# STRONG FELLER PROCESSES WITH MEASURE-VALUED DRIFTS

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Abstract. We construct a strong Feller process associated with $-\Delta + \sigma \cdot \nabla$, with drift $\sigma$ in a wide class of measures (weakly form-bounded measures, e.g. combining weak $L^d$ and Kato class measure singularities), by exploiting a quantitative dependence of the smoothness of the domain of an operator realization of $-\Delta + \sigma \cdot \nabla$ generating a holomorphic $C_0$-semigroup on $L^p(\mathbb{R}^d)$, $p > d-1$, on the value of the form-bound of $\sigma$. Our method admits extension to other types of perturbations of $-\Delta$ or $(-\Delta)^{\frac{\alpha}{2}}$, e.g. to yield new $L^p$-regularity results for Schrödinger operators with form-bounded measure potentials.

1. Let $\mathcal{L}^d$ be the Lebesgue measure on $\mathbb{R}^d$, $L^p = L^p(\mathbb{R}^d, \mathcal{L}^d)$, $L^{p,\infty} = L^{p,\infty}(\mathbb{R}^d, \mathcal{L}^d)$ and $W^{1,p} = W^{1,p}(\mathbb{R}^d, \mathcal{L}^d)$ the standard Lebesgue, weak Lebesgue and Sobolev spaces, $C^{0,\gamma} = C^{0,\gamma}(\mathbb{R}^d)$ the space of Hölder continuous functions ($0 < \gamma < 1$), $C_b = C_b(\mathbb{R}^d)$ the space of bounded continuous functions, endowed with the sup-norm, $C_\infty \subset C_b$ the closed subspace of functions vanishing at infinity, $\mathcal{W}^{s,p}$, $s > 0$, the Bessel space endowed with norm $\|u\|_{p,s} := \|g\|_p$, $u = (1-\Delta)^{-\frac{s}{2}} g$, $g \in L^p$, $\mathcal{W}^{-s,p}$ the dual of $\mathcal{W}^{s,p}$, and $\mathcal{S} = \mathcal{S}(\mathbb{R}^d)$ the L. Schwartz space of test functions. We denote by $\mathcal{B}(X,Y)$ the space of bounded linear operators between complex Banach spaces $X \to Y$, endowed with operator norm $\|\cdot\|_{X \to Y}$; $\mathcal{B}(X) := \mathcal{B}(X,X)$. Set $\|\cdot\|_{p \to q} := \|\cdot\|_{L^p \to L^q}$. We denote by $\overset{w}{\to}$ the weak convergence of $\mathbb{R}^d$- or $\mathbb{C}^d$-valued measures on $\mathbb{R}^d$, and the weak convergence in a given Banach space.

By $\langle u, v \rangle$ we denote the inner product in $L^2$,

$$\langle u, v \rangle = \langle u \bar{v} \rangle := \int_{\mathbb{R}^d} u \bar{v} \mathcal{L}^d \qquad (u, v \in L^2).$$

2. Let $d \geqslant 3$. The problem of constructing a Feller process having infinitesimal generator $-\Delta + b \cdot \nabla$, with singular drift $b : \mathbb{R}^d \to \mathbb{R}^d$, has been thoroughly studied in the literature (cf. [AKR, KR] and references therein), motivated by applications, as well as the search for the maximal (general) class of vector fields $b$ such that the associated process exists. This search culminated in the following classes of critical drifts:

Definition 1. A vector field $b : \mathbb{R}^d \to \mathbb{R}^d$ is said to belong to $\mathbf{F}_\delta$, the class of form-bounded vector fields, if $b$ is $\mathcal{L}^d$-measurable and there exists $\lambda = \lambda_\delta > 0$ such that

$$\|b(\lambda - \Delta)^{-\frac{1}{2}}\|_{2 \to 2} \leqslant \sqrt{\delta}.$$

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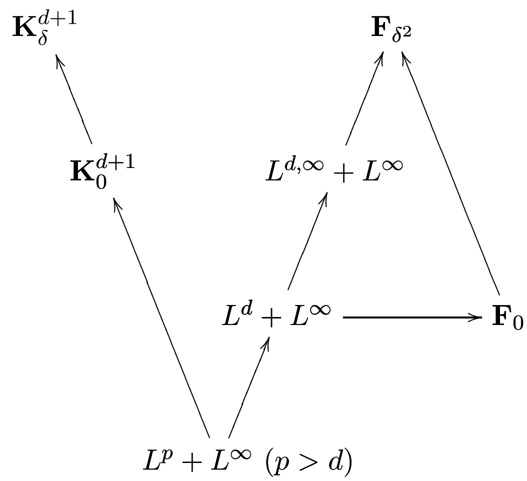



Here $\rightarrow$ stands for $\subsetneq$, inclusion of vector spaces.
The inclusions $L^d + L^\infty \subsetneq \mathbf{F}_0 := \bigcap_{\delta>0} \mathbf{F}_\delta$, $L^{d,\infty} + L^\infty \subsetneq \bigcup_{\delta>0} \mathbf{F}_\delta$ follow from the Sobolev embedding theorem, and the Strichartz inequality with sharp constants [KPS], respectively.

Definition 2. A vector field $b : \mathbb{R}^d \to \mathbb{R}^d$ is said to belong to the Kato class $\mathbf{K}_\delta^{d+1}$ if $b$ is $\mathcal{L}^d$-measurable and there exists $\lambda = \lambda_\delta > 0$ such that

$$\|b(\lambda - \Delta)^{-\frac{1}{2}}\|_{1\to 1} \leqslant \delta.$$

We have:

1) $b(x) = \sqrt{\delta}\frac{d-2}{2}x|x|^{-2} \in \mathbf{F}_\delta$ (Hardy inequality).

2) Also, if $|b(x)| \leqslant \mathbf{1}_{|x_1|<1}|x_1|^{s-1}$, where $0 < s < 1$, $x = (x_1, \dots, x_d)$, $\mathbf{1}_{|x_1|<1}$ is the characteristic function of $\{x : |x_1| < 1\}$, then $b \in \mathbf{K}_0^{d+1}$. An example of a $b \in \mathbf{K}_\delta^{d+1} \setminus \mathbf{K}_0^{d+1}$ can be obtained e.g. by modifying [AS, p. 250, Example 1][1]. Examples 1), 2) demonstrate that $\mathbf{K}_\delta^{d+1} \setminus \mathbf{F}_{\delta_1} \neq \varnothing$, and $\mathbf{F}_{\delta_1} \setminus \mathbf{K}_\delta^{d+1} \neq \varnothing$.

It is clear that

$$b \in \mathbf{F}_\delta \text{ (or } \mathbf{K}_\delta^{d+1}) \quad \Leftrightarrow \quad \varepsilon b \in \mathbf{F}_{\varepsilon\delta} \text{ (respectively, } \mathbf{K}_{\varepsilon\delta}^{d+1}), \quad \varepsilon > 0.$$

In particular, there exist $b \in \mathbf{F}_\delta$ ($\mathbf{K}_\delta^{d+1}$) such that $\varepsilon b \notin \mathbf{F}_0$ ($\mathbf{K}_0^{d+1}$) for any $\varepsilon > 0$ (cf. examples above). The vector fields in $\mathbf{F}_\delta \setminus \mathbf{F}_0$ and $\mathbf{K}_\delta^{d+1} \setminus \mathbf{K}_0^{d+1}$ have critical order singularities (i.e. sensitive to multiplication by a constant), at isolated points or along hypersurfaces, respectively.

Earlier, the Kato class $\mathbf{K}_\delta^{d+1}$, with $\delta > 0$ sufficiently small (but nevertheless allowed to be positive), has been recognized as 'the right one' for the existence of the Gaussian upper and lower bounds on the fundamental solution of $-\Delta + b\cdot\nabla$, see [S, Zh]; the Gaussian bounds yield an operator realization of $-\Delta + b\cdot\nabla$ generating a (contraction positivity preserving) $C_0$-semigroup in $C_\infty$ (moreover, in $C_b$), whose integral kernel is the transition probability function of a Feller process. In turn, $b \in \mathbf{F}_\delta$, $\delta < 4$, ensures that $-\Delta + b\cdot\nabla$ is dissipative in $L^p$, $p > \frac{2}{2-\sqrt{\delta}}$ [KS]; then, if $\delta < \min\{1, \big(\frac{2}{d-2}\big)^2\}$, the $L^p$-dissipativity allows to run a Moser-type iterative procedure of [KS], which takes $p \to \infty$ and

[1] The value of the relative bound $\delta$ plays a crucial role in the theory of $-\Delta + b\cdot\nabla$, e.g. if $\delta > 4$, then the uniqueness of solution of Cauchy problem for $\partial_t - \Delta + \sqrt{\delta}\frac{d-2}{2}x|x|^{-2}\cdot\nabla$ fails in $L^p$, see [KS, Example 7], see also comments below.

thus produces an operator realization of $-\Delta + b \cdot \nabla$ generating a $C_0$-semigroup in $C_\infty$, hence a Feller process.

The natural next step toward determining the general class of drifts $b$ 'responsible' for the existence of an associated Feller process is to consider $b = b_1 + b_2$, with $b_1 \in \mathbf{F}_{\delta_1}$, $b_2 \in \mathbf{K}^{d+1}_{\delta_2}$. Although it is not clear how to reconcile the dissipativity in $L^p$ and the Gaussian bounds, it turns out that neither of these properties is responsible for the existence of the process; in fact, the process exists for any $b$ in the following class [Ki]:

Definition 3. A vector field $b : \mathbb{R}^d \to \mathbb{R}^d$ is said to belong to $\mathbf{F}^{\frac{1}{2}}_\delta$, the class of *weakly* form-bounded vector fields, if $b$ is $\mathcal{L}^d$-measurable, and there exists $\lambda = \lambda_\delta > 0$ such that

$$\||b|^{\frac{1}{2}}(\lambda - \Delta)^{-\frac{1}{4}}\|_{2\to 2} \leqslant \sqrt{\delta}.$$

The class $\mathbf{F}^{\frac{1}{2}}_\delta$ has been introduced in [S2, Theorem 5.1]. We have

$$\mathbf{K}^{d+1}_\delta \subsetneq \mathbf{F}^{\frac{1}{2}}_\delta, \quad \mathbf{F}_{\delta^2} \subsetneq \mathbf{F}^{\frac{1}{2}}_\delta,$$

$$b \in \mathbf{F}_{\delta_1} \text{ and } \mathsf{f} \in \mathbf{K}^{d+1}_{\delta_2} \quad \Longrightarrow \quad b + \mathsf{f} \in \mathbf{F}^{\frac{1}{2}}_\delta,\ \sqrt{\delta} = \sqrt[4]{\delta_1} + \sqrt{\delta_2} \tag{1}$$

(see [Ki]). In [Ki], the construction of the process goes as follows: the starting object is an operator-valued function ($b \in \mathbf{F}^{\frac{1}{2}}_\delta$)

$$\begin{aligned}&\Theta_p(\zeta, b) := (\zeta - \Delta)^{-1} \\ &- (\zeta - \Delta)^{-\frac{1}{2}-\frac{1}{2q}} \underbrace{(\zeta - \Delta)^{-\frac{1}{2q'}}|b|^{\frac{1}{p'}}}_{\in\mathcal{B}(L^p)} \underbrace{\big(1 + b^{\frac{1}{p}} \cdot \nabla(\zeta - \Delta)^{-1}|b|^{\frac{1}{p'}}\big)^{-1}}_{\in\mathcal{B}(L^p)} \underbrace{b^{\frac{1}{p}} \cdot \nabla(\zeta - \Delta)^{-\frac{1}{2}-\frac{1}{2r}}}_{\in\mathcal{B}(L^p)} (\zeta - \Delta)^{-\frac{1}{2r'}},\end{aligned}$$

where $\operatorname{Re}\zeta > \frac{d}{d-1}\lambda_\delta$, $b^{\frac{1}{p}} := b|b|^{\frac{1}{p}-1}$, $p$ is in a bounded open interval determined by the form-bound $\delta$ (and expanding to $(1, \infty)$ as $\delta \downarrow 0$), and $1 < r < p < q$. Then (see [Ki] for details)

$$\Theta_p(\zeta, b) = (\zeta + \Lambda_p(b))^{-1},$$

where $\Lambda_p(b)$ is an operator realization of $-\Delta + b \cdot \nabla$ generating a holomorphic $C_0$-semigroup $e^{-t\Lambda_p(b)}$ on $L^p$, and the very definition of $\Theta_p(\zeta, b)$ implies that the domain of $\Lambda_p(b)$

$$D(\Lambda_p(b)) \subset \mathcal{W}^{1+\frac{1}{q},p}, \quad \text{for any } q > p.$$

The information about smoothness of $D(\Lambda_p(b))$ allows us to leap, by means of the Sobolev embedding theorem, from $L^p$, $p > d - 1$, to $C_\infty$, while moving the burden of the proof of convergence in $C_\infty$ (in the Trotter's approximation theorem) to $L^p$, a space having much weaker topology (locally). Then (see [Ki]) $\Theta_p(\mu, b)|_{\mathcal{S}} = (\mu + \Lambda_{C_\infty}(b))^{-1}|_{\mathcal{S}}$, where $\Lambda_{C_\infty}(b)$ is an operator realization of $-\Delta + b \cdot \nabla$ generating a contraction positivity preserving $C_0$-semigroup on $C_\infty$, hence a Feller process.

3. The primary goal of this note is to extend the method in [Ki] to weakly form-bounded measure drifts.

The study of measure perturbations of $-\Delta$ has a long history, see e.g. [AM, SV], where the $L^p$-regularity theory of $-\Delta$ (more generally, of a Dirichlet form) perturbed by a measure potential in the corresponding Kato class was developed, $1 \leqslant p < \infty$ (cf. Corollary 1 below).

Recently, [BC] constructed a strong Feller process associated with $-\Delta+\sigma\cdot\nabla$ with a $\mathbb{R}^d$-valued measure $\sigma$ in the Kato class $\bar{\mathbf{K}}^{d+1}_\delta$ (see definition below), for $\delta=0$, running perturbation-theoretic techniques in $C_b$, thus obtaining e.g. a Brownian motion drifting upward when penetrating certain fractal-like sets. We strengthen their result in Theorem 2 below.

DEFINITION 4. A $\mathbb{C}^d$-valued Borel measure $\sigma$ on $\mathbb{R}^d$ is said to belong to $\bar{\mathbf{F}}^{\frac{1}{2}}_\delta$, the class of weakly form-bounded measures, if there exists $\lambda=\lambda_\delta>0$ such that

$$\int_{\mathbb{R}^d}\bigg((\lambda-\Delta)^{-\frac{1}{4}}(x,y)f(y)dy\bigg)^2|\sigma|(dx)\leqslant\delta\|f\|_2^2,\quad f\in\mathcal{S}.,$$

where $|\sigma|:=|\sigma_1|+\dots+|\sigma_d|$ is the variation of $\sigma$. Clearly, $\mathbf{F}^{\frac{1}{2}}_\delta\subset\bar{\mathbf{F}}^{\frac{1}{2}}_\delta$.

DEFINITION 5. A $\mathbb{C}^d$-valued Borel measure $\sigma$ on $\mathbb{R}^d$ is said to belong to the Kato class $\bar{\mathbf{K}}^{d+1}_\delta$ if there exists $\lambda=\lambda_\delta>0$ such that

$$\sup_{x\in\mathbb{R}^d}\int_{\mathbb{R}^d}(\lambda-\Delta)^{-\frac{1}{2}}(x,y)|\sigma|(dy)\leqslant\delta.$$

See [BC] for examples of measures in $\bar{\mathbf{K}}^{d+1}_0$.

It is clear that $\mathbf{K}^{d+1}_\delta\subset\bar{\mathbf{K}}^{d+1}_\delta$. By Lemma 1 below, $\bar{\mathbf{K}}^{d+1}_\delta\subset\bar{\mathbf{F}}^{\frac{1}{2}}_\delta$.

The operator-valued function $\Theta_p(\zeta,\sigma)$, $\operatorname{Re}\zeta>\frac{d}{d-1}\lambda_\delta$ (see above), 'a candidate' for the resolvent of the desired operator realization of $-\Delta+\sigma\cdot\nabla$ generating a $C_0$-semigroup on $C_\infty$, is not well defined for a $\sigma$ having non-zero singular part. We modify the method in [Ki]. Also, in contrast to the setup of [Ki], a general $\sigma$ doesn't admit a monotone approximation by regular vector fields $v_k$ (i.e. by $v_k\mathcal{L}^d$), which complicates the proof of convergence $\Theta_2(\zeta,v_k\mathcal{L}^d)\overset{s}{\to}\Theta_2(\zeta,\sigma)$ in $L^2$, needed to carry out the method. We overcome this difficulty using an important variant of the Kato-Ponce inequality by [GO] (see also [BL]) (Proposition 5 below).

Our method depends on the fact that the operators $-\Delta$, $\nabla$ constituting $-\Delta+\sigma\cdot\nabla$ commute. In particular, our method admits a straightforward generalization to $(-\Delta)^{\frac{\alpha}{2}}+\sigma\cdot\nabla$, where $(-\Delta)^{\frac{\alpha}{2}}$ is the fractional Laplacian, $1<\alpha<2$, with measure $\sigma$ weakly form-bounded with respect to $\Delta^{\alpha-1}$, i.e.

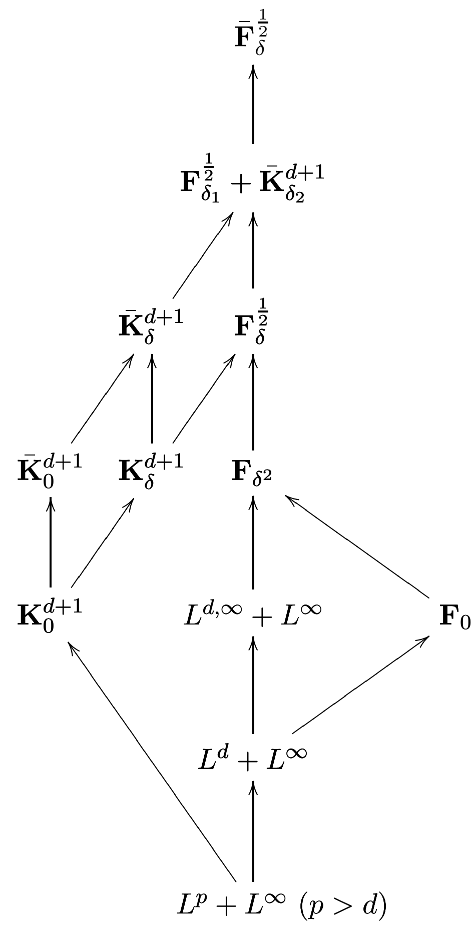
$$\int_{\mathbb{R}^d}\bigg((\lambda-\Delta)^{-\frac{\alpha-1}{4}}(x,y)f(y)dy\bigg)^2|\sigma|(dx)\leqslant\delta\|f\|_2^2,\quad f\in\mathcal{S}$$

for some $\lambda=\lambda_\delta>0$. (We note that the potential theory of operator $-\Delta^{\frac{\alpha}{2}}$ perturbed by a drift in the corresponding Kato class, as well as its associated process, attracted a lot of attention recently, see [BJ, CKS, KSo] and references therein.)

In Theorems 1, 2 (but not in Corollary 1) we assume that $\sigma$ admits an approximation by (weakly) form-bounded measures $\ll\mathcal{L}^d$ having the same form-bound $\delta$ (in fact, $\delta+\varepsilon$, for an arbitrarily small $\varepsilon>0$ independent of $k$). We verify this assumption for $\sigma=b\mathcal{L}^d+\nu$,

$$b\mathcal{L}^d\in\bar{\mathbf{F}}^{\frac{1}{2}}_{\delta_1},\qquad\nu\in\bar{\mathbf{K}}^{d+1}_{\delta_2},\qquad\sqrt{\delta}:=\sqrt{\delta_1}+\sqrt{\delta_2},$$

but do not address, in this note, the issue of constructing such an approximation for a general $\sigma$; we also do not address the issue (we believe, related) of constructing weakly form-bounded vector fields whose singularities are principally different from those of $\mathbf{F}_{\delta_1^2}+\mathbf{K}^{d+1}_{\delta_2}$ (cf. (1)).




The general classes of drifts studied in the literature in connection with operator $-\Delta + \sigma \cdot \nabla$.
Here we identify $b(x)$ with $b(x)\mathcal{L}^d$.

4. We proceed to precise formulations of our results.

Notation. Let

$$m_d := \inf_{\kappa>0} \sup_{\substack{x \neq y, \\ \operatorname{Re}\zeta>0}} \frac{|\nabla(\zeta - \Delta)^{-1}(x,y)|}{(\kappa^{-1}\operatorname{Re}\zeta - \Delta)^{-\frac{1}{2}}(x,y)} \tag{2}$$

(note that $m_d$ is bounded from above by $\pi^{\frac{1}{2}}(2e)^{-\frac{1}{2}}d^{\frac{d}{2}}(d-1)^{\frac{1-d}{2}} < \infty$, see [Ki, (A.1)]),

$$\mathcal{J} := \left(1 + \frac{1}{1+\sqrt{1-m_d\delta}}, 1 + \frac{1}{1-\sqrt{1-m_d\delta}}\right).$$

**Theorem 1** ($L^p$-theory of $-\Delta + \sigma \cdot \nabla$)**.** *Let $d \geqslant 3$. Assume that $\sigma$ is a $\mathbb{C}^d$-valued Borel measure in $\bar{\mathbf{F}}_\delta^{\frac{1}{2}}$ such that $\sigma = b\mathcal{L}^d + \nu$, where $b : \mathbb{R}^d \to \mathbb{C}^d$,*

$$b\mathcal{L}^d \in \bar{\mathbf{F}}_{\delta_1}^{\frac{1}{2}}, \qquad \nu \in \bar{\mathbf{K}}_{\delta_2}^{d+1}, \qquad \sqrt{\delta} := \sqrt{\delta_1} + \sqrt{\delta_2},$$

*or, more generally (see Lemma 1 below),* $\sigma \in \bar{\mathbf{F}}_\delta^{\frac{1}{2}}(\lambda)$ *is such that there exist* $v_k \in C_0^\infty(\mathbb{R}^d,\mathbb{C}^d)$, $v_k\mathcal{L}^d \in \bar{\mathbf{F}}_\delta^{\frac{1}{2}}(\lambda)$, $v_k\mathcal{L}^d \xrightarrow{w} \sigma$.

*If* $m_d\delta < 1$*, then for every* $p \in \mathcal{J}$*:*

*(i) There exists a holomorphic* $C_0$*-semigroup* $e^{-t\Lambda_p(\sigma)}$ *in* $L^p$ *such that, possibly after replacing* $v_k\mathcal{L}^d$*'s with a sequence of their convex combinations (also weakly converging to measure* $\sigma$*), we have*

$$e^{-t\Lambda_p(v_k\mathcal{L}^d)} \overset{s}{\to} e^{-t\Lambda_p(\sigma)} \text{ in } L^p,$$

*as* $k \uparrow \infty$*, where*

$$\Lambda_p(v_k\mathcal{L}^d) := -\Delta + v_k\cdot\nabla, \quad D(\Lambda_p(v_k\mathcal{L}^d)) = W^{2,p}.$$

*(ii) The resolvent set* $\rho(-\Lambda_p(\sigma))$ *contains a half-plane* $\mathcal{O} \subset \{\zeta \in \mathbb{C} : \operatorname{Re}\zeta > 0\}$*, and the resolvent* $(\zeta+\Lambda_p(\sigma))^{-1}$*,* $\zeta \in \mathcal{O}$*, admits an extension by continuity to a bounded linear operator in* $\mathcal{B}\left(\mathcal{W}^{-\frac{1}{r'},p},\mathcal{W}^{1+\frac{1}{q},p}\right)$*, where* $1 \leqslant r < \min\{2,p\}$*,* $\max\{2,p\} < q$*.*

*(iii) The domain of the generator* $D\big(\Lambda_p(\sigma)\big) \subset \mathcal{W}^{1+\frac{1}{q},p}$ *for every* $q > \max\{p,2\}$*.*

Remarks. **I.** If $\sigma \ll \mathcal{L}^d$, then the interval $\mathcal{J} \ni p$ in Theorem 1 can be extended, see [Ki] (in [Ki] we work directly in $L^p$, while in the proof of Theorem 1 we have to first prove our convergence results in $L^2$, and then transfer them to $L^p$ (Proposition 7), hence the more restrictive assumptions on $p$).

**II**. A straightforward modification of the proof of Theorem 1 yields:

**Corollary 1** ($L^p$-theory of $-\Delta+\Psi$)**.** *Let* $d \geqslant 3$*. Assume that* $\Psi$ *is a* $\mathbb{C}$*-valued Borel measure such that*

$$\int_{\mathbb{R}^d}\bigg((\lambda-\Delta)^{-\frac{1}{2}}(x,y)f(y)dy\bigg)^2|\Psi|(dx) \leqslant \delta\|f\|_2^2, \quad f \in \mathcal{S},$$

*for some* $\lambda = \lambda_\delta > 0$*. We write* $\Psi \in \bar{\mathbf{F}}_\delta(\Delta,\lambda)$*. Set* $V_k := \rho_k e^{\varepsilon_k\Delta}\Psi$*,* $\varepsilon_k \downarrow 0$*, where* $\rho_k \in C_0^\infty$*,* $0 \leqslant \rho_k \leqslant 1$*,* $\rho \equiv 1$ *in* $\{|x| \le k\}$*,* $\rho \equiv 0$ *in* $\{|x| \ge k+1\}$*, so that*

$$V_k\mathcal{L}^d \in \bar{\mathbf{F}}_\delta(\Delta,\lambda) \text{ for all } k, \qquad V_k\mathcal{L}^d \overset{w}{\to} \Psi \text{ as } k\uparrow\infty$$

*(see Lemma 2 below). If* $\delta < 1$*, then for every* $p \in \big(1+\frac{1}{1+\sqrt{1-\delta}},1+\frac{1}{1-\sqrt{1-\delta}}\big)$ *there exists a holomorphic* $C_0$*-semigroup* $e^{-t\Pi_p(\Psi)}$ *in* $L^p$ *such that*

$$e^{-t\Pi_p(V_k\mathcal{L}^d)} \overset{s}{\to} e^{-t\Pi_p(\Psi)} \text{ in } L^p,$$

*where* $\Pi_p(V_k\mathcal{L}^d) := -\Delta + V_k$*,* $D(\Pi_p(V_k\mathcal{L}^d)) = W^{2,p}$*, possibly after replacing* $V_k\mathcal{L}^d$*'s with a sequence of their convex combinations (also weakly converging to* $\Psi$*), and the domain of the generator* $D\big(\Pi_p(\Psi)\big) \subset \mathcal{W}^{\frac{1}{q},p}$*, for any* $q > \max\{2,p\}$*.*

Corollary 1 extends the results in [AM, SV] (applied to operator $-\Delta+\Psi$), where a real-valued $\Psi$ is assumed to be in the Kato class $\bar{\mathbf{K}}_\delta^d$ of measures (e.g. delta-function concentrated on a hypersurface). One disadvantage of Corollary 1, compared to [AM, SV], is that it requires $|\Psi| \leqslant \delta(\lambda-\Delta)$ (in the sense of quadratic forms) rather than $\Psi_- \leqslant \delta(\lambda-\Delta+\Psi_+)$, where $\Psi = \Psi_+ - \Psi_-$, $\Psi_+,\Psi_- \geqslant 0$.

The purpose of Theorem 1 is to prove

**Theorem 2** ($C_\infty$-theory of $-\Delta+\sigma\cdot\nabla$)**.** *Let $d \geqslant 3$. Assume that $\sigma$ is a $\mathbb{R}^d$-valued Borel measure in $\bar{\mathbf{F}}_\delta^{\frac{1}{2}}$ such that $\sigma = b\mathcal{L}^d+\nu$, where $b:\mathbb{R}^d\to\mathbb{R}^d$,*

$$b\mathcal{L}^d \in \bar{\mathbf{F}}_{\delta_1}^{\frac{1}{2}}, \qquad \nu \in \bar{\mathbf{K}}_{\delta_2}^{d+1}, \qquad \sqrt{\delta} := \sqrt{\delta_1}+\sqrt{\delta_2},$$

*or, more generally (see Lemma 1 below), $\sigma \in \bar{\mathbf{F}}_\delta^{\frac{1}{2}}(\lambda)$ is such that there exist $v_k \in C_0^\infty(\mathbb{R}^d,\mathbb{R}^d)$, $v_k\mathcal{L}^d \in \bar{\mathbf{F}}_\delta^{\frac{1}{2}}(\lambda)$, $v_k\mathcal{L}^d \xrightarrow{w} \sigma$.*

*If $m_d\delta < \frac{2d-5}{(d-2)^2}$, then:*

*$(i)$ There exists a positivity preserving contraction $C_0$-semigroup $e^{-t\Lambda_{C_\infty}(\sigma)}$ on $C_\infty$ such that , possibly after replacing $v_k\mathcal{L}^d$'s with a sequence of their convex combinations (also weakly converging to measure $\sigma$) we have*

$$e^{-t\Lambda_{C_\infty}(v_k\mathcal{L}^d)} \xrightarrow{s} e^{-t\Lambda_{C_\infty}(\sigma)} \text{ in } C_\infty, \quad t\geqslant 0,$$

*as $k\uparrow\infty$, where*

$$\Lambda_{C_\infty}(v_k\mathcal{L}^d) := -\Delta + v_k\cdot\nabla, \quad D(\Lambda_{C_\infty}(v_k\mathcal{L}^d)) = C^2\cap C_\infty.$$

*$(ii)$ [Strong Feller property] $(\mu+\Lambda_{C_\infty}(\sigma))^{-1}|_{\mathcal{S}}$ can be extended by continuity to a bounded linear operator in $\mathcal{B}(L^p, C^{0,\gamma})$, $\gamma<1-\frac{d-1}{p}$, for every $d-1<p<1+\frac{1}{1-\sqrt{1-m_d\delta}}$.*

*$(iii)$ The integral kernel $e^{-t\Lambda_{C_\infty}(\sigma)}(x,y)$ $(x,y\in\mathbb{R}^d)$ of $e^{-t\Lambda_{C_\infty}(\sigma)}$ determines the (sub-Markov) transition probability function of a Feller process.*

Remark. If $\sigma \ll \mathcal{L}^d$, then the constraint on $\delta$ in Theorem 2 can be relaxed, see [Ki], cf. Remark I above.

## 1. Approximating measures

1. **In Theorems 1 and 2.** Suppose $\sigma = b\mathcal{L}^d+\nu$, where $b:\mathbb{R}^d\to\mathbb{C}^d$, $b\mathcal{L}^d \in \bar{\mathbf{F}}_{\delta_1}^{\frac{1}{2}}(\lambda)$, and $\nu\in\bar{\mathbf{K}}_{\delta_2}^{d+1}(\lambda)$. The following statement is a part of Theorems 1 and 2.

**Lemma 1.** *There exist vector fields $v_k\in C_0^\infty(\mathbb{R}^d,\mathbb{C}^d)$, $k=1,2,\dots$ such that*
(1) *$v_k\mathcal{L}^d\in\bar{\mathbf{F}}_\delta^{\frac{1}{2}}(\lambda)$, $\sqrt{\delta}:=\sqrt{\delta_1}+\sqrt{\delta_2}$, for every $k$, and*
(2) *$v_k\mathcal{L}^d\xrightarrow{w}\sigma$ as $k\uparrow\infty$.*

*Proof.* We fix functions $\rho_k\in C_0^\infty$, $0\leqslant\rho_k\leqslant 1$, $\rho\equiv 1$ in $\{|x|\leq k\}$, $\rho\equiv 0$ in $\{|x|\geq k+1\}$, and define

$$v_k\mathcal{L}^d := b_k\mathcal{L}^d+\nu_k,$$

where, for some fixed $\varepsilon_k\downarrow 0$,

$$\nu_k := \rho_k e^{\varepsilon_k\Delta}\nu, \quad b_k := \rho_k e^{\varepsilon_k\Delta} b.$$

It is clear that $v_k\in C_0^\infty(\mathbb{R}^d,\mathbb{R}^d)$ and $v_k\mathcal{L}^d\xrightarrow{w}\sigma$ as $k\uparrow\infty$. Let us show that $\nu_k\in\bar{\mathbf{K}}_{\delta_2}^{d+1}(\lambda)$ for every $k$. Indeed, we have the following pointwise (a.e.) estimates on $\mathbb{R}^d$:

$$(\lambda-\Delta)^{-\frac{1}{2}}|\nu_k| \leqslant (\lambda-\Delta)^{-\frac{1}{2}}|e^{\varepsilon_k\Delta}\nu| \leqslant (\lambda-\Delta)^{-\frac{1}{2}}e^{\varepsilon_k\Delta}|\nu| = e^{\varepsilon_k\Delta}(\lambda-\Delta)^{-\frac{1}{2}}|\nu|.$$

Since $\|e^{\varepsilon_k \Delta}(\lambda-\Delta)^{-\frac{1}{2}}|\nu|\|_\infty \leqslant \|(\lambda-\Delta)^{-\frac{1}{2}}|\nu|\|_\infty$ and, in turn, $\|(\lambda-\Delta)^{-\frac{1}{2}}|\nu|\|_\infty \leqslant \delta_2$ ($\Leftrightarrow \nu \in \bar{\mathbf{K}}^{d+1}_{\delta_2}(\lambda)$), we have $\nu_k \in \bar{\mathbf{K}}^{d+1}_{\delta_2}(\lambda)$. By interpolation, $\nu_k \in \bar{\mathbf{F}}^{\frac{1}{2}}_{\delta_1}(\lambda)$. A similar argument yields $b_k \mathcal{L}^d \in \bar{\mathbf{F}}^{\frac{1}{2}}_{\delta_1}(\lambda)$. Thus, $v_k \mathcal{L}^d \in \bar{\mathbf{F}}^{\frac{1}{2}}_{\delta}(\lambda)$, for every $k$. □

2. **In Corollary 1.** Suppose $\Psi \in \bar{\mathbf{F}}_\delta(\Delta, \lambda)$. Select $\rho_k \in C_0^\infty$, $0 \leqslant \rho_k \leqslant 1$, $\rho \equiv 1$ in $\{|x| \leq k\}$, $\rho \equiv 0$ in $\{|x| \geq k+1\}$. Fix some $\varepsilon_k \downarrow 0$.

**Lemma 2.** *We have* $V_k := \rho_k e^{\varepsilon_k \Delta}\Psi \in C_0^\infty(\mathbb{R}^d)$, *and*

(1) $V_k \mathcal{L}^d \in \bar{\mathbf{F}}_\delta(\Delta, \lambda)$ *for every* $k$,

(2) $V_k \mathcal{L}^d \overset{w}{\to} \Psi$ *as* $k \uparrow \infty$.

*Proof.* Assertion (2) is immediate. Let us prove (1). It is clear that $V_k \mathcal{L}^d \in \bar{\mathbf{F}}_\delta(\Delta, \lambda)$ if and only if

$$\langle |V_k| \varphi, \varphi \rangle \leqslant \delta \langle (\lambda - \Delta)^{\frac{1}{2}} \varphi, (\lambda - \Delta)^{\frac{1}{2}} \varphi \rangle, \qquad \varphi \in \mathcal{S}.$$

We have $|V_k| = \rho_k e^{\varepsilon_k \Delta}|\Psi| \leqslant e^{\varepsilon_k \Delta}|\Psi|$, so

$$\begin{aligned}
\langle |V_k|\varphi, \varphi\rangle &\leqslant \langle e^{\varepsilon_k \Delta}|\Psi|\varphi, \varphi\rangle = \langle |\Psi|, e^{\varepsilon_k\Delta}(\varphi^2)\rangle \qquad \Big(\text{since } \Psi \in \bar{\mathbf{F}}_\delta(\Delta)\Big) \\
&\leqslant \delta \left\langle \left((\lambda - \Delta)^{\frac{1}{2}} [e^{\varepsilon_k \Delta}(\varphi^2)]^{\frac{1}{2}}\right)^2 \right\rangle = \delta \left\langle (\lambda - \Delta)[e^{\varepsilon_k \Delta}(\varphi^2)]^{\frac{1}{2}}, [e^{\varepsilon_k \Delta}(\varphi^2)]^{\frac{1}{2}} \right\rangle \\
&= \delta \langle e^{\varepsilon_k \Delta}\varphi^2\rangle + \delta \langle \nabla [e^{\varepsilon_k \Delta}(\varphi^2)]^{\frac{1}{2}}, \nabla [e^{\varepsilon_k \Delta}(\varphi^2)]^{\frac{1}{2}}\rangle \qquad \Big(\text{we are using } \langle e^{\varepsilon_k\Delta}\varphi^2\rangle = \langle \varphi^2\rangle\Big) \\
&= \delta\langle \varphi^2\rangle + \delta \langle (e^{\varepsilon_k \Delta}\varphi^2)^{-1}(e^{\varepsilon_k \Delta}\varphi \nabla\varphi)^2\rangle \qquad \Big(\text{by Hölder inequality}\Big) \\
&\leqslant \delta\langle\varphi^2\rangle + \delta\langle e^{\varepsilon_k\Delta}(\nabla\varphi)^2\rangle = \langle (\lambda - \Delta)^{\frac{1}{2}}\varphi, (\lambda-\Delta)^{\frac{1}{2}}\varphi\rangle,
\end{aligned}$$

as needed. □

## 2. Proof of Theorem 1

**Preliminaries. 1.** By Lemma 1, there exist vector fields $v_k \in C_0^\infty(\mathbb{R}^d, \mathbb{C}^d)$, $k = 1, 2, \dots$, such that $v_k \mathcal{L}^d \in \bar{\mathbf{F}}^{\frac{1}{2}}_{\delta}(\lambda)$, $\sqrt{\delta} := \sqrt{\delta_1} + \sqrt{\delta_2}$, and $v_k \mathcal{L}^d \overset{w}{\longrightarrow} \sigma$ as $k \uparrow \infty$.

**2.** Due to the strict inequality $m_d \delta < 1$, we may assume that the infimum $m_d$ (cf. (2)) is attained, i.e. there is $\kappa_d > 0$

$$|\nabla(\zeta - \Delta)^{-1}(x,y)| \leqslant m_d \Big(\kappa_d^{-1} \mathrm{Re}\,\zeta - \Delta\Big)^{-\frac{1}{2}}(x,y), \qquad x, y \in \mathbb{R}^d,\, x \neq y,\, \mathrm{Re}\,\zeta > 0.$$

**3.** Set $\mathcal{O} := \{\zeta \in \mathbb{C} : \mathrm{Re}\,\zeta \geqslant \kappa_d \lambda_\delta\}$,

**The method of proof.** We modify the method of [Ki]. Fix some $p \in \mathcal{J}$, and some $r, q$ satisfying $1 \leqslant r < \min\{2, p\} \leqslant \max\{2, p\} < q$. Our starting object is an operator-valued function

$$\Theta_p(\zeta, \sigma) := (\zeta - \Delta)^{-\frac{1}{2} - \frac{1}{2q}} \Omega_p(\zeta, \sigma, q, r)(\zeta - \Delta)^{-\frac{1}{2r'}} \in \mathcal{B}(L^p), \quad \zeta \in \mathcal{O},$$

which is 'a candidate' for the resolvent of the desired operator realization $\Lambda_p(\sigma)$ of $-\Delta + \sigma \cdot \nabla$ on $L^p$. Here

$$\Omega_p(\zeta,\sigma,q,r) := \left(\Omega_2(\zeta,\sigma,q,r)\Big|_{L^p\cap L^2}\right)^{\text{clos}}_{L^p} \in \mathcal{B}(L^p), \tag{3}$$

where, on $L^2$,

$$\Omega_2(\zeta,\sigma,q,r) := (\zeta-\Delta)^{-\frac{1}{2}\left(\frac{1}{2}-\frac{1}{q}\right)}(1+Z_2(\zeta,\sigma))^{-1}(\zeta-\Delta)^{-\frac{1}{2}\left(\frac{1}{2}-\frac{1}{r'}\right)} \in \mathcal{B}(L^2),$$

$$\begin{aligned} Z_2(\zeta,\sigma)h(x) &:= (\zeta-\Delta)^{-\frac{1}{4}}\sigma\cdot\nabla(\zeta-\Delta)^{-\frac{3}{4}}h(x) \\ &= \int_{\mathbb{R}^d}(\zeta-\Delta)^{-\frac{1}{4}}(x,y)\bigg(\int_{\mathbb{R}^d}\nabla(\zeta-\Delta)^{-\frac{3}{4}}(y,z)h(z)dz\bigg)\cdot\sigma(y)dy, \quad x\in\mathbb{R}^d, \quad h\in\mathcal{S}, \end{aligned}$$

and $\|Z_2\|_{2\to 2} < 1$, so $\Omega_2(\zeta,\sigma,q,r) \in \mathcal{B}(L^2)$, see Proposition 1 below. We prove that $\Omega_p(\zeta,\sigma,q,r) \in \mathcal{B}(L^p)$ in Proposition 6 below.

We show that $\Theta_p(\zeta,\sigma)$ is the resolvent of $\Lambda_p(\sigma)$ (assertion $(i)$ of Theorem 1) by verifying conditions of the Trotter approximation theorem:

1) $\Theta_p(\zeta,v_k\mathcal{L}^d) = (\zeta+\Lambda_p(v_k\mathcal{L}^d))^{-1}$, $\zeta\in\mathcal{O}$, where $\Lambda_p(v_k\mathcal{L}^d) := -\Delta + v_k\cdot\nabla$, $D(\Lambda_p(v_k\mathcal{L}^d)) = W^{2,p}$.

2) $\sup_{n\geqslant 1}\|\Theta_p(\zeta,v_k\mathcal{L}^d)\|_{p\to p} \leqslant C_p|\zeta|^{-1}$, $\zeta\in\mathcal{O}$.

3) $\mu\Theta_p(\zeta,v_k\mathcal{L}^d) \overset{s}{\to} 1$ in $L^p$ as $\mu\uparrow\infty$ uniformly in $k$.

4) $\Theta_p(\zeta,v_k\mathcal{L}^d) \overset{s}{\to} \Theta_p(\zeta,\sigma)$ in $L^p$ for some $\zeta\in\mathcal{O}$ as $k\uparrow\infty$ (possibly after replacing $v_k\mathcal{L}^d$'s with a sequence of their convex combinations, also weakly converging to measure $\sigma$), see Propositions 2 - 7 below for details.

We note that a priori in 1) the set of $\zeta$'s for which $\Theta_p(\zeta,v_k\mathcal{L}^d) = (\zeta+\Lambda_p(v_k\mathcal{L}^d))^{-1}$ may depend on $k$; the fact that it actually does not is the content of Proposition 3.

The proofs of 2), 3), contained in Proposition 2 and 4, are based on an explicit representation of $\Omega_p(\zeta,v_k\mathcal{L}^d,q,r)$, $k=1,2,\dots$, see formula (4) below. (The representation (4) doesn't exist if $\sigma$ has a non-zero singular part; we have to take a detour via $L^2$, (cf. (3)), which requires us to put somewhat more restrictive assumptions on $\delta$ (compared to [Ki], where the case of a $\sigma$ having zero singular part is treated).)

Next, 4) follows from $\Theta_2(\zeta,v_k\mathcal{L}^d) \overset{s}{\to} \Theta_2(\zeta,\sigma)$, combined with $\sup_n\|\Theta_p(\zeta,v_k\mathcal{L}^d)\|_{2(p-1)\to 2(p-1)} < \infty$ ($\Leftarrow$ 2)) and Hölder inequality, see Proposition 7. Our proof of $\Theta_2(\zeta,v_k\mathcal{L}^d) \overset{s}{\to} \Theta_2(\zeta,\sigma)$ (Proposition 5) uses the Kato-Ponce inequality by [GO].

Finally, we note that the very definition of the operator-valued function $\Theta_p(\zeta,\sigma)$ ensures smoothing properties $\Theta_p(\zeta,\sigma) \in \mathcal{B}\Big(\mathcal{W}^{-\frac{1}{r'},p},\mathcal{W}^{1+\frac{1}{q},p}\Big) \Rightarrow$ assertion $(ii)$. Assertion $(iii)$ is immediate from $(ii)$.

Now, we proceed to formulating and proving Propositions 1 - 7.

**Proposition 1.** *We have for every* $\zeta\in\mathcal{O}$

(1) $\|Z_2(\zeta,v_k\mathcal{L}^d)\|_{2\to 2} \leqslant \delta$ *for all* $k$.

(2) $\|Z_2(\zeta,\sigma)f\|_2 \leqslant \delta\|f\|_2$, *for all* $f\in\mathcal{S}$, *all* $k$.

*Proof.* (1) Define $H := |v_k|^{\frac{1}{2}}(\zeta-\Delta)^{-\frac{1}{4}}$, $S := v_k^{\frac{1}{2}}\nabla(\zeta-\Delta)^{-\frac{3}{4}}$ where $v_k^{\frac{1}{2}} := |v_k|^{-\frac{1}{2}}v_k$. Then $Z_2(\zeta, v_k\mathcal{L}^d) = H^*S$, and we have

$$\|Z_2(\zeta, v_k\mathcal{L}^d)\|_{2\to 2} \leqslant \|H\|_{2\to 2}\|S\|_{2\to 2} \leqslant \|H\|^2_{2\to 2}\|\nabla(\zeta-\Delta)^{-\frac{1}{2}}\|_{2\to 2} \leqslant \delta,$$

where $\|\nabla(\zeta-\Delta)^{-\frac{1}{2}}\|_{2\to 2} = 1$, and $\|H\|^2_{2\to 2} \leqslant \delta$ (cf. Lemma 1(1)).

(2) We have, for every $f$, $g \in \mathcal{S}$,

$$\begin{aligned}\langle g, Z_2(\zeta,\sigma)f\rangle =& \langle (\zeta-\Delta)^{-\frac{1}{4}}g, \sigma\cdot\nabla(\zeta-\Delta)^{-\frac{3}{4}}f\rangle \\ & \text{(here we are using } v_k\mathcal{L}^d \xrightarrow{w} \sigma) \\ &= \lim_k \langle (\zeta-\Delta)^{-\frac{1}{4}}g, v_k\cdot\nabla(\zeta-\Delta)^{-\frac{3}{4}}f\rangle \\ & \text{(here we are using assertion (1))} \\ & \leqslant \delta\|g\|_2\|f\|_2,\end{aligned}$$

i.e. $\|Z_2(\zeta,\sigma)f\|_2 \leqslant \delta\|f\|_2$, as needed. □

The natural extension of $Z_2(\zeta,\sigma)|_{\mathcal{S}}$ (by continuity) to $\mathcal{B}(L^2)$ will be denoted again by $Z_2(\zeta,\sigma)$. Since $\|Z_2(\zeta, v_k\mathcal{L}^d)\|_{2\to 2}, \|Z_2(\zeta,\sigma)\|_{2\to 2} \leqslant \delta < 1$, we have $\Omega_2(\zeta, v_k\mathcal{L}^d, q, r), \Omega_2(\zeta,\sigma,q,r) \in \mathcal{B}(L^2)$.

Set

$$\mathcal{I} := \left(\frac{2}{1+\sqrt{1-m_d\delta}}, \frac{2}{1-\sqrt{1-m_d\delta}}\right).$$

In the next few propositions, given a $p \in \mathcal{I}$, we assume $r, q$ satisfy $1 \leqslant r < \min\{2,p\} \leqslant \max\{2,p\} < q$.

The following proposition plays a principal role:

**Proposition 2.** *Let $p \in \mathcal{I}$. There exist constants $C_p$, $C_{p,q,r} < \infty$ such that for every $\zeta \in \mathcal{O}$*

(1) $\|\Omega_p(\zeta, v_k\mathcal{L}^d, q, r)\|_{p\to p} \leqslant C_{p,q,r}$ *for all $k$,*

(2) $\|\Omega_p(\zeta, v_k\mathcal{L}^d, \infty, 1)\|_{p\to p} \leqslant C_p|\zeta|^{-\frac{1}{2}}$ *for all $k$.*

*Proof.* Denote $v_k^{\frac{1}{p}} := |v_k|^{\frac{1}{p}-1}v_k$. Set:

$$\tilde{\Omega}_p(\zeta, v\mathcal{L}^d, q, r) := Q_p(q)(1+T_p)^{-1}G_p(r), \quad \zeta \in \mathcal{O}, \tag{4}$$

where

$$Q_p(q) := (\zeta-\Delta)^{-\frac{1}{2q'}}|v_k|^{\frac{1}{p'}}, \quad T_p := v_k^{\frac{1}{p}}\cdot\nabla(\zeta-\Delta)^{-1}|v|^{\frac{1}{p'}}, \quad G_p(r) := v_k^{\frac{1}{p}}\cdot\nabla(\zeta-\Delta)^{-\frac{1}{2}-\frac{1}{2r}},$$

are uniformly (in $k$) bounded in $\mathcal{B}(L^p)$, and, in particular, $\|T_p\|_{p\to p} \leqslant \frac{pp'}{4}m_d\delta$ (see the proof of [Ki, Prop. 1($i$)]), and $\frac{pp'}{4}m_d\delta < 1$ since $p \in \mathcal{I}$. It follows that $C_{p,q,r} := \sup_k \|\tilde{\Omega}_p(\zeta, v\mathcal{L}^d, q, r)\|_{p\to p} < \infty$. Now, $\tilde{\Omega}_p|_{L^2\cap L^p} = \Omega_2|_{L^2\cap L^p}$ (by expanding $(1+T_p)^{-1}$, $(1+Z_2)^{-1}$ in the K. Neumann series in $L^p$ and in $L^2$, respectively). Therefore, $\tilde{\Omega}_p = \Omega_p \Rightarrow$ assertion (1). The proof of assertion (2) follows closely the proof of [Ki, Prop. 1($ii$)]. □

Clearly, $\Theta_p(\zeta, v_k\mathcal{L}^d)$ does not depend on $q$, $r$. Taking $q = \infty$, $r = 1$, we obtain from Proposition 2:

$$\|\Theta_p(\zeta, v_k\mathcal{L}^d)\|_{p\to p} \leqslant C_p|\zeta|^{-1}, \quad \zeta \in \mathcal{O}. \tag{5}$$

**Proposition 3.** *Let $p \in \mathcal{I}$. For every $k=1,2,\dots$ $\mathcal{O} \subset \rho(-\Lambda_p(v_k\mathcal{L}^d))$, the resolvent set of $-\Lambda_p(v_k\mathcal{L}^d)$, and*

$$\Theta_p(\zeta, v_k\mathcal{L}^d) = (\zeta + \Lambda_p(v_k\mathcal{L}^d))^{-1}, \quad \zeta \in \mathcal{O},$$

*where $\Lambda_p(v_k\mathcal{L}^d) := -\Delta + v_k \cdot \nabla$, $D(\Lambda_{C_\infty}(v_k\mathcal{L}^d)) = W^{2,p}$.*

*Proof.* The proof repeats the proof of [Ki, Prop. 4]. □

**Proposition 4.** *For $p \in \mathcal{I}$, $\mu\Theta_p(\mu, v_k\mathcal{L}^d) \overset{s}{\rightarrow} 1$ in $L^p$ as $\mu \uparrow \infty$ uniformly in $k$.*

*Proof.* The proof repeats the proof of [Ki, Prop. 3]. □

**Proposition 5.** *There exists a sequence $\{\hat{v}_n\} \subset \mathrm{conv}\{v_k\} \subset C_0^\infty(\mathbb{R}^d, \mathbb{R}^d)$ such that*

$$\hat{v}_n\mathcal{L}^d \overset{w}{\longrightarrow} \sigma \text{ as } n \uparrow \infty, \tag{6}$$

*and*

$$\Omega_2(\zeta, \hat{v}_n\mathcal{L}^d, q, r) \overset{s}{\rightarrow} \Omega_2(\zeta, \sigma, q, r) \text{ in } L^2, \quad \zeta \in \mathcal{O}. \tag{7}$$

*Proof.* To prove (7), it suffices to establish convergence $Z_2(\zeta, \hat{v}_n\mathcal{L}^d) \overset{s}{\rightarrow} Z_2(\zeta, \sigma)$ in $L^2$, $\zeta \in \mathcal{O}$.

Let $\eta_r \in C_0^\infty$, $0 \leqslant \eta_r \leqslant 1$, $\eta_r \equiv 1$ on $\{x \in \mathbb{R}^d : |x| \leqslant r\}$ and $\eta_r \equiv 0$ on $\{x \in \mathbb{R}^d : |x| \geqslant r+1\}$.

*Claim* 1. *We have*

(*j*) $\|(\zeta - \Delta)^{-\frac{1}{4}}|v_k|(\zeta - \Delta)^{-\frac{1}{4}}\|_{2\to 2} \leqslant \delta$ *for all* $k$.

(*jj*) $\|(\zeta - \Delta)^{-\frac{1}{4}}|\sigma|(\zeta - \Delta)^{-\frac{1}{4}}f\|_2 \leqslant \delta\|f\|_2$, *for all* $f \in \mathcal{S}$.

*Proof.* Define $H := |v_k|^{\frac{1}{2}}(\zeta - \Delta)^{-\frac{1}{4}}$. We have $\|(\zeta - \Delta)^{-\frac{1}{4}}|v_k|(\zeta - \Delta)^{-\frac{1}{4}}\|_{2\to 2} = \|H^*H\|_{2\to 2} = \|H\|^2_{2\to 2} \leqslant \delta$, where $\|H\|^2_{2\to 2} \leqslant \delta$ ($\Leftrightarrow$ $v_k\mathcal{L}^d \in \bar{\mathbf{F}}_\delta^{\frac{1}{2}}(\lambda)$, cf. Lemma 1(1)), i.e. we have proved (*j*). An argument similar to the one in the proof of Proposition 1, but using assertion (*j*), yields (*jj*). □

*Claim* 2. *There exists a sequence $\{\hat{v}_n\} \subset \mathrm{conv}\{v_k\}$ such that* (6) *holds, and for every $r \geqslant 1$*

$$(\zeta - \Delta)^{-\frac{1}{4}}\eta_r(\hat{v}_n - \sigma) \cdot \nabla(\zeta - \Delta)^{-\frac{3}{4}} \overset{s}{\rightarrow} 0 \text{ in } L^2, \quad \mathrm{Re}\,\zeta \geqslant \lambda.$$

(*here and below we use shorthand $\hat{v}_n - \sigma := \hat{v}_n\mathcal{L}^d - \sigma$*).

*Proof of Claim 2.* In view of Claim 1(*j*), (*jj*), it suffices to establish this convergence over $\mathcal{S}$. Let $c(x) = e^{-x^2}$, so that $c \in \mathcal{S}$, $|(\zeta - \Delta)^{-\frac{1}{4}}c| > 0$ on $\mathbb{R}^d$.

*Step 1.* Let $r = 1$, so $\eta_r = \eta_1$. Let us show that there exists a sequence $\{v^1_{\ell_1}\} \subset \mathrm{conv}\{v_k\}$ such that

$$(\lambda - \Delta)^{-\frac{1}{4}}\eta_1(v^1_{\ell_1} - \sigma) \cdot \nabla(\lambda - \Delta)^{-\frac{3}{4}} \overset{s}{\rightarrow} 0 \text{ in } L^2 \text{ as } \ell_1 \uparrow \infty. \tag{8}$$

First, we show that

$$(\lambda - \Delta)^{-\frac{1}{4}}\eta_1(v_k - \sigma)(\lambda - \Delta)^{-\frac{1}{4}}c \overset{w}{\rightarrow} 0 \text{ in } L^2. \tag{9}$$

Indeed, by Claim 1(*j*), (*jj*), $\|(\lambda - \Delta)^{-\frac{1}{4}}\eta_1(v_k - \sigma)(\lambda - \Delta)^{-\frac{1}{4}}c\|_2 \leqslant 2\delta\|c\|_2$ for all $k$. Hence, there exists a subsequence of $\{v_k\}$ (without loss of generality, it is $\{v_k\}$ itself) such that $(\lambda - \Delta)^{-\frac{1}{4}}\eta_1(v_k -$

$\sigma)(\lambda-\Delta)^{-\frac{1}{4}}c \overset{w}{\rightarrow} h$ for some $h \in L^2$. Therefore, given any $f \in \mathcal{S}$, we have $\langle f, (\lambda-\Delta)^{-\frac{1}{4}}\eta_1(v_k-\sigma)(\lambda-\Delta)^{-\frac{1}{4}}c\rangle \rightarrow \langle f, h\rangle$. Along with that, since $v_k\mathcal{L}^d \overset{w}{\rightarrow} \sigma$, we also have

$$\langle f, (\lambda-\Delta)^{-\frac{1}{4}}\eta_1(v_k-\sigma)(\lambda-\Delta)^{-\frac{1}{4}}c\rangle = \langle (\lambda-\Delta)^{-\frac{1}{4}}f, \eta_1(v_k-\sigma)(\lambda-\Delta)^{-\frac{1}{4}}c\rangle \rightarrow 0,$$

i.e. $\langle f, h\rangle = 0$. Since $f \in \mathcal{S}$ was arbitrary, we have $h = 0$, which yields (9).

Now, in view of (9), by Mazur's Theorem, there exists a sequence $\{v^1_{\ell_1}\} \subset \operatorname{conv}\{v_k\}$ such that

$$(\lambda-\Delta)^{-\frac{1}{4}}\eta_1(v^1_{\ell_1}-\sigma)(\lambda-\Delta)^{-\frac{1}{4}}c \overset{s}{\rightarrow} 0 \text{ in } L^2. \tag{10}$$

We may assume without loss of generality that each $v^1_{\ell_1} \in \operatorname{conv}\{v_n\}_{n \geqslant \ell_1}$.

Next, set $\ell := \ell_1$, $\varphi_\ell := \eta_1(v^1_\ell - \sigma)$, $\Phi := (\lambda-\Delta)^{-\frac{1}{4}}c$, fix some $u \in \mathcal{S}$. We estimate:

$$\begin{aligned}
&\|(\lambda-\Delta)^{-\frac{1}{4}}\varphi_\ell \cdot \nabla(\lambda-\Delta)^{-\frac{3}{4}}u\|_2^2 \\
&= \left\langle \varphi_\ell \cdot \nabla(\lambda-\Delta)^{-\frac{3}{4}}u, (\lambda-\Delta)^{-\frac{1}{2}}\varphi_\ell \cdot \nabla(\lambda-\Delta)^{-\frac{3}{4}}u \right\rangle \\
&\left(\text{since } \varphi_\ell \equiv 0 \text{ on } \{|x| \geqslant 2\}, \text{ in the left multiple } \varphi_\ell = \varphi_\ell \Phi \frac{\eta_2}{\Phi}\right) \\
&= \left\langle \varphi_\ell \Phi \frac{\eta_2}{\Phi} \cdot \nabla(\lambda-\Delta)^{-\frac{3}{4}}u, (\lambda-\Delta)^{-\frac{1}{2}}\varphi_\ell \cdot \nabla(\lambda-\Delta)^{-\frac{3}{4}}u \right\rangle \\
&= \left\langle \varphi_\ell \Phi, \frac{\eta_2}{\Phi}\nabla(\lambda-\Delta)^{-\frac{3}{4}}u \left[(\lambda-\Delta)^{-\frac{1}{2}}\varphi_\ell \cdot \nabla(\lambda-\Delta)^{-\frac{3}{4}}u\right] \right\rangle \\
&\qquad \text{(here we are using in the left multiple that } \varphi_\ell = (\lambda-\Delta)^{\frac{1}{4}}(\lambda-\Delta)^{-\frac{1}{4}}\varphi_\ell) \\
&= \left\langle (\lambda-\Delta)^{-\frac{1}{4}}\varphi_\ell \Phi, (\lambda-\Delta)^{\frac{1}{4}}(fg_\ell) \right\rangle
\end{aligned}$$

where we set $f := \frac{\eta_2}{\Phi}\nabla(\lambda-\Delta)^{-\frac{3}{4}}u \in C_0^\infty(\mathbb{R}^d, \mathbb{R}^d)$, $g_\ell := (\lambda-\Delta)^{-\frac{1}{2}}\varphi_\ell \cdot \nabla(\lambda-\Delta)^{-\frac{3}{4}}u \in (\lambda-\Delta)^{-\frac{1}{4}}L^2$ (in view of Claim 1($j$), ($jj$)). Thus, in view of the above estimates,

$$\|(\lambda-\Delta)^{-\frac{1}{4}}\varphi_\ell \cdot \nabla(\lambda-\Delta)^{-\frac{3}{4}}u\|_2^2 \leqslant \|(\lambda-\Delta)^{-\frac{1}{4}}\varphi_\ell\Phi\|_2 \|(\lambda-\Delta)^{\frac{3}{4}}(fg_\ell)\|_2.$$

By the Kato-Ponce inequality of [GO, Theorem 1],

$$\|(\lambda-\Delta)^{\frac{1}{4}}(fg_\ell)\|_2 \leqslant C\Big(\|f\|_\infty \|(\lambda-\Delta)^{\frac{1}{4}}g_\ell\|_2 + \|(\lambda-\Delta)^{\frac{1}{4}}f\|_\infty \|g_\ell\|_2\Big),$$

for some $C = C(d) < \infty$. Clearly, $\|f\|_\infty$, $\|(\lambda-\Delta)^{\frac{1}{4}}f\|_\infty < \infty$, and $\|(\lambda-\Delta)^{\frac{1}{4}}g_\ell\|_2$, $\|g_\ell\|_2$ are uniformly (in $\ell$) bounded from above according to Claim 1($j$), ($jj$). Thus, in view of (10), we obtain (8) (recalling that $\ell_1 = \ell$, and $\varphi_{\ell_1} = \eta_1(v^1_{\ell_1} - \sigma)$).

*Step 2.* Now, we can repeat the argument of Step 1, but starting with sequence $\{v^1_{\ell_1}\}$ in place of $\{v_l\}$, thus obtaining a sequence $\{v^2_{\ell_2}\} \subset \operatorname{conv}\{v^1_{\ell_1}\}$ such that

$$(\lambda-\Delta)^{-\frac{1}{4}}\eta_2(v^2_{\ell_2}-\sigma) \cdot \nabla(\lambda-\Delta)^{-\frac{3}{4}} \overset{s}{\rightarrow} 0 \text{ in } L^2 \text{ as } \ell_2 \uparrow \infty.$$

We may assume without loss of generality that each $v^2_{\ell_2} \in \operatorname{conv}\{v^1_{\ell_1}\}_{\ell_1 \geqslant \ell_2}$. Therefore, we also have

$$(\lambda-\Delta)^{-\frac{1}{4}}\eta_1(v^2_{\ell_2}-\sigma) \cdot \nabla(\lambda-\Delta)^{-\frac{3}{4}} \overset{s}{\rightarrow} 0 \text{ in } L^2 \text{ as } \ell_2 \uparrow \infty.$$

Repeating this procedure $n-2$ times, we obtain a sequence $\{v^n_{\ell_n}\} \subset \mathrm{conv}\{v^{n-1}_{\ell_{n-1}}\}$ ($\subset \mathrm{conv}\{v_k\}$) such that

$$(\lambda-\Delta)^{-\frac{1}{4}}\eta_i(v^n_{\ell_n}-\sigma)\cdot\nabla(\lambda-\Delta)^{-\frac{3}{4}} \overset{s}{\to} 0 \text{ in } L^2 \text{ as } \ell_n\uparrow\infty, \quad 1\leqslant i\leqslant n.$$

*Step 3.* We set $\hat{v}_n := v^n_{\ell_n}$, $n\geqslant 1$, so for every $r\geqslant 1$

$$(\lambda-\Delta)^{-\frac{1}{4}}\eta_r(\hat{v}_n-\sigma)\cdot\nabla(\lambda-\Delta)^{-\frac{3}{4}} \overset{s}{\to} 0 \text{ in } L^2. \tag{11}$$

Since $v^n_{\ell_n}\in\mathrm{conv}\{v^{n-1}_{\ell_{n-1}}\}_{\ell_{n-1}\geqslant\ell_n}$, $v^{n-1}_{\ell_{n-1}}\in\mathrm{conv}\{v^{n-2}_{\ell_{n-2}}\}_{\ell_{n-2}\geqslant\ell_{n-1}}$, etc, we obtain that $v^n_{\ell_n}\in\mathrm{conv}\{v_k\}_{k\geqslant\ell_n}$, i.e. we also have (6). Finally, (11) combined with the resolvent identity yield

$$(\zeta-\Delta)^{-\frac{1}{4}}\eta_r(\hat{v}_n-\sigma)\cdot\nabla(\zeta-\Delta)^{-\frac{3}{4}} \overset{s}{\to} 0 \text{ in } L^2, \quad \mathrm{Re}\,\zeta\geqslant\lambda.$$

i.e. we have proved Claim 2. □

We are in a position to complete the proof of Proposition 5. Let us show that, for every $\zeta\in\mathcal{O}$

$$Z_2(\zeta,\hat{v}_n\mathcal{L}^d)g - Z_2(\zeta,\sigma)g = (\zeta-\Delta)^{-\frac{1}{4}}(\hat{v}_n-\sigma)\cdot\nabla(\zeta-\Delta)^{-\frac{3}{4}}g \overset{s}{\to} 0 \text{ in } L^2, \quad g\in\mathcal{S}.$$

Let us fix some $g\in\mathcal{S}$. We have

$$\begin{aligned}(\zeta-\Delta)^{-\frac{1}{4}}(\hat{v}_n-\sigma)\cdot\nabla(\zeta-\Delta)^{-\frac{3}{4}}g &= (\zeta-\Delta)^{-\frac{1}{4}}(\hat{v}_n-\eta_r\hat{v}_n)\cdot\nabla(\zeta-\Delta)^{-\frac{3}{4}}g \\ &+ (\zeta-\Delta)^{-\frac{1}{4}}(\eta_r\hat{v}_n-\eta_r\sigma)\cdot\nabla(\zeta-\Delta)^{-\frac{3}{4}}g \\ &+ (\zeta-\Delta)^{-\frac{1}{4}}(\eta_r\sigma-\sigma)\cdot\nabla(\zeta-\Delta)^{-\frac{3}{4}}g =: I_{1,r,n}+I_{2,r,n}+I_{3,r}.\end{aligned}$$

*Claim 3. Given any $\varepsilon>0$, there exists $r$ such that $\|I_{3,r}\|_2$, $\|I_{1,r,n}\|_2<\varepsilon$, for all $n$, $\zeta\in\mathcal{O}$.*

*Proof of Claim 3.* It suffices to prove $\|I_{1,r,n}\|_2<\varepsilon$ for all $n$. We will need the following elementary estimate: $|\nabla(\zeta-\Delta)^{-\frac{3}{4}}(x,y)|\leqslant M_d(\kappa_d^{-1}\mathrm{Re}\,\zeta-\Delta)^{-\frac{1}{4}}(x,y)$, $x,y\in\mathbb{R}^d$, $x\neq y$. We have

$$\begin{aligned}\|I_{1,r,n}\|_2 &= \|(\mathrm{Re}\,\zeta-\Delta)^{-\frac{1}{4}}(1-\eta_r)\hat{v}_n\cdot\nabla(\mathrm{Re}\,\zeta-\Delta)^{-\frac{3}{4}}g\|_2 \\ &\leqslant c_dM_d\|(\mathrm{Re}\,\zeta-\Delta)^{-\frac{1}{4}}(1-\eta_r)|\hat{v}_n|(\kappa_d^{-1}\mathrm{Re}\,\zeta-\Delta)^{-\frac{1}{4}}g\|_2 \\ &\leqslant c_dM_d\big\|(\mathrm{Re}\,\zeta-\Delta)^{-\frac{1}{4}}|\hat{v}_n|^{\frac{1}{2}}\big\|_{2\to2}\big\|(1-\eta_r)|\hat{v}_n|^{\frac{1}{2}}(\kappa_d^{-1}\mathrm{Re}\,\zeta-\Delta)^{-\frac{1}{4}}g\big\|_2\end{aligned}$$

We have $\big\|(\mathrm{Re}\,\zeta-\Delta)^{-\frac{1}{4}}|\hat{v}_n|^{\frac{1}{2}}\big\|_{2\to2}\leqslant\delta$ in view of Lemma 1(1). In turn,

$$\begin{aligned}&(1-\eta_r)|\hat{v}_n|^{\frac{1}{2}}(\kappa_d^{-1}\mathrm{Re}\,\zeta-\Delta)^{-\frac{1}{4}}g \\ &\quad = |\hat{v}_n|^{\frac{1}{2}}(\kappa_d^{-1}\mathrm{Re}\,\zeta-\Delta)^{-\frac{1}{4}}\,(\kappa_d^{-1}\mathrm{Re}\,\zeta-\Delta)^{\frac{1}{4}}(1-\eta_r)(\kappa_d^{-1}\mathrm{Re}\,\zeta-\Delta)^{-\frac{1}{4}}g,\end{aligned}$$

so

$$\big\|(1-\eta_r)|\hat{v}_n|^{\frac{1}{2}}(\kappa_d^{-1}\mathrm{Re}\,\zeta-\Delta)^{-\frac{1}{4}}g\big\|_2\leqslant\delta\|(\kappa_d^{-1}\mathrm{Re}\,\zeta-\Delta)^{\frac{1}{4}}(1-\eta_r)(\kappa_d^{-1}\mathrm{Re}\,\zeta-\Delta)^{-\frac{1}{4}}g\|_2,$$

where $\delta\|(\kappa_d^{-1}\mathrm{Re}\,\zeta-\Delta)^{\frac{1}{4}}(1-\eta_r)(\kappa_d^{-1}\mathrm{Re}\,\zeta-\Delta)^{-\frac{1}{4}}g\|_2\to0$ as $r\uparrow\infty$. The proof of Claim 3 is completed. □

Claim 2, which yields convergence $\|I_{2,r,n}\|_2 \to 0$ as $n \uparrow \infty$ for every $r$, and Claim 3, imply that

$$Z_2(\zeta, \hat{v}_n\mathcal{L}^d)g - Z_2(\zeta,\sigma)g \overset{s}{\to} 0 \text{ in } L^2, \quad g \in \mathcal{S}, \quad \zeta \in \mathcal{O},$$

which, in view of Claim 1($j$), ($jj$), yields $Z_2(\zeta, \hat{v}_n\mathcal{L}^d) - Z_2(\zeta,\sigma) \overset{s}{\to} 0$, $\zeta \in \mathcal{O}$, in $L^2$ ($\Rightarrow$(7)). By Claim 2, we also have (6). This completes the proof of Proposition 5. □

**Proposition 6.** *Let $p \in \mathcal{I}$. There exist constants $C_p$, $C_{p,q,r} < \infty$ such that for every $\zeta \in \mathcal{O}$*

(1) $\|\Omega_p(\zeta,\sigma,q,r)\|_{p\to p} \leqslant C_{p,q,r}$ *for all $k$,*

(2) $\|\Omega_p(\zeta,\sigma,\infty,1)\|_{p\to p} \leqslant C_p|\zeta|^{-\frac{1}{2}}$, *for all $k$.*

*Proof.* Immediate from Proposition 2 and Proposition 5. □

Now, we assume that $p \in \mathcal{J} \subsetneq \mathcal{I}$.

**Proposition 7.** *Let $\{\hat{v}_n\}$ be the sequence in Proposition 5. For any $p \in \mathcal{J}$,*

$$\Omega_p(\zeta, \hat{v}_n\mathcal{L}^d, q, r) \overset{s}{\to} \Omega_p(\zeta,\sigma,q,r) \text{ in } L^p, \quad \zeta \in \mathcal{O}.$$

*Proof.* Set $\Omega_p \equiv \Omega_p(\zeta,\sigma,q,r)$, $\Omega_p^n \equiv \Omega_p(\zeta,\hat{v}_n\mathcal{L}^d,q,r)$. Recall that since $p \in \mathcal{J}$, we have $2(p-1) \in \mathcal{I}$. Since $\Omega_p$, $\Omega_p^n \in \mathcal{B}(L^p)$, it suffices to prove convergence on $\mathcal{S}$. We have ($f \in \mathcal{S}$):

$$\|\Omega_p f - \Omega_p^n f\|_p^p \leqslant \|\Omega_p f - \Omega_p^n f\|_{2(p-1)}^{p-1}\|\Omega_p f - \Omega_p^n f\|_2. \tag{12}$$

Let us estimate the right-hand side in (12):

1) $\Omega_p f - \Omega_p^n f$ $\big(= \Omega_{2(p-1)}f - \Omega_{2(p-1)}^n f\big)$ is uniformly bounded in $L^{2(p-1)}$ by Proposition 2 and Proposition 6,

2) $\Omega_p f - \Omega_p^n f = \Omega_2 f - \Omega_2^n f \overset{s}{\to} 0$ in $L^2$ as $k \uparrow \infty$ by Proposition 5.

Therefore, by (12), $\Omega_p^n f \overset{s}{\to} \Omega_p f$ in $L^p$, as needed. □

This completes the proof of assertion ($i$), and thus the proof of Theorem 1.

## 3. Proof of Theorem 2

($i$) The approximating vector fields $v_k$ were constructed in Section 1. The proof repeats the proof of [Ki, Theorem 2]. Namely, we verify conditions of the Trotter approximation theorem for $\Lambda_{C_\infty}(v_k) := -\Delta + v_k\cdot\nabla$, $D(\Lambda_{C_\infty}(v_k)) = C^2 \cap C_\infty$:

1°) $\sup_n \|(\mu + \Lambda_{C_\infty}(v_k))^{-1}\|_{\infty\to\infty} \leqslant \mu^{-1}$, $\mu \geqslant \kappa_d\lambda_\delta$.

2°) $\mu(\mu + \Lambda_{C_\infty}(v_k))^{-1} \to 1$ in $C_\infty$ as $\mu \uparrow \infty$ uniformly in $n$.

3°) There exists $s$-$C_\infty$-$\lim_n(\mu + \Lambda_{C_\infty}(v_k))^{-1}$ for some $\mu \geqslant \kappa_d\lambda$.

1°) is immediate. Let us verify 2°) and 3°). Fix some $p \in \mathcal{J}$, $p > d-1$ (such $p$ exists since $m_d\delta < \frac{2d-5}{(d-2)^2}$), and let

$$\Theta_p(\mu,\sigma) := (\mu - \Delta)^{-\frac{1}{2}-\frac{1}{2q}}\Omega_p(\mu,\sigma,q,1) \in \mathcal{B}(L^p), \quad \mu \geqslant \kappa_d\lambda, \tag{13}$$

where $\max\{2,p\} < q$, see the proof of Theorem 1. We will be using the properties of $\Theta_p(\mu,\sigma)$ established there. Without loss of generality, we may assume that $\{v_k\}$ is the sequence constructed in Proposition 7, that is, $v_k \overset{w}{\to} \sigma$, and $\Omega_p(\mu, v_k\mathcal{L}^d, q, 1) \overset{s}{\to} \Omega_p(\mu,\sigma,q,1)$ in $L^p$ as $k \uparrow \infty$.

Given any $\gamma < 1 - \frac{d-1}{p}$ we can select $q$ sufficiently close to $p$ so that by the Sobolev embedding theorem,

$$(\mu - \Delta)^{-\frac{1}{2}-\frac{1}{2q}}[L^p] \subset C^{0,\gamma} \cap L^p, \quad \text{and} \quad (\mu - \Delta)^{-\frac{1}{2}-\frac{1}{2q}} \in \mathcal{B}(L^p, C_\infty).$$

Then Proposition 7 yields $\Theta_p(\mu, \hat{v}_n \mathcal{L}^d) f \overset{s}{\to} \Theta_p(\mu, \sigma) f$ in $C_\infty$, $f \in \mathcal{S}$, as $n \uparrow \infty$. The latter, combined with the next proposition and 1°), verifies condition 3°):

**Proposition 8.** *For every $k = 1, 2, \dots$, $\Theta_p(\mu, v_k \mathcal{L}^d)\mathcal{S} \subset \mathcal{S}$, and*

$$(\mu + \Lambda_{C_\infty}(v_k \mathcal{L}^d))^{-1}|_{\mathcal{S}} = \Theta_p(\mu, v_k \mathcal{L}^d)|_{\mathcal{S}}, \quad \mu \geqslant \kappa_d \lambda.$$

*Proof.* The proof repeats the proof of [Ki, Prop. 6]. □

**Proposition 9.** *$\mu \Theta_p(\mu, v_k) \overset{s}{\to} 1$ in $C_\infty$ as $\mu \uparrow \infty$ uniformly in $k$.*

*Proof.* The proof repeats the proof of [Ki, Prop. 8]. □

The last two proposition yield 2°). This completes the proof of assertion $(i)$.

$(ii)$ follows from the equality $\Theta_p(\mu, \sigma)|_{\mathcal{S}} = (\mu + \Lambda_{C_\infty}(C_\infty))^{-1}|_{\mathcal{S}}$ (by construction), representation (13), and the Sobolev embedding theorem.

$(iii)$ It follows from $(i)$ that $e^{-t\Lambda_{C_\infty}(\sigma)}$ is positivity preserving. The latter, 1°) and the Riesz-Markov-Kakutani representation theorem imply $(iii)$.

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