A note on the Erdős-Faber-Lovász Conjecture: quasigroups and complete digraphs*

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Abstract

A *decomposition* of a simple graph G is a pair (G, P) where P is a set of subgraphs of G, which partitions the edges of G in the sense that every edge of G belongs to exactly one subgraph in P. If the elements of P are induced subgraphs then the decomposition is denoted by $[G, P]$.

A *k-P-coloring* of a decomposition (G, P) is a surjective function that assigns to the edges of G a color from a k-set of colors, such that all edges of $H \in P$ have the same color, and, if $H_1, H_2 \in P$ with $V(H_1) \cap V(H_2) \neq \emptyset$ then $E(H_1)$ and $E(H_2)$ have different colors. The *chromatic index* $\chi'((G, P))$ of a decomposition (G, P) is the smallest number k for which there exists a k-P-coloring of (G, P).

The well-known Erdős-Faber-Lovász Conjecture states that any decomposition $[K_n, P]$ satisfies $\chi'([K_n, P]) \leq n$. We use quasigroups and complete digraphs to give a new family of decompositions that satisfy the conjecture.

1 Introduction

Erdős, Faber and Lovász, in 1972, conjectured the following (see [2]): "if $|A_i| = n$, $1 \leq i \leq n$, and $|A_i \cap A_j| \leq 1$, for $1 \leq i < j \leq n$, then one can color the elements of the union $\bigcup_{i=1}^{n} A_i$ by n colors, so that every set has elements of all the colors." This conjecture is called the Erdős-Faber-Lovász Conjecture (for short EFL), and this can be set in terms of decompositions (see [1, 3]).

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Conjecture 1.1. *If $[K_n, P]$ is a decomposition, then $\chi'([K_n, P]) \leq n$.*

In the following section we give a family of decompositions using finite quasigroups and complete digraphs satisfying Conjecture 1.1; this is a generalization of a previous result given in [1] and it is related with a result given in [3].

2 Quasigroups and digraphs

To begin with, we introduce definitions related to quasigroups, complete digraphs and linear-factorizations. A *digraph* D is a finite, non-empty set V (the *vertices* of D) together with a set A of ordered pairs of elements of V (the *arcs* of D). We denote by $|V|$ the *order* and by $|A|$ the *size* of D respectively.

A digraph D is called *symmetric* if whenever (u, v) is an arc of D then (v, u) is an arc of D –every graph can be interpreted as a symmetric digraph–. A *directed cycle* or a *d-gon* is a subdigraph with set of vertices $\{v_1, v_2, \dots, v_d\}$, such that their arcs are (v_d, v_1) and (v_i, v_{i+1}) for $i \in \{1, \dots, d-1\}$ and $d \geq 2$. A *loop* or a 1-*gon* is an arc joining a vertex with itself.

The *complete digraph* $\overrightarrow{K}^*_n$ has order n and size n^2 (n loops and $\binom{n}{2}$ 2-gons). A *linear-factor* of the complete digraph $\overrightarrow{K}^*_n$ is a subdigraph of order n and size n, such that it is a set of pairwise vertex-disjoint d-gons. A *linear-factorization* of $\overrightarrow{K}^*_n$ is a set of pairwise arc-disjoint linear-factors, such that these linear-factors induce a partition of the arcs, see Figure 1: c).

A *quasigroup* $(\mathcal{Q}_n, \cdot)$ is a set $\mathcal{Q}$ of n elements with a binary operation $\cdot$, such that for each x and y in $\mathcal{Q}$ there exist unique elements a and b in $\mathcal{Q}$ with $x \cdot a = y$ and $b \cdot x = y$.

Let $(\mathcal{Q}_n, \cdot)$ be a quasigroup and the complete digraph $\overrightarrow{K}^*_n$, such that its vertices are the elements of $\mathcal{Q}_n$. Afterwards, we color the arcs of $\overrightarrow{K}^*_n$ by n colors which are in a one-to-one correspondence with the elements of $\mathcal{Q}_n$ so that for any two vertices x and y in $\mathcal{Q}_n$ the arc (x, y) obtains the color corresponding to $a \in \mathcal{Q}_n$ for which $x \cdot a = y$ holds true. Then the resulting graph with the described coloring of arcs is called the *Cayley color graph* $C(\mathcal{Q}_n)$ of $\mathcal{Q}_n$. The Cayley color graph of a quasigroup is described in [4].

It is not hard to prove that the arcs colored by the same color in $C(\mathcal{Q}_n)$ induce a linear-factor of this digraph. An arc colored by the color corresponding color to some $a \in \mathcal{Q}_n$ outgoing from the vertex x leads into $x \cdot a$ in $C(\mathcal{Q}_n)$. The element $x \cdot a$ is exactly one for any x and any a of $\mathcal{Q}_n$.

Consequently, the Cayley color graph $C(\mathcal{Q}_n)$ can be considered as a linear-factorization $\mathcal{F}$ of $\overrightarrow{K}^*_n$ of n linear-factors. In [4] it was proved that any linear-factorization $\mathcal{F}$ of the complete digraph $\overrightarrow{K}^*_n$ and any one-to-one mapping of the vertex set of $\overrightarrow{K}^*_n$ onto the set of linear-factors of $\mathcal{F}$ determines a quasigroup $\mathcal{Q}_n$, such that the Cayley color graph $C(\mathcal{Q}_n)$ of $\mathcal{Q}_n$ can be considered $(\overrightarrow{K}^*_n, \mathcal{F})$, as described above.

Following, we relate the previous concepts with decompositions of complete graphs. Let $[K_n, P]$ be a decomposition P of K_n and let $\overrightarrow{K}_n$ be the symmetric

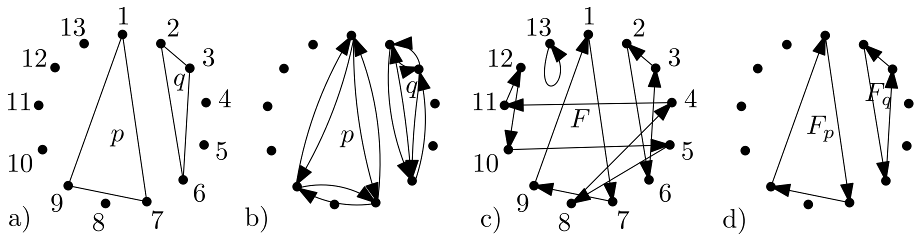

Figure 1: a) Two elements p and q of a decomposition of K_{13} into triangle arising from the cyclic Steiner System $STS(13)$. b) K_{13} as a symmetric digraph c) A linear-factor F for $n = 13$. The mapping $i \mapsto i + 1$ produces a linear-factorization. d) The restriction of F onto p and q.

complete digraph (without loops). We consider the decomposition $[\overrightarrow{K}_n, P]$ induced by $[K_n, P]$, that is, P is a set of subdigraphs of $\overrightarrow{K}_n$, which partitions the arcs of $\overrightarrow{K}_n$ in the sense that every arc of $\overrightarrow{K}_n$ belongs to exactly one subdigraph in P and every element of P is a symmetric complete subdigraph. The digraph $\overrightarrow{K}^*_n$ is $\overrightarrow{K}_n$ with the set L of n loops.

Now, we state and prove the main theorem:

Theorem 2.1. *Let $[\overrightarrow{K}_n, P]$ be a decomposition P of $\overrightarrow{K}_n$ arising from $[K_n, P]$ and let $(\overrightarrow{K}^*_n, \mathcal{F})$ be a linear-factorization $\mathcal{F}$ of $\overrightarrow{K}^*_n$. If there exists a function $h\colon P \to \mathcal{F}$, such that for any $p \in P$, $(A(p) \cup L) \cap A(h(p))$ is a linear-factor F_p of p^* –p with loops– and for any $p, q \in P$, $A(F_p) \cap A(F_q) = \emptyset$ then $\chi'([K_n, P]) \leq n$.*

Proof. Color the edges of an element p of P with $f(h(p))$ where f is a one-to-one mapping of a quasigroup $\mathcal{Q}$ onto the set of linear-factors of $\mathcal{F}$. The n-coloring is well-defined due to the fact that for any $p, q \in P$, $A(F_p) \cap A(F_q) = \emptyset$ and the result follows. □

We can explain Theorem 2.1 as following:

Let $(\overrightarrow{K}^*_n, \mathcal{F})$ be a linear-factorization $\mathcal{F}$ of $\overrightarrow{K}^*_n$. Then every decomposition P formed by complete subdigraphs obtained via some linear-factor f_0 of $\mathcal{F}$, meaning, the intersection of the arcs of $p \in P$ with the arcs of f_0 is a linear factor of p has a consequence that $\chi'([K_n, P]) \leq n$. Figure 1 illustrates Theorem 2.1 with an example for $n = 13$.

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