ be the canonical precontact algebra of the 2-precontact space $\underline{X}(\underline{B})$. Then the contact algebras $\underline{B}$ and $\underline{B'}$ are PCA-isomorphic.*

Proof. It follows from (15) and (12). □

Lemma 5.6. (Topological characterization of the connectedness of $\underline{B}$.) *Let $\underline{B} = (B, C)$ be a precontact algebra and $\underline{X}(\underline{B})$ be the canonical 2-precontact space of $\underline{B}$. Then $\underline{B}$ is connected iff $\underline{X}(\underline{B})$ is connected.*

Proof. Since $Clans(B, C) \equiv Clans(B, C^{\#})$ and $(B, C^{\#})$ is a contact algebra, we can use [5, Lemma 5.7(i3)] which implies that the 2-precontact space $\underline{X}(\underline{B})$ is connected iff the contact algebra $(B, C^{\#})$ is connected. Obviously, $(B, C^{\#})$ is connected iff $\underline{B}$ is connected. Thus, our assertion is proved. □

6 The Main Theorem

Theorem 6.1. (Representation theorem for precontact algebras.)

(a) *Let $\underline{B} = (B, C)$ be a precontact algebra and let $\underline{X}(\underline{B}) = (X, X_0, R)$ be the canonical 2-precontact space of $\underline{B}$. Then the function $g_B : (B, C) \longrightarrow 2^X$, defined in (13), is a PCA-isomorphism from (B, C) onto the canonical precontact algebra $(RC(X, X_0), C_{\underline{X}(\underline{B})})$ of $\underline{X}(\underline{B})$. The same function g_B is a PCA-isomorphism between contact algebras $(B, C^{\#})$ and $(RC(X, X_0), C_{(X,X_0)})$ (see Lemma 4.3(a) for $C_{(X,X_0)}$). The sets $RC(X)$ and $RC(X, X_0)$ coincide iff the precontact algebra $\underline{B}$ is complete. The algebra $\underline{B}$ satisfies the axiom $(Cref)$ (resp., $(Csym)$; (Ctr)) iff the 2-precontact space $\underline{X}(\underline{B})$ is reflexive (resp., symmetric; transitive). The algebra $\underline{B}$ is connected iff $\underline{X}(\underline{B})$ is connected.*

(b) *There exists a bijective correspondence between the class of all, up to PCA-isomorphism, (connected) precontact algebras and the class of all, up to PCS-isomorphism, (connected) 2-precontact spaces; namely, for every precontact algebra $\underline{B}$, the PCA-isomorphism class $[\underline{B}]$ of $\underline{B}$ corresponds to the PCS-isomorphism class $[\underline{X}(\underline{B})]$ of the canonical 2-precontact space $\underline{X}(\underline{B})$ of $\underline{B}$, and for every 2-precontact space $\underline{X}$, the PCS-isomorphism class $[\underline{X}]$ of $\underline{X}$ corresponds to the PCA-isomorphism class $[\underline{B}(\underline{X})]$ of the canonical precontact algebra $\underline{B}(\underline{X})$ of $\underline{X}$.*

Proof. (a) Using (14), we get that $\varphi \circ s_B = g_B$ (see (12) for φ and (5) for s_B). Now, we apply Proposition 5.5 for obtaining that

(16) $g_B : (B, C) \longrightarrow (RC(X, X_0), C_{\underline{X}(\underline{B})})$ is a PCA-isomorphism.

Further, using (11), we get that

(17) $g_B : (B, C^{\#}) \longrightarrow (RC(X, X_0), C_{(X,X_0)})$ is a CA-isomorphism.

By Lemma 4.3(a), we have that the sets $RC(X)$ and $RC(X, X_0)$ coincide iff X_0 is an extremally disconnected space. Since $X_0 = S(B)$, we have (by, e.g., [15, Proposition 7.21]) that X_0 is an extremally disconnected space iff B is a complete Boolean algebra. Hence, the sets $RC(X)$ and $RC(X, X_0)$ coincide iff the precontact algebra $\underline{B}$ is complete. All other assertion in (a) follow from Lemma 4.8 and (16).

(b) Let us denote by $\mathcal{PCA}$ the set of all, up to PCA-isomorphism, precontact algebras and by $\mathcal{PCS}$ the set of all, up to PCS-isomorphism, 2-precontact spaces. We will define two correspondences

$$\Phi_2 : \mathcal{PCA} \longrightarrow \mathcal{PCS} \quad \text{and} \quad \Psi_2 : \mathcal{PCS} \longrightarrow \mathcal{PCA}$$

and we will show that their compositions $\Phi_2 \circ \Psi_2$ and $\Psi_2 \circ \Phi_2$ are equal to the corresponding identities. We set, for every precontact algebra $\underline{B} = (B, C)$,

$$\Phi_2([\underline{B}]) = [\underline{X}(\underline{B})],$$

where $\underline{X}(\underline{B})$ is the canonical 2-precontact space of $\underline{B}$ (see Definition 5.1), $[\underline{B}]$ is the class of all precontact algebras which are PCA-isomorphic to the precontact algebra $\underline{B}$, and, analogously, $[\underline{X}(\underline{B})]$ is the class of all 2-precontact spaces which are PCS-isomorphic to the 2-precontact space $\underline{X}(\underline{B})$. Further, for every 2-precontact space $\underline{X} = (X, X_0, R)$, we set

$$\Psi_2([\underline{X}]) = [\underline{B}(\underline{X})],$$

where $\underline{B}(\underline{X})$ is the canonical precontact algebra of $\underline{X}$ (see Definition 4.5(b)). It is easy to see that the correspondences Φ_2 and Ψ_2 are well-defined.

Using (16), we get that for every precontact algebra $\underline{B} = (B, C)$, $\Psi_2(\Phi_2([\underline{B}])) = [\underline{B}]$. Thus we obtain that $\Psi_2 \circ \Phi_2 = id_{\mathcal{PCA}}$.

We will now prove that $\Phi_2 \circ \Psi_2 = id_{\mathcal{PCS}}$. Let $\underline{X} = (X, X_0, R)$ be a 2-precontact space. Set $(B, C) = (CO(X_0), C_R)$; then (B, C) is PCA-isomorphic to the canonical precontact algebra of $\underline{X}$ (see Definition 4.5(b)). Let $(\widehat{X}, \widehat{X}_0, \widehat{R})$ be the canonical 2-precontact space of (B, C) (see Definition 5.1 and (12)). Then $\widehat{X} = Clans(B, C)$, $\widehat{X}_0 = Ult(B)$ and $\widehat{R} = R_{(B,C)}$. For every $x \in X$, set

$$f(x) = \{a \in B \mid x \in \mathrm{cl}_X(a)\}.$$

Then $f(x) \neq \emptyset$ (by (PCS3)) and $f(x)$ is a clan in (B, C). Indeed, we have that: 1) $0 \notin f(x)$; 2) if $a \in f(x)$, $b \in B$ and $a \leq b$, then $x \in \mathrm{cl}_X(a) \subseteq \mathrm{cl}_X(b)$, and thus $b \in f(x)$; 3) if $a+b \in f(x)$ then $x \in \mathrm{cl}_X(a \cup b) = \mathrm{cl}_X(a) \cup \mathrm{cl}_X(b)$, and hence $x \in \mathrm{cl}_X(a)$ or $x \in \mathrm{cl}_X(b)$, i.e. $a \in f(x)$ or $b \in f(x)$; 4) if $a, b \in f(x)$ then $x \in \mathrm{cl}_X(a) \cap \mathrm{cl}_X(b)$, and thus, by (PCS4), $aC^{\#}b$. So, $f(x) \in \widehat{X}$. Hence, $f : X \longrightarrow \widehat{X}$. We will show that

$$f : (X, X_0, R) \longrightarrow (\widehat{X}, \widehat{X}_0, \widehat{R}), \quad x \mapsto f(x), \quad \text{is a PCS-isomorphism.}$$

Let $x, y \in X$ and $x \neq y$. Since, by (PCS1), X is a T_0-space, there exists an open subset U of X such that $|U \cap \{x, y\}| = 1$. Suppose that $x \in U$. Then $y \notin U$. According to (PCS3), there exists $a \in B$ such that $x \in X \setminus \mathrm{cl}_X(a) \subseteq U$. Thus $y \in \mathrm{cl}_x(a)$ and $x \notin \mathrm{cl}_X(a)$. Therefore, $a \in f(y) \setminus f(x)$. Hence $f(x) \neq f(y)$. If $y \in U$, then we argue analogously. So, f is a injection.

Let $\Gamma \in \widehat{X}$. Then, by (PCS5), there exists $x \in X$ such that $\Gamma = \{a \in B \mid x \in \mathrm{cl}_X(a)\}$. Hence $\Gamma = f(x)$. Thus f is a surjection. So, f is a bijection.

Let $a \in B$. Then $f(\mathrm{cl}_X(a)) = \{f(x) \mid x \in \mathrm{cl}_X(a)\} = \{f(x) \mid a \in f(x)\}$. Since f is a surjection, we get that $f(\mathrm{cl}_X(a)) = g_B(a)$ (see (13) for the notation $g_B(a)$). Since f is a bijection, we have also that $f^{-1}(g_B(a)) = \mathrm{cl}_X(a)$. Now, using (PCS3) and the fact that $\{g_B(a) \mid a \in B\}$ is a closed base of $\widehat{X}$ (see Definition 5.1), we get that f is a homeomorphism.

For every $x \in X_0$, we have that $f(x) = \{F \in CO(X_0) \mid x \in \mathrm{cl}_X(F)\} = \{F \in CO(X_0) \mid x \in F\} = u_x \in Ult(B) = \widehat{X}_0$ (see (6) for the notation u_x). Hence $f(X_0) \subseteq \widehat{X}_0$. For proving the inverse inclusion, let $u \in \widehat{X}_0$. Then $u \in Ult(CO(X_0))$. Now, by (PCS2), there exists $x \in X_0$ such that $x \in \bigcap u$. Then $u \subseteq u_x$ and, hence, $u = u_x$. Since, as we have already seen, $u_x = f(x)$, we get that $f(X_0) \supseteq \widehat{X}_0$. Therefore, $f(X_0) = \widehat{X}_0$.

Let $x, y \in X_0$. Then $f(x) = u_x$, $f(y) = u_y$. We have that $u_x, u_y \in \widehat{X}_0$ and $u_x \widehat{R} u_y \iff u_x \times u_y \subseteq C_R$. Hence, $(u_x \widehat{R} u_y) \iff$ (for every $F, G \in CO(X_0)$ such that $x \in F$ and $y \in G$, there exist $x' \in F$ and $y' \in G$ with $x' R y'$). Therefore, if xRy then, obviously, $f(x)\widehat{R}f(y)$. Let now $f(x)\widehat{R}f(y)$. Suppose that $x(-R)y$. Then $(x, y) \notin R$. Applying (PCS2), we get that there exist $F, G \in CO(X_0)$ such that $x \in F$, $y \in G$ and $(F \times G) \cap R = \emptyset$. Thus $F \in u_x$, $G \in u_y$ and for every $x' \in F$ and every $y' \in G$ we have that $x'(-R)y'$. This means that $F(-C_R)G$ and, hence, $u_x(-\widehat{R})u_y$, i.e. $f(x)(-\widehat{R})f(y)$, a contradiction. Therefore, xRy.

All this shows that f is a PCS-isomorphism. Hence $\Phi_2(\Psi_2([\underline{X}])) = [\underline{X}]$. Thus, $\Phi_2 \circ \Psi_2 = id_{\mathcal{PCS}}$. Therefore,

(18) $\Phi_2 : \mathcal{PCA} \longrightarrow \mathcal{PCS}$ is a bijection.

The statement for connected precontact algebras follows from (18) and Lemma 5.6. □

Corollary 6.2. *If $\underline{X} = (X, X_0, R)$ is a 2-precontact space then X is a compact space.*

Proof. By Theorem 6.1, there exists a precontact algebra $\underline{B} = (B, C)$ such that the 2-precontact space $\underline{X}$ is isomorphic to the canonical 2-precontact space $\underline{X}(\underline{B})$ of $\underline{B}$. Then, by [5, Lemma 5.7(i2)], X is a compact space (see also the proof of Proposition 5.3). □

7 2-Contact spaces and a new representation theorem for contact algebras

Proposition 7.1. *Let X_0 be a subspace of a topological space X. For every $F, G \in CO(X_0)$, set*

(19) $F\delta_{(X,X_0)}G$ *iff* $cl_X(F) \cap cl_X(G) \neq \emptyset$.

Then $(CO(X_0), \delta_{(X,X_0)})$ *is a contact algebra.*

Proof. Clearly, the relation $\delta_{(X,X_0)}$ satisfies the axioms (C0), (Cref) and (Csym) (see Definition 2.1 for them). It is easy to see that it satisfies the axiom (C+) as well. Hence, $(CO(X_0), \delta_{(X,X_0)})$ is a contact algebra. □

Definition 7.2. (2-Contact spaces.) (a) A topological pair (X, X_0) is called a *2-contact space* (abbreviated as CS) if the following conditions are satisfied:

$(CS1)$ X is a T_0-space;

$(CS2)$ X_0 is a Stone space;

$(CS3)$ $RC(X, X_0)$ is a closed base for X;

$(CS4)$ If $\Gamma \in Clans(CO(X_0), \delta_{(X,X_0)})$ (see (19) for the notation $\delta_{(X,X_0)}$) then there exists a point $x \in X$ such that $\Gamma = \Gamma_{x,X_0}$ (see (9) for Γ_{x,X_0}).

A 2-contact space (X, X_0) is called *connected* if the space X is connected.

(b) Let (X, X_0) be a 2-contact space. Then the contact algebra

$$\underline{B}^c(X, X_0) = (RC(X, X_0), C_{(X,X_0)})$$

(see Lemma 4.3(a) for the notation $C_{(X,X_0)}$) is said to be the *canonical contact algebra of the 2-contact space* (X, X_0).

(c) Let $\underline{B} = (B, C)$ be a contact algebra, $X = Clans(B, C)$, $X_0 = Ult(B)$ and τ be the topology on X described in Definition 5.1. Take the subspace topology on X_0. Then the pair

$$\underline{X}^c(\underline{B}) = (X, X_0)$$

is called the *canonical 2-contact space of the contact algebra* (B, C).

(d) Let (X, X_0) and $(\widehat{X}, \widehat{X}_0)$ be two 2-contact spaces. We say that (X, X_0) and $(\widehat{X}, \widehat{X}_0)$ are *CS-isomorphic* (or, simply, *isomorphic*) if there exists a homeomorphism $f : X \longrightarrow \widehat{X}$ such that $f(X_0) = \widehat{X}_0$.

Remark 7.3. Note that, by Lemma 4.3(a), the canonical contact algebra of a 2-contact space is indeed a contact algebra.

Example 7.4. Let X be a Stone space. Then the pair (X, X) is a 2-contact space. Indeed, the conditions (CS1)-(CS3) are obviously fulfilled. Let $B = CO(X)$. Obviously, $\delta_{(X,X)} = \rho_s^B$ (see Example 2.4 for ρ_s^B). Hence, using [5, Corollary 3.3] or arguing directly, we get that $Clans(B, \delta_{(X,X)}) = Ult(B) = \{u_x \mid x \in X\} = \{\Gamma_{x,X} \mid x \in X\}$. Thus, the condition (CS4) is also fulfilled.

Lemma 7.5. *For every 2-contact space (X, X_0) there exists a unique reflexive and symmetric binary relation R on X_0 such that (X, X_0, R) is a 2-precontact space.*

Proof. Let (X, X_0) be a 2-contact space. For every $x, y \in X_0$, set

(20) $xRy \iff ((\forall F \in u_x)(\forall G \in u_y)(\mathrm{cl}_X(F) \cap \mathrm{cl}_X(G) \neq \emptyset)),$

where $u_x = \{A \in CO(X_0) \mid x \in A\}$ and analogously for u_y (see (6)). Clearly, R is a reflexive and symmetric relation on X_0. We will show that for every $F, G \in CO(X_0)$,

(21) $\mathrm{cl}_X(F) \cap \mathrm{cl}_X(G) \neq \emptyset \iff FC_RG$

(recall that $FC_RG \iff ((\exists x \in F)(\exists y \in G)(xRy)))$. Indeed, let $F, G \in CO(X_0)$ and let $\mathrm{cl}_X(F) \cap \mathrm{cl}_X(G) \neq \emptyset$. Then $F\delta_{(X,X_0)}G$. Now, using Proposition 7.1 and Lemma 3.5(c), we get that there exists a clan Γ in $(CO(X_0), \delta_{(X,X_0)})$ such that $F, G \in \Gamma$. Then, by the axiom (CS4), there exists a point $z \in X$ such that $\Gamma = \Gamma_{z,X_0}$. Therefore, $\Gamma = \{A \in CO(X_0) \mid z \in \mathrm{cl}_X(A)\}$. Clearly, the principal filters $\mathcal{F}$ and $\mathcal{G}$ of the Boolean algebra $CO(X_0)$, which are generated, respectively, by F and G, are contained in Γ. Since Γ is a grill, Lemma 3.6 implies that there exist ultrafilters u and v of the Boolean algebra $CO(X_0)$ such that $u \cup v \subseteq \Gamma$, $\mathcal{F} \subseteq u$ and $\mathcal{G} \subseteq v$. Using the axiom (CS2), we obtain that there exist points $x, y \in X$ such that $\bigcap u = \{x\}$ and $\bigcap v = \{y\}$. Then, clearly, $u = u_x$ and $v = u_y$. So, we get that $z \in \mathrm{cl}_X(A)$, for every $A \in u_x \cup u_y$. Thus, xRy. Obviously, $x \in F$ and $y \in G$. Therefore, FC_RG. Conversely, let $F, G \in CO(X_0)$ and let FC_RG. Then $(\exists x \in F)(\exists y \in G)(xRy)$. Hence $F \in u_x$ and $G \in u_y$. Thus, $\mathrm{cl}_X(F) \cap \mathrm{cl}_X(G) \neq \emptyset$. So, (20) is proved. This shows that the triple (X, X_0, R) satisfies the axiom (PCS4). We obtain, as well, that $C_R = \delta_{(X,X_0)}$ and thus, using the axiom (CS4), we get that the axiom (PCS5) is satisfied as well.

Let us prove that the relation R is a closed relation on the space X_0. Indeed, let $x, y \in X_0$ and $(x, y) \notin R$. Then, by (20), there exist $F \in u_x$ and $G \in u_y$ such that $\mathrm{cl}_X(F) \cap \mathrm{cl}_X(G) = \emptyset$. This obviously implies that $(x, y) \in F \times G \subseteq X \times X \setminus R$. So, R is a closed relation. Hence, the triple (X, X_0, R) satisfies the axiom (PCS2). Since the axioms (PCS1) and (PCS3) are obviously satisfied, we get that (X, X_0, R) is a 2-precontact space.

Let (X, X_0, R') be a 2-precontact space and R' be a reflexive and symmetric relation on the set X_0. Then, by Proposition 2.5(b), $C_{R'} = (C_{R'})^{\#}$. Thus, using (11), we get that for every $F, G \in CO(X_0)$, $cl_X(F) \cap cl_X(G) \neq \emptyset$ iff $F(C_{R'})G$. Now, (21) implies that for every $F, G \in CO(X_0)$, $F(C_R)G \iff F(C_{R'})G$. Hence, $C_R \equiv C_{R'}$. Set $B = CO(X_0)$. Then we get that $\underline{B} = (B, C_R) = (B, C_{R'})$. Using (12), the proof of Theorem 6.1(b) and its notation, we obtain that: 1) $\Psi_2([(X, X_0, R')]) = [(B, C_R)] = [\underline{B}]$, 2) the map $f : (X, X_0, R') \longrightarrow \underline{X}(B, C_R)$ is a PCS-isomorphism and, for every $x \in X_0$, $f(x) = u_x$. Let $\widehat{R} = R_{(B,C_R)}$. Now, we get that for every $x, y \in X_0$,

$xR'y \iff f(x)\widehat{R}f(y) \iff u_x\widehat{R}u_y \iff u_x \times u_y \subseteq C_R \iff [(\forall F \in u_x)(\forall G \in u_y)(F(C_R)G)] \iff [(\forall F \in u_x)(\forall G \in u_y)(cl_X(F) \cap cl_X(G) \neq \emptyset)] \iff xRy$. Therefore, $R \equiv R'$. □

Corollary 7.6. *If $\underline{X} = (X, X_0)$ is a 2-contact space then X is a compact space.*

Proof. By Lemma 7.5, there exists a 2-precontact space (X, X_0, R). Then, by Corollary 6.2, X is a compact space. □

Proposition 7.7. *If (X, X_0, R) is a reflexive and symmetric 2-precontact space then (X, X_0) is a 2-contact space.*

Proof. Clearly, the conditions (CS1)-(CS3) are fulfilled. Let

$$\Gamma \in Clans(CO(X_0), \delta_{(X,X_0)}).$$

By Proposition 2.5(b), we have that $C_R = (C_R)^{\#}$. Now (11) implies that $C_R = \delta_{(X,X_0)}$. Hence, $\Gamma \in Clans(CO(X_0), C_R)$ and, by (PCS5), there exists $x \in X$ such that $\Gamma = \Gamma_{x,X_0}$. So, the condition (CS4) is satisfied. Therefore (X, X_0) is a 2-contact space. □

Proposition 7.8. *Let $\underline{B} = (B, C)$ be a contact algebra. Then the canonical 2-contact space $\underline{X}^c(\underline{B})$ is indeed a 2-contact space.*

Proof. By Proposition 5.3, the canonical 2-precontact space $\underline{X}(\underline{B}) = (X, X_0, R)$ is a 2-precontact space. Also, Lemma 3.5(d)(e) implies that (X, X_0, R) is a reflexive and symmetric 2-precontact space. Now, using Proposition 7.7, we get that (X, X_0) is a 2-contact space. Since $\underline{X}^c(\underline{B}) = (X, X_0)$, our assertion is proved. □

Theorem 7.9. (New representation theorem for contact algebras.)

(a) *Let (B, C) be a contact algebra and let (X, X_0) be the canonical 2-contact space of (B, C) (see Definition 7.2(c)). Then the function $g_B : B \longrightarrow 2^X$, defined in (13), is a CA-isomorphism from the algebra (B, C) onto the canonical contact algebra $(RC(X, X_0), C_{(X,X_0)})$ of (X, X_0). The sets $RC(X, X_0)$ and $RC(X)$ coincide iff the contact algebra (B, C) is complete. The contact algebra (B, C) is connected iff the 2-contact space (X, X_0) is connected.*

(b) *There exists a bijective correspondence between the class of all, up to CA-isomorphism, (connected) contact algebras and the class of all, up to CS-isomorphism, (connected) 2-contact spaces; namely, for every CA $\underline{B}$, the CA-isomorphism class $[\underline{B}]$ of $\underline{B}$ corresponds to the CS-isomorphism class $[\underline{X}^c(\underline{B})]$ of the canonical 2-contact space $\underline{X}^c(\underline{B})$ of $\underline{B}$, and for every 2-contact space (X, X_0), the CS-isomorphism class $[(X, X_0)]$ of (X, X_0) corresponds to the CA-isomorphism class $[\underline{B}^c(X, X_0)]$ of the canonical contact algebra $\underline{B}^c(X, X_0)$ of (X, X_0).*

Proof. (a) By Lemma 7.5, there exists a unique reflexive and symmetric binary relation R on X_0 such that (X, X_0, R) is a 2-precontact space. Since (B, C) is a contact algebra, we have that $C = C^{\#}$. Thus, using (17), we get that $g_B : (B, C) \longrightarrow (RC(X, X_0), C_{(X,X_0)})$ is a CA-isomorphism. The remaining assertions follow from Theorem 6.1(a).

(b) In this part of our proof, we will use the notation from the proof of Theorem 6.1(b).

Let $\mathcal{CA}$ be the class of all, up to CA-isomorphism, contact algebras. Let $\mathcal{CS}$ be the class of all, up to CS-isomorphism, 2-contact spaces. Let $2\mathcal{CS}$ be the class of all 2-contact spaces. Let $2\mathcal{PCS}$ be the class of all 2-precontact spaces (X, X_0, R) for which R is a reflexive and symmetric relation. Let $2\mathcal{PS}$ be the class of all, up to PCS-isomorphism, 2-precontact spaces (X, X_0, R) for which R is a reflexive and symmetric relation. Then, using Lemma 7.5 and Proposition 7.7, we get that the correspondence

$$\varphi : 2\mathcal{CS} \longrightarrow 2\mathcal{PCS}, \quad (X, X_0) \mapsto (X, X_0, R),$$

where the relation R is defined by the formula (20), is a bijection. It is clear that then the correspondence

$$\psi : \mathcal{CS} \longrightarrow 2\mathcal{PS}, \quad [(X, X_0)] \longrightarrow [\varphi(X, X_0)],$$

is a bijection as well. By Theorem 6.1(a), $\underline{B} = (B, C)$ is a contact algebra iff the 2-precontact space $\underline{X}(\underline{B}) = (X, X_0, R)$ is reflexive and symmetric, i.e., iff R is a reflexive and symmetric relation. Thus, if Φ_2' is the restriction of the correspondence Φ_2 to the subclass $\mathcal{CA}$ of the class $\mathcal{PCA}$, then

$$\Phi_2' : \mathcal{CA} \longrightarrow 2\mathcal{PS}$$

is a bijection. Therefore, the map

$$\Phi_2^c = \psi^{-1} \circ \Phi_2' : \mathcal{CA} \longrightarrow \mathcal{CS}, \quad [\underline{B}] \mapsto [\underline{X}^c(\underline{B})],$$

is a bijection. The assertion about connected contact algebras follows now from Theorem 6.1(b). □

We are now going to obtain an assertion from [5] (namely, [5, Theorem 5.1(ii)(for CAs)]) as a corollary of Theorem 7.9. This assertion concerns the class of C-semiregular spaces introduced in [5] (see Definition 7.17 below). We will also obtain some new facts about this class of spaces. We start with recalling and proving some preliminary assertions. Then we obtain a new theorem about the structure of C-semiregular spaces (see Theorem 7.19 below) and using it, we derive [5, Theorem 5.1(ii)(for CAs)] from Theorem 7.9 (see Corollary 7.23 below).

Lemma 7.10. ([5]) *Let X be a topological space. Then:*
(a) for every $x \in X$, σ_x is a clan of the contact algebra $(RC(X), C_X)$;
(b) if X is semiregular, then X is a T_0-space iff for every $x, y \in X$, $x \neq y$ implies that $\sigma_x \neq \sigma_y$ (see (9) for σ_x).

Lemma 7.11. ([4]) *Let* $(X, \mathcal{T})$ *be a topological space and* $U, V \in \mathcal{T}$. *Then*

$$\mathrm{int}(\mathrm{cl}(U) \cap \mathrm{cl}(V)) = \mathrm{int}(\mathrm{cl}(U \cap V)).$$

Corollary 7.12. *Let* $(X, \mathcal{T})$ *be a topological space and* $U, V \in \mathcal{T}$. *Then*

$$\mathrm{cl}(\mathrm{int}(\mathrm{cl}(U) \cap \mathrm{cl}(V))) = \mathrm{cl}(U \cap V).$$

Proof. Since $\mathrm{cl}(U \cap V)$ is a regular closed set, we get, using Lemma 7.11, that $\mathrm{cl}(\mathrm{int}(\mathrm{cl}(U) \cap \mathrm{cl}(V))) = \mathrm{cl}(\mathrm{int}(\mathrm{cl}(U \cap V))) = \mathrm{cl}(U \cap V)$. □

Definition 7.13. Let $(X, \mathcal{T})$ be a topological space and $x \in X$. The point x is said to be an *u-point* if for every $U, V \in \mathcal{T}$, $x \in \mathrm{cl}(U) \cap \mathrm{cl}(V)$ implies that $x \in \mathrm{cl}(U \cap V)$.

Proposition 7.14. *(a) A topological space* $(X, \mathcal{T})$ *is extremally disconnected iff every of its points is an u-point;*

(b) If X *is a dense subspace of a space* Y *and* $x \in X$, *then* x *is an u-point of* Y *iff* x *is an u-point of* X.

Proof. (a) Let X be extremally disconnected. Then, for every $U, V \in \mathcal{T}$, we have, by Lemma 7.11, that $\mathrm{cl}(U) \cap \mathrm{cl}(V) = \mathrm{int}(\mathrm{cl}(U) \cap \mathrm{cl}(V)) = \mathrm{int}(\mathrm{cl}(U \cap V)) = \mathrm{cl}(U \cap V)$. Hence, every point of X is an u-point.

Conversely, let every point of X be an u-point. Let $U \in \mathcal{T}$. Suppose that $\mathrm{cl}(U) \notin \mathcal{T}$. Then there exists $x \in \mathrm{cl}(U) \setminus \mathrm{int}(\mathrm{cl}(U))$. Hence $x \in X \setminus \mathrm{int}(\mathrm{cl}(U)) = \mathrm{cl}(X \setminus \mathrm{cl}(U))$. Set $V = X \setminus \mathrm{cl}(U)$. Then $V \in \mathcal{T}$ and $x \in \mathrm{cl}(U) \cap \mathrm{cl}(V)$. Since x is an u-point, we get that $x \in \mathrm{cl}(U \cap V) = \emptyset$, a contradiction. Hence, $\mathrm{cl}(U) \in \mathcal{T}$. So, X is extremally disconnected.

(b) Let x be an u-point of X. Let U, V be open subsets of Y and $x \in \mathrm{cl}_Y(U) \cap \mathrm{cl}_Y(V)$. Set $U' = U \cap X$ and $V' = V \cap X$. Then $x \in X \cap \mathrm{cl}_Y(U') \cap \mathrm{cl}_Y(V') = \mathrm{cl}_X(U') \cap \mathrm{cl}_X(V')$. Hence $x \in \mathrm{cl}_X(U' \cap V') = \mathrm{cl}_X(X \cap U \cap V) \subseteq \mathrm{cl}_Y(U \cap V)$. So, x is an u-point of Y.

Conversely, let x be an u-point of Y. Let U, V be open subsets of X and $x \in \mathrm{cl}_X(U) \cap \mathrm{cl}_X(V)$. There exist open subsets U' and V' of Y such that $U = U' \cap X$ and $V = V' \cap X$. Then $x \in \mathrm{cl}_Y(U') \cap \mathrm{cl}_Y(V')$. Hence $x \in X \cap \mathrm{cl}_Y(U' \cap V') = X \cap \mathrm{cl}_Y(X \cap U' \cap V') = X \cap \mathrm{cl}_Y(U \cap V) = \mathrm{cl}_X(U \cap V)$. Therefore, x is an u-point of X. □

Proposition 7.15. *Let* $(X, \mathcal{T})$ *be a topological space and* $x \in X$. *Then* x *is an u-point iff* σ_x *is an ultrafilter of the Boolean algebra* $RC(X)$ *(see (9) for* σ_x*).*

Proof. Since, by Lemma 7.10(a), σ_x is a grill of $RC(X)$, we have that σ_x is an ultrafilter of $RC(X)$ iff $(\forall F, G \in \sigma_x)(F.G \in \sigma_x)$. Hence, using Corollary 7.12, we get that: (σ_x is an ultrafilter of $RC(X)$) $\iff$ $((\forall F, G \in RC(X))[(x \in F \cap G) \to (x \in F.G)])$ $\iff$ $((\forall F, G \in RC(X))[(x \in F \cap G) \to (x \in \mathrm{cl}(\mathrm{int}(F \cap G)))])$ $\iff$ $((\forall U, V \in \mathcal{T})[(x \in \mathrm{cl}(U) \cap \mathrm{cl}(V)) \to (x \in \mathrm{cl}(\mathrm{int}(\mathrm{cl}(U) \cap \mathrm{cl}(V))))])$ $\iff$ $((\forall U, V \in \mathcal{T})[(x \in \mathrm{cl}(U) \cap \mathrm{cl}(V)) \to (x \in \mathrm{cl}(U \cap V))])$ $\iff$ (x is an u-point). □

Proposition 7.16. *Let X and Y be topological spaces and $x \in X$. If x is an u-point and $f : X \longrightarrow Y$ is an open map, then $f(x)$ is an u-point.*

Proof. Let U and V be open subsets of Y and $f(x) \in \mathrm{cl}_Y(U) \cap \mathrm{cl}_Y(V)$. We will show that $f(x) \in \mathrm{cl}_Y(U \cap V)$. We have that $x \in f^{-1}(\mathrm{cl}_Y(U))$ and $x \in f^{-1}(\mathrm{cl}_Y(V))$. Using [10, Exercise 1.4.C], we get that $x \in \mathrm{cl}_X(f^{-1}(U)) \cap \mathrm{cl}_X(f^{-1}(V))$. Since x is an u-point, we get, using again [10, Exercise 1.4.C], that $x \in \mathrm{cl}_X(f^{-1}(U) \cap f^{-1}(V)) = \mathrm{cl}_X(f^{-1}(U \cap V)) = f^{-1}(\mathrm{cl}_Y(U \cap V))$. Thus, $f(x) \in \mathrm{cl}_Y(U \cap V)$. Hence $f(x)$ is an u-point. □

Note that Proposition 7.14(a) and Proposition 7.16 imply the well-known fact that extremal disconnectedness is an invariant of open mappings (see [10, Exercise 6.2.H(b)]).

Definition 7.17. ([5]) A semiregular T_0-space (X, τ) is said to be *C-semiregular* if for every clan Γ in $(RC(X), C_X)$ there exists a point $x \in X$ such that $\Gamma = \sigma_x$ (see (9) for σ_x).

The next assertion was stated in [5] but it was left without proof there. For completeness, we will prove it here.

Proposition 7.18. ([5, Fact 4.1]) *Every C-semiregular space X is a compact space.*

Proof. Let $\mathcal{F} = \{F_a \mid \alpha \in A\}$ be a centered (= with finite intersection property) family of closed subsets of X. Since X is semiregular, for every $\alpha \in A$ there exists a subfamily $\mathcal{R}_\alpha$ of $RC(X)$ such that $F_\alpha = \bigcap \mathcal{R}_\alpha$. Let $\mathcal{R} = \bigcup\{\mathcal{R}_\alpha \mid \alpha \in A\}$. Then $\bigcap \mathcal{F} = \bigcap \mathcal{R}$ and $\mathcal{R}$ is a centered family. Thus there exists an ultrafilter u of the Boolean algebra $RC(X)$ containing $\mathcal{R}$. Since u is a clan in the contact algebra $(RC(X), C_X)$, there exists $x \in X$ such that $u = \sigma_x$. Therefore $x \in \bigcap u \subseteq \bigcap \mathcal{R} = \bigcap \mathcal{F}$. Hence, X is compact. □

Theorem 7.19. *For every C-semiregular space $(X, \mathcal{T})$, the set*

$$X_0 = \{x \in X \mid x \text{ is an u-point of } X\}$$

endowed with its subspace topology is a dense extremally disconnected compact Hausdorff subspace of $(X, \mathcal{T})$ and is the unique dense extremally disconnected compact Hausdorff subspace of $(X, \mathcal{T})$.

Proof. Set $(B, C) = (RC(X), C_X)$. Then (B, C) is a complete Boolean algebra. Let $(\widehat{X}, \widehat{X}_0)$ be the canonical 2-contact space of (B, C) (see Definition 7.2(c)). Since B is a complete Boolean algebra, its Stone space $\widehat{X}_0$ is extremally disconnected and $RC(\widehat{X}, \widehat{X}_0) = RC(\widehat{X})$. Using Lemma 7.10(b) and the fact that X is C-semiregular, we get that the map

$$f : X \longrightarrow \widehat{X}, \quad x \mapsto \sigma_x,$$

is a bijection. In the notation of Definition 5.1, we have that for every $F \in RC(X)$, $f(F) = \{f(x) \mid x \in F\} = \{\sigma_x \mid x \in F\} = \{\sigma_x \mid F \in \sigma_x\} = \{\Gamma \in \widehat{X} \mid F \in \Gamma\} = g_B(F)$.

Since f is a bijection, we obtain now that for every $F \in RC(X)$, $f^{-1}(g_B(F)) = F$. Using the fact that $RC(X)$ is a closed base of X and $\{g_B(F) \mid F \in RC(X)\}$ is a closed base of $\widehat{X}$, we get that f is a homeomorphism. Since $\widehat{X}_0 = Ult(RC(X))$, we get that $\widehat{X}_0 = \{\sigma_x \mid \sigma_x$ is an ultrafilter of $RC(X)\}$. Thus, Proposition 7.15 implies that $f^{-1}(\widehat{X}_0) = X_0$. Hence, $f(X_0) = \widehat{X}_0$. Therefore, (X, X_0) is a 2-contact space and $f : (X, X_0) \longrightarrow (\widehat{X}, \widehat{X}_0)$ is a CS-isomorphism. From this we get, in particular, that X_0 is a dense extremally disconnected compact Hausdorff subspace of $(X, \mathcal{T})$.

Let now X_0' be a dense extremally disconnected compact Hausdorff subspace of $(X, \mathcal{T})$. We will show that $X_0' = X_0$. Since X_0' is extremally disconnected, Proposition 7.14(a) implies that every point of X_0' is an u-point of X_0'. Using the fact that X_0' is dense in X, we get, by Proposition 7.14(b), that every point of X_0' is an u-point of X. Hence $X_0' \subseteq X_0$. Obviously, X_0' is a dense subspace of X_0. Since X_0 is Hausdorff and X_0' is compact, we get that $X_0' = X_0$. □

Corollary 7.20. *A compact Hausdorff space X is C-semiregular iff it is extremally disconnected.*

Proof. Let X be extremally disconnected.. Then $RC(X) = CO(X)$. Also, the sets $Clans(RC(X), C_X)$ and $Ult(CO(X))$ coincide. Indeed, let $\Gamma \in Clans(RC(X), C_X)$. Then $\Gamma \subseteq CO(X)$. Let $F, G \in \Gamma$. Since $F = (F \cap G) \cup (F \setminus G)$ and $F \cap G, F \setminus G \in CO(X) = RC(X)$, we have that $F \cap G \in \Gamma$ or $F \setminus G \in \Gamma$. Clearly, $(F \setminus G)(-C_X)G$. Hence $F \setminus G \notin \Gamma$. Therefore, $F \cap G \in \Gamma$. Thus, Γ is an ultrafilter of the Boolean algebra $CO(X)$. Then $|\bigcap \Gamma| = 1$. Let $\{x\} = \bigcap \Gamma$. Then, obviously, $\Gamma = \sigma_x$. Hence, X is a C-semiregular space.

Conversely, if X is C-semiregular then, by Theorem 7.19, X contains a dense extremally disconnected compact Hausdorff space Y. Since X is a Hausdorff space, we get that $X \equiv Y$. Therefore, X is extremally disconnected. □

Corollary 7.21. *If X is C-semiregular and $X_0 = \{x \in X \mid x$ is an u-point of $X\}$ then the pair (X, X_0) is a 2-contact space and X_0 is a dense extremally disconnected compact Hausdorff subspace of X; moreover, X_0 is the unique dense extremally disconnected compact Hausdorff subspace of X.*

Proof. By Theorem 7.19, X_0 is a dense extremally disconnected compact Hausdorff subspace of X and hence, the conditions (CS1) and (CS2) are fulfilled. Also, we obtain that $RC(X_0) = CO(X_0)$ and thus $RC(X, X_0) = RC(X)$; moreover, the map

$$e : (CO(X_0), \delta_{(X,X_0)}) \longrightarrow (RC(X), C_X), \quad F \mapsto \mathrm{cl}_X(F),$$

is a CA-isomorphism. Since X is semiregular, we get that the condition (CS3) is fulfilled. Let $\Gamma \in Clans(CO(X_0), \delta_{(X,X_0)})$. Then $e(\Gamma) \in Clans(RC(X), C_X)$ and, therefore, there exists $x \in X$ such that $e(\Gamma) = \sigma_x$. Since, obviously, $\sigma_x = \Gamma_{x,X_0}$, we get that the condition (CS4) is also fulfilled. Hence (X, X_0) is a 2-contact space.

The uniqueness assertion follows from Theorem 7.19. □

Lemma 7.22. *If (X, X_0) is a 2-contact space and X_0 is extremally disconnected, then X is C-semiregular.*

Proof. By (CS1), X is a T_0-space. Using Lemma 4.3(a) and (CS3), we get that X is a semiregular space and $RC(X, X_0) = RC(X)$. Since $CO(X_0) = RC(X_0)$, Lemma 4.1 implies that the map

$$e : (CO(X_0), \delta_{(X,X_0)}) \longrightarrow (RC(X), C_X), \quad F \mapsto \mathrm{cl}_X(F),$$

is a CA-isomorphism. Let $\Gamma \in Clans(RC(X), C_X)$ and $\Gamma' = e^{-1}(\Gamma)$. Then $\Gamma' \in Clans(CO(X_0), \delta_{(X,X_0)})$. Thus, by (CS4), there exists $x \in X$ such that $\Gamma' = \Gamma_{x,X_0}$. Then we get that $\Gamma = e(\Gamma') = e(\Gamma_{x,X_0}) = \sigma_x$. Therefore, X is a C-semiregular space. □

Corollary 7.23. ([5]) *There exists a bijective correspondence between the class of all, up to CA-isomorphism, (connected) complete contact algebras and the class of all, up to homeomorphism, (connected) C-semiregular spaces.*

Proof. Let $\mathcal{CCA}$ be the class of all, up to CA-isomorphism, complete contact algebras. Let $\mathcal{CCS}$ be the class of all, up to CS-isomorphism, 2-contact spaces (X, X_0) such that X_0 is extremally disconnected. Using Theorem 7.9(b) and the notation of its proof, we get that the map

$$\Phi_2^c : \mathcal{CA} \longrightarrow \mathcal{CS}, \quad [(B, C)] \mapsto [(X, X_0)],$$

where (X, X_0) is the canonical 2-precontact space of the contact algebra (B, C) (see Definition 7.2(c)), is a bijection. Then Theorem 7.9(a) implies that the restriction Φ_c of the correspondence Φ_2^c to the class $\mathcal{CCA}$ is a bijection between the later class and the class $\mathcal{CCS}$, i.e.,

$$\Phi_c : \mathcal{CCA} \longrightarrow \mathcal{CCS}$$

is a bijection. Let $\mathcal{CSR}$ be the class of all, up to homeomorphism, C-semiregular spaces. Then the map

$$\alpha : \mathcal{CCS} \longrightarrow \mathcal{CSR}, \quad [(X, X_0)] \mapsto [X],$$

is a bijection. Indeed, Lemma 7.22 implies that the correspondence α is well-defined; the fact that α is a bijection follows from Theorem 7.19. Now we get that the composition

$$\Phi_c' = \alpha \circ \Phi_c : \mathcal{CCA} \longrightarrow \mathcal{CSR}$$

is a bijection. The assertion about connected complete contact algebras follows now from Theorem 6.1(b). □

8 A connected version of the Stone Duality Theorem

The new representation theorems, presented in the previous section, permit us to obtain as particular cases the Stone Representation Theorem [22] and a new connected version of it. Let us start with the Stone Representation Theorem [22].

Proposition 8.1. *The Stone Representation Theorem follows from Theorem 7.9; namely, we obtain it equipping each Boolean algebra B with the smallest contact relation ρ_s^B on B (see Example 2.4 for ρ_s^B).*

Proof. Let B be a Boolean algebra. Then, by [5, Corollary 3.3], $Ult(B, \rho_s) = Clans(B, \rho_s)$. Hence, the canonical 2-contact space of the contact algebra $\underline{B} = (B, \rho_s)$ is the pair $\underline{X}^c(\underline{B}) = (X, X_0)$, where $X = X_0$ is the Stone space of B. Also, $RC(X, X_0) = CO(X_0)$ and $C_{(X,X_0)} = \rho_s^{B'}$, where $B' = CO(X_0)$. So, Theorem 7.9(a) reduces to the Stone Representation Theorem. Note that $RC(X_0) = CO(X_0)$ iff X_0 is extremally disconnected. Note as well that (B, ρ_s) is connected iff $B = \{0, 1\}$. Further, Theorem 7.9(b) reduced to the Stone Theorem that there exists a bijective correspondence between the class of all, up to Boolean isomorphism, Boolean algebras and the class of all, up to homeomorphism, Stone spaces. Finally, Corollary 7.23 and Corollary 7.20 imply that there exists a bijective correspondence between the class of all, up to Boolean isomorphism, complete Boolean algebras and the class of all, up to homeomorphism, compact Hausdorff extremally disconnected spaces. □

Now, equipping each Boolean algebra B with the largest contact relation ρ_l^B on B (see Example 2.4 for ρ_l^B), we will obtain a connected version of the Stone Representation Theorem. Further on we will extend it to a connected version of the Stone Duality Theorem.

Definition 8.2. (Stone 2-spaces.)

(a) A topological pair (X, X_0) is called a *Stone 2-space* (abbreviated as S2S) if it satisfies conditions (CS1)-(CS3) of Definition 7.2 and the following condition:

($S2S4$) If $\Gamma \in Grills(CO(X_0))$ then there exists a point $x \in X$ such that $\Gamma = \Gamma_{x,X_0}$ (see (9) for Γ_{x,X_0}).

(b) Let (X, X_0) and $(\widehat{X}, \widehat{X}_0)$ be two Stone 2-spaces. We say that (X, X_0) and $(\widehat{X}, \widehat{X}_0)$ are *S2S-isomorphic* (or, simply, *isomorphic*) if there exists a homeomorphism $f : X \longrightarrow \widehat{X}$ such that $f(X_0) = \widehat{X}_0$.

Proposition 8.3. *Let (X, X_0) be a Stone 2-space and $B = RC(X, X_0)$. Then:*
(a) $C_{(X,X_0)} = \rho_l^B$ (see Example 2.4 for ρ_l^B), and
(b) (X, X_0) is a 2-contact space.

Proof. (a) Clearly, $C_{(X,X_0)} \subseteq \rho_l^B$. Recall that for any $F, G \in CO(X_0)$,

$$\mathrm{cl}_X(F) C_{(X,X_0)} \mathrm{cl}_X(G) \iff \mathrm{cl}_X(F) \cap \mathrm{cl}_X(G) \neq \emptyset.$$

We will show that $\rho_l^B \subseteq C_{(X,X_0)}$. Let $F, G \in CO(X_0)$ and $F \neq \emptyset$, $G \neq \emptyset$ (i.e., $\mathrm{cl}_X(F), \mathrm{cl}_X(G) \in B$ and $\mathrm{cl}_X(F)\rho_l^B \mathrm{cl}_X(G)$). Obviously, there exist $u, v \in Ult(CO(X_0))$ such that $F \in u$ and $G \in v$. Then $\Gamma = u \cup v \in Grills(CO(X_0))$ (see, e.g., [5, Corollary 3.1]). Hence, by (S2S4), there exists a point $x \in X$ such that $\Gamma = \Gamma_{x,X_0}$. Then $x \in \mathrm{cl}_X(F) \cap \mathrm{cl}_X(G)$ and thus $\mathrm{cl}_X(F) C_{(X,X_0)} \mathrm{cl}_X(G)$. So, $\rho_l^B = C_{(X,X_0)}$.
(b) Since the map $e : (CO(X_0), \delta_{(X,X_0)}) \longrightarrow (RC(X, X_0), C_{(X,X_0)})$, $F \mapsto \mathrm{cl}_X(F)$, is a CA-isomorphism, we get, using (a) and [5, Example 3.1], that

$$Clans(CO(X_0), \delta_{(X,X_0)}) = Grills(CO(X_0)).$$

Thus, condition (CS4) follows from condition (S2S4). Hence (X, X_0) is a 2-contact space. □

Corollary 8.4. *Let (X, X_0) be a Stone 2-space. Then X is a compact connected T_0-space.*

Proof. According to Proposition 8.3(b), (X, X_0) is a 2-contact space. Hence, by Corollary 7.6, X is a compact space. Since the canonical CA $(RC(X, X_0), C_{(X,X_0)})$ of the 2-contact space (X, X_0) is connected (by Proposition 8.3(a)), Theorem 7.9(a) implies that the space X is connected. □

Definition 8.5. (The canonical Stone 2-space of a Boolean algebra and the canonical Boolean algebra of a Stone 2-space.)
(a) Let B be a Boolean algebra. Then the canonical 2-contact space of the contact algebra (B, ρ_l^B) is said to be the *canonical Stone 2-space of the Boolean algebra B* and is denoted by $\underline{X}^s(B)$.
(b) Let (X, X_0) be a Stone 2-space. Then the Boolean algebra $RC(X, X_0)$ is said to be the *canonical Boolean algebra of* (X, X_0).

Proposition 8.6. *The canonical Stone 2-space of a Boolean algebra B is indeed a Stone 2-space.*

Proof. Let $\underline{X}^c(B, \rho_l^B) = (X, X_0)$. Then, by Theorem 7.9(a), the contact algebras (B, ρ_l^B) and $(RC(X, X_0), C_{(X,X_0)})$ are CA-isomorphic. Since the contact algebras $(RC(X, X_0), C_{(X,X_0)})$ and $(CO(X_0), \delta_{(X,X_0)})$ are CA-isomorphic, we get that the contact algebras (B, ρ_l^B) and $(CO(X_0), \delta_{(X,X_0)})$ are CA-isomorphic. Then, by [5, Example 3.1], $Clans(CO(X_0), \delta_{(X,X_0)}) = Grills(CO(X_0))$. According to Proposition 7.8, (X, X_0) is a 2-contact space. Now we obtain that (X, X_0) is a Stone 2-space. □

Theorem 8.7. (A connected version of the Stone Representation Theorem.)

(a) *Let B be a Boolean algebra and let (X, X_0) be the canonical Stone 2-space of B (see Definition 8.5). Then the function $g_B : B \longrightarrow 2^X$, defined in (13), is a Boolean isomorphism from the Boolean algebra B onto the canonical Boolean algebra $RC(X, X_0)$ of (X, X_0). The sets $RC(X, X_0)$ and $RC(X)$ coincide iff the Boolean algebra B is complete.*

(b) *There exists a bijective correspondence between the class of all, up to Boolean isomorphism, Boolean algebras and the class of all, up to S2S-isomorphism, Stone 2-spaces; namely, for every Boolean algebra B, $[B] \mapsto [\underline{X}^s(B)]$, and for every Stone 2-space (X, X_0), $[(X, X_0)] \mapsto [RC(X, X_0)]$.*

Proof. (a) Since (X, X_0) is, by Definition 8.5(a), the canonical 2-contact space of the contact algebra (B, ρ_l^B), our assertion follows from Theorem 7.9(a).

(b) It follows from Theorem 7.9(b), Definition 8.5(a), Proposition 8.6 and Proposition 8.3. □

Definition 8.8. (Extremally connected spaces.) A semiregular T_0-space X is called an *extremally connected space* (abbreviated as ECS) if for every grill Γ in $RC(X)$ there exists a point $x \in X$ such that $\Gamma = \sigma_x$ (see (9) for σ_x).

Proposition 8.9. *Let X be an extremally connected space and $B = RC(X)$. Then:*
(a) $C_X = \rho_l^B$ (see Lemma 2.8 for C_X and Example 2.4 for ρ_l^B), and
(b) X is a C-semiregular space.

Proof. (a) Let $F, G \in B$ and $F \neq \emptyset$, $G \neq \emptyset$. Then, as in the proof of Proposition 8.3(a), we get that there exists a grill Γ in B such that $F, G \in \Gamma$. Then, by Definition 8.8, there exists $x \in X$ such that $\Gamma = \sigma_x$. Thus $x \in F \cap G$, i.e. $F C_X G$. Therefore, $\rho_l^B \subseteq C_X$. Since, obviously, $C_X \subseteq \rho_l^B$, we get that $C_X = \rho_l^B$.

(b) Since, by (a), $(B, C_X) = (B, \rho_l^B)$, [5, Example 3.1] implies that $Clans(B, C_X) = Grills(B)$. Thus X is a C-semiregular space. □

Corollary 8.10. *Let X be an extremally connected space and*

$$X_0 = \{x \in X \mid x \text{ is an u-point of } X\}.$$

Then (X, X_0) is a Stone 2-space and X_0 is a dense extremally disconnected compact Hausdorff subspace of X; moreover, X_0 is the unique dense extremally disconnected compact Hausdorff subspace of X.

Proof. According to Proposition 8.9(b), X is a C-semiregular space. Then, by Corollary 7.21, (X, X_0) is a 2-contact space and X_0 is a dense extremally disconnected compact Hausdorff subspace of X. Let $\Gamma \in Grills(CO(X_0))$. We have that $CO(X_0) = RC(X_0)$ and thus, by Lemma 4.1, the Boolean algebra $CO(X_0)$ is isomorphic to the Boolean algebra $RC(X)$. Now, using [5, Example 3.1] and Proposition 8.9(a), we get that $Grills(CO(X_0)) = Clans(CO(X_0), \rho_l^{CO(X_0)}) = Clans(CO(X_0), \delta_{(X,X_0)})$. Hence, by condition (CS4), there exists $x \in X$ such that $\Gamma = \Gamma_{x,X_0}$. Therefore, condition (S2S4) is satisfied and we obtain that (X, X_0) is a Stone 2-space.

The uniqueness assertion follows from Theorem 7.19. □

Lemma 8.11. *If (X, X_0) is a Stone 2-space and X_0 is extremally disconnected, then X is an extremally connected space.*

Proof. Let $B = RC(X)$. Since X_0 is an extremally disconnected space, we have that $RC(X_0) = CO(X_0)$. Then $RC(X, X_0) = B$. Now, Proposition 8.3(a) implies that $C_X = \rho_l^B$ and (X, X_0) is a 2-contact space. Then, by Lemma 7.22, X is a C-semiregular space. Since $(B, C_X) = (B, \rho_l^B)$, [5, Example 3.1] implies that X is an extremally connected space. □

Theorem 8.12. *There exists a bijective correspondence between the class of all, up to Boolean isomorphism, complete Boolean algebras and the class of all, up to homeomorphism, extremally connected spaces; namely, for every complete Boolean algebra B, $[B] \mapsto [X]$, where X is the first component of $\underline{X}^s(B)$, and for every extremally connected space X, $[X] \mapsto [RC(X)]$.*

Proof. It follows from Theorem 8.7(b), the Stone bijection between complete Boolean algebras and extremally disconnected Stone spaces, Lemma 8.11 and Corollary 8.10. □

Now we will obtain a connected version of the Stone Duality Theorem. We start with three simple assertions, the first of which is a slight generalization of [5, Proposition 4.1(iv)].

Lemma 8.13. *Let X be a topological space, B be a subalgebra of the Boolean algebra $(RC(X), +, ., *, \emptyset, X)$ (defined in 2.7), Γ be a grill of B, $x \in X$ and $\Gamma \subseteq \sigma_x^B$. Then $\nu_x^B \subseteq \Gamma$ (see (10) for ν_x^B and (9) for σ_x^B).*

Proof. We can suppose that $\Gamma \neq \sigma_x^B$. Let $F \in \sigma_x^B \setminus \Gamma$. Since $F + F^* = 1 \in \Gamma$, we get that $F^* \in \Gamma$. Suppose that $F \in \nu_x^B$. Then $x \in \mathrm{int}(F)$ and thus $x \notin F^*$. Since $\Gamma \subseteq \sigma_x^B$, we get a contradiction. Therefore, $F \notin \nu_x^B$. So, we obtain that $\nu_x^B \cap (\sigma_x^B \setminus \Gamma) = \emptyset$. Since $\nu_x^B \subseteq \sigma_x^B$, we get that $\nu_x^B \subseteq \Gamma$. □

Lemma 8.14. *Let X be a topological space, B be a subalgebra of the Boolean algebra $(RC(X), +, ., *, \emptyset, X)$ (defined in 2.7) and B be a closed base for the space X. Then, for every $x \in X$, the family $\mathcal{B}_x = \{\mathrm{int}(F) \mid F \in \nu_x^B\}$ is a base for X at the point x.*

Proof. Since B is a closed base for X, the family $\mathcal{B} = \{X \setminus F \mid F \in B\}$ is an open base for X. Further, for any $F \in B$, we have that $\mathrm{int}(F^*) = X \setminus F$. Thus $\mathcal{B} = \{\mathrm{int}(F^*) \mid F \in B\} = \{\mathrm{int}(F) \mid F \in B\}$. Hence, for every $x \in X$, the family $\{U \in \mathcal{B} \mid x \in U\}$ is a base for X at the point x. Clearly, $\{U \in \mathcal{B} \mid x \in U\} = \{\mathrm{int}(F) \mid F \in B, x \in \mathrm{int}(F)\} = \{\mathrm{int}(F) \mid F \in \nu_x^B\} = \mathcal{B}_x$. Therefore, for every $x \in X$, the family $\mathcal{B}_x$ is a base for X at the point x. □

Lemma 8.15. *Let A be a subalgebra of a Boolean algebra B and $\Gamma \in Grills(B)$. Then $\Gamma \cap A \in Grills(A)$.*

Proof. Clearly, if $u \in Ult(B)$ then $u \cap A \in Ult(A)$. Then, using [5, Corollary 3.1], we obtain our assertion. □

Definition 8.16. Let (X, X_0) and (X', X'_0) be two Stone 2-spaces and $f : X \longrightarrow X'$ be a continuous map. Then f is called a *2-map* if $f(X_0) \subseteq X'_0$.

The category of all Stone 2-spaces and all 2-maps between them will be denoted by **2Stone**.

The category of all Boolean algebras and all Boolean homomorphisms between them will be denoted by **Bool**.

Theorem 8.17. *The categories* **Bool** *and* **2Stone** *are dually equivalent.*

Proof. We will first define two contravariant functors

$$D^a : \mathbf{Bool} \longrightarrow \mathbf{2Stone} \quad \text{and} \quad D^t : \mathbf{2Stone} \longrightarrow \mathbf{Bool}.$$

Let $(X, X_0) \in |\mathbf{2Stone}|$. Define

$$D^t(X, X_0) = RC(X, X_0),$$

i.e. $D^t(X, X_0)$ is the canonical Boolean algebra of the Stone 2-space (X, X_0) (see Definition 8.5(b)). Hence $D^t(X, X_0) \in |\mathbf{Bool}|$.

Let $f \in \mathbf{2Stone}((X, X_0), (Y, Y_0))$. Define $D^t(f) : D^t(Y, Y_0) \longrightarrow D^t(X, X_0)$ by the formula

(22) $D^t(f)(\mathrm{cl}_Y(G)) = \mathrm{cl}_X(X_0 \cap f^{-1}(G)), \quad \forall G \in CO(Y_0).$

Set $\varphi_f = D^t(f)$. We will show that φ_f is a Boolean homomorphism between the Boolean algebras $RC(Y, Y_0)$ and $RC(X, X_0)$. Clearly, $\varphi_f(\emptyset) = \emptyset$ and $\varphi_f(Y) = X$. Let $F, G \in CO(Y_0)$. Then $\varphi_f(\mathrm{cl}_Y(F) + \mathrm{cl}_Y(G)) = \varphi_f(\mathrm{cl}_Y(F \cup G)) = \mathrm{cl}_X(X_0 \cap f^{-1}(F \cup G)) = \mathrm{cl}_X((X_0 \cap f^{-1}(F)) \cup (X_0 \cap f^{-1}(G))) = \varphi_f(\mathrm{cl}_Y(F)) + \varphi_f(\mathrm{cl}_Y(G))$. Also, using Lemma 4.1, we get that $\varphi_f((\mathrm{cl}_Y(F))^*) = \varphi_f(\mathrm{cl}_Y(Y_0 \setminus F) = \mathrm{cl}_X(X_0 \cap f^{-1}(Y_0 \setminus F)) = \mathrm{cl}_X(X_0 \cap (f^{-1}(Y_0) \setminus f^{-1}(F))) = \mathrm{cl}_X(X_0 \setminus (X_0 \cap f^{-1}(F))) = (\mathrm{cl}_X(X_0 \cap f^{-1}(F)))^* = (\varphi_f(\mathrm{cl}_Y(F))^*$. So, φ_f is a Boolean homomorphism, i.e. $D^t(f)$ is well-defined.

Now we will show that D^t is a contravariant functor. Clearly, $D^t(id_{X,X_0}) = id_{D^t(X,X_0)}$. Let $f \in \mathbf{2Stone}((X, X_0), (Y, Y_0))$ and $g \in \mathbf{2Stone}((Y, Y_0), (Z, Z_0))$. Then, for every $F \in CO(Z_0)$, $D^t(g \circ f)(\mathrm{cl}_Z(F)) = \mathrm{cl}_X(X_0 \cap (g \circ f)^{-1}(F)) = \mathrm{cl}_X(X_0 \cap f^{-1}(g^{-1}(F)))$ and $(D^t(f) \circ D^t(g))(\mathrm{cl}_Z(F)) = D^t(f)(\mathrm{cl}_Y(Y_0 \cap g^{-1}(F))) = \mathrm{cl}_X(X_0 \cap f^{-1}(Y_0 \cap g^{-1}(F))) = \mathrm{cl}_X(X_0 \cap f^{-1}(Y_0) \cap f^{-1}(g^{-1}(F))) = \mathrm{cl}_X(X_0 \cap f^{-1}(g^{-1}(F))) = D^t(g \circ f)(\mathrm{cl}_Z(F))$. So, D^t is a contravariant functor.

For every Boolean algebra A, set

$$D^a(A) = (X, X_0),$$

where (X, X_0) is the canonical Stone 2-space of the Boolean algebra A (see Definition 8.5(a)). Then Proposition 8.6 implies that $D^a(A) \in |\mathbf{2Stone}|$.

Let $\varphi \in \mathbf{Bool}(A, B)$. Let $D^a(A) = (X, X_0)$ and $D^a(B) = (Y, Y_0)$. Then we define the map

$$D^a(\varphi) : D^a(B) \longrightarrow D^a(A)$$

by the formula

(23) $D^a(\varphi)(\Gamma) = \varphi^{-1}(\Gamma), \quad \forall \Gamma \in Y.$

Set $f_\varphi = D^a(\varphi)$. Since every grill of a Boolean algebra B' is a union of ultrafilters of B' and every union of ultrafilters of B' is a grill of B' (see, e.g., [5, Corollary 3.1]), and the inverse image of an ultrafilter by a Boolean homomorphism between two Boolean algebras is again an ultrafilter, we get that $\forall \Gamma \in Y$, $f_\varphi(\Gamma) \in X$, i.e. $f_\varphi : Y \longrightarrow X$.

We will show that f_φ is a continuous function. Let $a \in A$. Then $g_A(a) = \{\Gamma \in X \mid a \in \Gamma\}$ is a basic closed subset of X (see Definition 8.5(a)). We will show that

(24) $f_\varphi^{-1}(g_A(a)) = g_B(\varphi(a))(= \{\Gamma' \in Y \mid \varphi(a) \in \Gamma'\})$.

Indeed, let $\Gamma' \in f_\varphi^{-1}(g_A(a))$. Then $f_\varphi(\Gamma') \in g_A(a)$. Thus $a \in \varphi^{-1}(\Gamma')$, i.e. $\varphi(a) \in \Gamma'$. So, $\Gamma' \in g_B(\varphi(a))$. Hence $f_\varphi^{-1}(g_A(a)) \subseteq g_B(\varphi(a))$. Conversely, let $\Gamma' \in g_B(\varphi(a))$, i.e. $\varphi(a) \in \Gamma'$. Then $a \in \varphi^{-1}(\Gamma') = f_\varphi(\Gamma')$. Hence $f_\varphi(\Gamma') \in g_A(a)$. Then $\Gamma' \in f_\varphi^{-1}(g_A(a))$. So, $f_\varphi^{-1}(g_A(a)) \supseteq g_B(\varphi(a))$. Thus the equation (24) is verified and we get that f_φ is a continuous function.

Let us now show that $f_\varphi(Y_0) \subseteq X_0$. Let $u' \in Y_0$. Then $u' \in Ult(B)$. Hence $f_\varphi(u') = \varphi^{-1}(u') \in Ult(A) = X_0$. Therefore, $f_\varphi(Y_0) \subseteq X_0$. So,

$$D^a(\varphi) \in \mathbf{2Stone}(D^a(B), D^a(A)).$$

Clearly, for every Boolean algebra B, $D^a(id_B) = id_{D^a(B)}$. Let $\varphi \in \mathbf{Bool}(A, B)$ and $\psi \in \mathbf{Bool}(B, B')$. Let $f_\varphi = D^a(\varphi)$, $f_\psi = D^a(\psi)$ and $D^a(B') = (Z, Z_0)$. Then, for every $\Gamma \in Z$, we have that $D^a(\psi \circ \varphi)(\Gamma) = (\psi \circ \varphi)^{-1}(\Gamma) = \varphi^{-1}(\psi^{-1}(\Gamma)) = f_\varphi(f_\psi(\Gamma)) = (D^a(\varphi) \circ D^a(\psi))(\Gamma)$. We get that D^a is a contravariant functor.

Let $(X, X_0) \in |\mathbf{2Stone}|$. Then $D^t(X, X_0) = RC(X, X_0)$. Set $B = RC(X, X_0)$. Let $D^a(B) = (Y, Y_0)$. Then $Y = Grills(B)$ and $Y_0 = Ult(B)$. By [5, Proposition 4.1(ii)], if $x \in X$ then $\sigma_x \in Clans(RC(X), C_X)$. Using Lemma 8.15, we get that, for every $x \in X$, $\sigma_x^B \in Clans(B, C_{(X,X_0)})$. According to Proposition 8.3(a), $C_{(X,X_0)} = \rho_l^B$ (see Example 2.4 for ρ_l^B). Hence, by [5, Example 3.1], $Clans(B, C_{(X,X_0)}) = Grills(B)$. Therefore, for every $x \in X$, $\sigma_x^B \in Grills(B)$. So, the following map is well-defined:

(25) $t_{(X,X_0)} : (X, X_0) \longrightarrow D^a(D^t(X, X_0)), \quad x \mapsto \sigma_x^{RC(X,X_0)}$,

and we will show that it is a homeomorphism. We start by proving that $t_{(X,X_0)}$ is a surjection. Let $\Gamma \in Y$. Then $\Gamma \in Grills(B)$. Using Lemma 4.1, we get that $\Gamma' = r_{X_0,X}(\Gamma) \in Grills(CO(X_0))$. Hence, by (S2S4), there exists $x \in X$ such that $\Gamma' = \Gamma_{x,X_0}$. Since, by Lemma 4.1, $\Gamma = e_{X_0,X}(\Gamma')$, we get that $\Gamma = \sigma_x^B = t_{(X,X_0)}(x)$. So, $t_{(X,X_0)}$ is a surjection. For showing that $t_{(X,X_0)}$ is a injection, let $x, y \in X$ and $x \neq y$. Since X is a T_0-space, there exists an open subset U of X such that $|U \cap \{x, y\}| = 1$. We can suppose, without loss of generality, that $x \in U$ and $y \notin U$. Since B is a closed base of X, there exists $F \in B$ such that $x \in X \setminus F \subseteq U$. Then $y \in F$ and $x \notin F$. Hence $F \in \sigma_y^B$ and $F \notin \sigma_x^B$, i.e. $t_{(X,X_0)}(x) \neq t_{(X,X_0)}(y)$. So, $t_{(X,X_0)}$ is a injection. Thus $t_{(X,X_0)}$ is a bijection. We will now prove that $t_{(X,X_0)}$ is a continuous map. We have that the family $\{g_B(F) = \{\Gamma \in Y \mid F \in \Gamma\} \mid F \in B\}$ is a closed base of Y. Let $F \in B$. We will show that

(26) $t_{(X,X_0)}^{-1}(g_B(F)) = F$.

Let $x \in F$. Set $t_{(X,X_0)}(x) = \Gamma$. Then $\Gamma = \sigma_x^B$. Since $F \in \Gamma$, we get that $\Gamma \in g_B(F)$. Thus $t_{(X,X_0)}(F) \subseteq g_B(F)$, i.e. $F \subseteq t^{-1}_{(X,X_0)}(g_B(F))$. Conversely, let $x \in t^{-1}_{(X,X_0)}(g_B(F))$. Set $\Gamma = t_{(X,X_0)}(x)$. Then $\Gamma \in g_B(F)$. Hence $F \in \Gamma$. Since $\Gamma = \sigma_x^B$, we get that $x \in F$. Hence $F \supseteq t^{-1}_{(X,X_0)}(g_B(F))$. So, $F = t^{-1}_{(X,X_0)}(g_B(F))$. This shows that $t_{(X,X_0)}$ is a continuous map. For showing that $t^{-1}_{(X,X_0)}$ is a continuous map, let $F \in B$. Using (26) and the fact that $t_{(X,X_0)}$ is a bijection, we get that $t_{(X,X_0)}(F) = g_B(F)$. Hence $(t^{-1}_{(X,X_0)})^{-1}(F) = g_B(F)$. This shows that $t^{-1}_{(X,X_0)}$ is a continuous map. So, $t_{(X,X_0)}$ is a homeomorphism.

We will now show that $t_{(X,X_0)}(X_0) = Y_0$. Let $x \in X_0$. Set $\Gamma = t_{(X,X_0)}(x)$. Then $\Gamma = \sigma_x^B$ and $r_{X_0,X}(\Gamma) = \{F \in CO(X_0) \mid x \in F\} = u_x^{X_0} \in Ult(CO(X_0)$. Then, by Lemma 4.1, $\Gamma = e_{X_0,X}(u_x^{X_0}) \in Ult(B) = Y_0$. Hence $t_{(X,X_0)}(X_0) \subseteq Y_0$. Let now $\Gamma \in Y_0$. Then $\Gamma \in Ult(B)$ and thus $u = r_{X_0,X}(\Gamma) \in Ult(CO(X_0))$. Clearly, there exist $x \in X_0$ such that $u = u_x^{X_0}$. Then $\Gamma = e_{X_0,X}(u_x^{X_0}) = \sigma_x^B = t_{(X,X_0)}(x)$. Therefore, $t_{(X,X_0)}(X_0) \supseteq Y_0$. We have proved that $t_{(X,X_0)}(X_0) = Y_0$. So, $t_{(X,X_0)}$ is a **2Stone**-isomorphism.

Let B be a Boolean algebra and let us set $(X, X_0) = D^a(B)$. Then $D^t(X, X_0) = RC(X, X_0)$ and, using Theorem 8.7(a), we get that the map

$$g_B : B \longrightarrow RC(X, X_0), \quad a \mapsto g_B(a) = \{\Gamma \in X \mid a \in \Gamma\},$$

is a Boolean isomorphism.

We will now show that

$$t : Id_{\mathbf{2Stone}} \longrightarrow D^a \circ D^t,$$

defined by $t(X, X_0) = t_{(X,X_0)}, \quad \forall (X, X_0) \in |\mathbf{2Stone}|$, is a natural isomorphism.

Let $f \in \mathbf{2Stone}((X, X_0), (Y, Y_0))$ and $\hat{f} = D^a(D^t(f))$. We have to show that $\hat{f} \circ t_{(X,X_0)} = t_{(Y,Y_0)} \circ f$. Set $\varphi_f = D^t(f)$, $A = RC(X, X_0)$ and $B = RC(Y, Y_0)$. Let $x \in X$. Then

$$(t_{(Y,Y_0)} \circ f)(x) = t_{(Y,Y_0)}(f(x)) = \sigma^B_{f(x)} = \{F \in B \mid f(x) \in B\}.$$

Further, $\hat{f}(t_{(X,X_0)}(x)) = \hat{f}(\sigma_x^A) = \varphi_f^{-1}(\sigma_x^A)$. Set $\Gamma' = \varphi_f^{-1}(\sigma_x^A)$. Then $\Gamma' = \{G \in B \mid \varphi_f(G) \in \sigma_x^X\} = \{\mathrm{cl}_Y(G_0) \mid G_0 \in CO(Y_0), x \in \mathrm{cl}_X(X_0 \cap f^{-1}(G_0))\}$. Let $G_0 \in CO(Y_0)$ and $\mathrm{cl}_Y(G_0) \in \Gamma'$. Then $x \in \mathrm{cl}_X(X_0 \cap f^{-1}(G_0))$ and thus $f(x) \in f(\mathrm{cl}_X(X_0 \cap f^{-1}(G_0)) \subseteq \mathrm{cl}_Y(f(X_0 \cap f^{-1}(G_0))) \subseteq \mathrm{cl}_Y(G_0)$. Therefore,

$$\Gamma' \subseteq \sigma^B_{f(x)}.$$

We have that $\Gamma' \in Grills(B)$. Hence $\Gamma'_r = r_{Y_0,Y}(\Gamma') \in Grills(CO(Y_0))$. Thus, by (S2S4), there exists $y \in Y$ such that $\Gamma'_r = \Gamma_{y,Y_0}$. Then

$$\Gamma' = \sigma_y^B.$$

Since $\Gamma' \subseteq \sigma^B_{f(x)}$, we get, by Lemma 8.13, that $\nu^B_{f(x)} \subseteq \Gamma'$. According to [5, Proposition 4.1], $\nu^B_{f(x)}$ is a filter of B. Hence, by Lemma 3.6, there exists an ultrafilter u of B such that $\nu^B_{f(x)} \subseteq u \subseteq \Gamma'$. Then $u \subseteq \sigma^B_y$ and since u is a grill of B, Lemma 8.13 implies that $\nu^B_y \subseteq u$. So, we obtained that $\nu^B_{f(x)} \cup \nu^B_y \subseteq u \subseteq \Gamma'$. Then, for every $F' \in \nu^B_{f(x)}$ and every $G' \in \nu^B_y$, we have that $F'.G' \neq 0$, i.e. $\mathrm{cl}_Y(\mathrm{int}_Y(F' \cap G')) \neq \emptyset$. Hence $\mathrm{int}_Y(F' \cap G') \neq \emptyset$ and thus $\mathrm{int}_Y(F') \cap \mathrm{int}_Y(G') \neq \emptyset$, for every $F' \in \nu^B_{f(x)}$ and every $G' \in \nu^B_y$. Since Y is a T_0-space, using Lemma 8.14, we get that $y = f(x)$. Therefore $\Gamma' = \sigma^B_{f(x)}$. Thus $\hat{f} \circ t_{(X,X_0)} = t_{(Y,Y_0)} \circ f$ and hence t is a natural isomorphism.

Finally, we will prove that

$$g : Id_{\mathbf{Bool}} \longrightarrow D^t \circ D^a, \text{ where } g(A) = g_A, \quad \forall A \in |\mathbf{Bool}|,$$

is a natural isomorphism.

Let $\varphi \in \mathbf{Bool}(A, B)$ and $\hat{\varphi} = D^t(D^a(\varphi))$. We have to prove that $g_B \circ \varphi = \hat{\varphi} \circ g_A$. Set $f = D^a(\varphi)$, $(X, X_0) = D^a(A)$ and $(Y, Y_0) = D^a(B)$. Then $\hat{\varphi} = D^t(f)(= \varphi_f)$. Let $a \in A$. Then $g_B(\varphi(a)) = \{\Gamma' \in Y \mid \varphi(a) \in \Gamma'\}$. Further, using (14), we get that

$$g_B(\varphi(a)) = \mathrm{cl}_Y(s_B(\varphi(a))) \quad \text{and} \quad g_A(a) = \mathrm{cl}_X(s_A(a)).$$

Thus

$$\hat{\varphi}(g_A(a)) = \mathrm{cl}_Y(Y_0 \cap f^{-1}(s_A(a))).$$

Let $u' \in Y_0 \cap f^{-1}(s_A(a))$. Then $u' \in Ult(B)$ and $f(u') \in s_A(a)$. Hence $\varphi^{-1}(u') \in s_A(a) = \{u \in Ult(A) \mid a \in u\}$. Thus $a \in \varphi^{-1}(u')$, i.e. $\varphi(a) \in u'$. Therefore $u' \in s_B(\varphi(a))$. So, $Y_0 \cap f^{-1}(s_A(a)) \subseteq s_B(\varphi(a))$. Conversely, let $u' \in s_B(\varphi(a))$. Then $u' \in Y_0$ and $\varphi(a) \in u'$. Hence $a \in \varphi^{-1}(u') = f(u')$. Thus $f(u') \in s_A(a)$. Therefore, $u' \in Y_0 \cap f^{-1}(s_A(a))$. So, $Y_0 \cap f^{-1}(s_A(a)) \supseteq s_B(\varphi(a))$ and we get that $Y_0 \cap f^{-1}(s_A(a)) = s_B(\varphi(a))$. Hence $\hat{\varphi}(g_A(a)) = \mathrm{cl}_Y(s_B(\varphi(a))) = g_B(\varphi(a))$. So, g is a natural isomorphism.

We have proved that (D^t, D^a, g, t) is a duality between the categories **2Stone** and **Bool**. □

Definition 8.18. (a) Let **ECS** be the category whose objects are all extremally connected spaces and whose morphisms are all continuous maps between the objects of **ECS** which preserve u-points (i.e., for every $X, Y \in |\mathbf{ECS}|$, $f \in \mathbf{ECS}(X, Y)$ iff f is a continuous map and for every u-point $x \in X$, $f(x)$ is an u-point of Y).

(b) Let **CBool** be the full subcategory of the category **Bool**, whose objects are all complete Boolean algebras.

Remark 8.19. (a) Clearly, **ECS** is indeed a category;

(b) Note that, according to Proposition 7.16, every open map between two objects of the category **ECS** is an **ECS**-morphism.

Theorem 8.20. *The categories* **CBool** *and* **ECS** *are dually equivalent.*

Proof. Let **2CStone** be the full subcategory of the category **2Stone**, whose objects are all Stone 2-spaces (X, X_0) for which X_0 is extremally disconnected. We will first show that the categories **2CStone** and **ECS** are isomorphic. Let us define two (covariant) functors

$$E_1 : \mathbf{ECS} \longrightarrow \mathbf{2CStone} \quad \text{and} \quad E_2 : \mathbf{2CStone} \longrightarrow \mathbf{ECS}.$$

Let $X \in |\mathbf{ECS}|$ and $X_0 = \{x \in X \mid x \text{ is an u-point of } X\}$. Then, by Corollary 8.10, $(X, X_0) \in |\mathbf{2CStone}|$ and we set

$$E_1(X) = (X, X_0).$$

Let $(X, X_0) \in |\mathbf{2CStone}|$. Then, by Lemma 8.11, $X \in |\mathbf{ECS}|$ and we set

$$E_2(X, X_0) = X.$$

Let $f \in \mathbf{ECS}(X, Y)$, $E_1(X) = (X, X_0)$ and $E_1(Y) = (Y, Y_0)$. Then, by the corresponding definitions, we get that f is continuous and $f(X_0) \subseteq Y_0$. Hence $f \in \mathbf{2CStone}(E_1(X), E_1(Y))$ and we set

$$E_1(f) = f.$$

Let $f \in \mathbf{2CStone}((X, X_0), (Y, Y_0))$. Then, by Lemma 8.11 and Corollary 8.10, we get that $X_0 = \{x \in X \mid x \text{ is an u-point of } X\}$ and $Y_0 = \{y \in Y \mid y \text{ is an u-point of } Y\}$. Since $f(X_0) \subseteq Y_0$, we obtain that $f \in \mathbf{ECS}(X, Y) = \mathbf{ECS}(E_2(X, X_0), E_2(Y, Y_0))$ and we set

$$E_2(f) = f.$$

Obviously, E_1 and E_2 are functors. If $X \in |\mathbf{ECS}|$ then $E_2(E_1(X)) = E_2(X, X_0) = X$. If $(X, X_0) \in |\mathbf{2CStone}|$ then $E_1(E_2(X, X_0)) = E_1(X)$. Using again Lemma 8.11 and Corollary 8.10, we get that $E_1(X) = (X, X_0)$. Hence $E_1(E_2(X, X_0)) = (X, X_0)$. Now it becomes obvious that $E_1 \circ E_2 = Id_{\mathbf{2CStone}}$ and $E_2 \circ E_1 = Id_{\mathbf{ECS}}$. So, the categories **ECS** and **2CStone** are isomorphic. Let EDStone be the class of all extremally disconnected Stone spaces. Then, using the Stone Theorem that $S(|\mathbf{CBool}|) =$ EDStone, we get that the restrictions $D^a_{|\mathbf{CBool}}$ and $D^t_{|\mathbf{2CStone}}$ of the duality functors D^a and D^t defined in the proof of Theorem 8.17, are duality functors between the categories **CBool** and **2CStone**. Setting

$$(27) \quad D^a_c = E_2 \circ D^a_{|\mathbf{CBool}} \quad \text{and} \quad D^t_c = D^t_{|\mathbf{2CStone}} \circ E_1,$$

we obtain that

$$D^a_c : \mathbf{CBool} \longrightarrow \mathbf{ECS} \quad \text{and} \quad D^t_c : \mathbf{ECS} \longrightarrow \mathbf{CBool}$$

are duality functors. □

9 On a class of compact T_0 extensions

Definition 9.1. An *extension* of a space X is a pair (Y, f), where Y is a space and $f : X \longrightarrow Y$ is a dense embedding of X into Y.

Two extensions $(Y_i, f_i),\ i = 1, 2$, of X are called *isomorphic* (or *equivalent*) if there exists a homeomorphism $\varphi : Y_1 \longrightarrow Y_2$ such that $\varphi \circ f_1 = f_2$. Clearly, the relation of isomorphism is an equivalence in the class of all extensions of X; the equivalence class of an extension (Y, f) of X will be denoted by $[(Y, f)]$.

We write
$$(Y_1, f_1) \leq (Y_2, f_2)$$
and say that the extension (Y_2, f_2) is *projectively larger* than the extension (Y_1, f_1) if there exists a continuous mapping $f : Y_2 \longrightarrow Y_1$ such that $f \circ f_2 = f_1$. This relation is a *preorder* (i.e., it is reflexive and transitive). Setting for every two extensions $(Y_i, f_i),\ i = 1, 2$, of a space X, $[(Y_1, f_1)] \leq [(Y_2, f_2)]$ iff $(Y_1, e_1) \leq (Y_2, e_2)$, we obtain a well-defined relation on the class of all, up to equivalence, extensions of X; obviously, it is also a preorder (see, e.g., [2]).

We write
$$(Y_1, f_1) \leq_{in} (Y_2, f_2)$$
and say that the extension (Y_2, f_2) is *injectively larger* than the extension (Y_1, f_1) if there exists a continuous mapping $f : Y_1 \longrightarrow Y_2$ such that $f \circ f_1 = f_2$ and f is a homeomorphism from Y_1 to the subspace $f(Y_1)$ of Y_2. This relation is a preorder. Setting for every two extensions $(Y_i, f_i),\ i = 1, 2$, of a space X, $[(Y_1, f_1)] \leq_{in} [(Y_2, f_2)]$ iff $(Y_1, e_1) \leq_{in} (Y_2, e_2)$, we obtain a well-defined relation on the class of all, up to equivalence, extensions of X; obviously, it is also a preorder (see, e.g., [2]).

Notation 9.2. Let Y be a space. We will denote by $CSR(Y)$ (resp., by $CCSR(Y)$) the class of all, up to equivalence, (connected) C-semiregular extensions of Y.

Recall that if B is a Boolean algebra, then we denote by $CRel(B)$ (resp., $CCRel$) the set of all (connected) contact relations on B. We define a relation " $\leq$" on the set $CRel(B)$ setting, for any $C_1, C_2 \in CRel(B)$, $C_1 \leq C_2 \iff C_1 \supseteq C_2$. We will denote again by " $\leq$" the restriction of the relation " $\leq$" to the set $CCRel$.

Theorem 9.3. *Let Y be an extremally disconnected compact Hausdorff space and $B = RC(Y)$. Then the ordered sets $(CRel(B), \leq)$ and $(CSR(Y), \leq)$, as well as the ordered sets $(CRel(B), \subseteq)$ and $(CSR(Y), \leq_{in})$, are isomorphic (see Definition 9.1 for the relations " $\leq$" and " $\leq_{in}$" on $CSR(Y)$). Also, the ordered sets $(CCRel(B), \leq)$ and $(CCSR(Y), \leq)$, as well as the ordered sets $(CCRel(B), \subseteq)$ and $(CCSR(Y), \leq_{in})$, are isomorphic.*

Proof. Let (X, f) be a C-semiregular extensions of Y. Set $X' = f(Y)$. Then, clearly, the map
$$e : (RC(X'), \delta_{(X,X')}) \longrightarrow (RC(X), C_X), \quad F \mapsto \mathrm{cl}_X(F),$$
is a CA-isomorphism (note that $RC(X') = CO(X')$). For every $F, G \in B$, set

(28) $FC_{(X,f)}G \iff \mathrm{cl}_X(f(F)) \cap \mathrm{cl}_X(f(G)) \neq \emptyset,$

i.e., $FC_{(X,f)}G \iff f(F)\delta_{(X,X')}f(G)$. Then, obviously, $(B, C_{(X,f)})$ is a contact algebra. Set

$$\varphi(X,f) = (B, C_{(X,f)}).$$

Clearly, two equivalent extension of Y define two coinciding contact relations on the Boolean algebra B. Thus we have that $\varphi([(X,f)]) = (B, C_{(X,f)})$ and, for simplicity, we will denote by the same letter φ the induced map on the set of equivalence classes of the C-semiregular extensions of Y.

Conversely, let C be a contact relation on the Boolean algebra B and let $(\widehat{X}, \widehat{X}_0)$ be the canonical 2-contact space of the complete contact algebra (B, C) (see Definition 7.2(c)). Then, by the definition of the space $\widehat{X}_0$ and the Stone Representation Theorem, we have that the map

$$\widehat{f} : Y \longrightarrow \widehat{X}, \quad y \mapsto u_y,$$

(see (6) for the notation u_y) is a homeomorphic embedding and $\widehat{f}(Y) = \widehat{X}_0$. Hence, $(\widehat{X}, \widehat{f})$ is an extension of the space Y. Using Lemma 7.22, we get that $\widehat{X}$ is a C-semiregular space. So, $(\widehat{X}, \widehat{f})$ is a C-semiregular extension of the space Y. Set

$$\psi(B,C) = (\widehat{X}, \widehat{f}).$$

Let (X, f_0) be a C-semiregular extensions of Y, $(B,C) = \varphi(X, f_0)$ and $(\widehat{X}, \widehat{f_0}) = \psi(B,C)$. We will show that (X, f_0) and $(\widehat{X}, \widehat{f_0})$ are isomorphic extensions of Y. As we have already seen, the map

$$e : (RC(f_0(Y)), \delta_{(X,f_0(Y))}) \longrightarrow (RC(X), C_X), \quad G \mapsto \mathrm{cl}_X(G), \quad \text{is a CA-isomorphism.}$$

Clearly, the map

$$\gamma^0_{f_0} : (B,C) \longrightarrow (RC(f_0(Y)), \delta_{(X,f_0(Y))}), \quad F \mapsto f_0(F), \quad \text{is a CA-isomorphism.}$$

Set $\gamma^0 = e \circ \gamma^0_{f_0}$. Then

$$\gamma^0 : (B,C) \longrightarrow (RC(X), C_X), \quad F \mapsto \mathrm{cl}_X(f_0(F)), \quad \text{is a CA-isomorphism.}$$

Thus the map

$$\gamma' : Clans(B,C) \longrightarrow Clans(RC(X), C_X), \quad \Gamma \mapsto \gamma^0(\Gamma), \quad \text{is a bijection.}$$

Since X is a C-semiregular space, Lemma 7.10 implies that the map

$$\kappa : X \longrightarrow Clans(RC(X), C_X), \quad x \mapsto \sigma_x, \quad \text{is a bijection.}$$

Let $\lambda = \kappa^{-1}$. Then we get that the map

$$f : \widehat{X} \longrightarrow X, \quad \Gamma \mapsto \lambda(\gamma'(\Gamma)), \quad \text{is a bijection}$$

(i.e., we set $f = \lambda \circ \gamma'$). We will show that $f \circ \widehat{f_0} = f_0$. Indeed, let $y \in Y$. Then $\widehat{f_0}(y) = u_y \in \widehat{X}$ and $f(\widehat{f_0}(y)) = f(u_y) = \lambda(\gamma'(u_y))$. Set $x = \lambda(\gamma'(u_y))$. Then

$\kappa(x) = \gamma^0(u_y)$, i.e., $\sigma_x = \{\mathrm{cl}_X(f_0(F)) \mid F \in B, y \in F\} = \{\mathrm{cl}_X(f_0(F)) \mid F \in B, f_0(y) \in f_0(F)\} = \{\mathrm{cl}_X(G) \mid G \in RC(f_0(Y)), f_0(y) \in G\} = e(u_{f_0(y)})$. Hence

(29) $e(u_{f_0(y)}) = \sigma_x$ and, thus, $r(\sigma_x) = u_{f_0(y)}$.

Suppose that $x \neq f_0(y)$. Since X is T_0 and semiregular, we get that there exists $F \in RC(X)$ such that $|F \cap \{x, f_0(y)\}| = 1$. If $x \in F$, then $f_0(y) \notin F$. Thus $F \in \sigma_x$ and $F \cap f_0(Y) \notin u_{f_0(y)}$. Since $r(F) = F \cap f_0(Y)$, we get a contradiction (see (29)). If $f_0(y) \in F$, then $x \notin F$. Thus $F \notin \sigma_x$ and $F \cap f_0(Y) \in u_{f_0(y)}$. Since $e(F \cap f_0(Y)) = F$ (by Lemma 4.1), we get a contradiction (see again (29)). Hence $x = f_0(y)$. Therefore,

$$f \circ \widehat{f_0} = f_0.$$

We will now show that f is a homeomorphism. Let $F \in B$. Then, using the notation of Definition 5.1, we obtain that $f(g_B(F)) = f(\{\Gamma \in \widehat{X} \mid F \in \Gamma\}) = \{f(\Gamma) \mid F \in \Gamma\} = \{f(\Gamma) \mid \gamma^0(F) \in \gamma^0(\Gamma)\} = \{f(\Gamma) \mid \mathrm{cl}_X(f_0(F)) \in \gamma'(\Gamma)\} = \{\lambda(\gamma'(\Gamma)) \mid \mathrm{cl}_X(f_0(F)) \in \kappa(\lambda(\gamma'(\Gamma)))\} = \{f(\Gamma) \mid \mathrm{cl}_X(f_0(F)) \in \sigma_{f(\Gamma)}\} = \{f(\Gamma) \mid f(\Gamma) \in \mathrm{cl}_X(f_0(F)) = \mathrm{cl}_X(f_0(F))$. So, $f(g_B(F)) = \mathrm{cl}_X(f_0(F))$, for every $F \in B$. Since f is a bijection, we also get that for every $F \in B$, $f^{-1}(\mathrm{cl}_X(f_0(F))) = g_B(F)$. Now, using the fact that $\{\mathrm{cl}_X(f_0(F)) \mid F \in B\} = RC(X)$ and that $RC(X)$ and $\{g_B(F)\ F \in B\}$ are closed bases of, respectively, X and $\widehat{X}$, we get that f is a homeomorphism. Therefore,

$$\psi(\varphi((X, f_0))) \text{ is isomorphic to } (X, f_0).$$

Let now C be a contact relation on the Boolean algebra B, $\psi(B, C) = (\widehat{X}, \widehat{f})$ and $\varphi(\psi(B, C)) = (B, \widehat{C})$. We will show that $C \equiv \widehat{C}$. We have that for every $F, G \in B$, $F\widehat{C}G \iff \mathrm{cl}_{\widehat{X}}(\widehat{f}(F)) \cap \mathrm{cl}_{\widehat{X}}(\widehat{f}(G)) \neq \emptyset$. Recall that the set $\{h_B(H) \mid H \in B\}$, where $h_B(H) = \{\Gamma \in \widehat{X} \mid H \notin \Gamma\}$, is an open base of $\widehat{X}$. Let us show that if $H \in B$ and $\Gamma \in \widehat{X}$, then

(30) $\Gamma \in \mathrm{cl}_{\widehat{X}}(\widehat{f}(H)) \iff H \in \Gamma.$

Indeed, using the fact that Γ satisfies condition (Clan2) (see Definition 3.1), we get that $(\Gamma \in \mathrm{cl}_{\widehat{X}}(\widehat{f}(H))) \iff$ (for every $P \in B\backslash\Gamma$, $h_B(P) \cap \{u_y \mid y \in H\} \neq \emptyset$) $\iff$ (for every $P \in B\backslash\Gamma$, there exists $y \in H$ such that $P \notin u_y$) $\iff$ (for every $P \in B\backslash\Gamma$, there exists $y \in H$ such that $y \notin P$) $\iff$ (for every $P \in B\backslash\Gamma$, $H \not\subseteq P$) $\iff$ $(H \in \Gamma)$. So, (30) is verified. Now, we get, using (30) and Lemma 3.5(c), that for every $F, G \in B$, $F\widehat{C}G \iff \exists\Gamma \in (\mathrm{cl}_{\widehat{X}}(\widehat{f}(F)) \cap \mathrm{cl}_{\widehat{X}}(\widehat{f}(G))) \iff (\exists\Gamma \in \widehat{X})(F, G \in \Gamma) \iff FCG$. Therefore $C \equiv \widehat{C}$.

So, the correspondence φ is a bijection between the set of all, up to equivalence, C-semiregular extensions of Y and the set of all contact relations on the Boolean algebra B. Let us show that φ is an isomorphism.

Let (X_i, f_i), $i = 1, 2$, be two C-semiregular extensions of Y and $[(X_1, f_1)] \geq [(X_2, f_2)]$ or $[(X_1, f_1)] \leq_{in} [(X_2, f_2)]$. Then there exists a continuous mapping $f : X_1 \longrightarrow X_2$ such that $f \circ f_1 = f_2$. Set $(B, C_i) = \varphi(X_i, f_i)$, $i = 1, 2$. Let $F, G \in B$ and $F(-C_2)G$. Then, by (28), $\mathrm{cl}_{X_2}(f_2(F)) \cap \mathrm{cl}_{X_2}(f_2(G)) = \emptyset$. Hence $f^{-1}(\mathrm{cl}_{X_2}(f_2(F))) \cap$

$f^{-1}(\mathrm{cl}_{X_2}(f_2(G))) = \emptyset$. Since $f(\mathrm{cl}_{X_1}(f_1(F))) \subseteq \mathrm{cl}_{X_2}(f(f_1(F))) = \mathrm{cl}_{X_2}(f_2(F))$, we get that $\mathrm{cl}_{X_1}(f_1(F)) \subseteq f^{-1}(\mathrm{cl}_{X_2}(f_2(F)))$. Analogously, $\mathrm{cl}_{X_1}(f_1(G)) \subseteq f^{-1}(\mathrm{cl}_{X_2}(f_2(G)))$. Thus $\mathrm{cl}_{X_1}(f_1(F)) \cap \mathrm{cl}_{X_1}(f_1(G)) = \emptyset$. Using once more (28), we get that $F(-C_1)G$. Therefore $C_1 \subseteq C_2$, i.e., $C_1 \geq C_2$.

Conversely, let C_1 and C_2 be two contact relations on B and $C_1 \geq C_2$, i.e., $C_1 \subseteq C_2$. Set $(X_i, f_i) = \psi(B, C_i)$, $i = 1, 2$. Then $\varphi^{-1}(B, C_i) = [(X_i, f_i)]$. By the definition of the map ψ, we have that for $i = 1, 2$, $X_i = Clans(B, C_i)$, the topology on X_i is generated by the closed base $\{\{\Gamma \in Clans(B, C_i) \mid F \in \Gamma\} \mid F \in B\}$, and $f_i(y) = u_y$, for every $y \in Y$. Since $C_1 \subseteq C_2$, we get that $Clans(B, C_1) \subseteq Clans(B, C_2)$. Now we define

$$f : X_1 \longrightarrow X_2, \quad \Gamma \mapsto \Gamma.$$

Then, for every $y \in Y$, $f(f_1(y)) = f(u_y) = u_y = f_2(y)$. Hence, $f \circ f_1 = f_2$. Since, for every $F \in B$, $f^{-1}(\{\Gamma \in X_2 \mid F \in \Gamma\}) = \{\Gamma \in X_1 \mid F \in \Gamma\}$, we get that f is a continuous map. Therefore, $[(X_1, f_1)] \geq [(X_2, f_2)]$. Let us note that f is even an embedding. Indeed, f is injective and for every $F \in B$, $f(\{\Gamma \in X_1 \mid F \in \Gamma\}) = f(X_1) \cap \{\Gamma \in X_2 \mid F \in \Gamma\}$. Hence, if $f' : X_1 \longrightarrow f(X_1)$ is the restriction of f and $g' = (f')^{-1} : f(X_1) \longrightarrow X_1$, then, for every $F \in B$, $(g')^{-1}(\{\Gamma \in X_1 \mid F \in \Gamma\}) = f(X_1) \cap \{\Gamma \in X_2 \mid F \in \Gamma\}$. So that, f is an embedding. Hence, $[(X_1, f_1)] \leq_{in} [(X_2, f_2)]$.

Therefore, φ is an isomorphism between the ordered sets $(CSR(Y), \leq)$ and $(CRel(B), \leq)$, and also between the ordered sets $(CSR(Y), \leq_{in})$ and $(CRel(B), \subseteq)$. Clearly, this implies that $CSR(Y)$ is a set and the preorders " $\leq$" and " $\leq_{in}$" on $CSR(Y)$, defined in Definition 9.1, are, in fact, orders.

Now, the assertions about connected contact relations on B follow immediately. □

Noting that the Stone space of a Boolean algebra B is extremally disconnected iff B is complete (see, e.g., [15]), the above theorem can be reformulated as follows:

Theorem 9.4. *Let B be a complete Boolean algebra and $Y = S(B)$ be its Stone space. Then the ordered sets $(CRel(B), \leq)$ and $(CSR(Y), \leq)$, as well as the ordered sets $(CRel(B), \subseteq)$ and $(CSR(Y), \leq_{in})$, are isomorphic (see Definition 9.1 for the relations " $\leq$" and " $\leq_{in}$" on $CSR(Y)$). Also, the ordered sets $(CCRel(B), \leq)$ and $(CCSR(Y), \leq)$, as well as the ordered sets $(CCRel(B), \subseteq)$ and $(CCSR(Y), \leq_{in})$, are isomorphic.*

Corollary 9.5. *Let Y be an extremally disconnected compact Hausdorff space and $B = RC(Y)$. Then the ordered sets $(CSR(Y), \leq)$ and $(CSR(Y), \leq_{in})$ have largest and smallest elements. The largest (resp., the smallest) element of the ordered set $(CSR(Y), \leq)$ coincides with the smallest (resp., the largest) element of $(CSR(Y), \leq_{in})$. For the largest element $[(\gamma Y, \gamma_Y)]$ of the ordered set $(CSR(Y), \leq_{in})$, we have that γY is an extremally connected space (in fact, $\gamma Y = D_c^a(B)$ (see (27) for D_c^a)). Also, if (cY, c) is a C-semiregular extension of Y and cY is an extremally connected space then the C-semiregular extensions (cY, c) and $(\gamma Y, \gamma_Y)$ of Y are equivalent.*

Proof. Clearly, by Theorem 9.3, the smallest (resp., the largest) element of the ordered set $(CSR(Y), \leq)$ is the largest (resp., the smallest) element of the ordered set $(CSR(Y), \leq_{in})$. So that we will regard only the ordered set $(CSR(Y), \leq_{in})$. By Example 2.4, the ordered set $(CRel(B), \subseteq)$, where $B = RC(Y)(= CO(Y))$, has largest and smallest elements. Thus, by Theorem 9.3, the ordered set $(CSR(Y), \leq_{in})$ also has largest and smallest elements. The fact that it has smallest element follows also from Corollary 7.20: this is the equivalence class of the extension (Y, id_Y) of Y. It is also obvious that it corresponds to the contact relation ρ_s on B (see the formula (28)). For the largest element $[(\gamma Y, \gamma)]$ of the ordered set $(CSR(Y), \leq_{in})$, we have that the map γ is defined by the formula $\gamma_Y(y) = u_y$, for every $y \in Y$ (see (6) for the notation u_y), and $\gamma Y = Clans(B, \rho_l) = Grills(B) = D_c^a(B)$. Thus γY is an extremally connected space (see Theorem 8.20).

Let (cY, c) be a C-semiregular extension of Y and let cY be an extremally connected space. By Proposition 8.9(a), we have that the standard contact relation C_{cY} on $RC(cY)$ coincides with the largest contact relation $\rho_l^{RC(cY)}$ on the Boolean algebra $RC(cY)$. Then, using (28), we obtain that the contact relation $C_{(cY,c)}$ on the Boolean algebra B, corresponding to the C-semiregular extension (cY, c) of Y (see the proof of Theorem 9.3), coincides with the largest contact relation ρ_l^B on the Boolean algebra B. Therefore, (cY, c) corresponds to ρ_l^B; thus (cY, c) and $(\gamma Y, \gamma_Y)$ are equivalent C-semiregular extensions of Y. □

Theorem 9.6. *Let X and Y be two extremally disconnected compact Hausdorff spaces, (cX, c) be an arbitrary C-semiregular extension of X and $f : X \longrightarrow Y$ be a continuous map. Then there exists a continuous map $f' : cX \longrightarrow \gamma Y$ such that $\gamma_Y \circ f = f' \circ c$ (see Corollary 9.5 for $(\gamma Y, \gamma_Y)$) (i.e., supposing that c and γ_Y are the embedding maps of X and Y in, respectively, cX and γY, we get that f can be extended to a continuous map $f' : cX \longrightarrow \gamma Y$). In particular, every continuous map $f : X \longrightarrow Y$ can be "extended" to a continuous map $\gamma f : \gamma X \longrightarrow \gamma Y$ (i.e., $\gamma_Y \circ f = \gamma f \circ \gamma_X$).*

Proof. Since $(cX, c) \leq_{in} (\gamma X, \gamma_X)$ (see Corollary 9.5), we can regard cX as a subspace of γX. Thus, it is enough to prove only that there exists a continuous map $\gamma f : \gamma X \longrightarrow \gamma Y$ such that $\gamma_Y \circ f = \gamma f \circ \gamma_X$. Regard the Boolean algebras $A = RC(X)$ and $B = RC(Y)$. By the Stone Duality, the map

$$\varphi_f = S(f) : B \longrightarrow A, \quad G \mapsto f^{-1}(G),$$

is a Boolean homomorphism. Hence, using Corollary 9.5 and Theorem 8.20, we get that $D_c^a(\varphi_f) : \gamma X \longrightarrow \gamma Y$ is a continuous map. Set $\gamma f = D_c^a(\varphi_f)$. We will show that $\gamma_Y \circ f = \gamma f \circ \gamma_X$. Let $x \in X$. Set $y = f(x)$. Using Corollary 9.5, (23) and (27), we get that $\gamma f(\gamma_X(x)) = \gamma f(u_x^A) = (\varphi_f)^{-1}(u_x^A)$ and $\gamma_Y(f(x)) = \gamma_Y(y) = u_y^B$. So, we have to show that $u_y^B = (\varphi_f)^{-1}(u_x^A)$. Let $G \in B$. Then we have that $G \in (\varphi_f)^{-1}(u_x^A) \iff \varphi_f(G) \in u_x^A \iff f^{-1}(G) \in u_x^A \iff x \in f^{-1}(G) \iff f(x) \in G \iff y \in G$. Therefore, $u_y^B = (\varphi_f)^{-1}(u_x^A)$. Thus, $\gamma_Y \circ f = \gamma f \circ \gamma_X$. □

Theorem 9.7. *Let X be an extremally disconnected compact Hausdorff space, Z be an extremally connected space, (cX, c) be an arbitrary C-semiregular extension of X and $f : X \longrightarrow Z$ be a continuous map such that, for every $x \in X$, $f(x)$ is an u-point* *of Z. Then there exists a continuous map $f' : cX \longrightarrow Z$ such that $f = f' \circ c$ (i.e., supposing that c is the embedding map of X in cX, we get that f can be extended to a continuous map $f' : cX \longrightarrow Z$). In particular, every open map $f : X \longrightarrow Z$ can be "extended" to a continuous map $f' : cX \longrightarrow Z$ (i.e., $f = f' \circ c$).*

Proof. Set $Y = \{z \in Z \mid z \text{ is an u-point of } Z\}$. Then, by Corollary 8.10, Y is a dense extremally disconnected compact Hausdorff subspace of Z. Setting $i_Y : Y \longrightarrow Z$ to be the embedding of Y in Z, we get (by Proposition 8.9(b)) that (Z, i_Y) is a C-semiregular extension of Y. Moreover, Corollary 9.5 implies that (Z, i_Y) and $(\gamma Y, \gamma_Y)$ are equivalent C-semiregular extensions of Y. Since $f(X) \subseteq Y$, our assertion follows now from Theorem 9.6.

Finally, note that if f is an open map, then, by Proposition 7.14(a) and Proposition 7.16, we have that, for every $x \in X$, $f(x)$ is an u-point of Z . □

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