# A sum form functional equation on a closed domain and its role in information theory

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Abstract. This paper is devoted to finding the general solutions of the functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} h(p_i q_j) = \sum_{i=1}^{n} h(p_i) + \sum_{j=1}^{m} k_j(q_j) + \lambda \sum_{i=1}^{n} h(p_i) \sum_{j=1}^{m} k_j(q_j)$$

valid for all complete probability distributions $(p_1, \dots, p_n)$, $(q_1, \dots, q_m)$, $0 \leqslant p_i \leqslant 1$, $0 \leqslant q_j \leqslant 1$, $i = 1, \dots, n$; $j = 1, \dots, m$, $\sum_{i=1}^{n} p_i = 1$, $\sum_{j=1}^{m} q_j = 1$; $n \geqslant 3$, $m \geqslant 3$ fixed integers; $\lambda \in \mathbb{R}$, $\lambda \neq 0$ and the mappings $h : I \to \mathbb{R}$, $k_j : I \to \mathbb{R}$, $j = 1, \dots, m$; $I = [0, 1]$, $\mathbb{R}$ denoting the set of all real numbers. A special case of the above functional equation was treated earlier by L. Losonczi and Gy. Maksa.

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## 1. Introduction

Let $\Gamma_n = \{(p_1,\ldots,p_n) : 0 \leqslant p_i \leqslant 1, i = 1,\ldots,n; \sum_{i=1}^{n} p_i = 1\},\ n = 2,3,\ldots$ denote the set of all discrete $n$-component complete probability distributions with nonnegative elements. Let $\mathbb{R}$ denote the set of all real numbers and

$$\begin{aligned} \Delta &= \{(x,y) : 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1,\ 0 \leqslant x+y \leqslant 1\},\ \text{the unit triangle}\,; \\ I &= \{x \in \mathbb{R} : 0 \leqslant x \leqslant 1\} = [0,1]\ ;\ I_0 = \{x \in \mathbb{R} : 0 < x < 1\}\,. \end{aligned}$$

A mapping $a : I \to \mathbb{R}$ is said to be additive on $I$ if

$$a(x+y) = a(x) + a(y)$$

holds for all $(x,y) \in \Delta$. A mapping $A : \mathbb{R} \to \mathbb{R}$ is said to be additive on $\mathbb{R}$ if

$$A(x+y) = A(x) + A(y) \tag{1.1}$$

holds for all $x \in \mathbb{R}$, $y \in \mathbb{R}$.

It is known [2] that every mapping $a : I \to \mathbb{R}$, additive on the unit triangle $\Delta$, has a unique additive extension $A : \mathbb{R} \to \mathbb{R}$ in the sense that $A$ satisfies the equation (1.1) for all $x \in \mathbb{R}$, $y \in \mathbb{R}$.

A mapping $M : I \to \mathbb{R}$ is said to be multiplicative on $I$ if

$$M(0) = 0 \tag{1.2}$$

$$M(1) = 1 \tag{1.3}$$

and

$$M(pq) = M(p)\,M(q) \tag{1.4}$$

holds for all $p \in I_0$, $q \in I_0$.

The functional equation (see [1])

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} f(p_i) + \sum_{j=1}^{m} f(q_j) + \lambda \sum_{i=1}^{n} f(p_i) \sum_{j=1}^{m} f(q_j) \tag{1.5}$$

where $f : I \to \mathbb{R}$, $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$, $\lambda = 2^{1-\alpha} - 1 \neq 0$ is useful in characterizing the entropy of degree $\alpha$ (see [3]) defined as

$$H_n^\alpha(p_1, \ldots, p_n) = (1 - 2^{1-\alpha})^{-1} \left( 1 - \sum_{i=1}^{n} p_i^\alpha \right), \tag{1.6}$$

where $H_n^\alpha : \Gamma_n \to \mathbb{R}$, $n = 2, 3, \ldots$ and $0^\alpha := 0$, $\alpha \neq 1$, $\alpha \in \mathbb{R}$. For $\lambda \in \mathbb{R}$, $\lambda \neq 0$, the general solutions of (1.5), for fixed integers $n \geqslant 3$, $m \geqslant 3$ and all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$ have been obtained in [6]. A generalization of (1.5) is the following functional equation (see [5])

$$\sum_{i=1}^{n} \sum_{j=1}^{m} f_{ij}(p_i q_j) = \sum_{i=1}^{n} h_i(p_i) + \sum_{j=1}^{m} k_j(q_j) + \lambda \sum_{i=1}^{n} h_i(p_i) \sum_{j=1}^{m} k_j(q_j) \tag{1.7}$$

with $f_{ij} : I \to \mathbb{R}$, $h_i : I \to \mathbb{R}$, $k_j : I \to \mathbb{R}$, $i = 1, \ldots, n$; $j = 1, \ldots, m$. For fixed integers $n \geqslant 3$, $m \geqslant 3$ and all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$, the measurable (in the sense of Lebesgue) solutions of (1.7) have been obtained in (see [5], Theorem 6 on p-69) but it seems that the general solutions of (1.7), for fixed integers $n \geqslant 3$, $m \geqslant 3$ and all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$ are still not known. As mentioned in [5], equations like (1.7) arise while characterizing measures of information concerned with two probability distributions. In this paper, we study the equation

$$\sum_{i=1}^{n} \sum_{j=1}^{m} h(p_i q_j) = \sum_{i=1}^{n} h(p_i) + \sum_{j=1}^{m} k_j(q_j) + \lambda \sum_{i=1}^{n} h(p_i) \sum_{j=1}^{m} k_j(q_j) \tag{1.8}$$

where $h : I \to \mathbb{R}$, $k_j : I \to \mathbb{R}$, $j = 1, \ldots, m$; $\lambda \in \mathbb{R}$, $\lambda \neq 0$ and $n \geqslant 3$, $m \geqslant 3$ are fixed integers. The functional equation (1.8) is a special case of (1.7).

If we define $f : I \to \mathbb{R}$ and $g_j : I \to \mathbb{R}$, $j = 1, \ldots, m$ as (with $\lambda \neq 0$)

$$f(x) = x + \lambda\, h(x) \quad \text{and} \quad g_j(x) = x + \lambda\, k_j(x) \tag{1.9}$$

for all $x \in I$, then (1.8) reduces to the functional equation

$$\sum_{i=1}^{n} \sum_{j=1}^{m} f(p_i q_j) = \sum_{i=1}^{n} f(p_i) \sum_{j=1}^{m} g_j(q_j)\,. \tag{1.10}$$

Also, (1.9) and (1.10) yield (1.8). Thus, if the general solutions of (1.10), for fixed integers $n \geqslant 3$, $m \geqslant 3$ and all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$ are known;

the corresponding general solutions of (1.8), for fixed integers $n \geqslant 3$, $m \geqslant 3$ and all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$ can be determined with the aid of (1.9).

We would like to mention that, on open domain, namely when $f : I_0 \to \mathbb{R}$, $h : I_0 \to \mathbb{R}$, $g_j : I_0 \to \mathbb{R}$, $k_j : I_0 \to \mathbb{R}$, $j = 1, \ldots, m$, the general solutions of (1.8) and (1.10) for fixed integers $n \geqslant 3$, $m \geqslant 3$ and all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$ have been found in [4]. The object of this paper is to determine the general solutions of (1.8) and (1.10), on the closed domain, namely when $f : I \to \mathbb{R}$, $h : I \to \mathbb{R}$, $g_j : I \to \mathbb{R}$, $k_j : I \to \mathbb{R}$, $j = 1, \ldots, m$; for fixed integers $n \geqslant 3$, $m \geqslant 3$ and all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$. While investigating these solutions, the functional equation

$$\sum_{i=1}^{n} \sum_{j=1}^{m} \varphi(p_i q_j) = \sum_{i=1}^{n} \varphi(p_i) \sum_{j=1}^{m} \varphi(q_j) + m(n-1)\, \varphi(0) \sum_{i=1}^{n} \varphi(p_i) \tag{1.11}$$

arises with $\varphi : I \to \mathbb{R}$, $n \geqslant 3$, $m \geqslant 3$ fixed integers and $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$.

To deal with equations (1.8), (1.10) and (1.11), we need the results and methods from [5] and [6].

## 2. Some preliminary results

We require the following two results in sections 3 and 4.

**Result 1.** [6]. Let $k \geqslant 3$ be a fixed integer and $c$ be a given constant. Suppose that a mapping $\psi : I \to \mathbb{R}$ satisfies the functional equation

$$\sum_{i=1}^{k} \psi(p_i) = c \tag{2.1}$$

for all $(p_1, \ldots, p_k) \in \Gamma_k$. Then there exists an additive mapping $B : \mathbb{R} \to \mathbb{R}$ such that

$$\psi(p) = B(p) - \frac{1}{k}\, B(1) + \frac{c}{k} \tag{2.2}$$

for all $p \in I$.

**Result 2.** [5]. If the mappings $\psi_j : I \to \mathbb{R}$, $j = 1, \ldots, m$ satisfy the functional equation

$$\sum_{j=1}^{m} \psi_j(q_j) = 0 \tag{2.3}$$

for an arbitrary but fixed integer $m \geqslant 3$ and all $(q_1, \ldots, q_m) \in \Gamma_m$, then there exists an additive mapping $A : \mathbb{R} \to \mathbb{R}$ and the constants $c_j$ $(j = 1, \ldots, m)$ such that

$$\psi_j(p) = A(p) + c_j \tag{2.4}$$

for all $p \in I$ and $j = 1, \ldots, m$ with

$$A(1) + \sum_{j=1}^{m} c_j = 0\,. \tag{2.5}$$

## 3. The functional equation (1.11)

In this section, we prove:

**Theorem 1.** *Let $n \geqslant 3$, $m \geqslant 3$ be fixed integers and $\varphi : I \to \mathbb{R}$ be a mapping which satisfies the functional equation* (1.11) *for all* $(p_1, \ldots, p_n) \in \Gamma_n$ *and* $(q_1, \ldots, q_m) \in \Gamma_m$. *Then $\varphi$ is of the form*

$$\varphi(p) = a(p) + \varphi(0) \tag{3.1}$$

*where $a : \mathbb{R} \to \mathbb{R}$ is an additive mapping with*

$$\left.\begin{array}{ll} \text{(i)} \quad a(1) = -\,nm\,\varphi(0) & \text{if } \ \varphi(1) + (n-1)\,\varphi(0) \neq 1 \\ \qquad\qquad \text{or} & \\ \text{(ii)} \quad a(1) = 1 - n\,\varphi(0) & \text{if } \ \varphi(1) + (n-1)\,\varphi(0) = 1 \end{array}\right\} \tag{3.2}$$

*or*

$$\varphi(p) = M(p) - B(p) \tag{3.3}$$

*where $B : \mathbb{R} \to \mathbb{R}$ is an additive mapping with $B(1) = 0$ and $M : I \to \mathbb{R}$ is multiplicative on $I$ in the sense that it satisfies* (1.2), (1.3) *and* (1.4) *for all* $p \in I_0$, $q \in I_0$.

**Proof.** Let us put $p_1 = 1$, $p_2 = \ldots = p_n = 0$ in (1.11). We obtain

$$[\varphi(1) + (n-1)\,\varphi(0) - 1]\left[\sum_{j=1}^{m} \varphi(q_j) + m(n-1)\,\varphi(0)\right] = 0 \tag{3.4}$$

for all $(q_1, \ldots, q_m) \in \Gamma_m$. We divide our discussion into two cases.

Case 1. $\varphi(1) + (n-1)\,\varphi(0) - 1 \neq 0$.

In this case, (3.4) reduces to

$$\sum_{j=1}^{m} \varphi(q_j) = -\,m(n-1)\,\varphi(0) \tag{3.5}$$

for all $(q_1, \ldots, q_m) \in \Gamma_m$. By Result 1, there exists an additive mapping $a : \mathbb{R} \to \mathbb{R}$ such that

$$\varphi(p) = a(p) - \frac{1}{m}\,a(1) - (n-1)\,\varphi(0) \tag{3.6}$$

for all $p \in I$. The substitution $p = 0$, in (3.6), gives

$$a(1) = -\,nm\,\varphi(0)\,. \tag{3.7}$$

From (3.6) and (3.7), (3.1) follows. Thus, we have obtained the solution (3.1) satisfying (i) in(3.2).

Case 2. $\varphi(1) + (n-1)\,\varphi(0) - 1 = 0$.

Let us write (1.11) in the form

$$\sum_{j=1}^{m}\left\{\sum_{i=1}^{n} \varphi(p_i q_j) - \varphi(q_j)\sum_{i=1}^{n} \varphi(p_i) - m(n-1)\varphi(0) q_j \sum_{i=1}^{n} \varphi(p_i)\right\} = 0\,. \tag{3.8}$$

Choose $(p_1, \ldots, p_n) \in \Gamma_n$ and fix it. Define $\psi : \Gamma_n \times I \to \mathbb{R}$ as

$$\psi(p_1, \ldots, p_n; q) = \sum_{i=1}^{n} \varphi(p_i q) - \varphi(q)\sum_{i=1}^{n}\varphi(p_i) - m(n-1)\varphi(0) q \sum_{i=1}^{n} \varphi(p_i) \tag{3.9}$$

for all $q \in I$. By Result 1, there exists a mapping $A_1 : \Gamma_n \times \mathbb{R} \to \mathbb{R}$, additive in the second variable, such that

$$\begin{aligned} &\sum_{i=1}^{n} \varphi(p_i q) - \varphi(q)\sum_{i=1}^{n} \varphi(p_i) - m(n-1)\,\varphi(0)\,q \sum_{i=1}^{n} \varphi(p_i) \\ &= \; A_1(p_1, \ldots, p_n; q) - \frac{1}{m}\,A_1(p_1, \ldots, p_n; 1) \end{aligned} \tag{3.10}$$

The substitution $q = 0$, in (3.10), gives

$$A_1(p_1, \ldots, p_n; 1) = m\,\varphi(0)\left[\sum_{i=1}^{n} \varphi(p_i) - n\right] \tag{3.11}$$

as $A_1(p_1, \ldots, p_n; 0)$. From (3.10) and (3.11), we obtain

$$\begin{aligned}&\sum_{i=1}^{n} \varphi(p_i q) - \varphi(q) \sum_{i=1}^{n} \varphi(p_i) - m(n-1)\,\varphi(0)\,q \sum_{i=1}^{n} \varphi(p_i)\\ &= \;\; A_1(p_1, \ldots, p_n; q) - \varphi(0) \sum_{i=1}^{n} \varphi(p_i) + n\,\varphi(0)\,. \end{aligned}\tag{3.12}$$

Since $(p_1, \ldots, p_n) \in \Gamma_n$ was chosen arbitrarily and then fixed, equation (3.12), indeed, holds for all $(p_1, \ldots, p_n) \in \Gamma_n$ and all $q \in I$.

Let $x \in I$ and $(r_1, \ldots, r_n) \in \Gamma_n$. Putting $q = x r_t$, $t = 1, \ldots, n$ in (3.12); adding the resulting $n$ equations and using the additivity of $A_1$ in the second variable, it follows that

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{t=1}^{n} \varphi(x p_i r_t) - \sum_{t=1}^{n} \varphi(x r_t) \sum_{i=1}^{n} \varphi(p_i) - m(n-1)\,\varphi(0)\,x \sum_{i=1}^{n} \varphi(p_i)\\ &= \;\; A_1(p_1, \ldots, p_n; x) - n\,\varphi(0) \sum_{i=1}^{n} \varphi(p_i) + n^2\,\varphi(0)\,. \end{aligned}\tag{3.13}$$

Also, if we put $q = x$ and $p_i = r_i$, $i = 1, \ldots, n$ in (3.12), we obtain

$$\begin{aligned}\sum_{t=1}^{n} \varphi(x r_t) \;\; &= \;\; \varphi(x) \sum_{t=1}^{n} \varphi(r_t) + m(n-1)\,\varphi(0)\,x \sum_{t=1}^{n} \varphi(r_t)\\ &\quad + A_1(r_1, \ldots, r_n; x) - \varphi(0) \sum_{t=1}^{n} \varphi(r_t) + n\,\varphi(0)\,. \end{aligned}\tag{3.14}$$

From (3.13) and (3.14), we can obtain the equation

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{t=1}^{n} \varphi(x p_i r_t) - [\varphi(x) + m(n-1)\,\varphi(0)\,x - \varphi(0)]\\ &\quad \times \sum_{i=1}^{n} \varphi(p_i) \sum_{t=1}^{n} \varphi(r_t) - n^2\,\varphi(0)\\ &\qquad = A_1(p_1, \ldots, p_n; x) + m(n-1)\,\varphi(0)\,x \sum_{i=1}^{n} \varphi(p_i)\\ &\qquad\quad + A_1(r_1, \ldots, r_n; x) \sum_{i=1}^{n} \varphi(p_i)\,. \end{aligned}\tag{3.15}$$

The symmetry of the left hand side of (3.15), in $p_i$ and $r_t$, $i = 1, \ldots, n$; $t = 1, \ldots, n$ gives rise to the equation

$$\begin{aligned} & A_1(p_1, \ldots, p_n; x) + m(n-1)\,\varphi(0)\,x \sum_{i=1}^{n} \varphi(p_i) + A_1(r_1, \ldots, r_n; x) \sum_{i=1}^{n} \varphi(p_i) \\ = \; & A_1(r_1, \ldots, r_n; x) + m(n-1)\,\varphi(0)\,x \sum_{t=1}^{n} \varphi(r_t) \\ & + A_1(p_1, \ldots, p_n; x) \sum_{t=1}^{n} \varphi(r_t) \end{aligned}$$

which can be written in the form

$$\begin{aligned} & [A_1(p_1, \ldots, p_n; x) + m(n-1)\,\varphi(0)\,x] \left[ \sum_{t=1}^{n} \varphi(r_t) - 1 \right] \\ = \; & [A_1(r_1, \ldots, r_n; x) + m(n-1)\,\varphi(0)\,x] \left[ \sum_{i=1}^{n} \varphi(p_i) - 1 \right]. \end{aligned} \tag{3.16}$$

Equation (3.16) holds for all $(r_1, \ldots, r_n) \in \Gamma_n$, $(p_1, \ldots, p_n) \in \Gamma_n$ and all $x \in I$.

Subcase 2.1. $\sum_{t=1}^{n} \varphi(r_t) - 1$ vanishes identically on $\Gamma_n$.

In this case,

$$\sum_{t=1}^{n} \varphi(r_t) = 1 \tag{3.17}$$

holds for all $(r_1, \ldots, r_n) \in \Gamma_n$. By Result 1, there exists an additive map $a : \mathbb{R} \to \mathbb{R}$ such that

$$\varphi(p) = a(p) - \frac{1}{n}\,a(1) + \frac{1}{n} \tag{3.18}$$

for all $p \in I$. The substitution $p = 0$, in (3.18), yields

$$a(1) = 1 - n\,\varphi(0)\,. \tag{3.19}$$

From (3.18) and (3.19), (3.1) follows again. Thus, we have obtained the solution (3.1) satisfying (ii) in (3.2).

Subcase 2.2. $\sum_{t=1}^{n} \varphi(r_t) - 1$ does not vanish identically on $\Gamma_n$.

Then, there exists a probability distribution $(r_1^*, \ldots, r_n^*) \in \Gamma_n$ such that

$$\sum_{t=1}^{n} \varphi(r_t^*) - 1 \neq 0\,. \tag{3.20}$$

Setting $r_1 = r_1^*, \ldots, r_n = r_n^*$ in (3.16), we obtain

$$\begin{aligned} & [A_1(p_1, \ldots, p_n; x) + m(n-1)\,\varphi(0)\,x] \left[\sum_{t=1}^{n} \varphi(r_t^*) - 1\right] \\ &= \quad [A_1(r_1^*, \ldots, r_n^*; x) + m(n-1)\,\varphi(0)\,x] \left[\sum_{i=1}^{n} \varphi(p_i) - 1\right] \end{aligned}$$

which gives, for all $x \in I$,

$$A_1(p_1, \ldots, p_n; x) = A(x) \left[\sum_{i=1}^{n} \varphi(p_i) - 1\right] - m(n-1)\,\varphi(0)\,x \tag{3.21}$$

where $A : \mathbb{R} \to \mathbb{R}$ is defined as

$$A(y) = \left[\sum_{t=1}^{n} \varphi(r_t^*) - 1\right]^{-1} [A_1(r_1^*, \ldots, r_n^*; y) + m(n-1)\,\varphi(0)\,y] \tag{3.22}$$

for all $y \in \mathbb{R}$. From (3.22), it is easy to verify that $A : \mathbb{R} \to \mathbb{R}$ is additive. Also, from (3.11) (with $p_i = r_i^*$, $i = 1, \ldots, n$) and (3.22), it is easy to derive

$$A(1) = m\,\varphi(0)\,. \tag{3.23}$$

From (3.12) and (3.21), it follows that

$$\begin{aligned} & \sum_{i=1}^{n} \varphi(p_i q) - \varphi(q) \sum_{i=1}^{n} \varphi(p_i) - m(n-1)\,\varphi(0)\,q \sum_{i=1}^{n} \varphi(p_i) \\ &= \quad A(q) \sum_{i=1}^{n} \varphi(p_i) - A(q) - m(n-1)\,\varphi(0)\,q - \varphi(0) \sum_{i=1}^{n} \varphi(p_i) + n\,\varphi(0) \end{aligned}$$

which, upon using (3.23), gives

$$\begin{aligned} & \sum_{i=1}^{n} [\varphi(p_i q) + A(p_i q) + m(n-1)\,\varphi(0)\,p_i q - \varphi(0)] \\ & \quad - [\varphi(q) + A(q) + m(n-1)\,\varphi(0)\,q - \varphi(0)] \\ & \quad \times \sum_{i=1}^{n} [\varphi(p_i) + A(p_i) + m(n-1)\,\varphi(0)\,p_i - \varphi(0)] \\ & \quad + [\varphi(q) + A(q) + m(n-1)\,\varphi(0)\,q - \varphi(0)]\,n(m-1)\,\varphi(0) = 0\,. \end{aligned} \tag{3.24}$$

Define a mapping $B : \mathbb{R} \to \mathbb{R}$ as

$$B(x) = A(x) + m(n-1)\,\varphi(0)\,x \tag{3.25}$$

for all $x \in \mathbb{R}$. Then, $B : \mathbb{R} \to \mathbb{R}$ is additive. Moreover, from (3.23) and (3.25), it follows that

$$B(1) = mn\,\varphi(0)\,. \tag{3.26}$$

With the help of (3.25), equation (3.24) can be written in the form

$$\begin{aligned}\sum_{i=1}^{n}[\varphi(p_iq) + B(p_iq) - \varphi(0)] - [\varphi(q) + B(q) - \varphi(0)] \\ \times \sum_{i=1}^{n}[\varphi(p_i) + B(p_i) - \varphi(0)] + n(m-1)\,\varphi(0)\,[\varphi(q) + B(q) - \varphi(0)] \\ = 0\,.\end{aligned} \tag{3.27}$$

Define a mapping $M : I \to \mathbb{R}$ as

$$M(x) = \varphi(x) + B(x) - \varphi(0) \tag{3.28}$$

for all $x \in I$. Notice that though $B : \mathbb{R} \to \mathbb{R}$ but, in (3.28), we are restricting its use only for all $x \in I$.

From (3.28), it is easy to see that (1.2) follows as $B(0) = 0$. Also, from (3.26), (3.28) and the fact that $\varphi(1) + (n-1)\,\varphi(0) = 1$, it follows that

$$M(1) = 1 + n(m-1)\,\varphi(0)\,. \tag{3.29}$$

Moreover, from (3.27) and (3.28), we get (for all $q \in I$)

$$\sum_{i=1}^{n} M(p_iq) - M(q)\sum_{i=1}^{n} M(p_i) + n(m-1)\,\varphi(0)\,M(q) = 0 \tag{3.30}$$

which can be written in the form

$$\sum_{i=1}^{n}[M(p_iq) - M(q)M(p_i) + n(m-1)\,\varphi(0)\,M(q)\,p_i] = 0\,. \tag{3.31}$$

By Result 1, there exists a mapping $E : \mathbb{R} \times I \to \mathbb{R}$, additive in the first variable, such that

$$M(pq) - M(p)M(q) + n(m-1)\,\varphi(0)\,M(q)\,p = E(p,q) - \frac{1}{n}\,E(1,q) \tag{3.31a}$$

for all $p \in I$, $q \in I$. The substitution $p = 0$ in (3.31a) and the use of (1.2) gives $E(1, q) = 0$ for all $q \in I$. Consequently, (3.31a) reduces to the equation

$$M(pq) - M(p)M(q) + n(m-1)\,\varphi(0)\,M(q)\,p = E(p,q) \tag{3.32}$$

for all $p \in I$, $q \in I$.

Now we prove that $n(m-1)\,\varphi(0) \neq 0$ is not possible.

If possible, suppose $n(m-1)\,\varphi(0) \neq 0$. Then, (3.29) gives $M(1) \neq 1$. Putting $q = 1$ in (3.30), using (3.29) and the fact that $M(1) - 1 \neq 0$, we get

$$\sum_{i=1}^{n} M(p_i) = M(1)$$

for all $(p_1, \ldots, p_n) \in \Gamma_n$. By Result 1, there exists an additive mapping $A_2 : \mathbb{R} \to \mathbb{R}$ such that

$$M(p) = A_2(p) - \frac{1}{n}\,A_2(1) + \frac{1}{n}\,M(1) \tag{3.33}$$

for all $p \in I$. The substitution $p = 0$, in (3.33), gives $A_2(1) = M(1)$ as $A_2(0) = 0$ and $M(0) = 0$. Hence

$$M(p) = A_2(p)$$

for all $p \in I$. Thus $M$ is additive on $I$. Now, from (3.20), (3.26), (3.28), (3.29) and the additivity of $M$ on $I$, we have

$$\begin{aligned} 1 \neq \sum_{t=1}^{n} \varphi(r_t^*) &= M(1) - B(1) + n\,\varphi(0) \\ &= 1 + n(m-1)\,\varphi(0) - nm\,\varphi(0) + n\,\varphi(0) = 1 \end{aligned}$$

a contradiction.

So, the only possibility is that $n(m-1)\,\varphi(0) = 0$. Since $n \geqslant 3$, $m \geqslant 3$ are fixed integers, it follows that $\varphi(0) = 0$ and hence $\varphi(1) = 1$. From this and (3.29), (1.3) follows. Since $\varphi(0) = 0$, equation (3.32) reduces to the equation

$$M(pq) - M(p)\,M(q) = E(p,q) \tag{3.34}$$

for all $p \in I$, $q \in I$. The left hand side of (3.34) is symmetric in $p$ and $q$. Hence $E(p,q) = E(q,p)$ for all $p \in I$, $q \in I$. Consequently, $E$ is also additive on $I$ in the second variable. We may assume that $E(p,\cdot)$ has been extended additively to the whole of $\mathbb{R}$.

Let $p \in I$, $q \in I$, $r \in I$. From (3.34), we have

$$\begin{aligned} E(pq,r) + M(r)\,E(p,q) &= M(pqr) - M(p)\,M(q)\,M(r) \\ &= E(qr,p) + M(p)\,E(q,r)\,. \end{aligned} \tag{3.35}$$

Now we prove that $E(p,q) = 0$ for all $p \in I$, $q \in I$. If possible, suppose there exists a $p^* \in I$ and a $q^* \in I$ such that $E(p^*,q^*) \neq 0$. Then, from (3.35)

$$M(r) = [E(p^*,q^*)]^{-1}\{E(q^*r,p^*) + M(p^*)E(q^*,r) - E(p^*q^*,r)\}$$

from which it follows that $M$ is additive on $I$. Now, making use of (3.20), (3.26), (3.28), (1.3), the additivity of $M$ and the fact that $\varphi(0) = 0$, we obtain

$$1 \neq \sum_{t=1}^{n} \varphi(r_t^*) = M(1) - B(1) + n\,\varphi(0) = 1 - mn\,\varphi(0) + n\,\varphi(0) = 1$$

a contradiction. Hence $E(p,q) = 0$ for all $p \in I$, $q \in I$. Now, (3.34) reduces to the equation

$$M(pq) = M(p)\,M(q) \tag{3.36}$$

for all $p \in I$, $q \in I$. From (3.36), (1.4) follows immediately for all $p \in I_0$, $q \in I_0$. Also, since $\varphi(0) = 0$, (3.28) reduces to (3.3) and (3.26) gives $B(1) = 0$. This completes the proof of Theorem 1. □

## 4. The functional equation (1.10)

In this section, we prove:

**Theorem 2.** *Let $n \geqslant 3$, $m \geqslant 3$ be fixed integers and $f : I \to \mathbb{R}$, $g_j : I \to \mathbb{R}$, $j = 1, \ldots, m$ be mappings which satisfy the functional equation* (1.10) *for all*

$(p_1, \ldots, p_n) \in \Gamma_n$ *and* $(q_1, \ldots, q_m) \in \Gamma_m$. *Then, any general solution of* (1.10) *is of the form*

$$f(p) = b(p)\,, \quad g_j \text{ any arbitrary real-valued mapping} \tag{4.1}$$

*where* $b : \mathbb{R} \to \mathbb{R}$ *is an additive mapping with* $b(1) = 0$ *or*

$$\left.\begin{aligned} &f(p) = [f(1) + (n-1)\,f(0)]\,a(p) + f(0) \\ &g_j(p) = a(p) + A^*(p) + g_j(0) \end{aligned}\right\} \tag{4.2}$$

*for all* $j = 1, \ldots, m$; *with* $a : \mathbb{R} \to \mathbb{R}$, $A^* : \mathbb{R} \to \mathbb{R}$ *being additive maps and*

$$\left.\begin{aligned} &a(1) = 1 - \frac{n\,f(0)}{f(1) + (n-1)\,f(0)}\,, && f(1) + (n-1)\,f(0) \neq 0 \\ &A^*(1) = -\sum_{j=1}^{m} g_j(0) + \frac{nm\,f(0)}{f(1) + (n-1)\,f(0)}\,, && f(1) + (n-1)\,f(0) \neq 0 \end{aligned}\right\} \tag{4.3}$$

*or*

$$\left.\begin{aligned} &f(p) = f(1)[M(p) - B(p)]\,, && f(1) \neq 0 \\ &g_j(p) = M(p) - B(p) + A^*(p) + g_j(0) \end{aligned}\right\} \tag{4.4}$$

*for all* $j = 1, \ldots, m$; *with* $B : \mathbb{R} \to \mathbb{R}$, $A^* : \mathbb{R} \to \mathbb{R}$ *being additive maps,* $B(1) = 0$, $A^*(1) = -\sum_{j=1}^{m} g_j(0)$ *and* $M : I \to \mathbb{R}$ *a multiplicative function in the sense that it satisfies* (1.2), (1.3) *and* (1.4) *for all* $p \in I_0$, $q \in I_0$.

**Proof.** Put $p_1 = 1$, $p_2 = \ldots = p_n = 0$ in (1.10). We obtain

$$\sum_{j=1}^{m} [f(q_j) + (n-1) f(0)] = [f(1) + (n-1)\,f(0)] \sum_{j=1}^{m} g_j(q_j) \tag{4.5}$$

for all $(q_1, \ldots, q_m) \in \Gamma_m$.

Case 1.

$$f(1) + (n-1)\,f(0) = 0\,. \tag{4.6}$$

Then, (4.5) reduces to the equation

$$\sum_{j=1}^{m} f(q_j) = -\,m(n-1)\,f(0) \tag{4.7}$$

valid for all $(q_1, \ldots, q_m) \in \Gamma_m$. By Result 1, there exists an additive mapping $b : \mathbb{R} \to \mathbb{R}$ such that

$$f(p) = b(p) - \frac{1}{m}\, b(1) - (n-1)\, f(0) \tag{4.8}$$

for all $p \in I$. The substitution $p = 0$, in (4.8), gives

$$b(1) = -\, nm\, f(0)\,. \tag{4.9}$$

From (4.8) and (4.9), it follows that

$$f(p) = b(p) + f(0) \tag{4.10}$$

for all $p \in I$. From (4.6), (4.9) and (4.10), using the fact that $n \geqslant 3$, $m \geqslant 3$ are fixed integers, it follows that

$$f(0) = 0\,. \tag{4.11}$$

From (4.9) and (4.11), it follows that

$$b(1) = 0\,. \tag{4.12}$$

Also, (4.10) and (4.11) give

$$f(p) = b(p) \tag{4.13}$$

for all $p \in I$. Also, from (1.10), (4.12), (4.13) and the additivity of $b : \mathbb{R} \to \mathbb{R}$, it follows that $g_j$ can be any arbitrary real-valued mapping. Thus, we have obtained the solution (4.1) in which $b$ satisfies (4.12).

Case 2. $f(1) + (n-1)\, f(0) \neq 0$.

In this case, (4.5) gives

$$\sum_{j=1}^{m} g_j(q_j) = [f(1) + (n-1)\, f(0)]^{-1} \sum_{j=1}^{m} [f(q_j) + (n-1)\, f(0)] \tag{4.14}$$

which can be written in the form

$$\sum_{j=1}^{m} \left\{ g_j(q_j) - [f(1) + (n-1)\, f(0)]^{-1} [f(q_j) + (n-1)\, f(0)] \right\} = 0\,. \tag{4.15}$$

This holds for all $(q_1, \ldots, q_m) \in \Gamma_m$. By Result 2, there exists an additive mapping $A^* : \mathbb{R} \to \mathbb{R}$ and constants $c_j$ $(j = 1, \ldots, m)$ such that

$$g_j(p) - [f(1) + (n-1)\, f(0)]^{-1} [f(p) + (n-1)\, f(0)] = A^*(p) + c_j \tag{4.16}$$

with

$$A^*(1) + \sum_{j=1}^{m} c_j = 0\,. \tag{4.17}$$

The substitution $p = 0$, in (4.16), gives

$$c_j = g_j(0) - [f(1) + (n-1) f(0)]^{-1}\, n f(0) \tag{4.18}$$

for $j = 1, \ldots, m$. From (4.17) and (4.18), we get $A^*(1)$ as mentioned in (4.3).

Also, from (4.16) and (4.18),

$$g_j(p) = [f(1) + (n-1)\, f(0)]^{-1} [f(p) - f(0)] + A^*(p) + g_j(0) \tag{4.19}$$

for $j = 1, \ldots, m$. Equation (4.19) tells us that if $f$ is known, then the corresponding form of $g_j(p)$, $j = 1, \ldots, m$, can be determined. To determine $f$, we eliminate $\sum_{j=1}^{m} g_j(q_j)$ from equations (1.10) and (4.14). We obtain the equation

$$\begin{aligned} \sum_{i=1}^{n} \sum_{j=1}^{m} f(p_i q_j) \;\; &= \;\; [f(1) + (n-1)\, f(0)]^{-1} \sum_{i=1}^{n} f(p_i) \sum_{j=1}^{m} f(q_j) \\ &\quad + [f(1) + (n-1)\, f(0)]^{-1} m(n-1)\, f(0) \sum_{i=1}^{n} f(p_i) \end{aligned} \tag{4.20}$$

valid for all $(p_1, \ldots, p_n) \in \Gamma_n$ and $(q_1, \ldots, q_m) \in \Gamma_m$.

Define a mapping $\varphi : I \to \mathbb{R}$ as

$$\varphi(x) = [f(1) + (n-1)\, f(0)]^{-1}\, f(x) \tag{4.21}$$

for all $x \in I$. Then (4.20) reduces to the functional (1.11) which also holds for all $(p_1, \ldots, p_n) \in \Gamma_n$ and $(q_1, \ldots, q_m) \in \Gamma_m$. Moreover, $\varphi$ satisfies the condition

$$\varphi(1) + (n-1)\,\varphi(0) = 1\,. \tag{4.22}$$

Also, from (4.21),

$$f(p) = [f(1) + (n-1)\,f(0)]\,\varphi(p) \tag{4.23}$$

for all $p \in I$ with $f(1) + (n-1)\,f(0) \neq 0$ and

$$\varphi(0) = \frac{f(0)}{f(1) + (n-1)\,f(0)}\,. \tag{4.24}$$

From, (4.19), (4.23), (4.24), (3.1) and (ii) in (3.2), the forms of $f(p)$, $g_j(p)$ and $a(1)$, as mentioned in (4.2) and (4.3), follow. Thus, we have obtained the solution (4.2), of (1.10), subject to $a(1)$ and $A^*(1)$ as mentioned in (4.3).

The form of $\varphi$, given by (3.3), with $B(1) = 0$, is also acceptable as in this case, $\varphi(0) = 0$, $\varphi(1) = 1$ and hence $\varphi(1) + (n-1)\,\varphi(0) = 1$. Now, from (4.24), $f(0) = 0$. The solution (4.4), of (1.10), follows from (4.23), (4.19), (3.3), (1.2), (1.3), (1.4) and the fact that $f(0) = 0$, $B(1) = 0$, $A^*(1) = -\sum_{j=1}^{m} g_j(0)$. This completes the proof of Theorem 2. □

## 5. The functional equation (1.8)

In this section, we prove:

**Theorem 3.** *Let $n \geqslant 3$, $m \geqslant 3$ be fixed integers and $h : I \to \mathbb{R}$, $k_j : I \to \mathbb{R}$, $j = 1, \ldots, m$ be mappings which satisfy the functional equation* (1.8) *for all $(p_1, \ldots, p_n) \in \Gamma_n$ and $(q_1, \ldots, q_m) \in \Gamma_m$ and $\lambda \neq 0$. Then, any general solution of* (1.8) *is of the form*

$$h(p) = \frac{1}{\lambda}\,[b(p) - p],\ \ k_j \textit{ any arbitrary real-valued mapping} \tag{5.1}$$

*where* $b : \mathbb{R} \to \mathbb{R}$ *is an additive mapping with* $b(1) = 0$ *or*

$$\left.\begin{aligned} h(p) &= \frac{1}{\lambda}\big\{[\lambda(h(1) + (n-1)\,h(0)) + 1]\,a(p) + \lambda\,h(0) - p\big\} \\ k_j(p) &= \frac{1}{\lambda}\big\{a(p) + A^*(p) + \lambda\,k_j(0) - p\big\} \end{aligned}\right\} \tag{5.2}$$

*for all* $j = 1, \ldots, m$*; with* $a : \mathbb{R} \to \mathbb{R}$*,* $A^* : \mathbb{R} \to \mathbb{R}$ *being additive maps and*

$$\left.\begin{aligned} a(1) &= 1 - \frac{n\lambda\,h(0)}{\lambda(h(1) + (n-1)\,h(0)) + 1}\,, \\ &\qquad\qquad \lambda(h(1) + (n-1)\,h(0)) + 1 \neq 0 \\ A^*(1) &= -\lambda\sum_{j=1}^{m} k_j(0) + \frac{nm\lambda\,h(0)}{\lambda(h(1) + (n-1)\,h(0)) + 1}\,, \\ &\qquad\qquad \lambda(h(1) + (n-1)\,h(0)) + 1 \neq 0\,. \end{aligned}\right\} \tag{5.3}$$

*or*

$$\left.\begin{aligned} h(p) &= \frac{1}{\lambda}\,\big\{[\lambda\,h(1) + 1][M(p) - B(p)] - p\big\}\,, \quad [\lambda\,h(1) + 1] \neq 0 \\ k_j(p) &= \frac{1}{\lambda}\,\big\{M(p) - B(p) + A^*(p) + \lambda\,k_j(0) - p\big\} \end{aligned}\right\} \tag{5.4}$$

*with* $B : \mathbb{R} \to \mathbb{R}$*,* $A^* : \mathbb{R} \to \mathbb{R}$ *being additive maps such that*

$$B(1) = 0\,, \quad A^*(1) = -\lambda\sum_{j=1}^{m} k_j(0) \tag{5.5}$$

*and* $M : I \to \mathbb{R}$ *a multiplicative function in the sense that it satisfies* (1.2)*,* (1.3) *and* (1.4) *for all* $p \in I_0$*,* $q \in I_0$*.*

**Proof.** Let us write (1.8) in the form

$$\sum_{i=1}^{n}\sum_{j=1}^{m}[\lambda\,h(p_iq_j) + p_iq_j] = \sum_{i=1}^{n}[\lambda\,h(p_i) + p_i]\sum_{j=1}^{m}[\lambda\,k_j(q_j) + q_j]\,. \tag{5.6}$$

Define the mappings $f : I \to \mathbb{R}$ and $g_j : I \to \mathbb{R}$, $j = 1, \ldots, m$ (with $\lambda \neq 0$), as in (1.9), for all $x \in I$. Then, (5.6) reduces to the functional equation (1.10) whose respective solutions are given by (4.1); (4.2) subject to the condition (4.3); and (4.4) subject to $B(1) = 0$, $A^*(1) = -\sum_{j=1}^{m} g_j(0)$; in which $b : \mathbb{R} \to \mathbb{R}$, $a : \mathbb{R} \to \mathbb{R}$, $A^* : \mathbb{R} \to \mathbb{R}$, $B : \mathbb{R} \to \mathbb{R}$ are all additive functions and $M : [0,1] \to \mathbb{R}$ is a multiplicative function. Now, making use of (1.9) along with (4.1); (4.2) subject to (4.3); and (4.4) subject to $B(1) = 0$ and $A^*(1) = -\sum_{j=1}^{m} g_j(0)$; the required solutions (5.1); (5.2) subject to (5.3); and (5.4) subject to (5.5); follow respectively. $\square$

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