# LINEAR INVISCID DAMPING FOR A CLASS OF MONOTONE SHEAR FLOW IN SOBOLEV SPACES

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ABSTRACT. In this paper, we prove the decay estimates of the velocity and  $H^1$  scattering for the 2D linearized Euler equations around a class of monotone shear flow in a finite channel. Our result is consistent with the decay rate predicted by Case in 1960.

#### 1. Introduction

In this paper, we consider the 2D incompressible Euler equations in a finite channel  $\{(x,y): x \in \mathbb{T}, y \in [0,1]\}$ :

(1.1) 
$$\begin{cases} \partial_t V + V \cdot \nabla V + \nabla P = 0, \\ \nabla \cdot V = 0, \\ V^2(t, x, 0) = V^2(t, x, 1) = 0, \\ V|_{t=0} = V_0(x, y). \end{cases}$$

where  $V=(V^1,V^2)$  and P denote the velocity and the pressure of the fluid respectively. Let  $\omega=\partial_x V^2-\partial_u V^1$  be the vorticity, which satisfies

$$(1.2) \omega_t + V \cdot \nabla \omega = 0.$$

It is well-known that the 2D incompressible Euler equations are globally well-posed for smooth data [6, 17]. However, the long time behaviour of the solution is widely open. We refer to [11, 8] for recent relevance results.

We are concerned with the asymptotic stability of the 2D linearized Euler equations around the shear flow (u(y), 0), which is a steady solution of 2D Euler equations. The linearized Euler equations around a shear flow (u(y), 0) take

(1.3) 
$$\begin{cases} \partial_t \omega + \mathcal{L}\omega = 0, \\ \omega|_{t=0} = \omega_0(x, y), \end{cases}$$

where  $\mathcal{L} = u(y)\partial_x + u''(y)\partial_x (-\Delta)^{-1}$ .

The stability of 2-D Euler equations is a very active field in Physics and Mathematics [9], especially for shear flows [22]. Rayleigh's inflection point theorem gives a necessary condition for linear stability of shear flow: u(y) has no infection points [20]. Arnold's theorem gives a sufficient condition for nonlinear Liapunov stability of shear flow [1]:

$$0 < c_1 \le \frac{u(y)}{u''(y)} \le c_2 < +\infty.$$

Lin [13] provided a large classes of unstable shear flows. We refer to [2, 14, 10, 24] and references therein for nonlinear instability.

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In 1907, Orr [19] observed that the velocity tends to zero as  $t \to +\infty$  for the linearized Euler equations around Couette flow (y,0). In this case, the linearized vorticity equation becomes

$$\omega_t + y \partial_x \omega = 0.$$

Thus,  $\omega(t, x, y) = \omega_0(x - ty, y)$ . Especially, in the case of the infinite channel  $\mathbb{T} \times \mathbf{R}$ , the velocity can be explicitly solved. Indeed, let  $\psi$  be the stream function, i.e.,  $(V^1, V^2) = (\partial_u \psi, -\partial_x \psi)$ . Hence,

$$(1.4) -\Delta \psi = \omega.$$

Since we deduce by taking Fourier transform to (1.4) that

$$\widehat{\psi}(t,\alpha,\xi) = \frac{\widehat{\omega}_0(\alpha,\xi+\alpha t)}{\alpha^2 + |\xi|^2},$$

which implies that  $V^1$  decays at  $t^{-1}$ , while  $V^2$  decays at  $t^{-2}$ . In the case of finite channel, Case [7] gave a formal proof of  $t^{-1}$  decay for the velocity. Recently, Lin and Zeng [16] present the optimal linear decay estimates of the velocity for the data in Sobolev space. More precisely, if  $\int_{\mathbb{T}} \omega_0(x,y) dx = 0$ , then it holds that

1. if 
$$\omega_0(x,y) \in H_x^{-1}H_y^1$$
, then

$$||V(t)||_{L^2} = O(\frac{1}{t}),$$

2. if 
$$\omega_0(x,y) \in H_x^{-1}H_y^2$$
, then

$$||V^2(t)||_{L^2} = O(\frac{1}{t^2}).$$

Such inviscid damping is surprising for a time reversible system. The basic mechanism leading to this phenomena is vorticity mixing driven by shear flow, which may be related to the appearance of coherent structures in 2D turbulence. This behaviour is similar to Landau damping [12], which predicted the rapid decay of the electric field of the linearized Vlasov equation around homogeneous equilibrium.

It is a very difficult problem to extend linear damping to nonlinear damping. Mouhot and Villani [18] made a breakthrough and proved nonlinear Landau damping for the perturbation in Gevrey class(see also [4]). Motivated by [18], Bedrossian and Masmoudi also proved the nonlinear inviscid damping of 2D Euler equations around Couette flow in infinite channel still for the perturbation in Gevrey class. Lin and Zeng [16, 15] also show that nonlinear damping is not true for the perturbation in low regularity Sobolev spaces.

The goal of this paper is to prove linear damping for the 2D Euler equations around general shear flow. In this case, there are few rigorous mathematical results. Case [7] gave the formal prediction for the decay of the velocity by using the Laplace transform and the leading singularity of the resolvent. Rosencrans and Sattinger [21] gave  $t^{-1}$  decay of the stream function with a continuous spectrum projection for analytic monotone shear flow. Stepin [23] proved  $t^{-\nu}(\nu < \mu_0)$  decay of the stream function for the monotone shear flow  $u(y) \in C^{2+\mu_0}(\mu_0 > \frac{1}{2})$  without inflection point. In a very interesting paper [5], Bouchet and Morita predicted similar decay estimates of the velocity for a class of stable shear flow with stationary streamlines by using the Laplace transform and an important observation: depletion phenomena of the vorticity at the stationary streamlines. More precisely, they formally proved that

$$\omega(t, x, y) \sim \omega_{\infty}(x, y) \exp(-iku(y)t) + O(t^{-\gamma})$$
 as  $t \to +\infty$ ,

where  $\omega_{\infty}(x, y_c) = 0$  at stationary points  $y_c$  of u(y).

In a recent paper, C. Zillinger [25] proved the same decay estimates as those of Couette flow given by Lin and Zeng [16] for a class of monotone shear flow in Sobolev spaces. However, his result imposed a strong assumption that  $L||u''||_{W^{3,\infty}}$  is small, where L is the wave-length with respect to x. Moreover, he also required that the initial vorticity vanishes on y = 0, 1 in the case of finite channel. Thus, the linear inviscid damping is still open for general monotone shear flow.

The main result of this paper is stated as follows.

**Theorem 1.1.** Let  $u(y) \in C^4([0,1])$  be a monotone function. Suppose that the linearized operator  $\mathcal{L}$  has no embedding eigenvalues. Assume that  $\int_{\mathbb{T}} \omega_0(x,y) dx = 0$  and  $P_{\mathcal{L}}\omega_0 = 0$ , where  $P_{\mathcal{L}}$  is the spectral projection to  $\sigma_d(\mathcal{L})$ . Then it holds that

1. if  $\omega_0(x,y) \in H_x^{-1}H_y^1$ , then

$$||V(t)||_{L^2} \le \frac{C}{\langle t \rangle} ||\omega_0||_{H_x^{-1} H_y^1};$$

2. if  $\omega_0(x,y) \in H_x^{-1}H_y^2$ , then

$$||V^2(t)||_{L^2} \le \frac{C}{\langle t \rangle^2} ||\omega_0||_{H_x^{-1} H_y^2};$$

3. if  $\omega_0(x,y) \in H_x^{-1}H_y^k$  for k=0,1, there exists  $\omega_\infty(x,y) \in H_x^{-1}H_y^k$  such that  $\|\omega(t, x + tu(y), y) - \omega_{\infty}\|_{L^{2}} \longrightarrow 0$  as  $t \to +\infty$ .

Let us give some remarks on Theorem 1.1.

- If u(y) has no inflection points, then  $\mathcal{L}$  has no eigenvalues. In Remark 6.4, we will present a sufficient condition on u(y) in the case when u(y) has inflection points so that  $\mathcal{L}$  has no eigenvalues.
- If the wave-length L with respect to x is suitably small, then  $\mathcal{L}$  has no embedding eigenvalues. In Lemma 6.1, we will present a sufficient and necessary condition on u(y) so that  $\mathcal{R}_{\alpha}$  has no embedding eigenvalues.
- By Zillinger's recent result [26], the  $L_x^2H_y^2$  norm of  $W(t,x,y) \triangleq \omega(t,x+tu(y),y)$  may blow up. Thus, it is in general unexpected for the  $H^2$  scattering.
- Our proof strongly relies on the monotonicity of u(y). Thus, it remains unknown whether the decay estimates predicted by Bouchet and Morita [5] for stable shear flow with stationary streamlines can be justified.
- Nonlinear inviscid damping is a challenging question even for the analytic perturbation.

#### 2. Sketch of the proof

Our proof is based on the representation formula of the solution

$$\widehat{\psi}(t,\alpha,y) = \frac{1}{2\pi i} \int_{\partial \Omega} e^{-i\alpha tc} (c - \mathcal{R}_{\alpha})^{-1} \widehat{\psi}(0,\alpha,y) dc,$$

where  $\mathcal{R}_{\alpha}$  is the Rayleigh operator defined by (3.3). Under the assumption that  $\mathcal{L}$  has no embedding eigenvalues and  $P_{\mathcal{L}}\omega_0 = 0$ , the asymptotic behaviour of the solution is only related to the continuous spectrum. Thus, we only need to study

$$\widehat{\psi}(t,\alpha,y) = \frac{1}{2\pi i} \int_{u(0)}^{u(1)} e^{-i\alpha tc} \lim_{\epsilon \to 0+} \left( (c - i\epsilon - \mathcal{R}_{\alpha})^{-1} - (c + i\epsilon - \mathcal{R}_{\alpha})^{-1} \right) \widehat{\psi}(0,\alpha,y) dc.$$

To establish the estimates of the resolvent  $(c-\mathcal{R}_{\alpha})^{-1}$ , we need to study the inhomogeneous Rayleigh equation

$$\begin{cases} \Phi'' - \alpha^2 \Phi - \frac{u''}{u - c} \Phi = f, \\ \Phi(0) = \Phi(1) = 0, \end{cases}$$

with  $f = \frac{\widehat{\omega}_0(\alpha, y)}{i\alpha(u-c)}$ . Indeed, it holds that

$$\lim_{\epsilon \to 0+} \left( (c - i\epsilon - \mathcal{R}_{\alpha})^{-1} - (c + i\epsilon - \mathcal{R}_{\alpha})^{-1} \right) \widehat{\psi}(0, \alpha, y)$$
$$= i\alpha \lim_{\epsilon \to 0+} \left( \Phi(y, c + i\epsilon) - \Phi(y, c - i\epsilon) \right) \triangleq i\alpha \widetilde{\Phi}(y, c).$$

Thus, we obtain

(2.1) 
$$\widehat{\psi}(t,\alpha,y) = \frac{1}{2\pi} \int_{u(0)}^{u(1)} e^{-i\alpha tc} \alpha \widetilde{\Phi}(y,c) dc.$$

Formally, if one can show that  $\widetilde{\Phi}(y,c) \in W^{2,1}$  in c, then integration by parts gives

$$\widehat{\psi}(t,\alpha,y) \sim O(t^{-2}) \int_{u(0)}^{u(1)} e^{-i\alpha tc} \partial_c^2 \widetilde{\Phi}(y,c) dc \sim O(t^{-2}).$$

One of main difficulties is that the solution  $\Phi(y,c)$  of the inhomogeneous Rayleigh equation has a singularity of order  $(y-y_c)\log|y-y_c|$  with  $y_c=u^{-1}(c)$  (see [7, 5]). Thus,  $\partial_c^2\widetilde{\Phi}(y,c)\sim \frac{1}{y-y_c}\notin L^1$ . This may be the main reason why the authors in [7, 21, 23] only obtained the  $O(t^{-1})$  decay of the stream function even in the analytic framework.

Indeed, in Section 6 and 7, we will show that  $\alpha \Phi(y,c) = 2\rho(c)\mu(c)\Gamma(y,c)$  with  $\rho(c) = (c-u(0))(u(1)-c)$  and

$$\Gamma(y,c) = \begin{cases} \phi(y,c) \int_0^y \frac{1}{\phi(z,c)^2} dz & 0 \le y < y_c, \\ \phi(y,c) \int_1^y \frac{1}{\phi(z,c)^2} dz & y_c < y \le 1, \end{cases}$$

where  $\phi(y,c)$  is the solution of the homogeneous Rayleigh equation:

$$\begin{cases} \phi'' - \alpha^2 \phi - \frac{u''}{u - c} \phi = 0, \\ \phi(y_c, c) = 0, \quad \phi'(y_c, c) = u'(y_c). \end{cases}$$

Thus,  $\phi(y,c)$  has the behaviour near  $y_c$ :

$$\phi(y,c) \sim u'(y_c)(y-y_c) + \frac{u''(y_c)}{2u'(y_c)}(y-y_c)^2 + o((y-y_c)^2),$$

which implies that  $\Gamma(y,c)$  has the behaviour near  $y_c$ :

$$\Gamma(y,c) \sim a + b(y - y_c) \log |y - y_c|,$$

for some constants a, b.

The goal of Section 4-Section 5 is to obtain various kinds of uniform estimates for the solution of the homogeneous Rayleigh equations. The assumption that u(y) is monotone plays an important role. In Section 8, we will establish the weighted  $H^2$  estimate of  $\mu(c)$ , where we need to assume that  $\mathcal{L}$  has no embedding eigenvalues.

Based on the solution formula (2.1), using the weighted  $H^2$  estimate of  $\mu(c)$  and  $L^p$  boundedness for various kinds of singular integral operators, we will establish the  $H_y^2$  estimate of

 $W(t,x,y)=\omega(t,x+tu(y),y)$  in Section 9. In fact, we only prove the weighted  $H_{y}^{2}$  estimate, while  $H_u^2$  bound is impossible in general.

With the uniform Sobolev estimates of the vorticity, the decay estimates can be deduced by following the dual argument introduced by Lin and Zeng [16].

In the appendix, we will establish  $L^p$  boundedness for various kinds of singular integral operators, which was used in Section 8 and Section 9.

#### 3. Spectrum of the linearized operator

In terms of the stream function  $\psi$ , the linearized Euler equations take

(3.1) 
$$\partial_t \Delta \psi + u(y) \partial_x \Delta \psi - u''(y) \partial_x \psi = 0.$$

Taking the Fourier transform in x, we get

$$(\partial_y^2 - \alpha^2)\partial_t \widehat{\psi} = i\alpha (u''(y) - u(y)(\partial_y^2 - \alpha^2))\widehat{\psi}.$$

Inverting the operator  $(\partial_y^2 - \alpha^2)$ , we find

$$-\frac{1}{i\alpha}\partial_t \widehat{\psi} = \mathcal{R}_{\alpha} \widehat{\psi},$$

where

(3.3) 
$$\mathcal{R}_{\alpha}\widehat{\psi} = -(\partial_y^2 - \alpha^2)^{-1} \left( u''(y) - u(\partial_y^2 - \alpha^2) \right) \widehat{\psi}.$$

It is easy to show that

$$\bigcup_{\alpha} \sigma_d(i\alpha \mathcal{R}_{\alpha}) = \sigma_d(\mathcal{L}).$$

Let us recall some classical results for the spectrum  $\sigma(\mathcal{R}_{\alpha})$  of the operator  $\mathcal{R}_{\alpha}$  (see [21, 23] for more details).

- 1. The spectrum  $\sigma(\mathcal{R}_{\alpha})$  is compact;
- 2. The continuous spectrum  $\sigma_c(\mathcal{R}_\alpha)$  is contained in the range Ran(u) of u(y);
- 3. The eigenvalues of  $\mathcal{R}_{\alpha}$  can not cluster except possibly along on Ran(u);
- 4. If u(y) has no infection points in [0,1], then  $\mathcal{R}_{\alpha}$  has no embedding eigenvalues;
- 5. If u(y) has inflection points, then  $\mathcal{R}_{\alpha}$  has no embedding eigenvalues for  $\alpha^2 > \alpha_{max}^2$ , where

$$\alpha_{max}^2 \stackrel{\text{def}}{=} - \inf_{y_c: u''(y_c) = 0} \inf_{\phi \in H_0^1(0,1)} \frac{\int_0^1 |\phi'(y)|^2 - \frac{u''(y)}{u(y) - u(y_c)} |\phi(y)|^2 dy}{\int_0^1 |\phi(y)|^2 dy}.$$

Proof of 4. Let  $\phi \in H^2(0,1) \cap H^1_0(0,1)$  be an eigenfunction of  $\mathcal{R}_{\alpha}$  with the eigenvalue  $c \in Ran(u)$ , i.e.,  $\mathcal{R}_{\alpha}\phi = c\phi$ , which can be reduced to the well-known Rayleigh equation:

$$(3.4) (u-c)(\phi'' - \alpha^2 \phi) - u'' \phi = 0.$$

If  $u''(u^{-1}(c)) \neq 0$ ,  $\phi(u^{-1}(c)) = 0$  by (3.4). Taking the inner product with  $\frac{\phi}{u-c}$  on both sides of (3.4), we obtain

$$\int_0^1 \phi'' \phi dy - \alpha^2 \int_0^1 |\phi|^2 dy - \int_0^1 u'' \frac{|\phi|^2}{u - c} dy = 0.$$

Integration by parts gives

$$-\int_0^1 |\phi' - u' \frac{\phi}{u - c}|^2 dy - \alpha^2 \int_0^1 |\phi|^2 dz = 0,$$

which implies that  $\phi \equiv 0$ . Thus,  $u''(u^{-1}(c)) = 0$  if c is an embedding eigenvalue.

*Proof of 5.* If c is an embedding eigenvalue with eigenfunction  $\phi$ , then we have

$$\int_0^1 |\phi'|^2 - \frac{u''}{u - c} |\phi|^2 dy + \alpha^2 \int_0^1 |\phi|^2 dy = 0.$$

However, for  $\alpha^2 > \alpha_{max}^2$ , we have

$$\int_0^1 |\phi'|^2 - \frac{u''}{u - c} |\phi|^2 dy + \alpha^2 \int_0^1 |\phi|^2 dy \neq 0.$$

In Lemma 6.1, we will present a sufficient and necessary condition on u(y) so that  $\mathcal{R}_{\alpha}$  has no embedding eigenvalues. In Remark 6.4, we will present a sufficient condition so that  $\mathcal{R}_{\alpha}$  has no eigenvalues, which in particular implies that  $\mathcal{R}_{\alpha}$  has no eigenvalues if u(y) has inflection points in [0,1] and  $\alpha^2 > \alpha_{max}^2$ .

Let  $\Omega$  be a simple connected domain including the spectrum  $\sigma(\mathcal{R}_{\alpha})$  of  $\mathcal{R}_{\alpha}$ . We have the following representation formula of the solution to (3.2):

(3.5) 
$$\widehat{\psi}(t,\alpha,y) = \frac{1}{2\pi i} \int_{\partial \Omega} e^{-i\alpha tc} (c - \mathcal{R}_{\alpha})^{-1} \widehat{\psi}(0,\alpha,y) dc.$$

Thus, the large time behaviour of the solution  $\hat{\psi}(t, \alpha, y)$  is reduced to the study of the resolvent  $(c - \mathcal{R}_{\alpha})^{-1}$ .

### 4. The homogeneous Rayleigh equation

To study the resolvent  $(c-\mathcal{R}_{\alpha})^{-1}$ , we first construct a smooth solution of the homogeneous Rayleigh equation on [0,1]:

$$(4.1) (u-c)(\phi'' - \alpha^2 \phi) - u'' \phi = 0,$$

where the complex constant c will be taken in four kinds of domains defined by

$$\begin{split} D_0 &\triangleq \left\{c \in [u(0), u(1)]\right\}, \\ D_{\epsilon_0} &\triangleq \left\{c = c_r + i\epsilon, \ c_r \in [u(0), u(1)], 0 < |\epsilon| < \epsilon_0\right\}, \\ B_{\epsilon_0}^l &\triangleq \left\{c = u(0) + \epsilon e^{i\theta}, \ 0 < \epsilon < \epsilon_0, \ \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}\right\}, \\ B_{\epsilon_0}^r &\triangleq \left\{c = u(1) - \epsilon e^{i\theta}, \ 0 < \epsilon < \epsilon_0, \ \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}\right\}, \end{split}$$

for some  $\epsilon_0 \in (0,1)$ . We denote

(4.2) 
$$\Omega_{\epsilon_0} \triangleq D_0 \cup D_{\epsilon_0} \cup B_{\epsilon_0}^l \cup B_{\epsilon_0}^r.$$

In the sequel, we always assume that  $u(y) \in C^4([0,1])$  and satisfies  $u'(y) \ge c_0$  for some  $c_0 > 0$ .

4.1. Functional space and the integral operator. Given  $|\alpha| \geq 1$ , let A be a constant larger than  $C|\alpha|$  with  $C \geq 1$  only depending on  $c_0$  and  $||u||_{C^4}$ .

**Definition 4.1.** For a function f(y,c) defined on  $[0,1] \times \Omega_{\epsilon_0}$ , we define

$$||f||_{X_0} \stackrel{def}{=} \sup_{(y,c)\in[0,1]\times D_0} \left| \frac{f(y,c)}{\cosh(A(y-y_c))} \right|,$$

$$||f||_X \stackrel{def}{=} \sup_{(y,c)\in[0,1]\times D_{\epsilon_0}} \left| \frac{f(y,c)}{\cosh(A(y-y_c))} \right|,$$

$$\begin{split} & \|f\|_{X_l} \overset{def}{=} \sup_{(y,c) \in [0,1] \times B_{\epsilon_0}^l} \left| \frac{f(y,c)}{\cosh(Ay)} \right|, \\ & \|f\|_{X_r} \overset{def}{=} \sup_{(y,c) \in [0,1] \times B_{\epsilon_0}^r} \left| \frac{f(y,c)}{\cosh(A(y-1))} \right|, \end{split}$$

where  $y_c = u^{-1}(c_r)$  with

$$(4.3) \quad c_r = Re \ c \quad for \ c \in D_{\epsilon_0}, \quad c_r = u(0) \quad for \ c \in B^l_{\epsilon_0}, \quad c_r = u(1) \quad for \ c \in B^r_{\epsilon_0}.$$

**Definition 4.2.** For a function f(y,c) defined on  $[0,1] \times \Omega_{\epsilon_0}$ , we define

$$||f||_{Y_C} \stackrel{def}{=} \sum_{k=0}^{2} \sum_{|\beta|=k} A^{-k} ||\partial^{\beta} f||_{X_0},$$

$$||f||_{Y} \stackrel{def}{=} ||f||_{X} + \frac{1}{A} (||\partial_x f||_{X} + ||\partial_c f||_{X}),$$

$$||f||_{Y_l} \stackrel{def}{=} ||f||_{X_l} + \frac{1}{A} (||\partial_x f||_{X_l} + ||\partial_c f||_{X_l}),$$

$$||f||_{Y_r} \stackrel{def}{=} ||f||_{X_r} + \frac{1}{A} (||\partial_x f||_{X_r} + ||\partial_c f||_{X_r}).$$

**Definition 4.3.** Let  $y_c = u^{-1}(c_r)$  with  $c_r$  defined by (4.3). The integral operator T is defined by

$$T \triangleq T_0 \circ T_{2,2} \stackrel{def}{=} \int_{u_0}^{y} \frac{1}{(u(y') - c)^2} \int_{u_0}^{y'} f(z, c) (u(z) - c)^2 dz dy',$$

where

$$T_0 f(y,c) \stackrel{def}{=} \int_{y_c}^y f(z,c) dz,$$

$$T_{k,j} f(y,c) \stackrel{def}{=} \frac{1}{(u(y)-c)^j} \int_{y_c}^y f(z,c) (u(z)-c)^k dz, \quad j \le k+1.$$

Lemma 4.4. There exists a constant C independent of A such that

$$||Tf||_{Y_C} \le \frac{C}{A^2} ||f||_{Y_C}, \quad ||Tf||_Y \le \frac{C}{A^2} ||f||_Y,$$
$$||Tf||_{Y_l} \le \frac{C}{A^2} ||f||_{Y_l}, \quad ||Tf||_{Y_r} \le \frac{C}{A^2} ||f||_{Y_r}.$$

*Proof.* We only prove the first inequality. The proof of the other inequalities is similar. A direct calculation shows

$$||T_0 f||_{X_0} = \sup_{(y,c) \in [0,1] \times D_0} \left| \frac{1}{\cosh A(y - y_c)} \int_{y_c}^{y} \frac{f(z,c)}{\cosh A(z - y_c)} \cosh A(z - y_c) dz \right|$$

$$\leq \sup_{(y,c) \in [0,1] \times D_0} \left| \frac{1}{\cosh A(y - y_c)} \int_{y_c}^{y} \cosh A(z - y_c) dz \right| ||f||_{X_0}$$

$$\leq \frac{1}{A} ||f||_{X_0},$$

and

$$||T_{2,2}f||_{X_0} \le \sup_{(y,c) \in [0,1] \times D_0} \left| \frac{y - y_c}{\cosh A(y - y_c)} \int_0^1 \cosh t A(y - y_c) dt \right| ||f||_{X_0}$$

$$\leq \frac{C}{A} ||f||_{X_0},$$

which imply

$$||Tf||_{X_0} \le \frac{C}{A^2} ||f||_{X_0}.$$

It is easy to see that

$$\partial_y Tf(y,c) = T_{2,2}f(y,c),$$
  
 $\partial_c Tf(y,c) = 2T_0 \circ T_{2,3}f(y,c) - 2T_0 \circ T_{1,2}f(y,c) + T\partial_c f(y,c),$ 

and

$$||T_{k,k+1}f||_{X_0} \le C \sup_{(y,c)\in[0,1]\times D_0} \left| \frac{1}{\cosh A(y-y_c)} \int_0^1 \cosh t A(y-y_c) dt \right| ||f||_{X_0}$$
  
$$\le C ||f||_{X_0},$$

which along with (4.4) yield

(4.5) 
$$\|\partial_{y,c}Tf\|_{X_0} \le \frac{C}{A} \|f\|_{X_0} + \frac{C}{A^2} \|\partial_c f\|_{X_0}.$$

Using the formula

$$T_{k,k+1}f(y,c) = \int_0^1 \frac{\left(\int_0^1 u'(y_c + st(y - y_c))ds\right)^k}{\left(\int_0^1 u'(y_c + s(y - y_c))ds\right)^{k+1}} f(y_c + t(y - y_c), c)t^k dt,$$

we can deduce that

$$\|\partial_{y,c}^2 T_{k,k+1} f\|_{X_0} \le C \sum_{|\beta| \le 2} \|\partial^{\beta} f\|_{X_0}.$$

Then by a similar argument leading to (4.5), we obtain

Putting (4.4)-(4.6) together, we conclude the first inequality.

## 4.2. Existence of the solution.

**Proposition 4.5.** There exists a solution  $\phi(y,c) \in C^1([0,1] \times \Omega_{\epsilon_0} \setminus D_0) \cap C([0,1] \times \Omega_{\epsilon_0})$  of the Rayleigh equation (4.1). Moreover, there exists  $\epsilon_0 > 0$  such that for any  $\epsilon \in [0,\epsilon_0)$  and  $(y,c) \in [0,1] \times \Omega_{\epsilon_0}$ ,

$$|\phi_1(y,c)| \ge \frac{1}{2}, \quad |\phi_1(y,c) - 1| \le C|u(y) - c|^2,$$

where  $\phi_1(y,c) = \frac{\phi(y,c)}{u(y)-c}$ , and the constant C may depend on  $\alpha$ .

The proof is based the following lemmas.

**Lemma 4.6.** Let  $c \in D_{\epsilon_0}$  and  $y_c = u^{-1}(c_r)$ . Then there exists a solution  $\phi(x,c) \in Y$  to the Rayleigh equation

$$\begin{cases} \phi'' - \alpha^2 \phi - \frac{u''}{u - c} \phi = 0, \\ \frac{\phi(y_c, c)}{u(y_c) - c} = 1, \quad \left(\frac{\phi(y, c)}{u(y) - c}\right)' \Big|_{y = y_c} = 0. \end{cases}$$

Moreover, there holds

$$\|\phi\|_Y \le C$$
,

where the constant C may depend on  $\alpha$ .

$$((u-c)^{2}\phi_{1}')' = \alpha^{2}\phi_{1}(u-c)^{2},$$

from which, we infer that

$$\phi_1(y,c) = 1 + \int_{y_c}^{y} \frac{\alpha^2}{(u(y') - c)^2} \int_{y_c}^{y'} \phi_1(z,c) (u(z) - c)^2 dz dy'.$$

This means that  $\phi_1(y,c)$  satisfies

$$\phi_1(y,c) = 1 + \alpha^2 T \phi_1(y,c).$$

It follows from Lemma 4.4 that the operator  $I - \alpha^2 T$  is invertible in the space Y. Thus,

$$\phi_1(y,c) = (I - \alpha^2 T)^{-1}1,$$

with the bound  $\|\phi_1\|_Y \leq C$ , hence  $\|\phi\|_Y \leq C$ .

In a similar way as in Lemma 4.6, we can show that

**Lemma 4.7.** Let  $c \in B_{\epsilon_0}^l$ . Then there exists a solution  $\phi(x,c) \in Y_l$  to the Rayleigh equation

$$\begin{cases} \phi'' - \alpha^2 \phi - \frac{u''}{u - c} \phi = 0, \\ \frac{\phi(0, c)}{u(0) - c} = 1, \quad \left(\frac{\phi(y, c)}{u(y) - c}\right)' \Big|_{y = 0} = 0. \end{cases}$$

Moreover, there holds

$$\|\phi\|_{Y_l} \leq C$$
,

where the constant C may depend on  $\alpha$ .

**Lemma 4.8.** Let  $c \in B_{\epsilon_0}^r$ . Then there exists a solution  $\phi(x,c) \in Y_r$  to the Rayleigh equation

$$\begin{cases} \phi'' - \alpha^2 \phi - \frac{u''}{u - c} \phi = 0, \\ \frac{\phi(1, c)}{u(1) - c} = 1, \quad \left(\frac{\phi(y, c)}{u(y) - c}\right)' \Big|_{y = 1} = 0. \end{cases}$$

Moreover, there holds

$$\|\phi\|_{Y_r} \leq C$$
,

where the constant C may depend on  $\alpha$ .

**Lemma 4.9.** Let  $c \in D_0$  and  $y_c = u^{-1}(c)$ . Then there exists a solution  $\phi(y,c) \in Y_C$  to the Rayleigh equation

$$\begin{cases} \phi'' - \alpha^2 \phi - \frac{u''}{u-c} \phi = 0, \\ \phi(y_c, c) = 0, \quad \phi'(y_c, c) = u'(y_c). \end{cases}$$

Moreover, there holds

$$\|\phi\|_{Y_C} \leq C$$

where the constant C may depend on  $\alpha$ .

*Proof.* We rewrite the Rayleigh equation (4.1) as

$$(\phi'(u-c) - \phi u')' = \alpha^2 \phi(u-c).$$

Using the boundary conditions  $\phi(x_0) = 0$  and  $\phi'(x_0) = u'(x_0)$ , we get

$$\phi'(y)(u(y) - c) - \phi(y)u'(x) = \int_{y_0}^{y} \alpha^2 \phi(z)(u(z) - c)dz,$$

which implies

$$\left(\frac{\phi(y,c)}{u(y)-u(y_c)}\right)' = \frac{\alpha^2}{(u(y)-u(y_c))^2} \int_{y_c}^y \phi(z)(u(z)-c)dz.$$

Let  $\phi_1(y,c) = \frac{\phi(y,c)}{u(y)-c}$ . So,  $\phi_1(y_c) = 1$  and

$$\phi_1(y,c) = 1 + \int_{y_c}^{y} \frac{\alpha^2}{(u(y') - u(y_c))^2} \int_{y_c}^{y'} \phi_1(z,c) (u(z) - u(y_c))^2 dz dy',$$

that is,

(4.7) 
$$\phi_1(y,c) = 1 + \alpha^2 T \phi_1(y,c).$$

Then Lemma 4.4 ensures the existence of the solution in  $Y_C$  to the equation (4.7) with the bound  $\|\phi_1\|_{Y_C} \leq C$ .

**Remark 4.10.** Since T is a positive operator,  $\phi_1(y,c)$  for  $c \in D_0$  given by (4.7) satisfies

$$\phi_1(y,c) \ge 1.$$

Moreover, the inverse of  $I - \alpha^2 T$  exists in  $X_0$  and has the infinite series representation

$$(I - \alpha^2 T)^{-1} = \sum_{k=0}^{\infty} \alpha^{2k} T^k.$$

Now we are in a position to prove Proposition 4.5. We define

$$\phi(y,c) \stackrel{\text{def}}{=} \begin{cases} \phi^0(y,c) & \text{for } c \in D_0, \\ \phi^{\pm}(y,c) & \text{for } c \in D_{\epsilon_0}, \\ \phi^l(y,c) & \text{for } c \in B_{\epsilon_0}^l, \\ \phi^r(y,c) & \text{for } c \in B_{\epsilon_0}^r, \end{cases}$$

where  $\phi^{\pm}$ ,  $\phi^l$ ,  $\phi^r$ ,  $\phi^0$  are given by Lemma 4.6, Lemma 4.7, Lemma 4.8 and Lemma 4.9 respectively.

By our constructions,  $\phi(y,c) \in C^1([0,1] \times \Omega_{\epsilon_0} \setminus D_0)$ . Notice that

$$Tf(y,c) = (y - y_c)^2 \iint_{[0,1]^2} f(y_c + st(y - y_c), c) K_0(s, t, y, c) dt ds,$$

where

$$K_0(s, t, y, c) = s \left( \frac{u(y_c + st(y - y_c), c) - c}{u(y_c + s(y - y_c), c) - c} \right)^2.$$

Using the fact that  $K_0 \in C([0,1]^3 \times \Omega_{\varepsilon_0})$  and  $|K_0| \leq s$ , we conclude that T maps  $C([0,1] \times \Omega_{\varepsilon_0})$  to  $C([0,1] \times \Omega_{\varepsilon_0})$ . Then by using the formula  $\phi_1(y,c) = \sum_{k=0}^{+\infty} \alpha^{2k} T^k 1$  for  $c \in \Omega_{\varepsilon_0}$  and that the convergence is uniform, we conclude that  $\phi_1(y,c) \in C([0,1] \times \Omega_{\varepsilon_0})$ , thus  $\phi(y,c) \in C([0,1] \times \Omega_{\varepsilon_0})$ . Furthermore, Remark 4.10 ensures that there exists  $\epsilon_0 > 0$  such that for any  $\epsilon \in [0,\epsilon_0)$  and  $(y,c) \in [0,1] \times \Omega_{\epsilon_0}$ ,

$$|\phi_1(y,c)| \ge \frac{1}{2}.$$

Using the integral representation of  $\phi_1(y,c)$ , we have

$$|\phi_1(y,c)-1| \le \alpha^2 \int_{y_c}^y \int_{y_c}^z |\phi_1(y',c)| \left| \frac{u(y')-c}{u(z)-c} \right|^2 dy' dz$$

$$\leq C|y - y_c|^2 \leq C|u(y) - u(y_c)|^2$$
  
 $\leq C|u(y) - c|^2.$ 

This completes the proof of Proposition 4.5.

#### 5. Uniform estimates for the homogeneous Rayleigh equation

The goal of this section is to establish some uniform estimates in wave number  $\alpha$  for the solution  $\phi(y,c)$  for  $c \in D_0$  of the Rayleigh equation given by Lemma 4.9. Let  $\phi_1(y,c) = \frac{\phi(y,c)}{u(y)-c}$  and  $y_c = u^{-1}(c)$  for  $c \in D_0$ .

Without loss of generality, we always assume  $\alpha \geq 1$  in the sequel.

#### 5.1. Uniform estimates in wave number.

**Proposition 5.1.** There exists a constant C independent of  $\alpha$  such that for any  $(y,c) \in [0,1] \times D_0$ ,

$$\frac{\sinh \alpha(y - y_c)}{C\alpha} \le \phi(y, c) \le \frac{C \sinh \alpha(y - y_c)}{\alpha},$$
$$\frac{\sinh \alpha(y - y_c)}{C\alpha(y - y_c)} \le \phi_1(y, c) \le \frac{C \sinh \alpha(y - y_c)}{\alpha(y - y_c)}.$$

Moreover, it holds that

$$|\partial_y^{\beta} \partial_c^{\gamma} \phi_1(y,c)| \le C \alpha^{\beta+\gamma} \phi_1(y,c)$$

for  $\beta + \gamma \leq 2$ .

The proof of the proposition needs the following two lemmas.

**Lemma 5.2.** Let the operator S be defined by

$$Sf(y,c) \stackrel{\text{def}}{=} \int_{y_c}^{y} \frac{dy'}{\sinh^2 \alpha(y'-y_c)} \int_{y_c}^{y'} \frac{\sinh^2 \alpha(z-y_c)u''(z)}{u(z)-u(y_c)} f(z,c) dz.$$

Then there exists a constant C independent of  $\alpha$  such that

$$||Sf||_{L^{\infty}([0,1]\times D_0)} \le C \frac{1+\ln\alpha}{\alpha} ||f||_{L^{\infty}([0,1]\times D_0)}.$$

Especially, there exists  $M_0 > 0$  such that for  $\alpha \geq M_0$ ,

$$||Sf||_{L^{\infty}([0,1]\times D_0)} \le \frac{1}{2}||f||_{L^{\infty}([0,1]\times D_0)}.$$

*Proof.* A direct calculation shows

$$||Sf||_{L^{\infty}([0,1]\times D_{0})} \leq \left| \int_{y_{c}}^{y} \frac{dy'}{\sinh^{2}\alpha(y'-y_{c})} \int_{y_{c}}^{y'} \frac{\sinh^{2}\alpha(z-y_{c})u''(z)}{u(z)-u(y_{c})} f(z,c)dz \right|$$

$$= \left| \int_{y_{c}}^{y} \frac{dy}{\sinh^{2}\alpha(y'-y_{c})} \int_{y_{c}}^{y'} \frac{\sinh^{2}\alpha(z-y_{c})}{z-y_{c}} \frac{u''(z)}{\int_{0}^{1} u'(y_{c}+(z-y_{c})t)dt} f(z,c)dz \right|$$

$$\leq C \frac{1}{\alpha} \int_{0}^{\alpha|y-y_{c}|} \frac{dy}{\sinh^{2}y'} \int_{0}^{|y'|} \frac{\sinh^{2}z}{z} dz ||f||_{L^{\infty}([0,1]\times D_{0})}$$

$$\leq C \frac{1}{\alpha} \int_{0}^{\alpha} \frac{\sinh^{2}z}{z} dz \int_{|z|}^{\infty} \frac{dy'}{\sinh^{2}y'} ||f||_{L^{\infty}([0,1]\times D_{0})}$$

$$= C \frac{1}{\alpha} \int_0^{\alpha} \frac{1 - e^{-2z}}{z} dz \|f\|_{L^{\infty}([0,1] \times D_0)}$$

$$\leq C \frac{1 + \ln \alpha}{\alpha} \|f\|_{L^{\infty}([0,1] \times D_0)}.$$

The second statement is direct.

**Lemma 5.3.** Given  $\alpha \geq 1$ , we denote  $\Sigma_{\alpha} \triangleq \{(y,c): |y-y_c| \leq \frac{1}{\alpha}, c = u(y_c)\}$ . Then there exists a constant C independent of  $\alpha$  such that

$$||T_0 f||_{L^{\infty}(\Sigma_{\alpha})} \le \frac{1}{\alpha} ||f||_{L^{\infty}(\Sigma_{\alpha})},$$
$$||T_{k,k} f||_{L^{\infty}(\Sigma_{\alpha})} \le \frac{C}{\alpha} ||f||_{L^{\infty}(\Sigma_{\alpha})}.$$

Moreover, for  $\beta + \gamma \leq 2$ ,

$$\|\partial_y^{\beta} \partial_c^{\gamma} T_{k,k+1} f\|_{L^{\infty}(\Sigma_{\alpha})} \le C \sum_{\beta_1 + \gamma_1 \le \beta + \gamma} \|\partial_x^{\beta_1} \partial_c^{\gamma_1} f\|_{L^{\infty}(\Sigma_{\alpha})}.$$

*Proof.* For  $(y,c) \in \Sigma_{\alpha}$ , we have

$$|T_0 f(y,c)| = \left| \int_{y_c}^y f(z,c) dz \right| \le |y - y_c| ||f||_{L^{\infty}(\Sigma_{\alpha})} \le \frac{1}{\alpha} ||f||_{L^{\infty}(\Sigma_{\alpha})},$$

and

$$|T_{k,k}f(y,c)| = \left| \frac{1}{(u(y)-c)^k} \int_{y_c}^{y} (u(z)-c)^k f(z,c) dz \right|$$

$$\leq C|y-y_c| ||u'||_{L^{\infty}}^k ||f||_{L^{\infty}(\Sigma_{\alpha})} \leq \frac{C}{\alpha} ||f||_{L^{\infty}(\Sigma_{\alpha})}.$$

Using the formula

$$T_{k,k+1}f(y,c) = \int_0^1 f(y_c + t(y - y_c), c) \frac{\left(\int_0^1 u'(y_c + st(y - y_c))ds\right)^k}{\left(\int_0^1 u'(y_c + s(y - y_c))ds\right)^{k+1}} t^k dt,$$

a direct calculation gives the third inequality of the lemma. We omit the details here.  $\Box$ 

Now let us turn to the proof of Proposition 5.1.

*Proof.* Let  $M_0$  be given by Lemma 5.2. The case of  $\alpha \leq M_0$  is a direct consequence of Lemma 4.9, which implies that

$$1 \le |\phi_1(y,c)| \le C, \quad C^{-1}|y-y_c| \le |\phi(y,c)| \le C|y-y_c|, \quad \left|\partial_y^\beta \partial_c^\gamma \phi_1(y,c)\right| \le C$$

for any  $(y,c) \in [0,1] \times D_0$ , where the constant C depends on  $M_0$ .

Thus, it suffices to consider the case of  $\alpha \geq M_0$ . It is easy to check that the solution  $\phi(y,c)$  satisfies

$$\left(\sinh^2\alpha(y-y_c)\left(\frac{\phi(y,c)}{\sinh\alpha(y-y_c)}\right)'\right)' = \sinh\alpha(y-y_c)\frac{u''}{u(y)-u(y_c)}\phi(y,c).$$

Let  $\widetilde{\phi}_1(y,c) = \frac{\alpha\phi(y,c)}{\sinh\alpha(y-y_c)}$ , then  $\widetilde{\phi}_1(y,c)$  satisfies

$$\widetilde{\phi}_1(y,c) = u'(y_c) + \int_{y_c}^y \frac{dy'}{\sinh^2 \alpha(y'-y_c)} \int_{y_c}^{y'} \frac{\sinh^2 \alpha(z-y_c)u''(z)}{u(z)-u(y_c)} \widetilde{\phi}_1(z,c)dz.$$

Lemma 5.2 ensures that

$$\widetilde{\phi}_1(y,c) = (I-S)^{-1}(u'(y_c))$$

with the bound

$$C^{-1} \le \widetilde{\phi}_1(y,c) \le C$$
 for any  $(y,c) \in [0,1] \times D_0$ ,

which implies the first part of the proposition.

Next we prove the derivative estimates. Thanks to  $u'(y) \ge c_0$ , it holds that for any y, z with  $|y - y_c| \le |z - y_c|$ ,

$$(u(y) - c)^2 \le (u(z) - c)^2, \quad c = u(y_c).$$

Thus, for  $(y, c) \in \Sigma_{\alpha}$ ,

$$|Tf(y,c)| \le \int_{y_c}^{y} \frac{1}{(u(z)-c)^2} \int_{y_c}^{z} (u(y')-c)^2 ||f||_{L^{\infty}(\Sigma_{\alpha})} dy' dz$$

$$\le \int_{y_c}^{y} (z-y_c) dz ||f||_{L^{\infty}(\Sigma_{\alpha})}$$

$$\le \frac{1}{2\alpha^2} ||f||_{L^{\infty}(\Sigma_{\alpha})}.$$

This means that  $\alpha^2 T$  is a contraction mapping from  $L^{\infty}(\Sigma_{\alpha})$  to  $L^{\infty}(\Sigma_{\alpha})$ . Thus,

$$\|\phi_1\|_{L^{\infty}(\Sigma_{\alpha})} \le \|(I - \alpha^2 T)^{-1} 1\|_{L^{\infty}(\Sigma_{\alpha})} \le 2.$$

On the other hand, we have

$$\partial_y \phi_1(y,c) = \alpha^2 T_{2,2} \phi_1(y,c), 
\partial_{yy} \phi_1(y,c) = -2\alpha^2 u'(y) T_{2,3} \phi_1(y,c) + \phi(y,c),$$

then it follows from Lemma 5.3 that for any  $(y,c) \in \Sigma_{\alpha}$ ,

$$|\partial_y \phi_1(y,c)| \le C\alpha, \quad |\partial_{yy} \phi_1(y,c)| \le C\alpha^2.$$

Now we deal with the derivative estimate in c variable. Recalling that

$$\phi_1(y,c) = 1 + \alpha^2 T \phi_1(y,c),$$
  
$$\partial_c T f = 2T_0 T_{2,3} f - 2T_0 T_{1,2} f + T \partial_c f,$$

we obtain

$$\partial_c \phi_1(y,c) = 2\alpha^2 T_0 T_{2,3} \phi_1(y,c) - 2\alpha^2 T_0 T_{1,2} \phi_1(y,c) + \alpha^2 T \partial_c \phi_1(y,c),$$

$$\partial_y \partial_c \phi_1(y,c) = 2\alpha^2 T_{2,3} \phi_1(y,c) - 2\alpha^2 T_{1,2} \phi_1(y,c) + \alpha^2 T_{2,2} \partial_c \phi_1(y,c),$$

and

$$\begin{split} \partial_c^2 \phi_1(y,c) &= 2\alpha^2 \partial_c T_0 T_{2,3} \phi_1(y,c) - 2\alpha^2 \partial_c T_0 T_{1,2} \phi_1(y,c) + \alpha^2 \partial_c T \partial_c \phi_1(y,c) \\ &= 2\alpha^2 T_0 \partial_c T_{2,3} \phi_1(y,c) - 2\alpha^2 T_0 \partial_c T_{1,2} \phi_1(y,c) + \frac{\alpha^2}{3u'(y_c)} \\ &+ 2\alpha^2 T_0 T_{2,3} \partial_c \phi_1(y,c) - 2\alpha^2 T_0 T_{1,2} \partial_c \phi_1(y,c) + \alpha^2 T \partial_c^2 \phi_1(y,c), \end{split}$$

which along with Lemma 5.3 ensure that

$$\|\partial_{c}\phi_{1}\|_{L^{\infty}(\Sigma_{\alpha})} = \|(I - \alpha^{2}T)^{-1}(2\alpha^{2}T_{0}T_{2,3}\phi_{1}(y,c) - 2\alpha^{2}T_{0}T_{1,2}\phi_{1})\|_{L^{\infty}(\Sigma_{\alpha})}$$

$$\leq C(\alpha^{2}\|T_{0}T_{2,3}\phi_{1}\|_{L^{\infty}(\Sigma_{\alpha})} + \alpha^{2}\|T_{0}T_{1,2}\phi_{1}\|_{L^{\infty}(\Sigma_{\alpha})})$$

$$\leq C\alpha\|\phi_{1}\|_{L^{\infty}(\Sigma_{\alpha})} \leq C\alpha,$$

and

$$\|\partial_y \partial_c \phi_1\|_{L^{\infty}(\Sigma_{\alpha})} + \|\partial_c^2 \phi_1\|_{L^{\infty}(\Sigma_{\alpha})} \le C\alpha^2.$$

Finally, let us consider the case of  $|y-y_c| \geq \frac{1}{\alpha}$ . Using the fact that

$$|\phi_1(y,c)| \le C \frac{\sinh \alpha(y-y_c)}{\alpha(y-y_c)} \le C \cosh \alpha(y-y_c),$$

$$C \sinh \alpha |y - y_c| \ge \cosh \alpha (y - y_c) - 1$$
 for  $|y - y_c| \ge \frac{1}{\alpha}$ ,

we infer that

$$\begin{aligned} |\partial_y \phi_1(y,c)| &= \left| \frac{\alpha^2}{(u(y) - u(y_c))^2} \int_{y_c}^y \phi_1(z) (u(z) - u(y_c))^2 dz \right| \\ &\leq C \left| \frac{\alpha}{(y - y_c)^2} \int_{y_c}^y \sinh \alpha (z - y_c) (z - y_c) dz \right| \\ &\leq C \frac{|\cosh \alpha (y - y_c) - 1|}{|y - y_c|} \leq C \alpha \phi_1(y,c). \end{aligned}$$

Using the equation

$$\phi_1'' + \frac{2u'}{u - c}\phi_1' = \alpha^2 \phi_1,$$

we obtain

$$|\partial_y^2 \phi_1| \le C\alpha^2 \phi_1.$$

To estimate the derivative in c variable, we introduce  $f = \frac{\partial_y \phi_1}{\phi_1}$  and  $g = \frac{\partial_c \phi_1}{\phi_1}$ . It is easy to see that  $\partial_c f = \partial_y g$  and

$$(\phi^2 \partial_c f)' + 2\phi_1^2 u' f = 0,$$

which gives

$$\partial_c f(y,c) = \partial_y g(y,c) = \frac{-2}{\phi(y,c)^2} \int_{u_c}^{y} \phi_1^2(z,c) u'(z) f(z,c) dz = \frac{-2}{\phi(y,c)^2} T_0(\phi_1^2 u' f),$$

For  $|y-y_c| \ge \frac{1}{\alpha}$ ,  $\phi_1^2 - 1 \sim \phi_1^2$  and  $\partial_y \phi_1(y,c)$  has the same sign as  $y-y_c$ . Then

$$|\partial_{y}g(y,c)| = |\partial_{c}f(y,c)| \le \frac{C}{\phi(y,c)^{2}} \int_{y_{c}}^{y} \phi_{1}(z,c)\partial_{z}\phi_{1}(z,c)dz$$
$$\le C \frac{\phi_{1}(y,c)^{2} - 1}{\phi(y,c)^{2}} \le \frac{C}{|y - y_{c}|^{2}},$$

from which, it follows that

$$|g(y,c)| \le \frac{C}{|y-y_c|} \le C\alpha,$$

and thus,

$$|\partial_c \phi_1| \le C\alpha \phi_1, \quad |\partial_y \partial_c \phi_1| \le C\alpha^2 \phi_1.$$

We have

$$\partial_c^2 f(y,c) = \partial_y \partial_c g(y,c) = -2 \int_{y_c}^y \partial_c \left( \frac{\phi_1(z,c)^2}{\phi(y,c)^2} f(z,c) \right) u'(z) dz,$$

which implies

$$|\partial_c^2 f(y,c)| \le \frac{C\alpha}{|y-y_c|^2}.$$

Thus, we obtain

$$|\partial_c g(y,c)| \le \frac{C\alpha}{|y-y_c|} \le C\alpha^2, \quad |\partial_c^2 \phi_1| \le C\alpha^2 \phi_1.$$

The proof is finished.

**Lemma 5.4.** There exists a constant C independent of  $\alpha$  such that for any  $(y,c) \in [0,1] \times D_0$ ,

$$\frac{1}{C} \left( \frac{\sinh \alpha(y - y_c)}{\alpha(y - y_c)} - 1 \right) \le \phi_1(y, c) - 1 \le C \left( \frac{\sinh \alpha(y - y_c)}{\alpha(y - y_c)} - 1 \right),$$
$$|\phi_1(y, c) - 1| \le C \alpha^2 (y - y_c)^2 \phi_1(y, c),$$

where  $c = u(y_c)$ .

*Proof.* We have

$$\phi_{1}(y,c) - 1 = \int_{y_{c}}^{y} \frac{\alpha^{2}}{(u(y') - u(y_{c}))^{2}} \int_{y_{c}}^{y'} \phi_{1}(z)(u(z) - u(y_{c}))^{2} dz dy'$$

$$\geq C^{-1} \int_{y_{c}}^{y} \frac{\alpha}{(y' - y_{c})^{2}} \int_{y_{c}}^{y'} \sinh \alpha (z - y_{c})(z - y_{c}) dz dy'$$

$$= C^{-1} \int_{0}^{\alpha (y - y_{c})} \frac{\cosh z}{z} - \frac{\sinh z}{z^{2}} dz$$

$$= C^{-1} \left( \frac{\sinh \alpha (y - y_{c})}{\alpha (y - y_{c})} - 1 \right) \geq 0,$$

and

$$\phi_{1}(y,c) - 1 = \int_{y_{c}}^{y} \frac{\alpha^{2}}{(u(y') - u(y_{c}))^{2}} \int_{y_{c}}^{y'} \phi_{1}(z)(u(z) - u(y_{c}))^{2} dz dy'$$

$$\leq C \int_{y_{c}}^{y} \frac{\alpha}{(y' - y_{c})^{2}} \int_{y_{c}}^{y'} \sinh \alpha (z - y_{c})(z - y_{c}) dz dy'$$

$$= C \int_{0}^{\alpha (y - y_{c})} \frac{\cosh z}{z} - \frac{\sinh z}{z^{2}} dz$$

$$= C \left( \frac{\sinh \alpha (y - y_{c})}{\alpha (y - y_{c})} - 1 \right).$$

This gives the first inequality.

Thanks to  $\partial_u \phi_1(y,c) \geq 0$  for  $y \geq y_c$  and  $\partial_u \phi_1(y,c) \leq 0$  for  $y \leq y_c$ , we have

$$\phi_1(y,c) \ge \phi_1(z,c), \quad |u(z) - c| \le |u(y) - c|$$

for  $|y - y_c| \ge |z - y_c|$ . Then we get

$$|\phi_1(y,c) - 1| \le \alpha^2 \int_{y_c}^y \int_{y_c}^z \phi_1(y',c) \frac{(u(y') - c)^2}{(u(z) - c)^2} dy' dz$$
  

$$\le C\alpha^2 |y - y_c|^2 \phi_1(y,c) \le C\alpha^2 |u(y) - u(y_c)|^2 \phi_1(y,c).$$

The second inequality is proved.

**Remark 5.5.** We also need a more precise estimate for  $\phi_1(y,c) - 1$ :

$$\phi_1(y,c) - 1 = \alpha^2 (y - y_c)^2 \iint_{[0,1]^2} \phi_1(y_c + st(y - y_c), c) K(y_c + s(y - y_c), y_c, t) s dt ds$$
  
=  $\alpha^2 (y - y_c)^2 \mathcal{T}(\phi_1)(y, c)$ ,

where  $y_c = u^{-1}(c)$  and

$$K(y, y_c, t) = \frac{\left(\int_0^1 u'(y_c + st(y - y_c))ds\right)^2}{\left(\int_0^1 u'(y_c + s(y - y_c))ds\right)^2} t^2.$$

Using the fact that  $||K||_{C^2_{y,y_c,t}} \leq C$  and  $\phi_1(y,c) \geq \phi_1(z,c)$  for  $|y-y_c| \geq |z-y_c|$ , we get by Proposition 5.1 that

$$|\mathcal{T}(\phi_1)(y,c)| \leq C\phi_1(y,c),$$

$$|\partial_y \mathcal{T}(\phi_1)(y,c)| + |\partial_c \mathcal{T}(\phi_1)(y,c)| \leq C\alpha\phi_1(y,c),$$

$$|\partial_y^2 \mathcal{T}(\phi_1)(y,c)| + |\partial_c^2 \mathcal{T}(\phi_1)(y,c)| + |\partial_{yc} \mathcal{T}(\phi_1)(y,c)| \leq C\alpha^2\phi_1(y,c).$$

## 5.2. Uniform estimates for good derivative.

**Proposition 5.6.** There exists a constant C independent of  $\alpha$  such that for any  $(z,c) \in D_0 \times D_0$ ,

$$\left| (\partial_z + \partial_c)\phi_1(u^{-1}(z), c) \right| \le C\alpha^2 (y - y_c)^2 \phi_1,$$
  
$$\left| (\partial_z + \partial_c)^2 \phi_1(u^{-1}(z), c) \right| \le C\alpha^2 (y - y_c)^2 \cosh \alpha (y - y_c),$$

where  $y_c = u^{-1}(c)$  and  $y = u^{-1}(z)$ .

*Proof.* A direct calculation gives

$$\begin{split} &(\partial_z + \partial_c)\phi_1(u^{-1}(z),c) \\ &= \frac{\alpha^2}{(z-c)^2} \int_{y_c}^{u^{-1}(z)} (u(y')-c)^2 \phi_1(y',c) dy' \frac{1}{u'(u^{-1}(z))} \\ &+ 2\alpha^2 \int_{y_c}^{u^{-1}(z)} \frac{1}{(u(y')-c)^3} \int_{y_c}^{y'} (u(t)-c)^2 \phi_1(t,c) dt dy' \\ &- 2\alpha^2 \int_{y_c}^{u^{-1}(z)} \frac{1}{(u(y')-c)^2} \int_{y_c}^{y'} (u(t)-c) \phi_1(t,c) dt dy' \\ &+ \alpha^2 T(\partial_c \phi_1)(u^{-1}(z),c) \\ &= \frac{\alpha^2}{(z-c)^2} \int_{y_c}^{u^{-1}(z)} (u(y')-c)^2 \phi_1(y',c) dy' \frac{1}{u'(u^{-1}(z))} + \alpha^2 T(\partial_c \phi_1)(u^{-1}(z),c) \\ &- \alpha^2 \int_{y_c}^{u^{-1}(z)} \partial_{y'} \left(\frac{1}{(u(y')-c)^2}\right) \frac{1}{u'(y')} \int_{y_c}^{y'} (u(t)-c)^2 \phi_1(t,c) dt dy' \\ &- \alpha^2 \int_{y_c}^{u^{-1}(z)} \frac{1}{(u(y')-c)^2} \int_{y_c}^{y'} \partial_t \left((u(t)-c)^2\right) \frac{\phi_1(t,c)}{u'(t)} dt dy', \end{split}$$

Integration by parts and taking into consideration the boundary condition, we obtain

$$(\partial_z + \partial_c)\phi_1(u^{-1}(z), c)$$

$$\begin{split} &= \frac{\alpha^2}{(z-c)^2} \int_{y_c}^{u^{-1}(z)} (u(y')-c)^2 \phi_1(y',c) dy' \frac{1}{u'(u^{-1}(z))} + \alpha^2 T(\partial_c \phi_1)(u^{-1}(z),c) \\ &- \alpha^2 \Big( \frac{1}{(u(y')-c)^2} \Big) \frac{1}{u'(y')} \int_{y_c}^{y'} (u(z')-c)^2 \phi_1(z',c) dz' \Big|_{y_c}^{u^{-1}(z)} \\ &+ \alpha^2 \int_{y_c}^{u^{-1}(z)} \Big( \frac{1}{(u(y')-c)^2} \Big) \partial_{y'} \Big( \frac{1}{u'(y')} \int_{y_c}^{y'} (u(z')-c)^2 \phi_1(z',c) dz' \Big) dy' \\ &- \alpha^2 \int_{y_c}^{u^{-1}(z)} \frac{1}{(u(y')-c)^2} (u(z')-c)^2 \frac{\phi_1(z',c)}{u'(z')} \Big|_{y_c}^{y'} dy' \\ &+ \alpha^2 \int_{y_c}^{u^{-1}(z)} \frac{1}{(u(y')-c)^2} \int_{y_c}^{y'} (u(z')-c)^2 \partial_y \Big( \frac{\phi_1(z',c)}{u'(z')} \Big) dz' dy' \\ &= \alpha^2 T(\partial_c \phi_1) (u^{-1}(z),c) + \alpha^2 \int_{y_c}^{u^{-1}(z)} \frac{1}{(u(y')-c)^2} \partial_{y'} \Big( \frac{1}{u'(y')} \Big) \int_{y_c}^{y'} (u(z')-c)^2 \phi_1(z',c) dz' dy' \\ &+ \alpha^2 \int_{y_c}^{u^{-1}(z)} \frac{1}{(u(y')-c)^2} \int_{y_c}^{y'} (u(z')-c)^2 \partial_y \Big( \frac{\phi_1(z',c)}{u'(z')} \Big) dz' dy' \\ &= \alpha^2 T(\partial_c \phi_1) (u^{-1}(z),c) + \alpha^2 T_0 \Big( \partial_y \Big( \frac{1}{u'} \Big) T_{2,2}(\phi_1) \Big) (u^{-1}(z),c) \\ &+ \alpha^2 T \Big( \partial_y \Big( \frac{\phi_1}{u'} \Big) (u^{-1}(z),c) + \alpha^2 T_0 \Big( \partial_y \Big( \frac{1}{u'} \Big) T_{2,2}(\phi_1) \Big) (u^{-1}(z),c) \\ &+ \alpha^2 T \Big( \phi_1 \partial_y \Big( \frac{1}{u'} \Big) \Big) (u^{-1}(z),c). \end{split}$$

This shows that

$$(I - \alpha^2 T) \left( \partial_z \phi_1 + \partial_c \phi_1 \right) (u^{-1}(z), c)$$
  
=  $\alpha^2 T_0 \left( \partial_y \left( \frac{1}{u'} \right) T_{2,2}(\phi_1) \right) (u^{-1}(z), c) + \alpha^2 T \left( \phi_1 \partial_y \left( \frac{1}{u'} \right) \right) (u^{-1}(z), c),$ 

where the right hand side is bounded by  $\alpha^2 T \phi_1$ . Recall that T is a positive operator. Thus, we have

$$\begin{aligned} \left| (\partial_z + \partial_c) \phi_1(u^{-1}(z), c) \right| \\ &\leq \left| (I - \alpha^2 T)^{-1} \left[ \alpha^2 T_0 \left( \partial_y \left( \frac{1}{u'} \right) T_{2,2}(\phi_1) \right) (u^{-1}(z), c) \right. \right. \\ &+ \alpha^2 T \left( \phi_1 \partial_y \left( \frac{1}{u'} \right) \right) (u^{-1}(z), c) \right] \right| \\ &\leq C \left| (I - \alpha^2 T)^{-1} \alpha^2 T \phi_1 \right| \\ &\leq C \left| \alpha^2 T (I - \alpha^2 T)^{-2} 1 \right| \leq C \left| \sum_{k=1}^{\infty} k \alpha^{2k} T^k 1 \right|. \end{aligned}$$

$$(5.1)$$

Using the fact that  $(u(z)-c)^2 \le (u(y')-c)^2$  for  $|z-y_c| \le |y'-y_c|$ , we obtain

$$|T1| = \left| \int_{y_c}^{y} \frac{1}{(u(y') - c)^2} \int_{y_c}^{y'} (u(z) - c)^2 dz dy' \right|$$

$$\leq \frac{1}{2}(y - y_c)^2.$$

A simple iteration gives

$$|T^k 1| \le \frac{1}{(2k)!} (y - y_c)^{2k},$$

which implies

$$\left| \sum_{k=1}^{\infty} k \alpha^{2k} T^k 1 \right| \leq C \alpha (y - y_c) \sum_{k=1}^{\infty} \frac{1}{(2k-1)!} (\alpha (y - y_c))^{2k-1}$$

$$\leq C \alpha (y - y_c) \sinh \alpha (y - y_c)$$

$$\leq C \alpha^2 (y - y_c)^2 \phi_1,$$

which together with (5.1) gives the first inequality.

Now we prove the second inequality. We have

$$(\partial_z + \partial_c)^2 \phi_1(u^{-1}(z), c)$$

$$= \alpha^2 (\partial_z + \partial_c) T(\partial_c \phi_1) (u^{-1}(z), c) + \alpha^2 (\partial_z + \partial_c) T_0 \left( \partial_y \left( \frac{1}{u'} \right) T_{2,2}(\phi_1) \right) (u^{-1}(z), c)$$

$$+ \alpha^2 (\partial_z + \partial_c) T \left( \partial_y \left( \frac{\phi_1}{u'} \right) \right) (u^{-1}(z), c).$$

Let F(y,c) be a function with  $\frac{F(y,c)}{(u(y)-c)^2} \to 0$  as  $y \to y_c$ . Then we have

$$\begin{split} &(\partial_z + \partial_c) \int_{y_c}^{u^{-1}(z)} \frac{1}{(u(y') - c)^2} F(y', c) dy' \\ &= \frac{F(u^{-1}(z), c)}{(z - c)^2} (u^{-1})'(z) - \frac{F(y', c)}{(u(y') - c)^2 u'(y')} \Big|_{y_c}^{u^{-1}(z)} \\ &+ \int_{y_c}^{u^{-1}(z)} \frac{1}{(u(y') - c)^2} \left( \partial_y \left( \frac{1}{u'(y')} \right) F(y', c) \right) dy' + \int_{y_c}^{u^{-1}(z)} \frac{1}{(u(y') - c)^2} \left( \frac{1}{u'(y')} \partial_y F(y', c) \right) dy' \\ &+ \int_{y_c}^{u^{-1}(z)} \frac{1}{(u(y') - c)^2} \partial_c F(y', c) dy' \\ &= \int_{y_c}^{u^{-1}(z)} \frac{F(y', c)}{(u(y') - c)^2} \partial_y \left( \frac{1}{u'(y')} \right) dy' + \int_{y_c}^{u^{-1}(z)} \frac{1}{(u(y') - c)^2} \left( \frac{1}{u'(y')} \partial_y F(y', c) \right) dy' \\ &+ \int_{y_c}^{u^{-1}(z)} \frac{1}{(u(y') - c)^2} \partial_c F(y', c) dy'. \end{split}$$

Similarly, we have

$$(\partial_z + \partial_c) \int_{y_c}^{u^{-1}(z)} f(y', c) (u(y') - c)^2 dy'$$

$$= f(u^{-1}(z), c) (z - c)^2 (u^{-1})'(z) + \int_{y_c}^{u^{-1}(z)} \partial_c f(y', c) (u(y') - c)^2 dy'$$

$$- \int_{y_c}^{u^{-1}(z)} \frac{f(y', c)}{u'(y')} \partial_y ((u(y') - c)^2) dy'$$

$$= \int_{y_c}^{u^{-1}(z)} \partial_c f(y',c) (u(y')-c)^2 dy' + \int_{y_c}^{u^{-1}(z)} \partial_y \left(\frac{f(y',c)}{u'(y')}\right) (u(y')-c)^2 dy'.$$

With the above equalities, we can deduce that

$$(\partial_{z} + \partial_{c})^{2} \phi_{1}(u^{-1}(z), c)$$

$$= 2\alpha^{2} \int_{y_{c}}^{u^{-1}(z)} \partial_{y} \left(\frac{1}{u'(y')}\right) T_{2,2}(\partial_{c}\phi_{1})(y', c) dy' + \alpha^{2} \int_{y_{c}}^{u^{-1}(z)} T_{2,2}(\partial_{c}^{2}\phi_{1})(y', c) dy'$$

$$+ \alpha^{2} \int_{y_{c}}^{u^{-1}(z)} T_{2,2}\phi_{1}(y', c) \left(\partial_{y}\left(\frac{1}{u'(y')}\right)\right)^{2} dy' + \alpha^{2} \int_{y_{c}}^{u^{-1}(z)} T_{2,2}\phi_{1}(y', c) \frac{1}{u'(y')} \partial_{y}^{2}\left(\frac{1}{u'(y')}\right) dy'$$

$$+ 2\alpha^{2} \int_{y_{c}}^{u^{-1}(z)} \partial_{y}\left(\frac{1}{u'(y')}\right) T_{2,2}\left(\partial_{y}\left(\frac{\phi_{1}}{u'}\right)\right)(y', c) dy' + 2\alpha^{2} \int_{y_{c}}^{u^{-1}(z)} T_{2,2}\left(\partial_{c}\partial_{y}\left(\frac{\phi_{1}}{u'}\right)\right)(y', c) dy'$$

$$+ \alpha^{2} \int_{y_{c}}^{u^{-1}(z)} T_{2,2}\left(\partial_{y}\left(\frac{1}{u'}\partial_{y}\left(\frac{\phi_{1}}{u'}\right)\right)\right)(y', c) dy'.$$

Note that

$$\begin{aligned} \partial_c^2 \phi_1 + \partial_y \left( \frac{1}{u'} \partial_y \left( \frac{\phi_1}{u'} \right) \right) + 2 \partial_c \partial_y \left( \frac{\phi_1}{u'} \right) \\ &= (\partial_z + \partial_c)^2 \phi_1 (u^{-1}(z), c) + 2 \partial_y \left( \frac{1}{u'(y)} \right) (\partial_z + \partial_c) \phi_1 (u^{-1}(z), c) \\ &+ \partial_y^2 \left( \frac{1}{u'(y)} \right) \frac{\phi_1}{u'(y)} + \left( \partial_y \left( \frac{1}{u'(y)} \right) \right)^2 \phi_1. \end{aligned}$$

Thus, we obtain

$$\begin{split} &(\partial_{z} + \partial_{c})^{2} \phi_{1}(u^{-1}(z), c) \\ &= \alpha^{2} T \Big( (\partial_{z} + \partial_{c})^{2} \phi_{1}(u^{-1}(z), c) \Big) (u^{-1}(z), c) + 2\alpha^{2} T_{0} \Big( \partial_{y} \Big( \frac{1}{u'(y')} \Big) T_{2,2}((\partial_{z} + \partial_{c}) \phi_{1}(u^{-1}(z), c)) \Big) \\ &+ \alpha^{2} \int_{y_{c}}^{u^{-1}(z)} T_{2,2} \phi_{1}(y', c) \Big( \partial_{y} \Big( \frac{1}{u'(y')} \Big) \Big)^{2} dy' + \alpha^{2} \int_{y_{c}}^{u^{-1}(z)} T_{2,2} \phi_{1}(y, c) \frac{1}{u'(y)} \partial_{y}^{2} \Big( \frac{1}{u'(y)} \Big) dy' \\ &+ 2\alpha^{2} \int_{y_{c}}^{u^{-1}(z)} \partial_{y} \Big( \frac{1}{u'(y')} \Big) T_{2,2} \Big( \phi_{1} \partial_{y} \Big( \frac{1}{u'} \Big) \Big) dy' + 2\alpha^{2} T \Big( \partial_{y} \Big( \frac{1}{u'} \Big) (\partial_{z} + \partial_{c}) \phi_{1}(u^{-1}(z), c) \Big) \\ &+ \alpha^{2} T \Big[ \partial_{y}^{2} \Big( \frac{1}{u'} \Big) \frac{\phi_{1}}{u'} + \Big( \partial_{y} \Big( \frac{1}{u'} \Big) \Big)^{2} \phi_{1} \Big]. \end{split}$$

Then by using the estimate

$$|(\partial_y + \partial_c)\phi_1(u^{-1}(y), c)| \le C|(I - \alpha^2 T)^{-1}\alpha^2 T\phi_1|,$$

and the fact that T is a positive operator, we deduce that

$$\begin{aligned} &|(\partial_{y} + \partial_{c})^{2} \phi_{1}(u^{-1}(y), c)| \\ &\leq C \left| (I - \alpha^{2}T)^{-1} \alpha^{2}T \phi_{1} \right| + C \left| (I - \alpha^{2}T)^{-1} \alpha^{2}T (I - \alpha^{2}T)^{-1} \alpha^{2}T \phi_{1} \right| \\ &\leq C \left| \alpha^{2}T (I - \alpha^{2}T)^{-2}1 \right| + C \left| (\alpha^{2}T)^{2} (I - \alpha^{2}T)^{-3}1 \right| \\ &\leq C \alpha (y - y_{c}) \sinh \alpha (y - y_{c}) + C \left| \sum_{k=0}^{\infty} (k+1)(k+2)(\alpha^{2}T)^{k+2}1 \right| \end{aligned}$$

$$\leq C(\alpha(y-y_c)\sinh\alpha(y-y_c) + \alpha^2(y-y_c)^2\cosh\alpha(y-y_c)).$$

This deduces the second inequality.

## 6. The inhomogeneous Rayleigh equation

In this section, we consider the inhomogeneous Rayleigh equation

(6.1) 
$$\begin{cases} \Phi'' - \alpha^2 \Phi - \frac{u''}{u - c} \Phi = f, \\ \Phi(0) = \Phi(1) = 0. \end{cases}$$

Let  $\phi(y,c)$  be the solution of the homogeneous Rayleigh equation given by Proposition 4.5, and  $\phi_1(y,c) \triangleq \frac{\phi(y,c)}{u(y)-c}$ . We denote

(6.2) 
$$II_2 \triangleq \text{p.v.} \int_0^1 \frac{u'(y) - u'(y_c)}{(u(y) - c)^2} dy,$$

(6.3) 
$$II_3 \triangleq \int_0^1 \frac{1}{(u(y) - c)^2} \left(\frac{1}{\phi_1(y, c)^2} - 1\right) dy,$$

for  $c \in D_0$  and  $y_c = u^{-1}(c)$ .

The following lemma gives a sufficient and necessary condition so that  $\mathcal{R}_{\alpha}$  has no embedding eigenvalues.

**Lemma 6.1.** Let  $A(c) = u(0) - u(1) - \rho(c)II_2 + u'(y_c)\rho(c)II_3$  and  $B(c) = \pi \rho(c)\frac{u''(y_c)}{u'(y_c)^2}$ , where  $\rho(c) = (c - u(0))(u(1) - c)$ . Then  $c \in D_0$  is an embedding eigenvalue of  $\mathcal{R}_{\alpha}$  if and only if  $A(c)^2 + B(c)^2 = 0$ .

*Proof.* Let  $c = u(y_c)$  be an embedding eigenvalue of  $\mathcal{R}_{\alpha}$ . We know that  $u''(u^{-1}(c)) = 0$ , thus B(c) = 0. Next we show A(c) = 0. In such case, A(c) = 0 is equivalent to the Wronskian  $W[\varphi_1, \varphi_2; c] = 0$ , where  $\varphi_1(y, c)$  and  $\varphi_2(y, c)$  are two linearly independent solutions of the Rayleigh equation

(6.4) 
$$\varphi'' - \alpha^2 \varphi - \frac{u''}{u - c} \varphi = 0.$$

Indeed, thanks to  $u''(y_c) = 0$ , we can construct a smooth solution  $\varphi(y, c)$  of (6.4) with the boundary conditions  $\varphi(0, c) = 0$  and  $\varphi'(0, c) = 1$ . Moreover, we have

$$W[\varphi_1, \varphi_2; c] = \varphi(1, c).$$

Let  $\phi(y,c)$  be a solution of (6.4) given by Lemma 4.9 and  $\phi_1(y,c) = \frac{\phi(y,c)}{u(y)-c}$ . Then  $\varphi(y,c)$  has the following representation formula

$$\varphi(y,c) = \phi_1(0,c)\overline{\varphi}(y,c),$$

where  $\overline{\varphi}(y,c)$  is given by

$$\overline{\varphi}(y,c) = \frac{\phi_1(y,c)}{u'(y_c)}(u(y) - u(0)) 
+ \frac{\phi_1(y,c)}{u'(y_c)}(u(y) - c)(u(0) - c) \int_0^y \frac{u'(y_c) - u'(z)}{(u(z) - c)^2} dz 
+ \phi_1(y,c)(u(y) - c)(u(0) - c) \int_0^y \frac{1}{(u(z) - c)^2} \left(\frac{1}{\phi_1(z,c)^2} - 1\right) dz.$$

Then we find that  $\varphi(1,c)=0$  is equivalent to

$$\begin{split} 0 = & \varphi(1,c) = \phi_1(0,c)\overline{\varphi}(y,c) \\ = & \frac{\phi_1(0,c)\phi_1(1,c)}{u'(y_c)}(u(1)-u(0)) - \frac{\phi_1(0,c)\phi_1(1,c)}{u'(y_c)}\rho(c) \int_0^1 \frac{u'(y_c)-u'(y)}{(u(y)-c)^2} dy \\ & - \phi_1(0,c)\phi_1(1,c)\rho(c) \int_0^1 \frac{1}{(u(y)-c)^2} \Big(\frac{1}{\phi_1(y,c)^2}-1\Big) dy \\ = & - \frac{\phi_1(0,c)\phi_1(1,c)}{u'(y_c)} \mathbf{A}(c). \end{split}$$

It remains to show that if  $A(c)^2 + B(c)^2 = 0$ , then c must be an embedding eigenvalue of  $\mathcal{R}_{\alpha}$ . The equality  $A(c)^2 + B(c)^2 = 0$  implies that  $u''(y_c) = 0$  and  $\varphi(1,c) = 0$ . Then  $\varphi(y,c)$  is a non zero solution of (6.4) with boundary conditions  $\varphi(0,c) = \varphi(1,c) = 0$ . This means that c is an embedding eigenvalue of  $\mathcal{R}_{\alpha}$  with eigenfunction  $\varphi(y,c)$ .

**Remark 6.2.** For  $c \in \mathbb{C} \setminus D_0$ , let  $\varphi(y,c)$  be the solution of (6.4) with  $\varphi(0,c) = 0$  and  $\varphi'(0,c) = 1$ , then  $\varphi(y,c)$  is analytic in c, and  $\lim_{c \to \infty} \varphi(y,c) = \frac{\sinh(\alpha y)}{\alpha}$ . For  $c \in \Omega_{\epsilon_0} \setminus D_0$ , we have

$$\varphi(y,c) = \phi(0,c)\phi(y,c) \int_0^y \frac{1}{\phi(y',c)^2} dy',$$

in particular,

$$\varphi(1,c) = \phi(0,c)\phi(1,c) \int_0^1 \frac{1}{\phi(y,c)^2} dy.$$

**Lemma 6.3.** If  $\mathcal{R}_{\alpha}$  has no embedding eigenvalues, then there exists  $\epsilon_1 > 0$  such that for any  $c \in \Omega_{\epsilon_1} \setminus D_0$ ,  $\varphi(1, c) \neq 0$ .

*Proof.* Let  $c_r$  be defined by (4.3). We claim the following uniform convergence:

1. for  $c_{\epsilon} \in \{Im \, c > 0\} \cap D_{\epsilon_0}$ ,

$$\rho(c_{\epsilon}) \int_{0}^{1} \frac{1}{\phi(y, c_{\epsilon})^{2}} dy \longrightarrow \frac{1}{u'(y_{c})} (A(c_{r}) - iB(c_{r})) \text{ as } c_{\epsilon} \to c_{r};$$

2. for  $c_{\epsilon} \in \{Im \, c < 0\} \cap D_{\epsilon_0}$ ,

$$\rho(c_{\epsilon}) \int_{0}^{1} \frac{1}{\phi(y, c_{\epsilon})^{2}} dy \longrightarrow \frac{1}{u'(y_{c})} (\mathbf{A}(c_{r}) + i\mathbf{B}(c_{r})) \text{ as } c_{\epsilon} \to c_{r};$$

3. for  $c_{\epsilon} \in B_{\epsilon_0}^l \cup B_{\epsilon_0}^r$ ,

$$\rho(c_{\epsilon}) \int_0^1 \frac{1}{\phi(y, c_{\epsilon})^2} dy \longrightarrow \frac{u(0) - u(1)}{u'(y_c)} \text{ as } c_{\epsilon} \to c_r.$$

By definition of  $\rho(c)$  and  $\phi_1(y,c)$ , we have

$$\varphi(1,c) = -\rho(c)\phi_1(0,c)\phi_1(1,c) \int_0^1 \frac{1}{\phi(y,c)^2} dy \text{ for } c \in \Omega_{\epsilon_0} \setminus D_0.$$

It is easy to show that A(c) and B(c) are continuous in  $D_0$ . This along with the above claims implies that  $\varphi(1,c)$  is continuous in  $\Omega_{\epsilon_0}$ . Thus, the conclusion follows from Lemma 6.1.

Now let us first prove claim 1. Let  $c_{\epsilon} = c + i\epsilon$  with  $\epsilon \neq 0$  and  $c \in D_0$  and  $y_c = u^{-1}(c)$ . We write

$$\rho(c_{\epsilon}) \int_{0}^{1} \frac{1}{\phi(y, c_{\epsilon})^{2}} dy$$

$$= \rho(c_{\epsilon}) \int_{0}^{1} \frac{dy}{(u(y) - c_{\epsilon})^{2} \phi_{1}(y, c_{\epsilon})^{2}}$$

$$= \rho(c_{\epsilon}) \int_{0}^{1} \frac{dy}{(u(y) - c_{\epsilon})^{2}} + \rho(c_{\epsilon}) \int_{0}^{1} \frac{1}{(u(y) - c_{\epsilon})^{2}} \left(\frac{1}{\phi_{1}(y, c_{\epsilon})^{2}} - 1\right) dy.$$

By Lebesgue dominated convergence theorem, we get

$$\lim_{\epsilon \to 0} \int_0^1 \frac{1}{(u(y) - c_{\epsilon})^2} \left( \frac{1}{\phi_1(y, c_{\epsilon})^2} - 1 \right) dy = \int_0^1 \frac{1}{(u(y) - c)^2} \left( \frac{1}{\phi_1(y, c)^2} - 1 \right) dy = II_3.$$

Thanks to  $u'(y_c) > c > 0$ , we have

$$\rho(c_{\epsilon}) \int_{0}^{1} \frac{dy}{(u(y) - c_{\epsilon})^{2}} \\
= \frac{1}{u'(y_{c})} \rho(c_{\epsilon}) \left( \int_{0}^{1} \frac{u'(y)dy}{(u(y) - c_{\epsilon})^{2}} - \int_{0}^{1} \frac{u'(y) - u'(y_{c})}{(u(y) - c_{\epsilon})^{2}} dy \right) \\
\triangleq \frac{1}{u'(y_{c})} (I_{1} - I_{2}).$$

Thanks to  $\left|\frac{1}{(u(y)-c_{\epsilon})^2}\right|>c\epsilon^2>0$ , we get

$$I_{1} = -\frac{(c_{\epsilon} - u(0))(u(1) - c_{\epsilon})}{u(y) - c_{\epsilon}} \Big|_{0}^{1} = \frac{(c_{\epsilon} - u(0))(u(1) - c_{\epsilon})}{u(0) - c_{\epsilon}} - \frac{(c_{\epsilon} - u(0))(u(1) - c_{\epsilon})}{u(1) - c_{\epsilon}}$$
$$= u(0) - u(1).$$

Set 
$$g(y) = u'(y) - u'(y_c) - \frac{u''(y_c)}{u'(y_c)^2} u'(y) (u(y) - c)$$
. Then  $g(y_c) = 0$  and  $g'(y_c) = 0$ .

$$I_{2} = \rho(c_{\epsilon}) \int_{0}^{1} \frac{u'(y) - u'(y_{c})}{(u(y) - c_{\epsilon})^{2}} dy$$

$$= \rho(c_{\epsilon}) \int_{0}^{1} \frac{g(y)}{(u(y) - c_{\epsilon})^{2}} dy + \rho(c_{\epsilon}) \frac{u''(y_{c})}{u'(y_{c})^{2}} \int_{0}^{1} \frac{u'(y)(u(y) - c)}{(u(y) - c_{\epsilon})^{2}} dy.$$

Because of  $|g'(y)| = |g'(y) - g'(y_c)| \le C|y - y_c|$ , we get

$$\int_0^1 \frac{g(y)}{(u(y) - c_{\epsilon})^2} dy \to p.v. \int_0^1 \frac{g(y)}{(u(y) - c)^2} dy.$$

Thus, we deduce that as  $\epsilon \to 0+$ 

$$\rho(c_{\epsilon}) \int_{0}^{1} \frac{u'(y)(u(y) - c)}{(u(y) - c_{\epsilon})^{2}} dy = \rho(c_{\epsilon}) \int_{u(0) - c_{\epsilon}}^{u(1) - c_{\epsilon}} \frac{x + i\epsilon}{x^{2}} dx$$

$$= \rho(c_{\epsilon}) \ln \frac{u(1) - c_{\epsilon}}{u(0) - c_{\epsilon}} + i\epsilon (u(0) - u(1))$$

$$\longrightarrow (c - u(0))(u(1) - c) \ln \frac{u(1) - c}{c - u(0)} + i\pi (c - u(0))(u(1) - c)$$

$$= \text{p.v.} \rho(c) \int_{0}^{1} \frac{u'(y)(u(y) - c)}{(u(y) - c)^{2}} dy + i\pi \rho(c).$$

This shows that

$$I_2 \longrightarrow \rho(c)II_2 + i\pi \frac{u''(y_c)}{u'(y_c)^2}\rho(c)$$
 as  $\epsilon \to 0 + .$ 

So, we have as  $\epsilon \to 0+$ ,

$$\rho(c_{\epsilon}) \int_{0}^{1} \frac{1}{\phi(y, c_{\epsilon})^{2}} dy \longrightarrow \frac{1}{u'(y_{c})} \Big( u(0) - u(1) - \rho(c) \Pi_{2} - i\pi \frac{u''(y_{c})}{u'(y_{c})^{2}} \rho(c) \Big) + \rho(c) \Pi_{3}.$$

This shows claim 1, and the proof of claim 2 is similar.

Let us prove claim 3. Let  $c_{\epsilon} = u(0) + \epsilon e^{i\theta}$  with  $\epsilon > 0$ . Similarly, we write

$$\rho(c_{\epsilon}) \int_{0}^{1} \frac{1}{\phi(y, c_{\epsilon})^{2}} dy$$

$$= \rho(c_{\epsilon}) \int_{0}^{1} \frac{dy}{(u(y) - c_{\epsilon})^{2}} + \rho(c_{\epsilon}) \int_{0}^{1} \frac{1}{(u(y) - c_{\epsilon})^{2}} \left(\frac{1}{\phi_{1}(y, c_{\epsilon})^{2}} - 1\right) dy.$$

Because  $\frac{1}{(u(y)-c_{\epsilon})^2} \left(\frac{1}{\phi_1(y,c_{\epsilon})^2}-1\right)$  is uniformly bounded, the second term tends to zero. For the first term, we write as above

$$\rho(c_{\epsilon}) \int_0^1 \frac{dy}{(u(y) - c_{\epsilon})^2} \triangleq \frac{1}{u'(0)} (I_1 - I_2).$$

It's easy to see that

$$I_1 = u(0) - u(1).$$

For  $I_2$ , we have

$$|I_{2}| = \left| \epsilon e^{i\theta} (u(1) - u(0) - \epsilon e^{i\theta}) \int_{u(0)}^{u(1)} \frac{u'(u^{-1}(z)) - u'(0)}{(z - u(0) - \epsilon e^{i\theta})^{2}} (u^{-1})'(z) dz \right|$$

$$\leq C \epsilon \int_{0}^{u(1) - u(0)} \frac{z}{\sqrt{(z + \epsilon \cos \theta)^{2} + \epsilon^{2} \sin^{2} \theta}} dz$$

$$\leq C \epsilon |\ln \epsilon| \to 0 \quad \text{as } \epsilon \to 0.$$

For the case of  $c_{\epsilon} \in B_{\epsilon_0}^l$ , the proof is similar.

**Remark 6.4.** If A(c) < 0 and B(c) = 0 for  $c \in D_0$ , then  $\mathcal{R}_{\alpha}$  has no eigenvalue.

Indeed, under this assumption we can find  $0 < \epsilon_1 < 1$  such that for any  $c \in \overline{\Omega_{\epsilon_1}} \setminus D_0$ ,  $\varphi(1,c) \notin (-\infty,0]$ . Since  $\varphi(1,c)$  is analytic for  $c \in \mathbb{C} \setminus D_0$ , and  $\varphi(1,c) \to \frac{\sinh \alpha}{\alpha}$  for  $c \to \infty$ , we can find R > 2 such that  $\varphi(1,c) \in \mathbb{C} \setminus (-\infty,0]$  for  $|c| \geq R$ . Then by residue theorem, the number of roots of  $\varphi(1,c)$  for  $c \in \mathbb{C} \setminus D_0$  equals to the number of roots of  $\varphi(1,c)$  for  $c \in B_R \setminus \overline{\Omega_{\epsilon_1}}$ , which equals to (count multiplicity)

$$\frac{1}{2\pi i} \oint_{\partial B_R} -\oint_{\partial \Omega_{\epsilon_1}} \frac{\partial_c \varphi(1,c)}{\varphi(1,c)} dc = \frac{1}{2\pi i} \oint_{\partial B_R} -\oint_{\partial \Omega_{\epsilon_1}} \partial_c \ln \varphi(1,c) dc.$$

Let  $z = \varphi(1,c)$ . For  $c \in \partial B_R$ ,  $\ln z$  is the analytic function defined in  $\mathbb{C} \setminus (-\infty,0]$  such that  $\ln z = \ln |z| + i \arg z$  for |z| > 0,  $-\pi < \arg z < \pi$ . For  $c \in \partial \Omega_{\epsilon_1}$ , by our assumption, if we take  $\epsilon_1$  small enough, such that  $\ln(z) = \ln |z| + i \arg z$  for |z| > 0,  $-\pi < \arg z < \pi$ . Thus, we deduce that

$$\frac{1}{2\pi i} \oint_{\partial B_R} - \oint_{\partial \Omega_{\epsilon_1}} \partial_c \ln \varphi(1,c) dc = 0.$$

**Proposition 6.5.** Let  $c \in \Omega_{\epsilon_0}$  with  $\epsilon_1$  as in Lemma 6.3. Suppose that  $\mathcal{R}_{\alpha}$  has no embedding eigenvalues. Then we have the following representation formula for the solution of (6.1):

$$\Phi(y,c) = \phi(y,c) \int_0^y \frac{1}{\phi(z,c)^2} \int_{y_c}^z \phi(y',c) f(y',c) dy' dz + \mu(c) \phi(y,c) \int_0^y \frac{1}{\phi(y',c)^2} dy',$$

where  $y_c = u^{-1}(c_r)$  with  $c_r$  defined by (4.3) and

$$\mu(c) = -\frac{\int_0^1 \frac{1}{\phi(z,c)^2} \int_{y_c}^z \phi f(y',c) dy' dz}{\int_0^1 \frac{1}{\phi(y,c)^2} dy}.$$

*Proof.* Let  $\varphi(y)$  be a solution of the homogenous Rayleigh equation

$$\varphi'' - \alpha^2 \varphi - \frac{u''}{u - c} \varphi = 0,$$

then the inhomogeneous Rayleigh equation

$$\begin{cases} \Phi'' - \alpha^2 \Phi - \frac{u''}{u - c} \Phi = f, \\ \Phi(0) = \Phi(1) = 0, \end{cases}$$

is equivalent to

$$\begin{cases} \left(\varphi^2\left(\frac{\Phi}{\varphi}\right)'\right)' = f\varphi, \\ \psi(0) = \psi(1) = 0. \end{cases}$$

This gives our result by integration and noting that  $\mu(c)$  is well-defined by Lemma 6.3.

Next we study the convergence of the solution  $\psi(y,c)$  with  $f = \frac{\omega_0(y)}{i\alpha(u-c)}$ . Let  $c \in D_0$  and  $y_c = u^{-1}(c)$ . We introduce

$$\Phi_{\pm}(y,c) \triangleq \begin{cases} \phi \int_{0}^{y} \frac{1}{\phi(z,c)^{2}} \int_{y_{c}}^{z} \phi f(y',c) dy' dz + \mu_{\pm}(c) \phi \int_{0}^{y} \frac{1}{\phi(y',c)^{2}} dy' & 0 \leq y \leq y_{c}, \\ \phi \int_{1}^{y} \frac{1}{\phi(z,c)^{2}} \int_{y_{c}}^{z} \phi f(y',c) dy' dz + \mu_{\pm}(c) \phi \int_{1}^{y} \frac{1}{\phi(y',c)^{2}} dy' & y_{c} \leq y \leq 1, \end{cases}$$

and

$$\Phi_l(y) \triangleq \phi(y, u(0)) \int_1^y \frac{1}{\phi(z, u(0))^2} \int_0^z \phi(y', u(0)) f(y', u(0)) dy' dz, 
\Phi_r(y) \triangleq \phi(y, u(1)) \int_0^y \frac{1}{\phi(z, u(1))^2} \int_1^z \phi(y', u(1)) f(y', u(1)) dy' dz,$$

where

$$\mu_{+}(c) = \frac{1}{\alpha} \frac{iu'(y_c)\rho(c)\Pi_1 - \rho(c)\frac{\omega_0(y_c)}{u'(y_c)}\pi}{u(0) - u(1) - \rho(c)\Pi_2 - i\pi\rho(c)\frac{u''(y_c)}{u'(y_c)^2} + u'(y_c)\rho(c)\Pi_3},$$

$$\mu_{-}(c) = \frac{1}{\alpha} \frac{iu'(y_c)\rho(c)\Pi_1 + \rho(c)\frac{\omega_0(y_c)}{u'(y_c)}\pi}{u(0) - u(1) - \rho(c)\Pi_2 + i\pi\rho(c)\frac{u''(y_c)}{u'(y_c)^2} + u'(y_c)\rho(c)\Pi_3},$$

with

(6.5) 
$$II_1(\omega_0) = \text{p.v.} \int_0^1 \frac{\int_{y_c}^z \omega_0(y')\phi_1(y',c)dy'}{\phi(z,c)^2} dz.$$

**Lemma 6.6.** Suppose that f(y,c) is a Lipschitz function in  $y \in [0,1]$  with

$$|f(y,c) - f(z,c)| \le C|y-z|$$
 for  $(y,z,c) \in [0,1]^2 \times D_{\epsilon_0}$ ,

and  $f(y, c_{\epsilon})$  uniformly converges to f(y, c) for  $c_{\epsilon} = c + i\epsilon, c \in D_0$  as  $\epsilon \to 0$ . Then it holds that for  $0 \le y \le y_c$ ,

$$(y-y_c+i\epsilon)\int_0^y \frac{f(z,c_\epsilon)}{(z-y_c+i\epsilon)^2}dz \longrightarrow (y-y_c)\int_0^y \frac{f(z,c)}{(z-y_c)^2}dz$$

as  $\epsilon$  tends to zero.

*Proof.* Thanks to  $0 \le z \le y \le y_c$ , then  $|z - y_c| \ge |y - y_c|$  and

$$\left| \frac{(f(z, c_{\epsilon}) - f(y_c, c_{\epsilon}))(y - y_c + i\epsilon)}{(z - y_c + i\epsilon)^2} \right| \le C \frac{|z - y_c|(|y - z_0|^2 + \epsilon^2)^{1/2}}{|z - y_c|^2 + \epsilon^2} \le C,$$

which together with Lebesgue dominated convergence theorem gives

$$(y - y_c + i\epsilon) \int_0^y \frac{f(z, c_\epsilon) - f(y_c, c_\epsilon)}{(z - y_c + i\epsilon)^2} dz \longrightarrow (y - y_c) \int_0^y \frac{f(z, c) - f(y_c, c)}{(z - y_c)^2} dz.$$

On the other hand, we have

$$(y - y_c + i\epsilon) \int_0^y \frac{1}{(y - y_c + i\epsilon)^2} dz = -\frac{y - y_c + i\epsilon}{-y_c + i\epsilon} + 1$$
$$\longrightarrow (y - y_c) \int_0^y \frac{1}{(z - y_c)^2} dz$$

as  $\epsilon$  tends to zero. Then the lemma follows easily.

**Proposition 6.7.** Let  $\Phi(y,c)$  be a solution of (6.1) given by Proposition 6.5 with  $f = \frac{\omega_0(y)}{i\alpha(u-c)}$  for  $\omega_0 \in L^2(0,1)$ . If  $\mathcal{R}_{\alpha}$  has no embedding eigenvalues, then it holds that

1. for any  $(y, y_c) \in [0, 1] \times [0, 1]$ ,

$$\lim_{\epsilon \to 0+} \Phi(y, c_{\epsilon}) = \Phi_{+}(y, u(y_c)), \quad \lim_{\epsilon \to 0-} \Phi(y, c_{\epsilon}) = \Phi_{-}(x, u(y_c)),$$

where  $c_{\epsilon} = c + i\epsilon$  and  $c = u(y_c)$ .

2. for any  $y \in [0, 1]$ ,

$$\lim_{\epsilon \to 0+} \Phi(y, c_{\epsilon}^{l}) = \Phi_{l}(y), \quad \lim_{\epsilon \to 0+} \Phi(y, c_{\epsilon}^{r}) = \Phi_{r}(y),$$

where 
$$c_{\epsilon}^l = u(0) + \epsilon e^{i\theta}$$
 and  $c_{\epsilon}^r = u(1) - \epsilon e^{i\theta}$  for  $\theta \in [\frac{\pi}{2}, \frac{3\pi}{2}]$ .

*Proof.* Let us first prove the first statement. To show the convergence of  $\mu(c_e)$ , we consider

$$\mu(c_{\epsilon}) = -\frac{\int_0^1 \frac{1}{\phi(z,c_{\epsilon})^2} \int_{y_c}^z \phi f(y,c_{\epsilon}) dy dz}{\int_0^1 \frac{1}{\phi(y,c_{\epsilon})^2} dy}$$
$$= -\frac{\rho(c_{\epsilon}) \int_0^1 \frac{1}{\phi(z,c_{\epsilon})^2} \int_{y_c}^z \phi f(y,c_{\epsilon}) dy dz}{\rho(c_{\epsilon}) \int_0^1 \frac{1}{\phi(y,c_{\epsilon})^2} dy}.$$

By the claims in the proof of Lemma 6.3, it suffices to deal with the numerator of  $\mu(c)$ , which is decomposed as

$$\rho(c_{\epsilon}) \int_{0}^{1} \frac{1}{\phi(z, c_{\epsilon})^{2}} \int_{y_{\epsilon}}^{z} \phi f(y, c_{\epsilon}) dy dz = \rho(c_{\epsilon}) \int_{0}^{1} \frac{1}{\phi(z, c_{\epsilon})^{2}} \int_{y_{\epsilon}}^{z} \phi_{1}(y, c_{\epsilon}) \frac{\omega_{0}(y)}{i\alpha} dy dz$$

$$= \rho(c_{\epsilon}) \int_{0}^{1} \frac{1}{\phi(z, c_{\epsilon})^{2}} \int_{y_{c}}^{z} (\phi_{1}(y, c_{\epsilon}) - 1) \frac{\omega_{0}(y)}{i\alpha} dy dz$$

$$+ \rho(c_{\epsilon}) \int_{0}^{1} \frac{\int_{y_{c}}^{y} \frac{\omega_{0}(y)}{i\alpha} dy}{(u(z) - c_{\epsilon})^{2}} (\frac{1}{\phi_{1}(z, c_{\epsilon})^{2}} - 1) dz$$

$$+ \rho(c_{\epsilon}) \int_{0}^{1} \frac{\int_{y_{c}}^{z} \frac{\omega_{0}(y)}{i\alpha} dy}{(u(z) - c_{\epsilon})^{2}} dz.$$

Using the inequality  $|\phi_1(y, c_{\epsilon}) - 1| \le C(|y - y_c|^2 + \epsilon^2)$  and Lebesgue dominated convergence theorem, we get

$$\rho(c_{\epsilon}) \int_{0}^{1} \frac{1}{\phi(z, c_{\epsilon})^{2}} \int_{y_{c}}^{z} (\phi_{1}(y, c_{\epsilon}) - 1) \frac{\omega_{0}(y)}{i\alpha} dy dz$$

$$\longrightarrow \rho(c) \int_{0}^{1} \frac{1}{\phi(z, c)^{2}} \int_{y_{c}}^{z} (\phi_{1}(y, c) - 1) \frac{\omega_{0}(y)}{i\alpha} dy dz,$$

and

$$\rho(c_{\epsilon}) \int_0^1 \frac{\int_{y_c}^y \frac{\omega_0(y)}{i\alpha} dy}{(u(z) - c_{\epsilon})^2} \left(\frac{1}{\phi_1(z, c_{\epsilon})^2} - 1\right) dz \longrightarrow \rho(c) \int_0^1 \frac{\int_{y_c}^z \frac{\omega_0(y)}{i\alpha} dy}{(u(z) - c)^2} \left(\frac{1}{\phi_1(z, c)^2} - 1\right) dz$$

as  $\epsilon$  tends to zero. Similar to the proof of  $I_2$  in Lemma 6.3, we have

$$\rho(c_{\epsilon}) \int_0^1 \frac{\int_{x_0}^z \frac{\omega_0(y)}{i\alpha} dy}{(u(z) - c_{\epsilon})^2} dz \to \rho(c) \int_0^1 \frac{\int_{x_0}^z \frac{\omega_0(y)}{i\alpha} dy}{(u(z) - c)^2} dz + \rho(c) \frac{\omega_0(y_c)}{i\alpha u'(y_c)^2} \pi i.$$

as  $\varepsilon \to 0+$ . Thus, we obtain as  $\epsilon \to 0+$ .

$$\rho(c_{\epsilon}) \int_{0}^{1} \frac{1}{\phi(z, c_{\epsilon})^{2}} \int_{y_{c}}^{z} \phi f(y, c_{\epsilon}) dy dz$$

$$\longrightarrow \text{p.v.} \rho(c) \int_{0}^{1} \frac{\int_{y_{c}}^{z} \frac{\omega_{0}(y)\phi_{1}(y, c)}{i\alpha} dy}{\phi(z, c)^{2}} dz + \rho(c) \frac{\omega_{0}(y_{c})}{i\alpha u'(y_{c})^{2}} \pi i$$

$$= -\frac{1}{\alpha} \left( i\rho(c) \Pi_{1}(\omega_{0}) - \rho(c) \frac{\omega_{0}(y_{c})}{u'(y_{c})^{2}} \pi \right).$$

Similarly, we have that  $\epsilon \to 0-1$ 

$$\rho(c_{\epsilon}) \int_{0}^{1} \frac{1}{\phi(z, c_{\epsilon})^{2}} \int_{u_{\epsilon}}^{z} \phi f(y, c_{\epsilon}) dy dz \longrightarrow -\frac{1}{\alpha} \Big( i \rho(c) \Pi_{1} + \rho(c) \frac{\omega_{0}(y_{c})}{u'(y_{c})^{2}} \pi \Big).$$

This shows by Lemma 6.1 that

(6.6) 
$$\mu(c_{\epsilon}) \longrightarrow \mu_{\pm}(c) \quad as \quad \epsilon \to 0 \pm .$$

Next we show the convergence of  $\psi(y, c_{\epsilon})$ . Using the formula

$$\phi(y, c_{\epsilon}) = (u(y) - u(y_c) - i\epsilon)\phi_1(y, c_{\epsilon})$$
$$= ((y - y_c) \int_0^1 u'(y_c + t(y - y_c))dt - i\epsilon)\phi_1(y, c_{\epsilon}),$$

we deduce from Lemma 6.6 that as  $\epsilon \to 0$ ,

$$\phi(y, c_{\epsilon}) \int_0^y \frac{1}{\phi(z, c_{\epsilon})^2} dy \longrightarrow \phi(y, c) \int_0^y \frac{1}{\phi(y, c)^2} dy,$$

$$\phi(y, c_{\epsilon}) \int_{0}^{y} \frac{1}{\phi(z, c_{\epsilon})^{2}} \int_{y_{c}}^{z} \phi f(y', c_{\epsilon}) dy' dz \longrightarrow \phi(y, c) \int_{0}^{y} \frac{1}{\phi(z, c)^{2}} \int_{y_{c}}^{z} \phi f(y', c) dy' dz,$$

for  $y_c \ge y \ge 0$ , and

$$\phi(y, c_{\epsilon}) \int_{1}^{y} \frac{1}{\phi(y, c_{\epsilon})^{2}} dy \longrightarrow \phi(y, c) \int_{1}^{y} \frac{1}{\phi(y', c)^{2}} dy',$$

$$\phi(y, c_{\epsilon}) \int_{1}^{y} \frac{1}{\phi(z, c_{\epsilon})^{2}} \int_{y_{c}}^{z} \phi f(y', c_{\epsilon}) dy dz \longrightarrow \phi \int_{1}^{x} \frac{1}{\phi(z, c)^{2}} \int_{y_{c}}^{z} \phi f(y', c) dy' dz,$$

for  $y_c \leq y \leq 1$ . This finishes the proof of the first statement.

The second statement is similar just by noting that in this case, we have

$$\mu(c_{\epsilon}^l) \longrightarrow 0, \quad \mu(c_{\epsilon}^r) \longrightarrow 0$$

as  $\epsilon \to 0+$ .

#### 7. Representation formula of the solution

Let  $\widehat{\psi}(t,\alpha,y)$  be the solution of the linearized Euler equations

$$\frac{1}{i\alpha}\partial_t \widehat{\psi} = (\partial_y^2 - \alpha^2)^{-1} (u''(y) - u(y)(\partial_y^2 - \alpha^2))\widehat{\psi} = -\mathcal{R}_\alpha \widehat{\psi}$$

with the initial data

$$\psi(0, \alpha, y) = (\alpha^2 - \partial_y^2)^{-1} \widehat{\omega}_0(\alpha, y),$$

where  $\widehat{\omega}_0(\alpha, y) = \mathcal{F}_x \omega_0$ . We know from (3.5) that

$$\widehat{\psi}(t,\alpha,y) = \frac{1}{2\pi i} \int_{\partial \Omega} e^{-i\alpha tc} (c - \mathcal{R}_{\alpha})^{-1} \widehat{\psi}(0,\alpha,y) dc,$$

where  $\Omega$  is a simple connected domain including the spectrum  $\sigma(\mathcal{R}_{\alpha})$  of  $\mathcal{R}_{\alpha}$ .

Note that  $P_{i\alpha\mathcal{R}_{\alpha}}\widehat{\psi}(0,\alpha,y)=0$  if  $P_{\mathcal{L}}\omega_0=0$ . Thus, we have

$$\widehat{\psi}(t,\alpha,y) = \frac{1}{2\pi} \int_{\partial \Omega_{\epsilon}} e^{-i\alpha tc} (c - \mathcal{R}_{\alpha})^{-1} \widehat{\psi}(0,\alpha,y) dc,$$

where  $\Omega_{\epsilon}$  defined by (4.2) with  $\epsilon$  sufficiently small.

Let  $\Phi(\alpha, y, c)$  be the solution of (6.1) with  $f(\alpha, y, c) = \frac{\widehat{\omega}_0(\alpha, y)}{i\alpha(u-c)}$ . It is easy to see that

$$(c - \mathcal{R}_{\alpha})^{-1} \widehat{\psi}(0, \alpha, y) = i\alpha \Phi(\alpha, y, c).$$

This gives

(7.1) 
$$\widehat{\psi}(t,\alpha,y) = \frac{1}{2\pi} \int_{\partial\Omega_{\epsilon}} \alpha \Phi(\alpha,y,c) e^{-i\alpha ct} dc.$$

Since  $\mathcal{L}(\text{thus}, \mathcal{R}_{\alpha})$  has no embedding eigenvalues, Lemma 6.3 ensures that there are no eigenvalues of  $\mathcal{R}_{\alpha}$  in  $\Omega_{\epsilon}$  for  $\epsilon$  small enough. Then it follows from Proposition 6.7 that

$$\widehat{\psi}(t,\alpha,y) = \frac{1}{2\pi} \int_{\partial\Omega_{\epsilon}} \alpha \Phi(\alpha,y,c) e^{-i\alpha ct} dc$$

$$= \lim_{\epsilon \to 0^{+}} \frac{1}{2\pi} \int_{\partial\Omega_{\epsilon}} \alpha \Phi(\alpha,y,c) e^{-i\alpha ct} dc$$

$$= \frac{1}{2\pi} \int_{u(0)}^{u(1)} \alpha \widetilde{\Phi}(y,c) e^{-i\alpha ct} dc,$$
(7.2)

where

$$\widetilde{\Phi}(y,c) = \begin{cases} (\mu_{-}(c) - \mu_{+}(c))\phi(y,c) \int_{0}^{y} \frac{1}{\phi(z,c)^{2}} dz & 0 \le y < y_{c}, \\ (\mu_{-}(c) - \mu_{+}(c))\phi(y,c) \int_{1}^{y} \frac{1}{\phi(z,c)^{2}} dz & y_{c} < y \le 1, \end{cases}$$

with  $y_c = u(c)$ , and  $\phi(y, c)$  is the solution of (4.1) given by Proposition 4.5. We denote

$$A \triangleq u(0) - u(1) - \rho(c) \Pi_2 + u'(y_c) \rho(c) \Pi_3,$$

$$B \triangleq \pi \rho(c) \frac{u''(y_c)}{u'(y_c)^2},$$

$$C \triangleq \rho(c) \frac{\widehat{\omega}_0(\alpha, y_c)}{u'(y_c)} \pi, \quad D \triangleq u'(y_c) \rho(c) \Pi_1(\widehat{\omega}_0),$$

where  $\rho(c) = (c - u(0))(u(1) - c)$ , and  $II_1, II_2$  and  $II_3$  are given by (6.5), (6.2) and (6.3) respectively. Then we have

(7.3) 
$$\mu_{-}(c) - \mu_{+}(c) = \frac{2}{\alpha} \frac{AC + BD}{A^{2} + B^{2}} \triangleq \frac{2}{\alpha} \rho(c) \mu(c).$$

8. Sobolev regularity of  $\mu(c)$ 

In this section, we study the regularity of  $\mu(c)$  defined by (7.3). The result is stated as follows.

**Proposition 8.1.** With the same assumptions as in Theorem 1.1, there exists a constant C independent of  $\alpha$  such that

1.  $L^2$  estimates

$$\|\rho\mu\|_{L^2} \le \frac{C}{\alpha} \|\widehat{\omega}_0(\alpha, \cdot)\|_{L^2},$$
  
$$\|\mu\|_{L^2} \le C \|\widehat{\omega}_0(\alpha, \cdot)\|_{L^2}.$$

2.  $W^{1,2}$  estimates

$$\|\partial_c(\rho\mu)\|_{L^2} \le C\|\widehat{\omega}_0(\alpha,\cdot)\|_{H^1},$$
  
$$\|\partial_c\mu\|_{L^2} \le C(1+\alpha)\|\widehat{\omega}_0(\alpha,\cdot)\|_{H^1}.$$

3.  $W^{2,2}$  estimates

$$\|\partial_c^2(\rho\mu)\|_{L^2} \le C(1+\alpha)\|\widehat{\omega}_0(\alpha,\cdot)\|_{H^2}, \\ \|\rho\partial_c^2(\rho\mu)\|_{L^2} \le C\|\widehat{\omega}_0(\alpha,\cdot)\|_{H^2}.$$

*Proof.* Let us first estimate II<sub>1</sub>, II<sub>2</sub> and II<sub>3</sub>. Recall that

$$\begin{split} & \text{II}_{1}(\widehat{\omega}_{0}) = \text{p.v.} \int_{0}^{1} \frac{\int_{y_{c}}^{z} \widehat{\omega}_{0}(\alpha, y') \phi_{1}(y', c) dy'}{\phi(z, c)^{2}} dz, \\ & \text{II}_{2} = \text{p.v.} \int_{0}^{1} \frac{u'(y) - u'(y_{c})}{(u(y) - c)^{2}} dy, \\ & \text{II}_{3} = \int_{0}^{1} \frac{1}{(u(y) - c)^{2}} \left(\frac{1}{\phi_{1}(y, c)^{2}} - 1\right) dy. \end{split}$$

By a change of variable,  $II_1$  can be written as

$$II_1(\widehat{\omega}_0)(c) = \mathcal{L}_1(\widehat{\omega}_0(\alpha, u^{-1})(u^{-1})').$$

Then Lemma 11.4, Lemma 11.9 and Lemma 11.10 ensure that

(8.1) 
$$\left\| \operatorname{II}_{1}(\widehat{\omega}_{0}) \right\|_{L^{p}} \leq C \left\| \widehat{\omega}_{0}(\alpha, \cdot) \right\|_{L^{p}},$$

(8.2) 
$$\|\partial_c(\rho \mathrm{II}_1(\widehat{\omega}_0))\|_{L^p} \leq C \|\widehat{\omega}_0(\alpha,\cdot)\|_{W^{1,p}},$$

(8.3) 
$$\left\|\partial_c^2(\rho^2 \Pi_1(\widehat{\omega}_0))\right\|_{L^p} \le C \|\widehat{\omega}_0(\alpha, \cdot)\|_{W^{2,p}}.$$

Let  $y_c = u^{-1}(c)$  in the sequel. We rewrite  $II_2$  as

$$\begin{split} & \text{II}_{2}(c) = \text{p.v.} \int_{0}^{1} \frac{\int_{y_{c}}^{z} u''(y) dy}{(u(z) - c)^{2}} dz \\ & = -\frac{1}{u'(y_{c})} \int_{0}^{1} \frac{(\int_{0}^{1} u''(z + t(z - y_{c})) dt)^{2}}{(\int_{0}^{1} u'(z + t(z - y_{c})) dt)^{2}} dz \\ & + \frac{1}{u'(y_{c})} \text{p.v.} \int_{0}^{1} \frac{-u'(z) \int_{y_{c}}^{z} u''(y) dy}{(u(z) - c)^{2}} dz \\ & \triangleq & \text{II}_{2,1} + \frac{1}{u'(y_{c})} \text{II}_{2,2}. \end{split}$$

It's easy to see that

$$\|\mathrm{II}_{2,1}\|_{L^{\infty}} + \|\partial_c \mathrm{II}_{2,1}\|_{L^{\infty}} + \|\partial_c^2 \mathrm{II}_{2,1}\|_{L^{\infty}} \le C.$$

We rewrite  $II_{2,2}$  as

$$\begin{split} \mathrm{II}_{2,2}(c) &= \int_{u(0)}^{u(1)} \frac{\int_{c}^{c'} u''(u^{-1}(z))(u^{-1})'(z)dz}{(c'-c)^{2}} dc' \\ &= -\chi_{D_{0}}(c)H(u''(u^{-1})(u^{-1})'\chi_{D_{0}})(c) \\ &+ \frac{1}{c-u(0)} \int_{u(0)}^{c} u''(u^{-1}(z))(u^{-1})'(z)dz - \frac{1}{u(1)-c} \int_{c}^{u(0)} u''(u^{-1}(z))(u^{-1})'(z)dz, \end{split}$$

from which and Lemma 11.5, it follows that

$$\|\mathrm{II}_{2,2}\|_{L^p} + \|\partial_c(\rho\mathrm{II}_{2,2})\|_{L^p} + \|\partial_c^2(\rho^2\mathrm{II}_{2,2})\|_{L^p} \le C.$$

Thus, we deduce that for any  $p \in (1, \infty)$ ,

(8.4) 
$$\|\mathrm{II}_2\|_{L^p} + \|\partial_c(\rho\mathrm{II}_2)\|_{L^p} + \|\partial_c^2(\rho^2\mathrm{II}_2)\|_{L^p} \le C,$$

(8.5) 
$$\|\rho II_2\|_{L^{\infty}} + \|\partial_c(\rho^2 II_2)\|_{L^{\infty}} \le C.$$

We get by Lemma 5.4 that

$$\begin{split} |\mathrm{II}_3(c)| &= \int_0^1 \frac{1}{(u(y)-c)^2} \Big(1 - \frac{1}{\phi_1(y,c)^2}\Big) dy \\ &\geq C^{-1} \int_0^1 \frac{1}{(y-y_c)^2} \Big(\frac{\phi_1^2(y,c)-1}{\phi_1(y,c)^2}\Big) dy \\ &\geq C^{-1} \alpha \int_{-\alpha y_c}^{\alpha(1-y_c)} \frac{1}{\sinh^2 y} \Big(\frac{\sinh y}{y} - 1\Big) dy \\ &\geq C^{-1} \alpha \int_0^{1/2} \frac{1}{\sinh^2 y} \Big(\frac{\sinh y}{y} - 1\Big) dy \geq C^{-1} \alpha, \end{split}$$

and

$$|II_{3}(c)| = \int_{0}^{1} \frac{1}{(u(y) - c)^{2}} \left(1 - \frac{1}{\phi_{1}(y, c)^{2}}\right) dy$$

$$\leq C\alpha \int_{-\alpha y_{c}}^{\alpha(1 - y_{c})} \frac{1}{y \sinh y} \left(\frac{\sinh y}{y} - 1\right) dy$$

$$\leq C\alpha \left(1 + \int_{|y| > 1} \frac{1}{y^{2}} + \frac{1}{y \sinh y} dy\right) \leq C\alpha.$$

This gives

(8.6) 
$$C^{-1}\alpha \le |\mathrm{II}_3(c)| \le C\alpha.$$

By Remark 5.5, we have

$$\partial_{c} \Pi_{3}(c) = \partial_{c} \int_{0}^{1} \frac{1}{(u(y) - c)^{2}} \left(\frac{1}{\phi_{1}(y, c)^{2}} - 1\right) dy$$

$$= \int_{0}^{1} -\partial_{y} \left(\frac{1}{(u(y) - c)^{2}}\right) \frac{1}{u'(y)} \left(\frac{1}{\phi_{1}(y, c)^{2}} - 1\right) dy$$

$$+ \int_{0}^{1} \frac{1}{(u(y) - c)^{2}} \partial_{c} \left(\frac{1}{\phi_{1}(y, c)^{2}} - 1\right) dy$$

$$= \frac{\alpha^{2} \mathcal{T}(\phi_{1})(y, c)(\phi_{1}(y, c) + 1)}{\phi_{1}(y, c)^{2} u'(y_{c}) \int_{0}^{1} u'(y_{c} + t(y - y_{c})) dt} \Big|_{y=0}^{1}$$

$$+ \int_{0}^{1} \frac{1}{(u(y) - c)^{2}} \left(\partial_{c} + \frac{1}{u'(y)} \partial_{y}\right) \left(\frac{1}{\phi_{1}(y, c)^{2}} - 1\right) dy$$

$$+ \int_{0}^{1} \frac{1}{(u(y) - c)^{2}} \left(\frac{1}{u'(y)}\right)' \left(\frac{1}{\phi_{1}(y, c)^{2}} - 1\right) dy,$$

and

$$\begin{split} \partial_c^2 \Pi_3(c) &= \partial_c \left( \frac{\alpha^2 \mathcal{T}(\phi_1)(y,c)(\phi_1(y,c)+1)}{\phi_1(y,c)^2 u'(y_c) \int_0^1 u'(y_c+t(y-y_c)) dt} \Big|_{y=0}^1 \right) \\ &+ \int_0^1 -\partial_y \left( \frac{1}{(u(y)-c)^2} \right) \frac{1}{u'(y)} \left( \partial_c + \frac{1}{u'(y)} \partial_y \right) \left( \frac{1}{\phi_1(y,c)^2} \right) dy \\ &+ \int_0^1 -\partial_y \left( \frac{1}{(u(y)-c)^2} \right) \frac{1}{u'(y)} \left( \frac{1}{u'(y)} \right)' \left( \frac{1}{\phi_1(y,c)^2} - 1 \right) dy \\ &+ \int_0^1 \left( \frac{1}{(u(y)-c)^2} \right) \partial_c \left( \partial_c + \frac{1}{u'(y)} \partial_y \right) \left( \frac{1}{\phi_1(y,c)^2} \right) dy \\ &+ \int_0^1 \frac{1}{(u(y)-c)^2} \left( \frac{1}{u'(y)} \right)' \partial_c \left( \frac{1}{\phi_1(y,c)^2} - 1 \right) dy \\ &= \partial_c \left( \frac{\alpha^2 \mathcal{T}(\phi_1)(y,c)(\phi_1(y,c)+1)}{\phi_1(y,c)^2 u'(y_c) \int_0^1 u'(y_c+t(y-y_c)) dt} \Big|_{y=0}^1 \right) \\ &- \frac{1}{(u(y)-c)^2} \frac{1}{u'(y)} \left( \partial_c + \frac{1}{u'(y)} \partial_y \right) \left( \frac{1}{\phi_1(y,c)^2} \right) \Big|_{y=0}^1 \\ &- \frac{1}{(u(y)-c)^2} \frac{1}{u'(y)} \left( \frac{1}{u'(y)} \right)' \left( \frac{1}{\phi_1(y,c)^2} - 1 \right) \Big|_{y=0}^1 \end{split}$$

$$+ \int_{0}^{1} \frac{1}{(u(y) - c)^{2}} \left(\frac{1}{u'(y)}\right)' \left(\partial_{c} + \frac{1}{u'(y)}\partial_{y}\right) \left(\frac{1}{\phi_{1}(y, c)^{2}}\right) dy$$

$$+ \int_{0}^{1} \frac{1}{(u(y) - c)^{2}} \left(\partial_{c} + \frac{1}{u'(y)}\partial_{y}\right)^{2} \left(\frac{1}{\phi_{1}(y, c)^{2}}\right) dy$$

$$+ \int_{0}^{1} \frac{1}{(u(y) - c)^{2}} \left(\frac{1}{u'(y)} \left(\frac{1}{u'(y)}\right)'\right)' \left(\frac{1}{\phi_{1}(y, c)^{2}} - 1\right) dy$$

$$+ \int_{0}^{1} \frac{1}{(u(y) - c)^{2}} \left(\frac{1}{u'(y)}\right)' \left(\frac{1}{u'(y)}\partial_{y} + \partial_{c}\right) \left(\frac{1}{\phi_{1}(y, c)^{2}}\right) dy.$$

Then by Remark 5.5 again, Proposition 5.1 and Proposition 5.6, we obtain

(8.7) 
$$\|\partial_c \operatorname{II}_3\|_{L^{\infty}} \le C\alpha^2, \quad \|\partial_c^2 \operatorname{II}_3\|_{L^{\infty}} \le C\alpha^3, \quad \|\partial_c(\rho^2 \partial_c \operatorname{II}_3)\|_{L^{\infty}} \le C\alpha.$$

Now let us turn to the estimate of  $\mu(c)$ .

$$\begin{split} \mu(c) = & \frac{\pi \rho(c) \Pi_3(c) \widehat{\omega}_0(\alpha, y_c)}{\mathbf{A}^2 + \mathbf{B}^2} - \frac{\pi \rho(c) \Pi_2(c) \widehat{\omega}_0(\alpha, y_c)}{u'(y_c) (\mathbf{A}^2 + \mathbf{B}^2)} \\ & + \frac{\pi (u(0) - u(1)) \widehat{\omega}_0(\alpha, y_c)}{u'(y_c) (\mathbf{A}^2 + \mathbf{B}^2)} + \frac{\pi u''(y_c) \rho \Pi_1(\widehat{\omega}_0)(c)}{u'(y_c) (\mathbf{A}^2 + \mathbf{B}^2)}. \end{split}$$

By Lemma 6.1,  $A^2 + B^2 \ge C^{-1}$ . Thus, by (8.6), we get

$$A^2 + B^2 \ge C^{-1}((1 + \alpha \rho)^2 + \rho^2).$$

Thus, by (8.1), (8.6) and (8.5), we get

$$\|\mu\|_{L^{2}} \leq C\left(1 + \frac{1}{\alpha}\|\mathrm{II}_{3}\|_{L^{\infty}} + \|\rho\mathrm{II}_{2}\|_{L^{\infty}}\right)\|\widehat{\omega}_{0}(\alpha, \cdot)\|_{L^{2}} + C\|\mathrm{II}_{1}\|_{L^{2}} \leq C\|\widehat{\omega}_{0}(\alpha, \cdot)\|_{L^{2}}.$$

Using the inequality  $(1 + \alpha \rho)^2 + \rho^2 \ge C^{-1}(\rho \alpha)^k$  for k = 1, 2, we can deduce that

$$\|\rho\mu\|_{L^2} \leq \frac{C}{\alpha} \|\widehat{\omega}_0(\alpha,\cdot)\|_{L^2}.$$

The  $H^1$  estimate of  $\mu(c)$  is similar. Here we just show the estimate of one term.

$$\partial_c \left( \frac{\rho \mathrm{II}_1(\widehat{\omega}_0)(c)}{\mathrm{A}^2 + \mathrm{B}^2} \right) = \frac{\partial_c (\rho \mathrm{II}_1(\widehat{\omega}_0)(c))}{\mathrm{A}^2 + \mathrm{B}^2} - \frac{\rho \mathrm{II}_1(\widehat{\omega}_0)(c)\partial_c (\mathrm{A}^2 + \mathrm{B}^2)}{(\mathrm{A}^2 + \mathrm{B}^2)^2}.$$

We infer from (8.2) and (8.6) that

$$\left\| \frac{\partial_c(\rho \Pi_1(\widehat{\omega}_0)(c))}{\mathbf{A}^2 + \mathbf{B}^2} \right\|_{L^2} \le C \|\widehat{\omega}_0(\alpha, \cdot)\|_{L^2}.$$

Notice that

$$\begin{aligned} |\partial_{c}(\mathbf{A}^{2} + \mathbf{B}^{2})| \leq & \left| 2\mathbf{A} \left( \partial_{c}(u'(y_{c})\rho(c)\mathbf{I}\mathbf{I}_{3}) - \partial_{c}(\rho(c)\mathbf{I}\mathbf{I}_{2}) \right) + 2\mathbf{B}\partial_{c}\mathbf{B} \right| \\ \leq & C(\mathbf{A}^{2} + \mathbf{B}^{2})^{1/2} \left( 1 + |\partial_{c}(\rho(c)\mathbf{I}\mathbf{I}_{3})| + |\rho\mathbf{I}\mathbf{I}_{3}| + |\partial_{c}(\rho(c)\mathbf{I}\mathbf{I}_{2})| \right) \end{aligned}$$

Thus, by (8.1), (8.7), (8.4) and Sobolev embedding, we get

$$\begin{split} & \left\| \frac{\rho \Pi_{1}(\widehat{\omega}_{0})(c)\partial_{c}(\mathbf{A}^{2} + \mathbf{B}^{2})}{(\mathbf{A}^{2} + \mathbf{B}^{2})^{2}} \right\|_{L^{2}} \\ & \leq C \left\| \frac{\partial_{c}(\rho(c)\Pi_{2})}{(\mathbf{A}^{2} + \mathbf{B}^{2})^{3/2}} \right\|_{L^{4}} \|\rho \Pi_{1}(\widehat{\omega}_{0})\|_{L^{4}} + C \left( \frac{1}{\alpha} \|\partial_{c}\Pi_{3}\|_{L^{\infty}} + \|\Pi_{3}\|_{L^{\infty}} \right) \|\rho \Pi_{1}(\widehat{\omega}_{0})\|_{L^{2}} \\ & \leq C\alpha \|\widehat{\omega}_{0}(\alpha, \cdot)\|_{H^{1}}, \end{split}$$

which gives

$$\left\| \partial_c \left( \frac{\rho \Pi_1(\widehat{\omega}_0)(c)}{\mathbf{A}^2 + \mathbf{B}^2} \right) \right\|_{L^2} \le C \alpha \|\widehat{\omega}_0(\alpha, \cdot)\|_{H^1}.$$

The  $H^2$  estimate of  $\mu(c)$  is similar. We left it to the interested readers.

#### 9. Uniform Sobolev estimates of the vorticity

Recall that  $\widehat{\omega}(t, \alpha, y)$  satisfies

$$\begin{cases} \partial_t \widehat{\omega} + i\alpha u \widehat{\omega} + i\alpha u'' \widehat{\psi} = 0, \\ \widehat{\omega}|_{t=0} = \mathcal{F}\omega_0(\alpha, y). \end{cases}$$

This is equivalent to

$$\left(e^{i\alpha t u(y)}\widehat{\omega}(t,\alpha,y)\right)_t = -i\alpha e^{i\alpha t u(y)}u''(y)\widehat{\psi}(t,\alpha,y).$$

Integration in t gives

$$e^{i\alpha t u(y)}\widehat{\omega}(t,\alpha,y) = \widehat{\omega}_0(\alpha,y) - i\alpha u''(y) \int_0^t e^{i\tau u(y)} \widehat{\psi}(\tau,\alpha,y) d\tau.$$

We denote  $W(t, x, y) \triangleq \omega(t, x + u(y)t, y)$ . We find that

$$\widehat{W}(t,\alpha,y) = e^{i\alpha t u(y)} \widehat{\omega}(t,\alpha,y).$$

Then we get by (7.2) that

$$\widehat{W}(t,\alpha,y) = \widehat{\omega}_0(\alpha,y) - \frac{u''(y)}{\pi} \int_{u(0)}^{u(1)} \frac{(e^{it(u(y)-c)\alpha} - 1)\rho(c)\mu(c)\Gamma(y,c)}{u(y)-c} dc$$

$$\triangleq \widehat{\omega}_0(\alpha,y) - \frac{u''(y)}{\pi} \mathbf{T}(\mu)(t,y),$$

where

$$\Gamma(y,c) = \begin{cases} \phi(y,c) \int_0^y \frac{1}{\phi(z,c)^2} dz \triangleq \Gamma_0(y,c) & 0 \le y < y_c, \\ \phi(y,c) \int_1^y \frac{1}{\phi(z,c)^2} dz \triangleq \Gamma_1(y,c) & y_c < y \le 1. \end{cases}$$

We have the following uniform estimates in Sobolev space for W(t, x, y).

**Proposition 9.1.** With the same assumptions as in Theorem 1.1, it holds that

$$\begin{split} & \|W(t)\|_{H_x^{-1}L_y^2} \leq C \|w_0\|_{H_x^{-1}L_y^2}, \\ & \|W(t)\|_{H_x^{-1}H_y^1} \leq C \|w_0\|_{H_x^{-1}H_y^1}, \\ & \|\rho(u(y))\partial_y^2 W(t)\|_{H_x^{-1}L_y^2} \leq C \|w_0\|_{H_x^{-1}H_y^2}. \end{split}$$

*Proof.* The proof is split into three steps.

Step 1.  $L^2$  estimate

Let  $\varphi(y) \in C_0^{\infty}(0,1)$  with  $\|\varphi\|_{L^2} = 1$ . A direct calculation gives

$$\int_0^1 \mathbf{T}(\mu)(t,y)\varphi(y)dy$$

$$\begin{split} &= \int_{0}^{1} \int_{u(0)}^{u(1)} \frac{(e^{it(u(y)-c)\alpha}-1)\rho(c)\mu(c)\Gamma(y,c)}{u(y)-c} dc\varphi(y) dy \\ &= \int_{0}^{1} \int_{u(0)}^{u(y)} \frac{(e^{it(u(y)-c)\alpha}-1)\rho(c)\mu(c)\Gamma(y,c)}{u(y)-c} dc\varphi(y) dy \\ &+ \int_{0}^{1} \int_{u(y)}^{u(1)} \frac{(e^{it(u(y)-c)\alpha}-1)\rho(c)\mu(c)\Gamma(y,c)}{u(y)-c} dc\varphi(y) dy \\ &= \int_{u(0)}^{u(1)} \rho(c)\mu(c) \int_{y_{c}}^{1} \frac{(e^{it(u(y)-c)\alpha}-1)\Gamma_{1}(y,c)\varphi(y)}{u(y)-c} dy dc \\ &+ \int_{u(0)}^{u(1)} \rho(c)\mu(c) \int_{y_{c}}^{y_{c}} \frac{(e^{it(u(y)-c)\alpha}-1)\Gamma_{0}(y,c)\varphi(y)}{u(y)-c} dy dc \\ &= \int_{u(0)}^{u(1)} \rho(c)\mu(c) \int_{y_{c}}^{1} (e^{it(u(y)-c)\alpha}-1)\phi_{1}(y,c) \int_{1}^{y} \frac{1}{\phi(z,c)^{2}} dz\varphi(y) dy dc \\ &+ \int_{u(0)}^{u(1)} \rho(c)\mu(c) \int_{0}^{y_{c}} (e^{it(u(y)-c)\alpha}-1)\phi_{1}(y,c) \int_{0}^{y} \frac{1}{\phi(z,c)^{2}} dz\varphi(y) dy dc \\ &= \int_{u(0)}^{u(1)} \rho(c)\mu(c) \int_{0}^{1} \frac{-1}{\phi(z,c)^{2}} \int_{y_{c}}^{z} (e^{it(u(y)-c)\alpha}-1)\varphi(y)\phi_{1}(y,c) dy dz dc \\ &= \int_{u(0)}^{u(1)} \rho(c)\mu(c) e^{-i\alpha ct} \mathbb{T}\Big((u^{-1})', \varphi \circ u^{-1} e^{it\alpha z}(u^{-1})', \phi_{1}(u^{-1},c), \frac{1}{\phi_{1}(u^{-1},c)^{2}}\Big)(c) dc \\ &+ \int_{u(0)}^{u(1)} \rho(c)\mu(c) \mathbb{T}\Big((u^{-1})', \varphi \circ u^{-1}(u^{-1})', \phi_{1}(u^{-1},c), \frac{1}{\phi_{1}(u^{-1},c)^{2}}\Big)(c) dc. \end{split}$$

Then by Lemma 11.7 and Proposition 8.1, we get

$$\int_{0}^{1} \mathbf{T}(\mu)(t,y)\varphi(y)dy \leq C \|\rho(c)\mu(c)\|_{L^{2}} \|\varphi\|_{L^{2}} \leq C \|\widehat{\omega}_{0}(\alpha,\cdot)\|_{L^{2}}.$$

This gives

$$||W(t)||_{H_x^{-1}L_x^2} \le C||\omega_0||_{H_x^{-1}L_x^2}.$$

Step 2.  $H^1$  estimate

Let  $\varphi(y) \in C_0^{\infty}(0,1)$  with  $\|\varphi\|_{L^2} = 1$ . Then we have

$$\int_{0}^{1} \partial_{y} \mathbf{T}(\rho\mu)(t,y)\varphi(y)dy = -\int_{0}^{1} \mathbf{T}(\rho\mu)(t,y)\varphi'(y)dy 
= \int_{u(0)}^{u(1)} \rho(c)\mu(c) \int_{0}^{1} \frac{1}{\phi(y',c)^{2}} \int_{y_{c}}^{y'} (e^{it(u(y)-c)\alpha} - 1)\varphi'(y)\phi_{1}(y,c)dydy'dc 
= \int_{u(0)}^{u(1)} \rho(c)\mu(c)\mathbb{T}\Big((u^{-1})', (\varphi \circ u^{-1})', e^{it\alpha(z-c)}\phi_{1}(u^{-1},c), \frac{1}{\phi_{1}(u^{-1},c)^{2}}\Big)(c)dc 
- \int_{u(0)}^{u(1)} \rho(c)\mu(c)\mathbb{T}\Big((u^{-1})', (\varphi \circ u^{-1})', \phi_{1}(u^{-1},c), \frac{1}{\phi_{1}(u^{-1},c)^{2}}\Big)(c)dc.$$

We get by Lemma 11.2 that

$$\begin{split} &\int_{0}^{1} \partial_{y} \mathbf{T}(\rho\mu(\cdot))(t,y)\varphi(y)dy \\ &= -\int_{u(0)}^{u(1)} \rho(c)\mu(c)\partial_{c}\mathcal{L}_{1}(\varphi \circ u^{-1})(c)dc + \int_{u(0)}^{u(1)} \rho(c)\mu(c)\mathcal{L}_{2}(\varphi \circ u^{-1})(c)dc \\ &+ \int_{u(0)}^{u(1)} \rho(c)\mu(c)\partial_{c} \left(e^{-it\alpha c}\mathcal{L}_{1}(\varphi \circ u^{-1}e^{it\alpha z})(c)\right)dc \\ &- \int_{u(0)}^{u(1)} \rho(c)\mu(c)e^{-it\alpha c}\mathcal{L}_{2}(\varphi \circ u^{-1}e^{it\alpha z})(c)dc \\ &+ \int_{u(0)}^{u(1)} \rho(c)\mu(c)e^{-it\alpha c}\mathcal{B}(\varphi \circ u^{-1}e^{it\alpha z})(c)dc - \int_{u(0)}^{u(1)} \rho(c)\mu(c)\mathcal{B}(\varphi \circ u^{-1})(c)dc \\ &= \int_{u(0)}^{u(1)} \partial_{c}(\rho(c)\mu(c))\mathcal{L}_{1}(\varphi \circ u^{-1})(c)dc + \int_{u(0)}^{u(1)} \rho(c)\mu(c)\mathcal{L}_{2}(\varphi \circ u^{-1})(c)dc \\ &- \int_{u(0)}^{u(1)} \partial_{c}(\rho(c)\mu(c))\left(e^{-it\alpha c}\mathcal{L}_{1}(\varphi \circ u^{-1}(z)e^{it\alpha z})(c)\right)dc \\ &- \int_{u(0)}^{u(1)} \rho(c)\mu(c)e^{-it\alpha c}\mathcal{L}_{2}(\varphi \circ u^{-1}e^{it\alpha z})(c)dc \\ &+ \int_{u(0)}^{u(1)} \rho(c)\mu(c)e^{-it\alpha c}\mathcal{B}(\varphi \circ u^{-1}(z)e^{it\alpha z})(c)dc - \int_{u(0)}^{u(1)} \rho(c)\mu(c)\mathcal{B}(\varphi \circ u^{-1})(c)dc. \end{split}$$

Thus, by Lemma 11.10, Lemma 11.9 and Proposition 8.1, we obtain

$$\int_{0}^{1} \partial_{y} \mathbf{T}(\rho \mu(\cdot))(t, y) \varphi(y) dy$$

$$\leq \|\partial_{c}(\rho(c)\mu(c))\|_{L^{2}} \Big( \|\mathcal{L}_{1}(\varphi \circ u^{-1})\|_{L^{2}} + \|\mathcal{L}_{1}(e^{it\alpha z}\varphi \circ u^{-1})\|_{L^{2}} \Big) + C\|\rho\mu\|_{L^{2}} \Big( \|\mathcal{L}_{2}(\varphi \circ u^{-1})\|_{L^{2}} + \|\mathcal{L}_{2}(e^{it\alpha z}\varphi \circ u^{-1})\|_{L^{2}} + \|\mathcal{B}(\varphi \circ u^{-1})\|_{L^{2}} + \|\mathcal{B}(e^{it\alpha z}\varphi \circ u^{-1})\|_{L^{2}} \Big)$$

$$\leq C\|\partial_{c}(\rho(c)\mu(c))\|_{L^{2}} + C\|\rho\mu\|_{L^{2}} \leq C\|\widehat{\omega_{0}}(\alpha, \cdot)\|_{H^{1}}.$$

This gives

$$||W(t)||_{H_x^{-1}H_y^1} \le C||\omega_0||_{H_x^{-1}H_y^1}.$$

## Step 3. $H^2$ estimate

Let  $\varphi(y) \in C_0^{\infty}(0,1)$  with  $\|\varphi\|_{L^2} = 1$ . Then we have

$$\begin{split} &\int_0^1 \rho(u(y)) \partial_y^2 \mathbf{T}(\rho \mu(\cdot))(t,y) \varphi(y) dy \\ &= \int_0^1 \mathbf{T}(\rho \mu(\cdot))(t,y) \partial_y^2 \big( \varphi(y) \rho(u(y)) \big) dy \\ &= -\int_{u(0)}^{u(1)} \rho(c) \mu(c) \int_0^1 \frac{1}{\phi(y',c)^2} \int_{y_c}^{y'} \big( e^{it(u(y)-c)\alpha} - 1 \big) \partial_y^2 \big( \varphi(y) \rho(u(y)) \big) \phi_1(y,c) dy dy' dc, \end{split}$$

where

$$\int_{0}^{1} \frac{1}{\phi^{2}(z,c)} \int_{x_{0}}^{z} (e^{it(u(y)-c)\alpha} - 1)\partial_{y}^{2}(\varphi(y)\rho(u(y)))\phi_{1}(y,c)dydz 
= \mathbb{T}\Big((u^{-1})', \left(\partial_{y}^{2}(\varphi(y)\rho(u(y)))\circ u^{-1}\right)(u^{-1})', (e^{it(z-c)\alpha} - 1)\phi_{1}(u^{-1}(z),c), \frac{1}{\phi_{1}(u^{-1}(z),c)^{2}}\Big).$$

We denote  $F(z,c) = \phi_1(u^{-1}(z),c)$ ,  $G(z,c) = \frac{1}{\phi_1(u^{-1}(z),c)^2}$  and  $h(z) = \varphi(u^{-1}(z))\rho(z)$ . Using the fact that

$$(f \circ g)' = (f' \circ g)g', \quad (f \circ g)'' = (f'' \circ g)(g')^2 + \frac{(f \circ g)'g''}{g'},$$

we deduce from Lemma 11.1 and Lemma 11.2 that

$$\begin{split} &\rho(c)\int_{0}^{1}\frac{1}{\phi^{2}(z,c)}\int_{x_{0}}^{z}(e^{it(u(y)-c)\alpha}-1)\partial_{y}^{2}\left(\varphi(y)\rho(u(y))\right)\phi_{1}(y,c)dydz\\ &=\rho(c)\mathbb{T}\Big((u^{-1})',\,h'',\,(e^{it(z-c)\alpha}-1)\frac{1}{(u^{-1})'}F,\,G\Big)\\ &-\rho(c)\mathbb{T}\Big((u^{-1})',\,h',\,(e^{it(z-c)\alpha}-1)\frac{(u^{-1})''}{((u^{-1})')^{2}}F,\,G\Big)\\ &=\rho(c)\partial_{c}\mathbb{T}\Big((u^{-1})',\,h',\,(e^{it(z-c)\alpha}-1)\frac{1}{(u^{-1})'}F,\,G\Big)\\ &-\rho(c)\mathbb{T}\Big((u^{-1})',\,h',\,(e^{it(z-c)\alpha}-1)(\partial_{y}+\partial_{c})\left(\frac{1}{(u^{-1})'}F\right),\,G\Big)\\ &-\rho(c)\mathbb{T}\Big((u^{-1})',\,h',\,(e^{it(z-c)\alpha}-1)\frac{1}{(u^{-1})'}F,\,(\partial_{z}+\partial_{c})G\Big)\\ &-\rho(c)\mathbb{T}\Big((u^{-1})'',\,h',\,(e^{it(z-c)\alpha}-1)\frac{1}{(u^{-1})'}F,\,G\Big)\\ &+\mathbb{B}_{0}\Big((u^{-1})',\,h',\,(e^{it(z-c)\alpha}-1)\frac{1}{(u^{-1})'}F,\,G\Big)\\ &+\mathbb{B}_{1}\Big((u^{-1})',\,h',\,(e^{it(z-c)\alpha}-1)\frac{1}{(u^{-1})'}F,\,G\Big)\\ &+\rho(c)\mathbb{T}\Big((u^{-1})',\,h,\,(e^{it(z-c)\alpha}-1)\frac{(u^{-1})''}{((u^{-1})')^{2}}F,\,G\Big)\\ &+\rho(c)\mathbb{T}\Big((u^{-1})',\,h,\,(e^{it(z-c)\alpha}-1)\frac{(u^{-1})''}{((u^{-1})')^{2}}F,\,(\partial_{z}+\partial_{c})G\Big)\\ &+\rho(c)\mathbb{T}\Big((u^{-1})'',\,h,\,(e^{it(z-c)\alpha}-1)\frac{(u^{-1})''}{((u^{-1})')^{2}}F,\,G\Big)\\ &-\mathbb{B}_{0}\Big((u^{-1})',\,h,\,(e^{it(z-c)\alpha}-1)\frac{(u^{-1})''}{((u^{-1})')^{2}}F,\,G\Big)\\ &-\mathbb{B}_{0}\Big((u^{-1})',\,h,\,(e^{it(z-c)\alpha}-1)\frac{(u^{-1})''}{((u^{-1})')^{2}}F,\,G\Big). \end{split}$$

This gives

$$\rho(c) \int_{0}^{1} \frac{1}{\phi^{2}(z,c)} \int_{x_{0}}^{z} (e^{it(u(y)-c)\alpha} - 1) \partial_{y}^{2} (\varphi(y)\rho(u(y))) \phi_{1}(y,c) dy dz 
= \rho(c) \partial_{c} \mathbb{T} \Big( (u^{-1})', h', (e^{it(z-c)\alpha} - 1) \frac{1}{(u^{-1})'} F, G \Big) 
+ \mathbb{B}_{0} \Big( (u^{-1})', h', (e^{it(z-c)\alpha} - 1) \frac{1}{(u^{-1})'} F, G \Big) + \mathbb{B}_{1} \Big( (u^{-1})', h', (e^{it(z-c)\alpha} - 1) \frac{1}{(u^{-1})'} F, G \Big) 
+ \Xi_{1}(h) + \rho(c) \partial_{c} \Xi_{2}(h),$$

where

$$\begin{split} \Xi_{1}(h) &= -e^{-it\alpha c} \mathcal{B} \Big( h e^{itz} \frac{(u^{-1})''}{((u^{-1})')^{2}} \Big) + \mathcal{B} \Big( h \frac{(u^{-1})''}{((u^{-1})')^{2}} \Big) \\ &+ \rho(c) e^{-it\alpha c} \mathcal{L}_{2} \Big( h e^{itz} \frac{(u^{-1})''}{((u^{-1})')^{2}} \Big) - \rho(c) \mathcal{L}_{2} \Big( h \frac{(u^{-1})''}{((u^{-1})')^{2}} \Big) \\ &+ \rho(c) e^{-it\alpha c} \mathcal{L}_{1} \Big( h e^{itz} \Big( \frac{(u^{-1})''}{((u^{-1})')^{2}} \Big)' \Big) - \rho(c) \mathcal{L}_{1} \Big( h \Big( \frac{(u^{-1})''}{((u^{-1})')^{2}} \Big)' \Big) \\ &+ \rho(c) e^{-it\alpha c} \mathcal{L}_{3} \Big( h e^{itz} \frac{1}{(u^{-1})'} \Big) - \rho(c) \mathcal{L}_{3} \Big( h \frac{1}{(u^{-1})'} \Big) \\ &- e^{-it\alpha c} \mathcal{B}_{1} \Big( h(z) e^{itz} \frac{1}{(u^{-1})'} \Big) + \mathcal{B}_{1} \Big( h(z) \frac{1}{(u^{-1})'} \Big) \\ &+ \rho(c) e^{-it\alpha c} \mathcal{L}_{1} \Big( h(z) e^{itz} \Big( \frac{1}{(u^{-1})'} \Big)'' \Big) - \rho(c) \mathcal{L}_{1} \Big( h \Big( \frac{1}{(u^{-1})'} \Big)'' \Big), \end{split}$$

and

$$\Xi_{2}(h) = e^{-it\alpha c} \mathcal{L}_{1}\left(h(z)e^{itz}\frac{(u^{-1})''}{((u^{-1})')^{2}}\right) - \mathcal{L}_{1}\left(h\frac{(u^{-1})''}{((u^{-1})')^{2}}\right) - e^{-it\alpha c} \mathcal{L}_{2}\left(he^{itz}\frac{1}{(u^{-1})'}\right) - \mathcal{L}_{2}\left(h\frac{1}{(u^{-1})'}\right) - e^{-it\alpha c} \mathcal{L}_{2}\left(he^{itz}\left(\frac{1}{(u^{-1})'}\right)'\right) - \mathcal{L}_{2}\left(h\left(\frac{1}{(u^{-1})'}\right)'\right).$$

By Lemma 11.10 and Lemma 11.9, we have

Using Lemma 11.3, we write

$$\mathbb{B}_{0}\Big((u^{-1})', h', (e^{it(z-c)\alpha} - 1)\frac{1}{(u^{-1})'}F, G\Big)$$

$$+ \mathbb{B}_{1}\Big((u^{-1})', h', (e^{it(z-c)\alpha} - 1)\frac{1}{(u^{-1})'}F, G\Big)$$

$$= \partial_{c}\mathbb{B}_{0}\Big((u^{-1})', h(e^{it(z-c)\alpha} - 1)\frac{1}{(u^{-1})'}, F, G\Big)$$

$$+ \partial_{c}\mathbb{B}_{1}\Big((u^{-1})', h(e^{it(z-c)\alpha} - 1)\frac{1}{(u^{-1})'}, F, G\Big)$$

$$- \mathbb{B}_{0}\Big((u^{-1})', h, (e^{it(z-c)\alpha} - 1)\frac{1}{(u^{-1})'}F, \partial_{c}G\Big)$$

$$-\mathbb{B}_{1}\Big((u^{-1})', h, (e^{it(z-c)\alpha} - 1)\frac{1}{(u^{-1})'}F, \partial_{c}G\Big)$$

$$-\mathbb{B}_{0}\Big((u^{-1})', h(e^{it(z-c)\alpha} - 1)\frac{1}{(u^{-1})'}, (\partial_{z} + \partial_{c})F, \partial_{c}G\Big)$$

$$-\mathbb{B}_{1}\Big((u^{-1})', h(e^{it(z-c)\alpha} - 1)\frac{1}{(u^{-1})'}, (\partial_{z} + \partial_{c})F, \partial_{c}G\Big)$$

$$-\mathbb{B}_{0}\Big((u^{-1})', h(e^{it(z-c)\alpha} - 1)(\frac{1}{(u^{-1})'})', F, \partial_{c}G\Big)$$

$$-\mathbb{B}_{1}\Big((u^{-1})', h(e^{it(z-c)\alpha} - 1)(\frac{1}{(u^{-1})'})', F, \partial_{c}G\Big)$$

$$+\frac{(u^{-1})'(u(0))G(u(0), c)}{c - u(0)} \int_{u(0)}^{c} h(z)(e^{it(z-c)\alpha} - 1)\frac{1}{(u^{-1})'}F(z, c)dz$$

$$+\frac{(u^{-1})'(u(1))G(u(1), c)}{u(1) - c} \int_{c}^{u(1)} h(z)(e^{it(z-c)\alpha} - 1)\frac{1}{(u^{-1})'}F(z, c)dz$$

$$\triangleq \partial_{c}\Xi_{4}(h) + \Xi_{3}(h).$$

It follows from Lemma 11.8 that

(9.2) 
$$\|\Xi_3(h)\|_{L^2} \le C\|\varphi\|_{L^2}, \quad \|\Xi_4(h)\|_{L^2} \le \frac{C}{\alpha}\|\varphi\|_{L^2}.$$

Using Lemma 11.1, we write

$$\begin{split} &\rho(c)\partial_{c}\mathbb{T}\Big((u^{-1})',\,h',\,(e^{it(z-c)\alpha}-1)\frac{1}{(u^{-1})'}F,\,G\Big)\\ &=\rho(c)\partial_{c}^{2}\mathbb{T}\Big((u^{-1})',\,h(e^{it(z-c)\alpha}-1)\frac{1}{(u^{-1})'},\,F,\,G\Big)\\ &-\rho(c)\partial_{c}\mathbb{T}\Big((u^{-1})',\,h,\,(e^{it(z-c)\alpha}-1)(\partial_{c}+\partial_{y})\big(\frac{1}{(u^{-1})'}F\big),\,G\Big)\\ &-\rho(c)\partial_{c}\mathbb{T}\Big((u^{-1})',\,h,\,(e^{it(z-c)\alpha}-1)\frac{1}{(u^{-1})'}F,\,(\partial_{c}+\partial_{z})G\Big)\\ &-\rho(c)\partial_{c}\mathbb{T}\Big((u^{-1})'',\,h,\,(e^{it(z-c)\alpha}-1)\frac{1}{(u^{-1})'}F,\,G\Big)\\ &+\rho(c)\partial_{c}\frac{1}{\rho}\mathbb{B}_{0}\Big((u^{-1})',\,h,\,(e^{it(z-c)\alpha}-1)\frac{1}{(u^{-1})'}F,\,G\Big)\\ &+\rho(c)\partial_{c}\frac{1}{\rho}\mathbb{B}_{1}\Big((u^{-1})',\,h,\,(e^{it(z-c)\alpha}-1)\frac{1}{(u^{-1})'}F,\,G\Big)\\ &=\rho(c)\partial_{c}^{2}[\mathcal{L}_{1},\rho]\Big(\varphi(e^{it(z-c)\alpha}-1)\frac{1}{(u^{-1})'}\Big)+\rho(c)\partial_{c}^{2}\Big(\rho\mathcal{L}_{1}\Big(\varphi(e^{it(z-c)\alpha}-1)\frac{1}{(u^{-1})'}\Big)\Big)\\ &-\rho(c)\partial_{c}\mathcal{L}_{2}\Big(h(e^{it(z-c)\alpha}-1)\frac{1}{(u^{-1})'}\Big)-\rho(c)\partial_{c}\mathcal{L}_{1}\Big(h(e^{it(z-c)\alpha}-1)\big(\frac{1}{(u^{-1})'}\big)'\Big)\\ &+\rho(c)\partial_{c}\frac{1}{\rho}\mathcal{B}\Big(h(e^{it(z-c)\alpha}-1)\frac{1}{(u^{-1})'}\Big)\\ &\triangleq\rho(c)\partial_{c}\Xi_{5}(\varphi)+\rho\partial_{c}^{2}(\rho\Xi_{6}(\varphi))+\rho\partial_{c}\Xi_{7}(h). \end{split}$$

This together with Lemma 11.11, Lemma 11.10 and Lemma 11.9 shows that

Thus, we obtain

$$\int_{0}^{1} \rho(u(y))\partial_{y}^{2} \mathbf{T}(\rho\mu(\cdot))(t,y)\varphi(y)dy$$

$$= -\int_{u(0)}^{u(1)} \mu(c) \Big(\Xi_{1}(h) + \rho(c)\partial_{c}\Xi_{2}(h) + \Xi_{3}(h) + \partial_{c}\Xi_{4}(h)$$

$$+ \rho(c)\partial_{c}\Xi_{5}(\varphi) + \rho\partial_{c}^{2}(\rho\Xi_{6}(\varphi)) + \rho\partial_{c}\Xi_{7}(h)\Big)dc$$

$$= -\int_{u(0)}^{u(1)} \mu(c)\Xi_{1}(h) - \partial_{c}(\rho\mu)\Xi_{2}(h) + \mu(c)\Xi_{3}(h) - \Xi_{4}(h)\partial_{c}\mu$$

$$- \partial_{c}(\rho\mu)\Xi_{5}(\varphi) + \rho(c)\partial_{c}^{2}(\rho\mu)\Xi_{6}(\varphi) - \partial_{c}(\rho\mu)\Xi_{7}(h)dc,$$

which along with (9.1)-(9.3) and Proposition 8.1 yields

$$\|\rho(u(y))\partial_y^2 \mathbf{T}(\rho\mu)(t,y)\|_{L^2} \le C\|\widehat{\omega}_0(\alpha,\cdot)\|_{H^2}.$$

This implies

$$\|\rho(u(\cdot))\partial_y^2 W(t)\|_{H_x^{-1}H_y^2} \le C\|\omega_0\|_{H_x^{-1}H_y^2}.$$

The proof of the proposition is completed.

## 10. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1.

10.1. **Decay estimates.** With the uniform Sobolev estimates of the vorticity, the decay estimates of the velocity are similar to [16]. For the completeness, we present a proof. By the elliptical estimate, we get

$$||V||_{L^2} < C||\omega||_{H^{-1}}.$$

We get by duality that

$$||V(t)||_{L^{2}} \leq \sup_{\varphi \in C_{0}^{\infty}, \|\varphi\|_{H^{1}} \leq 1} \left| \int_{0}^{1} \int_{-\pi}^{\pi} \varphi \omega(t) dx dy \right|$$

$$(10.1) \qquad \leq C \sup_{\varphi \in C_{0}^{\infty}, \|\varphi\|_{H^{1}} \leq 1} \left| \sum_{|\alpha| \neq 0} \int_{0}^{1} \overline{\widehat{\varphi}(\alpha, y)} \widehat{W}(t, \alpha, y) e^{-i\alpha u(y)t} dy \right|.$$

Integration by parts gives

$$\begin{split} \int_{0}^{1} \overline{\widehat{\varphi}(\alpha,y)} \widehat{W}(t,\alpha,y) e^{-i\alpha u(y)t} dy &= \frac{e^{-i\alpha u(y)t}}{-i\alpha t u'(y)} \overline{\widehat{\varphi}(\alpha,y)} \widehat{W}(t,\alpha,y) \Big|_{y=0}^{1} \\ &- \int_{0}^{1} \frac{e^{-i\alpha u(y)t}}{-i\alpha t} \partial_{y} \Big( \overline{\frac{\widehat{\varphi}(\alpha,y)}{u'(y)}} \widehat{W}(t,\alpha,y) \Big) dy \\ &= - \int_{0}^{1} \frac{e^{-i\alpha u(y)t}}{-i\alpha t} \partial_{y} \Big( \overline{\frac{\widehat{\varphi}(\alpha,y)}{u'(y)}} \widehat{W}(t,\alpha,y) \Big) dy, \end{split}$$

from which and Proposition 9.1, we infer that

$$||V(t)||_{L^{2}} \leq Ct^{-1} \sup_{\varphi \in C_{0}^{\infty}, \|\varphi\|_{H^{1}} \leq 1} ||W(t)||_{H_{x}^{-1}H_{y}^{1}} ||\varphi||_{L_{x}^{2}H_{y}^{1}}$$
$$\leq Ct^{-1} ||\omega_{0}||_{H_{x}^{-1}H_{y}^{1}}.$$

Also we get from (10.1) that

$$\begin{split} \|V(t)\|_{L^{2}} &\leq C \sup_{\varphi \in C_{0}^{\infty}, \ \|\varphi\|_{H^{1}} \leq 1} \|W(t)\|_{H^{-1}_{x}L^{2}_{y}} \|\varphi\|_{H^{1}_{x}L^{2}_{y}} \\ &\leq C \|\omega_{0}\|_{H^{-1}_{x}L^{2}_{y}}. \end{split}$$

Recall that 
$$-\triangle V^2=\partial_x\omega$$
 and  $V^2(x,0)=V^2(1,y)=0$ . We define 
$$-\Delta\vartheta=V^2,\quad \vartheta(t,x,0)=\vartheta(t,x,1)=0.$$

We get by the elliptic estimate that

$$\|\vartheta\|_{H^2} \le C\|V^2\|_{L^2}.$$

Integration by parts gives

$$||V^{2}||_{L^{2}}^{2} = \iint \partial_{x} \omega \vartheta dx dy = \sum_{|\alpha| \neq 0} \int_{0}^{1} i\alpha e^{-i\alpha u(y)t} \widehat{W}(t, \alpha, y) \overline{\widehat{\vartheta}}(t, \alpha, y) dy$$

$$= \sum_{|\alpha| \neq 0} \int_{0}^{1} \frac{e^{-i\alpha u(y)t}}{t} \partial_{y} \Big( \frac{\widehat{W}(t, \alpha, y) \overline{\widehat{\vartheta}}(t, \alpha, y)}{u'(y)} \Big) dy$$

$$= \sum_{|\alpha| \neq 0} \frac{e^{-i\alpha u(y)t}}{i\alpha t^{2} u'} \partial_{y} \Big( \frac{\widehat{W}(t, \alpha, y) \overline{\widehat{\vartheta}}(t, \alpha, y)}{u'(y)} \Big) \Big|_{y=0}^{1}$$

$$- \sum_{|\alpha| \neq 0} \int_{0}^{1} \frac{e^{-i\alpha u(y)t}}{i\alpha t^{2}} \partial_{y} \Big( \frac{1}{u'} \partial_{y} \Big( \frac{\widehat{W}(t, \alpha, y) \overline{\widehat{\vartheta}}(t, \alpha, y)}{u'(y)} \Big) \Big) dy.$$

By Sobolev embedding and Proposition 9.1, the first term on the right hand side is bounded by

$$Ct^{-2} \sum_{|\alpha| \neq 0} \frac{1}{\alpha} \|\widehat{W}(t, \alpha, \cdot)\|_{H_y^1} \|\vartheta(t, \alpha, \cdot)\|_{H_y^2} \leq Ct^{-2} \|\omega_0\|_{H_x^{-1}H_y^2} \|V^2\|_{L^2}.$$

Using Proposition 9.1 again and Hardy inequality, we get

$$\begin{split} \left\| \partial_y \left( \frac{1}{u'} \partial_y \left( \frac{\widehat{W}(t, \alpha, y) \overline{\widehat{\vartheta}}(t, \alpha, y)}{u'(y)} \right) \right) \right\|_{L^1_y} &\leq C \| \rho(u(y)) \partial_y^2 \widehat{W}(t, \alpha, y) \|_{L^2_y} \left\| \frac{\vartheta(t, \alpha, y)}{\rho(u(y))} \right\|_{L^2_y} \\ &+ C \| \partial_y \widehat{W}(t, \alpha, y) \|_{L^2_y} \left\| \vartheta(t, \alpha, y) \right\|_{H^1_y} \\ &+ C \| \widehat{W}(t, \alpha, y) \|_{L^2_y} \left\| \vartheta(t, \alpha, y) \right\|_{H^2_y} \\ &\leq C \| \omega_0(\alpha, \cdot) \|_{H^2_y} \left\| \vartheta(t, \alpha, y) \right\|_{H^2_y}. \end{split}$$

This implies that the second term is bounded by

$$Ct^{-2}\|\omega_0\|_{H_x^{-1}H_y^2}\|V^2\|_{L^2}.$$

Thus, we show that

$$||V^2||_{L^2} \le Ct^{-2}||\omega_0||_{H_x^{-1}H_y^2}.$$

10.2. **Scattering.** We only prove  $H^1$  scattering. The proof of  $L^2$  scattering is similar. Suppose that  $\omega_0 \in H_x^{-1}H_y^1$ . Let  $\{\omega_0^{(n)}\}_{n\geq 1} \subseteq H_x^{-1}H_y^2$  be a sequence such that  $\omega_0^{(n)} \longrightarrow \omega_0$  in  $H_x^{-1}H_y^1$ . Let  $\omega^{(n)}(t,x,y)$  be the solution to linearized Euler equation with initial data  $\omega_0^{(n)}(x,y)$ . Then  $\omega^{(n)} \longrightarrow \omega$  in  $L_t^{\infty}(H_x^{-1}H_y^1)$ . Thanks to  $W_t^{(n)} = u''(y)\partial_x\psi^{(n)}$ , we get

$$W^{(n)}(t,x,y) = \omega^{(n)}(0,x,y) - \int_0^t u''(y) \partial_x \psi^{(n)}(s,x+su(y),y) ds.$$

By the decay estimate, we have

$$\|\partial_x \psi^{(n)}(s, x, y)\|_{L^2_{x,y}} \le \frac{C}{\langle t \rangle^2} \|\omega_0^{(n)}\|_{H^{-1}_x H^2_y}.$$

Letting  $t \to \infty$ , we get

$$W^{(n)}(t,x,y) \longrightarrow \omega^{(n)}(0,x,y) - \int_0^\infty u''(y)\partial_x \psi^{(n)}(s,x+su(y),y)ds = \omega_\infty^{(n)} \quad \text{in } H_x^{-1}L_y^2.$$

By Proposition 9.1,  $\{W^{(n)}\}_{n\geq 1}$  is a Cauchy sequence in  $L^{\infty}_t(H^{-1}_xH^1_y)$ . Thus,  $\omega^{(n)}_{\infty}$  is a Cauchy sequence in  $H^{-1}_xH^1_y$ , then  $\omega^{\infty} = \lim_{n\to\infty}\omega^{(n)}_{\infty} \in H^{-1}_xH^1_y$ .

## 11. Appendix

11.1. Multilinear singular integral operators. The multilinear operator  $\mathbb{T}$  is defined by

$$\mathbb{T}(f, g, F, G)(c) \stackrel{\text{def}}{=} \text{p.v.} \int_{u(0)}^{u(1)} \frac{\int_{c}^{c'} g(z)F(z, c)dz}{(c - c')^2} G(c', c)f(c')dc',$$

where f, g are functions defined on  $D_0$  and F, G are functions defined on  $D_0 \times D_0$ . The multilinear operator  $\mathbb{B}_0$  and  $\mathbb{B}_1$  are defined by

$$\mathbb{B}_{0}(f,g,F,G)(c) \stackrel{\text{def}}{=} \frac{(u(1)-c)f(u(0))}{c-u(0)} \int_{u(0)}^{c} g(z)F(z,c)G(u(0),c)dz,$$

$$\mathbb{B}_{1}(f,g,F,G)(c) \stackrel{\text{def}}{=} \frac{(c-u(0))f(u(1))}{u(1)-c} \int_{c}^{u(1)} g(z)F(z,c)G(u(1),c)dz.$$

Lemma 11.1. It holds that

$$\partial_{c}(\rho(c)\mathbb{T}(f,g,F,G)(c)) = \rho(c)\mathbb{T}(f,g',F,G)(c) + \rho(c)\mathbb{T}(f,g,(\partial_{z}+\partial_{c})F,G)(c) + \rho(c)\mathbb{T}(f,g,F,(\partial_{z}+\partial_{c})G)(c) + \rho(c)\mathbb{T}(f',g,F,G)(c) + \rho'(c)\mathbb{T}(f,g,F,G)(c) - \mathbb{B}_{0}(f,g,F,G)(c) - \mathbb{B}_{1}(f,g,F,G)(c).$$

*Proof.* Thanks to

$$\rho(c) - \rho(c') = (c' - c)(c' + c - u(0) - u(1)),$$

we get

$$\begin{split} &\rho(c)\mathbb{T}(f,g,F,G)(c) \\ &= \text{p.v.} \int_{u(0)}^{u(1)} \frac{(c'+c-u(0)-u(1))\int_{c}^{c'}g(z)F(z,c)dz}{c'-c} G(c',c)f(c')dc' \\ &+ \text{p.v.} \int_{u(0)}^{u(1)} \frac{\int_{c}^{c'}g(z)F(z,c)dz}{(c'-c)^2} G(c',c)f(c')\rho(c')dc'. \end{split}$$

A direct calculation shows

$$\partial_{c} \left( \frac{\int_{c}^{c'} g(z) F(z, c) dz}{c' - c} \right)$$

$$= -\partial_{c'} \left( \frac{\int_{c}^{c'} g(z) F(z, c) dz}{c' - c} \right) + \frac{\int_{c}^{c'} g'(z) F(z, c) dz}{c' - c} + \frac{\int_{c}^{c'} g(z) (\partial_{c} + \partial_{z}) F(z, c) dz}{c' - c},$$

and

$$\partial_{c} \left( \frac{\int_{c}^{c'} g(z)F(z,c)dz}{(z-c)^{2}} \right) = -\partial_{c'} \left( \frac{\int_{c}^{c'} g(z)F(z,c)dz}{(c'-c)^{2}} \right) + \frac{\int_{c}^{c'} g'(z)F(z,c)dz}{(c'-c)^{2}} + \frac{\int_{c}^{c'} g(z)(\partial_{c} + \partial_{z})F(z,c)dy}{(c'-c)^{2}}.$$

Thus, we obtain

$$\begin{split} &\partial_{c} \left( \rho(c) \mathbb{T}(f,g,F,G)(c) \right) \\ =& \text{p.v.} \int_{u(0)}^{u(1)} \frac{\int_{c}^{c'} g(z) F(z,c) dz}{(c'-c)} (\partial_{c} + \partial_{c'}) \Big( G(c',c) f(c')(c'+c-u(0)-u(1)) \Big) dc' \\ &+ \text{p.v.} \int_{u(0)}^{u(1)} \frac{\int_{c}^{c'} g(z) F(z,c) dz}{(c'-c)^{2}} (\partial_{c} + \partial_{c'}) \Big( G(c',c) f(c') \rho(c') \Big) dc' \\ &+ \text{p.v.} \int_{u(0)}^{u(1)} \frac{\int_{c}^{c'} g(z) (\partial_{c} + \partial_{z}) F(z,c) dz}{(c'-c)} G(c',c) f(c')(c'+c-u(0)-u(1)) dc' \\ &+ \text{p.v.} \int_{u(0)}^{u(1)} \frac{\int_{c}^{c'} g(z) (\partial_{c} + \partial_{z}) F(z,c) dz}{(c'-c)^{2}} G(c',c) f(c') \rho(c') dc' \\ &+ \text{p.v.} \int_{u(0)}^{u(1)} \frac{\int_{c}^{c'} g'(z) F(z,c) dz}{(c'-c)} G(c',c) f(c')(c'+c-u(0)-u(1)) dc' \\ &+ \text{p.v.} \int_{u(0)}^{u(1)} \frac{\int_{c}^{c'} g'(z) F(z,c) dz}{(c'-c)^{2}} G(c',c) f(c') \rho(c') dc' \\ &- \frac{(c-u(0)) f(u(1))}{u(1)-c} \int_{c}^{u(1)} g(z) F(z,c) G(u(1),c) dz \\ &+ \frac{(c-u(1)) f(u(0))}{c-u(0)} \int_{c(0)}^{c} g(z) F(z,c) G(u(0),c) dz. \end{split}$$

Then the lemma follows by using

and  $\rho(c) - \rho(c') = (c' - c)(c' + c - u(0) - u(1)).$ 

By Lemma 11.1 and  $(\partial_z + \partial_c)e^{it\alpha(z-c)} = 0$ , we can deduce that

Lemma 11.2. It holds that

$$\rho(c)\mathbb{T}(f, g', e^{it\alpha(z-c)}F, G)(c) = \rho(c)\partial_c \left(e^{-it\alpha c}\mathbb{T}(f, ge^{it\alpha z}, F, G)\right)(c) - \rho(c)e^{-it\alpha c}\mathbb{T}(f, ge^{it\alpha z}, (\partial_z + \partial_c)F, G)(c)$$

$$-\rho(c)e^{-it\alpha c}\mathbb{T}(f, ge^{it\alpha z}, F, (\partial_z + \partial_c)G)(c)$$

$$-\rho(c)e^{-it\alpha c}\mathbb{T}(f', ge^{it\alpha z}, F, G)(c)$$

$$+e^{-it\alpha c}\mathbb{B}_0(f, ge^{it\alpha z}, F, G)(c) + e^{-it\alpha c}\mathbb{B}_1(f, ge^{it\alpha z}, F, G)(c).$$

Lemma 11.3. It holds that

$$\partial_{c}\mathbb{B}_{0}(f,g,F,G)(c) = \mathbb{B}_{0}(f,g,F,\partial_{c}G)(c) - \frac{f(u(0))G(u(0),c)}{c-u(0)} \int_{u(0)}^{c} g(z)F(z,c)dz + \mathbb{B}_{0}(f,g',F,G)(c) + \mathbb{B}_{0}(f,g,(\partial_{z}+\partial_{c})F,G)(c),$$
$$\partial_{c}\mathbb{B}_{1}(f,g,F,G)(c) = \mathbb{B}_{1}(f,g,F,\partial_{c}G)(c) - \frac{f(u(1))G(u(0),c)}{u(1)-c} \int_{c}^{u(1)} g(z)F(z,c)dz + \mathbb{B}_{1}(f,g',F,G)(c) + \mathbb{B}_{1}(f,g,(\partial_{z}+\partial_{c})F,G)(c).$$

*Proof.* The lemma follows by a direct calculation.

Next let us introduce some linear operators. Let  $\phi(y,c)$  be the solution of (4.1) given by Proposition 4.5 and  $\phi_1(y,c) = \frac{\phi(y,c)}{u(y)-c}$ . We define

$$\mathcal{L}_{1}(g)(c) \stackrel{\text{def}}{=} \mathbb{T}\Big((u^{-1})', g, \phi_{1}(u^{-1}, c), \frac{1}{\phi_{1}(u^{-1}, c)^{2}}\Big)(c),$$

$$\mathcal{L}_{2}(g)(c) \stackrel{\text{def}}{=} \mathbb{T}\Big((u^{-1})', g, (\partial_{z} + \partial_{c})\phi_{1}(u^{-1}, c), \frac{1}{\phi_{1}(u^{-1}, c)^{2}}\Big)(c)$$

$$+ \mathbb{T}\Big((u^{-1})', g, \phi_{1}(u^{-1}, c), (\partial_{z} + \partial_{c})\Big(\frac{1}{\phi_{1}(u^{-1}, c)^{2}}\Big)\Big)(c)$$

$$+ \mathbb{T}\Big((u^{-1})'', g, \phi_{1}(u^{-1}, c), \frac{1}{\phi_{1}(u^{-1}, c)^{2}}\Big)(c),$$

$$\mathcal{L}_{3}(g)(c) \stackrel{\text{def}}{=} \mathbb{T}\Big((u^{-1})', g, (\partial_{z} + \partial_{c})^{2}\phi_{1}(u^{-1}, c), \frac{1}{\phi_{1}(u^{-1}, c)^{2}}\Big)(c)$$

$$+ \mathbb{T}\Big((u^{-1})'', g, \phi_{1}(u^{-1}, c), (\partial_{z} + \partial_{c})^{2}\Big(\frac{1}{\phi_{1}(u^{-1}, c)^{2}}\Big)\Big)(c)$$

$$+ \mathbb{T}\Big((u^{-1})''', g, \phi_{1}(u^{-1}, c), \frac{1}{\phi_{1}(u^{-1}, c)^{2}}\Big)(c)$$

$$+ 2\mathbb{T}\Big((u^{-1})'', g, (\partial_{z} + \partial_{c})\phi_{1}(u^{-1}, c), \frac{1}{\phi_{1}(u^{-1}, c)^{2}}\Big)(c)$$

$$+ 2\mathbb{T}\Big((u^{-1})', g, (\partial_{z} + \partial_{c})\phi_{1}(u^{-1}, c), (\partial_{z} + \partial_{c})\Big(\frac{1}{\phi_{1}(u^{-1}, c)^{2}}\Big)\Big)(c).$$

We also introduce the linear operators

$$\mathcal{B}(g)(c) \stackrel{\text{def}}{=} \mathbb{B}_0 \Big( (u^{-1})', g, \phi_1(u^{-1}, c), \frac{1}{\phi_1(u^{-1}, c)^2} \Big)(c)$$

$$+ \mathbb{B}_1 \Big( (u^{-1})', g, \phi_1(u^{-1}, c), \frac{1}{\phi_1(u^{-1}, c)^2} \Big)(c),$$

$$\mathcal{B}_1(g)(c) \stackrel{\text{def}}{=} \mathbb{B}_0 \Big( (u^{-1})', g, (\partial_z + \partial_c) \phi_1(u^{-1}, c), \frac{1}{\phi_1(u^{-1}, c)^2} \Big)(c)$$

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$$+ \mathbb{B}_{1}\Big((u^{-1})', g, (\partial_{z} + \partial_{c})\phi_{1}(u^{-1}, c), \frac{1}{\phi_{1}(u^{-1}, c)^{2}}\Big)(c)$$

$$+ \mathbb{B}_{0}\Big((u^{-1})', g, \phi_{1}(u^{-1}, c), (\partial_{z} + \partial_{c})\Big(\frac{1}{\phi_{1}(u^{-1}, c)^{2}}\Big)\Big)(c)$$

$$+ \mathbb{B}_{1}\Big((u^{-1})', g, \phi_{1}(u^{-1}, c), (\partial_{z} + \partial_{c})\Big(\frac{1}{\phi_{1}(u^{-1}, c)^{2}}\Big)\Big)(c)$$

$$+ \mathbb{B}_{0}\Big((u^{-1})'', g, \phi_{1}(u^{-1}, c), \frac{1}{\phi_{1}(u^{-1}, c)^{2}}\Big)(c)$$

$$+ \mathbb{B}_{1}\Big((u^{-1})'', g, \phi_{1}(u^{-1}, c), \frac{1}{\phi_{1}(u^{-1}, c)^{2}}\Big)(c).$$

Lemma 11.4. It holds that

$$\partial_c (\rho \mathcal{L}_1(g)) = \rho(c) \mathcal{L}_1(g') + \rho'(c) \mathcal{L}_1(g) + \rho(c) \mathcal{L}_2(g) + \mathcal{B}(g),$$
  
$$\rho \partial_c \mathcal{L}_2(g) = \rho \mathcal{L}_2(g') + \rho \mathcal{L}_3(g) - \mathcal{B}_1(g),$$

and

$$\partial_c^2(\rho^2 \mathcal{L}_1(g)) = 2\rho''\rho\mathcal{L}_1(g) + 4\rho'\rho\mathcal{L}_1(g') + 2\rho'\rho'\mathcal{L}_1(g) + 4\rho'\rho\mathcal{L}_2(g) - 2\rho'\mathcal{B}(g) + \rho^2\mathcal{L}_1(g'') + 2\rho^2\mathcal{L}_2(g') - \rho\mathcal{B}(g') + \rho^2\mathcal{L}_3(g) - \rho\mathcal{B}_1(g) - \partial_c(\rho\mathcal{B}(g)).$$

*Proof.* The lemma can be proved by using Lemma 11.1. We omit the details.  $\Box$ 

11.2. Boundedness of singular integral operators. Let H be the Hilbert operator which is defined by

$$Hf(c) = \text{p.v.} \int \frac{f(c')}{c - c'} dc'.$$

The maximal Hilbert transform  $H^*$  is defined by

$$H^*f(c) = \sup_{\epsilon > 0} |H^{\epsilon}f(c)|, \quad H^{\epsilon}f(c) = \int_{|c-c'| \ge \epsilon} \frac{f(c')}{c - c'} dc'.$$

We denote

$$Hf(c) \triangleq \chi_{D_0}(c)H(f\chi_{D_0})(c).$$

**Lemma 11.5.** Suppose that  $f \in H^2(D_0)$ , then it holds that for any  $p \in (1, \infty)$ ,

$$\|\mathbf{H}f\|_{L^{p}} \leq C\|f\|_{L^{p}},$$
  
$$\|\rho\mathbf{H}f\|_{W^{1,p}} \leq C\|f\|_{W^{1,p}},$$
  
$$\|\rho^{2}\mathbf{H}f\|_{W^{2,p}} \leq C\|f\|_{W^{2,p}}.$$

Proof. Thanks to

$$\begin{split} [\rho, \mathbf{H}] f(c) &= \int_{u(0)}^{u(1)} \frac{\rho(c) - \rho(c')}{c - c'} f(c') dc' \\ &= \int_{u(0)}^{u(1)} (u(0) + u(1) - c - c') f(c') dc', \end{split}$$

we deduce that

(11.1) 
$$\|\partial_c[\rho, \mathbf{H}]f\|_{L^p} + \|\partial_c^2[\rho^2, \mathbf{H}]f\|_{L^p} \le C\|f\|_{L^p}.$$

We have

$$\partial_{c}\left(\text{p.v.} \int_{u(0)}^{u(1)} \frac{\rho(c')f(c')}{c - c'} dc'\right) = \partial_{c}\left(\text{p.v.} \int_{c - u(1)}^{c - u(0)} \frac{\rho(c - z)f(c - z)}{z} dz\right)$$

$$= \text{p.v.} \int_{c - u(1)}^{c - u(0)} \frac{(\rho f)'(c - z)}{z} dz = \text{p.v.} \int_{u(0)}^{u(1)} \frac{(\rho f)'(c')}{c - c'} dc',$$

from which and  $L^p$  boundedness of Hilbert transform, we infer that

$$\left\| \partial_c \left( \text{p.v.} \int_{u(0)}^{u(1)} \frac{\rho(c')f(c')}{c - c'} dc' \right) \right\|_{L^p} \le C \|(\rho f)'\|_{L^p},$$

$$\left\| \partial_c^2 \left( \text{p.v.} \int_{u(0)}^{u(1)} \frac{\rho(c')^2 f(c')}{c - c'} dc' \right) \right\|_{L^p} \le C_0 \|(\rho^2 f)''\|_{L^p},$$

which along with (11.1) gives the lemma.

**Lemma 11.6.** There exists a constant C independent of  $\epsilon \in (0, (u(1) - u(0))/10)$  such that for any  $p \in (1, \infty)$ 

$$\left\| \sup_{\epsilon} \left| \int_{\{|c-c'| > \epsilon\} \cap [u(0), u(1)]} \frac{\int_{c}^{c'} f(z) dz}{(c-c')^2} dc' \right| \right\|_{L^{p}} \le C \|f\|_{L^{p}},$$

$$\left\| p.v. \int_{u(0)}^{u(1)} \frac{\int_{c}^{c'} f(z) dz}{(c-c')^2} dc' \right\|_{L^{p}} \le C \|f\|_{L^{p}}.$$

*Proof.* For  $u(0) \le c < u(0) + \epsilon$ , we have

$$\int_{c+\epsilon}^{u(1)} \frac{\int_{c}^{c'} f(z)dz}{(c-c')^2} dc' = \int_{c+\epsilon}^{u(1)} -\partial_{c'} \left(\frac{1}{c'-c}\right) \int_{c}^{c'} f(z)dz dc' 
= -\frac{1}{u(1)-c} \int_{c}^{u(1)} f(z)dz + \frac{1}{\epsilon} \int_{c}^{c+\epsilon} f(z)dz + \int_{c+\epsilon}^{u(1)} \frac{f(z)}{z-c} dz.$$

For  $u(0) + \epsilon \le c \le u(1) - \epsilon$ , we have

$$\int_{c+\epsilon}^{u(1)} + \int_{u(0)}^{c-\epsilon} \frac{\int_{c}^{c'} f(z)dz}{(c'-c)^{2}} dc' 
= -\int_{c+\epsilon}^{u(1)} + \int_{u(0)}^{c-\epsilon} \partial_{c'} \left(\frac{1}{c'-c}\right) \int_{c}^{c'} f(z)dz dc' 
= -\frac{1}{u(1)-c} \int_{c}^{u(1)} f(z)dz + \frac{1}{\epsilon} \int_{c}^{c+\epsilon} f(z)dz + \int_{c+\epsilon}^{u(1)} + \int_{u(0)}^{c-\epsilon} \frac{f(z)}{z-c} dz 
- \frac{1}{\epsilon} \int_{c-\epsilon}^{c} f(z)dz + \frac{1}{c-u(0)} \int_{u(0)}^{c} f(z)dz.$$

For  $u(1) - \epsilon < c \le u(1)$ ,

$$\begin{split} \int_{u(0)}^{c-\epsilon} \frac{\int_{c}^{c'} f(z)dz}{(c'-c)^2} dc' &= \int_{u(0)}^{c-\epsilon} -\partial_{c'} \left(\frac{1}{c'-c}\right) \int_{c}^{c'} f(z)dzdc' \\ &= \frac{1}{c-u(0)} \int_{u(0)}^{c} f(z)dz - \frac{1}{\epsilon} \int_{c-\epsilon}^{c} f(z)dz + \int_{u(0)}^{c-\epsilon} \frac{f(z)}{z-c}dz. \end{split}$$

Thus, we obtain

$$\begin{split} \int_{\{|c-c'|>\epsilon\}\cap[u(0),u(1)]} \frac{\int_{c}^{c'} f(z)dz}{(c-c')^2} dc' &= -\chi_{D_0}(c) H^{\epsilon}(f\chi_{D_0})(c) \\ &+ \Big(\frac{1}{c-u(0)} \int_{u(0)}^{c} f(z)dz - \frac{1}{\epsilon} \int_{c-\epsilon}^{c} f(z)dz\Big) \chi_{[\epsilon,u(1)]}(c) \\ &- \Big(\frac{1}{u(1)-c} \int_{c}^{u(1)} f(z)dz - \frac{1}{\epsilon} \int_{c}^{c+\epsilon} f(z)dz\Big) \chi_{[u(0),u(1)-\epsilon]}(c). \end{split}$$

Taking  $\epsilon \to 0+$  and using the fact that  $\frac{1}{\epsilon} \int_{x-\epsilon}^{x} f(y) dy - \frac{1}{\epsilon} \int_{x}^{x+\epsilon} f(y) dy \to 0$ , we deduce

p.v. 
$$\int_{u(0)}^{u(1)} \frac{\int_{c}^{c'} f(z)dz}{(c-c')^2} dc' = -\chi_{D_0}(c)H(f\chi_{D_0})(c) + \frac{1}{c-u(0)} \int_{u(0)}^{c} f(z)dz - \frac{1}{u(1)-c} \int_{c}^{u(1)} f(z)dz.$$

With the above identities, the boundedness in  $L^p$  follows from  $L^p$  boundedness of Hilbert transform, maximal Hilbert transform and Hardy-Littlewood maximal function.

## 11.3. Boundedness of multilinear singular integral operators.

**Lemma 11.7.** There exists a constant C independent of  $\alpha$  such that for any  $p \in (1, \infty)$ ,

(11.2) 
$$\left\| \mathbb{T}\left(f, g, \phi_1(u^{-1}, c), \frac{1}{\phi_1(u^{-1}, c)^2}\right) \right\|_{L^p} \le C(\|f\|_{L^\infty} + \|f'\|_{L^\infty}) \|g\|_{L^p}.$$

*Proof.* We write

$$\mathbb{T}\left(f,g,\phi_{1}(u^{-1},c),\frac{1}{\phi_{1}(u^{-1},c)^{2}}\right)(c) 
= \int_{u(0)}^{u(1)} \frac{\int_{c}^{c'} g(z)(\phi_{1}(u^{-1}(z),c)-1)dz}{(c'-c)^{2}\phi_{1}(u^{-1}(c'),c)^{2}} f(c')dc' 
+ \int_{u(0)}^{u(1)} \frac{\int_{c}^{c'} g(z)dz}{(c'-c)^{2}} \frac{f(c')-f(c)}{\phi_{1}(u^{-1}(c'),c)^{2}} dc' 
+ f(c) \int_{\{|c'-c| \leq \frac{1}{\alpha}\} \cap D_{0}} \frac{\int_{c}^{c'} g(z)dz}{(c'-c)^{2}} \frac{(\phi_{1}(u^{-1}(c'),c)+1)(\phi_{1}(u^{-1}(c'),c)-1)}{\phi_{1}(u^{-1}(c'),c)^{2}} dc' 
+ f(c) \int_{\{|c'-c| > \frac{1}{\alpha}\} \cap D_{0}} \frac{\int_{c}^{c'} g(z)dz}{(c'-c)^{2}\phi_{1}(u^{-1}(c'),c)^{2}} dc' 
+ f(c) p.v. \int_{u(0)}^{u(1)} \frac{\int_{c}^{c'} g(z)dz}{(c'-c)^{2}} dc' - f(c) p.v. \int_{\{|c'-c| > \frac{1}{\alpha}\} \cap D_{0}} \frac{\int_{c}^{c'} g(z)dz}{(c'-c)^{2}} dc' 
\triangleq I_{1} + \cdots I_{6}.$$

**Step 1.** Estimate of  $I_1$ 

We infer from Remark 5.5 that

$$I_1 = \alpha^2 \int_{u(0)}^{u(1)} \frac{\int_c^{c'} g(z)(u^{-1}(z) - u^{-1}(c))^2 \mathcal{T}(\phi_1)(u^{-1}(z), c) dz}{(c' - c)^2 \phi_1(u^{-1}(c'), c)^2} f(c') dc'.$$

For 
$$|z-c| \le |c'-c|$$
, we have  $(u^{-1}(z) - u^{-1}(c))^2 \le C(c'-c)^2$ . And by Remark 5.5,  $\mathcal{T}(\phi_1)(u^{-1}(z),c) \le C\phi_1(u^{-1}(z),c) \le C\phi_1(u^{-1}(c'),c)$ .

Then we infer that

$$|I_1| \le C\alpha^2 ||f||_{L^{\infty}} \int_{u(0)}^{u(1)} \int_c^{c'} |g(z)| dz \frac{1}{\phi_1(u^{-1}(c'), c)} dc',$$

from which and Proposition 5.1, it follows that

$$|I_{1}| \leq C\alpha^{2} ||f||_{L^{\infty}} ||g||_{L^{\infty}} \int_{0}^{1} \frac{|y - y_{c}|}{\phi_{1}(y, c)} dy$$

$$\leq C \left( \int_{0}^{\alpha(1 - y_{c})} \frac{y^{2}}{\sinh y} dy + \int_{0}^{\alpha y_{c}} \frac{y^{2}}{\sinh y} dy \right) ||f||_{L^{\infty}} ||g||_{L^{\infty}}$$

$$\leq C ||f||_{L^{\infty}} ||g||_{L^{\infty}}.$$

And by Proposition 5.1 again, we get

$$||I_{1}||_{L^{1}} \leq C||f||_{L^{\infty}} \left( \int_{0}^{1} \int_{0}^{y_{c}} \alpha^{2} \int_{z}^{y_{c}} |g(u(y''))| \frac{\alpha(z - y_{c})}{\sinh \alpha(z - y_{c})} dy'' dz dy_{c} \right)$$

$$+ \int_{0}^{1} \int_{y_{c}}^{1} \alpha^{2} \int_{y_{c}}^{z} |g(u(y''))| \frac{\alpha(z - y_{c})}{\sinh \alpha(z - y_{c})} dy'' dz dy_{c} \right)$$

$$\leq C\alpha^{2} ||f||_{L^{\infty}} \int_{0}^{1} |g(u(y''))| \int_{0}^{y''} \int_{y''}^{1} \frac{\alpha(z - y_{c})}{\sinh \alpha(z - y_{c})} dy_{c} dz dy''$$

$$+ C\alpha^{2} ||f||_{L^{\infty}} \int_{0}^{1} |g(u(y''))| \int_{0}^{y''} \int_{y''}^{1} \frac{\alpha(z - y_{c})}{\sinh \alpha(z - y_{c})} dz dy_{c} dy''.$$

Using the fact that

$$\frac{z}{\sinh z} \le 2 \frac{\sinh \frac{z}{2}}{\sinh z} \le \frac{1}{\cosh \frac{z}{2}} \le \frac{1}{e^{\frac{z}{2}}}, \ z > 0.$$

we deduce that

$$\left|\alpha^{2} \int_{0}^{y} \int_{y}^{1} \frac{\alpha(z - y_{c})}{\sinh \alpha(z - y_{c})} dy_{c} dz\right| \leq \int_{\alpha y}^{+\infty} \int_{0}^{\alpha y} e^{\frac{1}{2}(z - y_{c})} dz dy_{c}$$
$$\leq 2 \int_{\alpha y}^{+\infty} e^{\frac{1}{2}\alpha(y - y_{c})} - e^{-\frac{1}{2}y_{c}} dy_{c} \leq 8.$$

This shows that

$$||I_1||_{L^1} \le C||f||_{L^\infty}||g||_{L^1}.$$

Then by the interpolation, we get

$$||I_1||_{L^p} \le C||f||_{L^\infty}||g||_{L^p}.$$

**Step 2.** Estimate of  $I_2$ 

Thanks to  $\phi_1(u^{-1}(z),c)^2 \ge 1$ , we have

$$|I_2| \le C \int_{u(0)}^{u(1)} \frac{\int_c^{c'} |g(z)| dz}{c' - c} \frac{\|f'\|_{L^{\infty}}}{\phi_1(u^{-1}(c'), c)^2} dc'$$

$$\leq C \|f'\|_{L^{\infty}} \int_{u(0)}^{u(1)} \frac{\int_{c}^{c'} |g(z)| dz}{c' - c} dc' \leq C \|f'\|_{L^{\infty}} \|g\|_{L^{\infty}}.$$

For  $L^1$  estimate, we write

$$\begin{split} \int_{u(0)}^{u(1)} \frac{\int_{c}^{c'} |g(z)| dz}{c' - c} dc' &= \int_{u(0)}^{c} \frac{\int_{c}^{c'} |g(z)| dz}{c' - c} dc' + \int_{c}^{u(1)} \frac{\int_{c}^{c'} |g(z)| dz}{c' - c} dc' \\ &= \int_{u(0)}^{c} |g(z)| \int_{u(0)}^{z} \frac{1}{c - c'} dc' dz + \int_{c}^{u(1)} |g(z)| \int_{z}^{u(1)} \frac{1}{c' - c} dc' dz \\ &= \int_{u(0)}^{c} |g(z)| \ln \frac{c - u(0)}{c - z} dz + \int_{c}^{u(1)} |g(z)| \ln \frac{u(1) - c}{z - c} dz, \end{split}$$

from which, it follows that

$$\begin{split} \left\| \int_{u(0)}^{u(1)} \frac{\int_{c}^{c'} |g(z)| dz}{c' - c} dc' \right\|_{L^{1}} &\leq \int_{u(0)}^{u(1)} \int_{u(0)}^{c} |g(z)| \ln \frac{c - u(0)}{c - z} dz + \int_{c}^{u(1)} |g(z)| \ln \frac{u(1) - c}{z - c} dz dc \\ &\leq \int_{u(0)}^{u(1)} |g(z)| \int_{z}^{u(1)} \ln \frac{c - u(0)}{c - z} dc dz + \int_{u(0)}^{u(1)} |g(z)| \int_{u(0)}^{z} \ln \frac{u(1) - c}{z - c} dc dz \\ &\leq C \|g\|_{L^{1}}. \end{split}$$

This gives

$$||I_2||_{L^1} \le C||f'||_{L^\infty}||g||_{L^1}.$$

By the interpolation, we get

$$||I_2||_{L^p} \leq C||f'||_{L^\infty}||g||_{L^p}.$$

**Step 3.** Estimate of  $I_3$ 

By Lemma 5.4,  $|\phi_1(u^{-1}(c'),c)-1| \leq \alpha^2(c'-c)^2\phi_1(u^{-1}(c'),c)$ . Then we have

$$|I_{3}| \leq C\alpha^{2} ||f||_{L^{\infty}} \int_{\{0 \leq c' - c \leq \frac{1}{\alpha}\} \cap D_{0}} \int_{c}^{c'} |g(z)| dz dc'$$

$$+ C\alpha^{2} ||f||_{L^{\infty}} \int_{\{0 \leq c - c' \leq \frac{1}{\alpha}\} \cap D_{0}} \int_{c'}^{c} |g(z)| dz dc'$$

$$\leq C ||f||_{L^{\infty}} \alpha^{2} \int_{c}^{\min \left\{c + \frac{1}{\alpha}, u(1)\right\}} \int_{c}^{c'} |g(z)| dz dc'$$

$$+ C ||f||_{L^{\infty}} \alpha^{2} \int_{\max \left\{c - \frac{1}{\alpha}, u(0)\right\}}^{c} \int_{c'}^{c} |g(z)| dz dc',$$

from which, it follows that

$$||I_3||_{L^{\infty}} \le C||f||_{L^{\infty}} ||\alpha^2 \int_{\{|c'-c| \le \frac{1}{\alpha}\} \cap D_0} ||g||_{L^{\infty}} |c'-c|dc'||_{L^{\infty}}$$
  
$$\le C||f||_{L^{\infty}} ||g||_{L^{\infty}},$$

and

$$||I_3||_{L^1} \le C\alpha^2 ||f||_{L^{\infty}} \left( \int_{u(0)}^{u(1)} \int_c^{\min\left\{c + \frac{1}{\alpha}, u(1)\right\}} \int_c^{c'} |g(z)| dz dc' dc \right)$$

$$\begin{split} &+ \int_{u(0)}^{u(1)} \int_{\max \left\{ c - \frac{1}{\alpha}, u(0) \right\}}^{c} \int_{c'}^{c} |g(z)| dz dc' dc \bigg) \\ \leq & C \alpha^{2} \|f\|_{L^{\infty}} \bigg( \int_{u(0)}^{u(1)} |g(z)| \int_{z - \frac{1}{\alpha}}^{z} \Big( c + \frac{1}{\alpha} - z \Big) dc dz \\ &+ \int_{u(0)}^{u(1)} |g(z)| \int_{z}^{z + \frac{1}{\alpha}} \Big( \frac{1}{\alpha} + z - c \Big) dc dz \bigg) \\ \leq & C \|f\|_{L^{\infty}} \|g\|_{L^{1}}. \end{split}$$

Thus by the interpolation, we get

$$||I_3||_{L^p} \le C||f||_{L^\infty}||g||_{L^p}.$$

# Step 4. Estimate of $I_4$

We infer from Proposition 5.1 that

$$\frac{1}{\phi_1(u^{-1}(c'),c)} \ge C^{-1} \frac{\alpha(u^{-1}(c') - u^{-1}(c))}{\sinh \alpha(u^{-1}(c') - u^{-1}(c))},$$

which gives

$$|I_{4}| \leq C \|f\|_{L^{\infty}} \int_{|c'-c| \geq \frac{1}{\alpha}} \|g\|_{L^{\infty}} \frac{\alpha^{2} |u^{-1}(c') - u^{-1}(c)|}{\sinh^{2} \alpha (u^{-1}(c') - u^{-1}(c))} dc'$$

$$\leq C \|f\|_{L^{\infty}} \|g\|_{L^{\infty}} \int_{1}^{\infty} \frac{z}{\sinh^{2} z} dz$$

$$\leq C \|f\|_{L^{\infty}} \|g\|_{L^{\infty}},$$

and

$$||I_4||_{L^1} \leq C\alpha^2 ||f||_{L^{\infty}} \int_{u(0)}^{u(1)} \int_{\{|z-c| \geq \frac{1}{\alpha}\} \cap \{z \geq c\} \cap [u(0), u(1)]} \int_c^z |g(c')| \frac{1}{\phi_1(u^{-1}(z), c)^2} dc' dz dc$$

$$+ C\alpha^2 ||f||_{L^{\infty}} \int_{u(0)}^{u(1)} \int_{\{|z-c| \geq \frac{1}{\alpha}\} \cap \{z < c\} \cap [u(0), u(1)]} \int_z^c |g(c')| \frac{1}{\phi_1(u^{-1}(z), c)^2} dc' dz dc.$$

By a change of variable, the first term is bounded by

$$C\|f\|_{L^{\infty}} \int_{0}^{1} \int_{\{|u(y'')-c| \geq \frac{1}{\alpha}\} \cap \{u(y'') \geq c\} \cap [0,1]} \int_{y_{c}}^{y''} |g(u(y'))| \frac{\alpha^{2}}{\phi_{1}(y'',c)^{2}} dy' dy'' dy_{c}$$

$$\leq C\|f\|_{L^{\infty}} \int_{0}^{1} |g(u(y'))| \int_{0}^{y'} \int_{y'}^{1} \frac{\alpha^{2}}{\phi_{1}(y'',c)^{2}} dy'' dy_{c} dy'$$

$$\leq C\|f\|_{L^{\infty}} \|g\|_{L^{1}},$$

where we used the fact that  $\phi_1(y'',c) \ge C^{-1} \frac{\sinh \alpha(y''-y_c)}{\alpha(y''-y_c)}$  so that

$$\left| \int_{0}^{y'} \int_{y'}^{1} \frac{\alpha^{2}}{\phi_{1}^{2}(y'',c)} dy'' dy_{c} \right| \leq C \int_{0}^{y'} \int_{y'}^{1} \frac{\alpha^{4}(y''-y_{c})^{2}}{\sinh \alpha (y''-y_{c})^{2}} dy'' dy_{c}$$

$$\leq C \int_{0}^{\alpha y'} \int_{\alpha y'}^{+\infty} e^{-1/2(y-y_{c})} dy dy_{c} \leq C.$$

The estimate of the second term is the same. This gives

$$||I_4||_{L^1} \le C||f||_{L^\infty}||g||_{L^1}.$$

Thus, the interpolation gives

$$||I_4||_{L^p} \le C||f||_{L^\infty}||g||_{L^p}.$$

**Step 5.** Estimates of  $I_5$  and  $I_6$ 

Lemma 11.6 ensures that

$$||I_5||_{L^p} + ||I_6||_{L^p} \le C||f||_{L^\infty}||g||_{L^p}.$$

Putting the estimates in Step 1–Step 5 together, we conclude the lemma.

**Lemma 11.8.** There exists a constant C independent of  $\alpha$  such that for any  $p \in (1, \infty)$ ,

$$\|\mathbb{B}_{0}(f, \rho g, F, \partial_{c}G)\|_{L^{p}} + \|\mathbb{B}_{1}(f, \rho g, F, \partial_{c}G)\|_{L^{p}} + \|\mathbb{B}_{0}(f, q, (\partial_{z} + \partial_{c})F, G)\|_{L^{p}} + \|\mathbb{B}_{1}(f, q, (\partial_{z} + \partial_{c})F, G)\|_{L^{p}} \leq C\|f\|_{L^{\infty}}\|q\|_{L^{p}},$$

where  $F(z,c) = \phi_1(u^{-1}(z),c)$  and  $G(z,c) = \frac{1}{\phi_1(u^{-1}(z),c)^2}$ . We also have

$$\|\mathcal{B}(\rho g)\|_{L^p} \le \frac{1}{\alpha} \|g\|_{L^p}.$$

*Proof.* We only prove the estimate of  $\mathbb{B}_0$ , the estimate of  $\mathbb{B}_1$  is similar. We get by Proposition 5.1 that

$$\left| \mathbb{B}_0(f, \rho g, F, \partial_c G)(c) \right| \le C \|f\|_{L^{\infty}} \frac{\alpha}{\phi_1(0, c)} \int_{u(0)}^c |g(z)| dz,$$

from which, it follows that

$$\|\mathbb{B}_{0}(f, \rho g, F, \partial_{c}G)\|_{L^{\infty}} \leq C \sup_{y \in [0,1]} \frac{\alpha^{2}y^{2}}{\sinh \alpha y} \|f\|_{L^{\infty}} \|g\|_{L^{\infty}}$$

$$\leq C \|f\|_{L^{\infty}} \|g\|_{L^{\infty}},$$

$$\|\mathbb{B}_{0}(f, \rho g, F, \partial_{c}G)\|_{L^{1}} \leq C \|f\|_{L^{\infty}} \int_{0}^{1} |g(z)| dz \int_{0}^{1} \frac{\alpha^{2}y}{\sinh \alpha y} dy$$

$$\leq \|f\|_{L^{\infty}} \|g\|_{L^{1}}.$$

We get by Proposition 5.6 that

$$|\mathbb{B}_0(f, g, (\partial_z + \partial_c)F, G)(c)| \le C||f||_{L^{\infty}} \frac{\alpha^2(c - u(0))}{\phi_1(0, c)} \int_{u(0)}^c |g(z)| dz,$$

which implies that

$$\|\mathbb{B}_{0}(f, g, (\partial_{z} + \partial_{c})F, G)\|_{L^{\infty}} \leq \|f\|_{L^{\infty}} \|g\|_{L^{\infty}}, \|\mathbb{B}_{0}(f, g, (\partial_{z} + \partial_{c})F, G)\|_{L^{1}} \leq C\|f\|_{L^{\infty}} \|g\|_{L^{1}}.$$

Then  $L^p$  estimate follows by the interpolation.

Notice that

$$|\alpha \mathcal{B}(\rho g)| \le C \frac{\alpha}{\phi_1(0,c)} \int_{u(0)}^c |g(z)| dz + C \frac{\alpha}{\phi_1(1,c)} \int_c^{u(1)} |g(z)| dz.$$

Then  $L^p$  estimate of  $\mathcal{B}(\rho g)$  can be deduced in a similar way.

**Lemma 11.9.** There exists a constant C independent of  $\alpha$  such that for any  $p \in (1, \infty)$ ,

$$\|\mathcal{B}(g)\|_{L^p} + \|\partial_c(\rho\mathcal{B}(g))\|_{L^p} + \|\rho^{-1}\mathcal{B}(\rho g)\|_{L^p} + \|\mathcal{B}_1(g)\|_{L^p} \le C\|g\|_{L^p}.$$

If g(u(0)) = g(u(1)) = 0, then we have

$$\|\rho \mathcal{B}(g')\|_{L^p} \le C\|g\|_{L^p}.$$

*Proof.* The estimate of  $\|\mathcal{B}(g)\|_{L^p}$  follows from  $L^p$  boundedness of Hardy-Littlewood maximal function.

Step 1. Estimate of  $\|\partial_c(\rho\mathcal{B}(g))\|_{L^p}$ 

It suffices to consider  $\mathbb{B}_0$ . By the definition of  $\mathbb{B}_0$ , we have

$$\partial_{c}\left(\rho(c)\mathbb{B}_{0}\left(f,g,\phi_{1}(u^{-1}(\cdot),c),\frac{1}{\phi_{1}(0,c)^{2}}\right)\right)$$

$$=2(c-u(1))f(u(0))\int_{u(0)}^{c}g(z)\frac{\phi_{1}(u^{-1}(z),c)}{\phi_{1}(0,c)^{2}}dz+\frac{(u(1)-c)^{2}f(u(0))g(c)}{\phi_{1}(0,c)^{2}}$$

$$+(u(1)-c)^{2}f(u(0))\int_{u(0)}^{c}g(z)\partial_{c}\left(\frac{\phi_{1}(u^{-1}(z),c)}{\phi_{1}(0,c)^{2}}\right)dz.$$

The  $L^p$  estimate of the first two terms are obvious. So we only consider the last term, which is bounded by

$$C\|f\|_{L^{\infty}} \int_{0}^{y_{c}} |g(u(y))| \frac{\alpha}{\phi_{1}(0,c)} dy \le C\|f\|_{L^{\infty}} \frac{\alpha^{2} y_{c}}{\sinh \alpha y_{c}} \int_{0}^{y_{c}} |g(u(y))| dy,$$

where we used Proposition 5.1. Thus,  $L^{\infty}$  norm of the last term is bounded by

$$C \sup_{y \in [0,1]} \frac{\alpha^2 y^2}{\sinh \alpha y} \|g\|_{L^{\infty}} \|f\|_{L^{\infty}},$$

while  $L^1$  norm is bounded by

$$C\|f\|_{L^{\infty}} \int_{0}^{1} |g(u(y))| \int_{u_{c}}^{1} \frac{\alpha^{2} y_{c}}{\sinh \alpha y_{c}} dy_{c} dy \leq C\|f\|_{L^{\infty}} \|g\|_{L^{1}}.$$

This implies by the interpolation that

$$\|\partial_c(\rho\mathcal{B}(q))\|_{L^p} < C\|q\|_{L^p}.$$

Step 2. Estimate of  $\|\rho^{-1}\mathcal{B}(\rho g)\|_{L^p}$ 

Using the fact that

(11.3) 
$$\phi_1(u^{-1}(z), c) \le \phi_1(u^{-1}(c'), c) \quad \text{for } |z - c| \le |c' - c|,$$

we deduce that

$$\left| \rho^{-1} \mathcal{B}(\rho g)(c) \right| \leq \frac{C}{(c - u(0))^2} \int_{u(0)}^{c} g(z)(z - u(0))(u(1) - z) dz$$

$$+ \frac{C}{(u(1) - c)^2} \int_{c}^{u(1)} g(z)(z - u(0))(u(1) - z) dz$$

$$\leq \frac{C}{c - u(0)} \int_{u(0)}^{c} g(z) dz + \frac{C}{u(1) - c} \int_{c}^{u(1)} g(z) dz,$$

from which and  $L^p$  boundedness of Hardy-Littlewood maximal function, we infer that

$$\|\rho^{-1}\mathcal{B}(\rho g)\|_{L^p} \le C\|g\|_{L^p}.$$

Step 3. Estimate of  $\|\mathcal{B}_1(g)\|_{L^p}$ 

By Proposition 5.6, we have

$$\left| (\partial_z + \partial_c)\phi_1(u^{-1}(z), c) \right| \le C_0 \alpha^2 (z - c)^2 \phi_1(u^{-1}(z), c),$$

$$\left| (\partial_z + \partial_c) \left( \frac{1}{\phi_1(u^{-1}(z), c)^2} \right) \right| \le C_0 \alpha^2 (z - c)^2 \frac{1}{\phi_1(u^{-1}(z), c)^2},$$

which along with (11.3) gives So we have

$$\mathcal{B}_1(g) \le \frac{C}{c - u(0)} \int_{u(0)}^c |\varphi(y)| dy + \frac{C}{u(1) - c} \int_c^{u(1)} |\varphi(y)| dy,$$

from which and  $L^p$  boundedness of Hardy-Littlewood maximal function, we infer that

$$\|\mathcal{B}_1(g)\|_{L^p} \le C\|g\|_{L^p}.$$

Step 4. Estimate of  $\|\rho \mathcal{B}(g')\|_{L^2}$ 

Using the boundary condition of g, we get

$$\begin{split} \rho(c)\mathcal{B}(g') = &(u(1)-c)^2(u^{-1})'(u(0))\int_{u(0)}^c g'(z)\frac{\phi_1(u^{-1}(z),c)}{\phi_1(0,c)^2}dz \\ &+ (c-u(0))^2(u^{-1})'(u(1))\int_c^{u(1)} g'(z)\frac{\phi_1(u^{-1}(z),c)}{\phi_1(1,c)^2}dz \\ = &- (u(1)-c)^2(u^{-1})'(u(0))\int_c^c g(z)\frac{\partial_z\phi_1(u^{-1}(z),c)}{\phi_1(0,c)^2}dz \\ &- (c-u(0))^2(u^{-1})'(u(1))\int_c^{u(1)} g(z)\frac{\partial_z\phi_1(u^{-1}(z),c)}{\phi_1(1,c)^2}dz \\ &+ (u(1)-c)^2(u^{-1})'(u(0))g(c)\frac{1}{\phi_1(0,c)^2} \\ &- (c-u(0))^2(u^{-1})'(u(1))g(c)\frac{1}{\phi_1(1,c)^2}. \end{split}$$

Then by Proposition 5.1, we obtain

$$|\rho(c)\mathcal{B}(g')(c)| \le C|g(c)| + C\int_0^{y_c} |g(u(z))| \frac{\alpha \cosh \alpha(z - y_c)}{\phi_1(0, c)^2} dz + C\int_{y_c}^1 |g(u(z))| \frac{\alpha \cosh \alpha(z - y_c)}{\phi_1(1, c)^2} dz.$$

with  $c = u(y_c)$ . Thus, we have

$$\|\rho \mathcal{B}(g')\|_{L^{\infty}} \le C\|g\|_{L^{\infty}} + C\|g\|_{L^{\infty}} \sup_{c \in D_0} \left( \frac{\sinh \alpha y_c}{\phi_1(0,c)^2} + \frac{\sinh \alpha (1-y_c)}{\phi_1(1,c)^2} \right)$$
  
$$\le C\|g\|_{L^{\infty}}.$$

On the other hand, we have

$$\|\rho(c)\mathcal{B}(g')\|_{L^{1}} \leq C\|g\|_{L^{1}} + C\int_{0}^{1} \int_{0}^{y_{c}} |g(z)| \frac{\alpha \cosh \alpha(z - y_{c})}{\phi_{1}(0, c)^{2}} dz dy_{c}$$

$$+ C\int_{0}^{1} \int_{y_{c}}^{1} |g(z)| \frac{\alpha \cosh \alpha(z - y_{c})}{\phi_{1}(1, c)^{2}} dz dy_{c}$$

$$\leq C\|g\|_{L^{1}} + C\|g\|_{L^{1}} \int_{0}^{\infty} \frac{z^{2} \cosh z}{\sinh^{2}(z)} dz$$

$$\leq C \|g\|_{L^1}$$
.

By the interpolation, we get

$$\|\rho(c)\mathcal{B}(g')\|_{L^p} \le C\|g\|_{L^p}.$$

The proof is completed.

**Lemma 11.10.** There exists a constant C independent of  $\alpha$  such that for any  $p \in (1, \infty)$ ,

$$\|\mathcal{L}_1(g)\|_{L^p} + \|\mathcal{L}_2(g)\|_{L^p} + \|\mathcal{L}_3(g)\|_{L^p} \le C\|g\|_{L^p}.$$

*Proof.* The estimate of  $\|\mathcal{L}_1(g)\|_{L^2}$  follows from Lemma 11.7. Now we consider  $\mathcal{L}_2(g)$ . It suffices to consider the first two terms in the definition of  $\mathcal{L}_2(g)$ . By Proposition 5.6, both terms are bounded by

$$C\alpha^2 \int_{u(0)}^{u(1)} \left| \int_c^{c'} |g(z)| dz \right| \frac{1}{\phi_1(u^{-1}(c'), c)} dc'.$$

With this, the proof is the same as  $I_1$  in the proof of Lemma 11.7. Next we consider  $\mathcal{L}_3(g)$ . It suffice to estimate the following terms

$$\mathbb{T}\Big((u^{-1})', g, (\partial_z + \partial_c)^2 \phi_1(u^{-1}, c), \frac{1}{\phi_1(u^{-1}, c)^2}\Big)(c) \\
+ \mathbb{T}\Big((u^{-1})', g, \phi_1(u^{-1}, c), (\partial_z + \partial_c)^2 \Big(\frac{1}{\phi_1(u^{-1}, c)^2}\Big)\Big)(c) \triangleq \mathcal{L}_2^1(g).$$

By Proposition 5.6, we have

$$|(\partial_z + \partial_c)^2 \phi_1(u^{-1}(z), c)| \le C\alpha^2 (u^{-1}(z) - u^{-1}(c))^2 \cosh \alpha (u^{-1}(z) - u^{-1}(c)),$$

and

$$\left| (\partial_z + \partial_c) \left( \frac{1}{\phi_1^2(u^{-1}(z), c)} \right) \right| \le C\alpha^2 (u^{-1}(z) - u^{-1}(c))^2 \frac{\cosh \alpha (u^{-1}(z) - u^{-1}(c))}{\phi_1^3(u^{-1}(z), c)},$$

which imply

$$|\mathcal{L}_{2}^{1}(g)(c)| \leq C\alpha^{2} \int_{u(0)}^{u(1)} \left| \int_{c}^{c'} |g(z)| dz \right| \frac{\cosh \alpha (u^{-1}(c') - u^{-1}(c))}{\phi_{1}^{2}(u^{-1}(c'), c)} dc'.$$

Thus, we obtain

$$\|\mathcal{L}_{2}^{1}(g)\|_{L^{\infty}} \leq C \int_{0}^{\infty} \frac{z^{3} \cosh z}{\sinh^{2} z} dz \|g\|_{L^{\infty}} \leq C \|g\|_{L^{\infty}},$$

and

$$\begin{split} \|\mathcal{L}_{2}^{1}(g)\|_{L^{1}} &\leq C\alpha^{2} \int_{0}^{1} |g(y)| \int_{0}^{y} \int_{y}^{1} \frac{\alpha^{2}(z-y_{c})^{2} \cosh \alpha(z-y_{c})}{\sinh^{2} \alpha(z-y_{c})} dy_{c} dz dy \\ &+ C\alpha^{2} \int_{0}^{1} |g(y)| \int_{0}^{y} \int_{y}^{1} \frac{\alpha^{2}(z-y_{c})^{2} \cosh \alpha(z-y_{c})}{\sinh^{2} \alpha(z-y_{c})} dz dy_{c} dy \\ &\leq C \|g\|_{L^{1}}, \end{split}$$

where we used  $\frac{z^2\cosh z}{\sinh^2 z} \leq \frac{C_0}{e^{z/2}}$  for  $z \geq 0$ . By the interpolation, we get

$$\|\mathcal{L}_{2}^{1}(g)\|_{L^{p}} \leq C\|g\|_{L^{p}}.$$

This completes the proof of the lemma.

**Lemma 11.11.** There exists a constant C independent of  $\alpha$  such that for any  $p \in (1, \infty)$ ,

$$\|[\mathcal{L}_1, \rho](g')\|_{L^p} + \|\partial_c[\mathcal{L}_1, \rho](g)\|_{L^p} \le C\|g\|_{L^p}.$$

*Proof.* We have

$$\partial_{c}[\mathcal{L}_{1}, \rho]g = \mathcal{L}_{1}((\rho g)') + \mathcal{L}_{2}(\rho g) - \frac{1}{\rho}\mathcal{B}(\rho g)$$
$$- \rho'\mathcal{L}_{1}(g) - \rho\mathcal{L}_{1}(g') - \rho\mathcal{L}_{2}(g) + \mathcal{B}(g)$$
$$= [\mathcal{L}_{1}, \rho](g') + \mathcal{L}_{1}(\rho'g) + \mathcal{L}_{2}(\rho g) - \frac{1}{\rho}\mathcal{B}(\rho g)$$
$$- \rho'\mathcal{L}_{1}(g) - \rho\mathcal{L}_{2}(g) + \mathcal{B}(g).$$

Thus, by Lemma 11.9 and Lemma 11.10, it suffices to consider  $[\mathcal{L}_1, \rho](g')$ . Since

$$\begin{split} [\mathcal{L}_{1},\rho](g')(c) &= \int_{u(0)}^{u(1)} \frac{\int_{c}^{c'}(\rho(z)-\rho(c))g'(z)\phi_{1}(u^{-1}(z),c)dz}{(c'-c)^{2}} \frac{(u^{-1})'(c')}{\phi_{1}(u^{-1}(c'),c)^{2}} dc' \\ &= p.v. \int_{u(0)}^{u(1)} \frac{g(c')}{c'-c} \frac{(u(0)+u(1)-c'-c)(u^{-1})'(c')}{\phi_{1}(u^{-1}(c'),c)} dc' \\ &- \int_{u(0)}^{u(1)} \frac{\int_{c}^{c'}(\rho(z)-\rho(c))g(z)\partial_{z}\phi_{1}(u^{-1}(z),c)dz}{(c'-c)^{2}} \frac{(u^{-1})'(c')}{\phi_{1}(u^{-1}(c'),c)^{2}} dc' - \mathcal{L}_{1}(\rho'g) \\ &\triangleq \Theta_{1} + \Theta_{2} - \mathcal{L}_{1}(\rho'g). \end{split}$$

Let  $f(c',c) = (u(0) + u(1) - c' - c)(u^{-1})'(c')$ .  $\Theta_1$  can be rewritten as

$$p.v. \int_{u(0)}^{u(1)} \frac{g(c')}{c'-c} \frac{(u(0)+u(1)-c'-c)(u^{-1})'(c')}{\phi_1(u^{-1}(c'),c)} dc'$$

$$= (u(0)+u(1)-2c)(u^{-1})'(c)p.v. \int_{u(0)}^{u(1)} \frac{g(c')}{c'-c} dc'$$

$$- (u(0)+u(1)-2c)(u^{-1})'(c)p.v. \int_{D_0 \cap \{|c'-c|>\frac{1}{\alpha}\}} \frac{g(c')}{c'-c} dc'$$

$$+ \int_{u(0)}^{u(1)} \frac{g(c')}{c'-c} \frac{f(c',c)-f(c,c)}{\phi_1(u^{-1}(c'),c)} dc'$$

$$+ \int_{D_0 \cap \{|c'-c|>\frac{1}{\alpha}\}} \frac{g(c')}{c'-c} \frac{f(c,c)}{\phi_1(u^{-1}(c'),c)} dc'$$

$$- f(c,c) \int_{D_0 \cap \{|c'-c|<\frac{1}{\alpha}\}} \frac{g(c')}{c'-c} \frac{\phi_1(u^{-1}(c'),c)-1}{\phi_1(u^{-1}(c'),c)} dc'.$$

The  $L^p$  boundedness of the first two terms follows from  $L^p$  boundedness of Hilbert transform. The third term is trivial since  $|f(c',c)-f(c,c)| \leq C|c'-c|$ . The fourth term is bounded by

$$C \int_{[0,1] \cap \{|x-x_0| > \frac{c_0}{\alpha}\}} |g(u(y))| \frac{\alpha^2 |y-y_c|}{\sinh \alpha (y-y_c)} dy.$$

Thanks to  $K_1(z) = \alpha \frac{\alpha z}{\sinh \alpha z} \chi_{[c_0, +\infty]}(\alpha z) \in L^1$ , we get

$$\left\| \int_{D_0 \cap \{|c'-c| > \frac{1}{\alpha}\}} \frac{g(c')}{c'-c} \frac{f(c,c)}{\phi_1(u^{-1}(c'),c)} dc' \right\|_{L^p} \le C \||g \circ u| * K_1\|_{L^p}$$

$$\le C \|g\|_{L^p}.$$

The last term is bounded by

$$C \int_{D_0 \cap \{|c'-c| \le \frac{1}{\alpha}\}} g(c') \alpha^2 |c'-c| dc'$$

Thanks to  $K_2(z) = \alpha^2 z \chi_{[0,1]}(\alpha z) \in L^1$ , we get

$$\left\| \int_{D_0 \cap \{|c'-c| \le \frac{1}{\alpha}\}} \frac{g(c')}{c'-c} \frac{\phi_1(u^{-1}(c'),c) - 1}{\phi_1(u^{-1}(c'),c)} dc' \right\|_{L^p} \le C_0 \||g| * K_2 \|_{L^p}$$

$$\le C_0 \|g\|_{L^p}.$$

This shows that

$$\|\Theta_1\|_{L^p} \le C\|g\|_{L^p}.$$

For  $\Theta_2$  term, we have

$$|\Theta_2(c)| \le C\alpha \int_0^1 \frac{\int_{y_c}^y |g(u(z))| dz}{y - y_c} \frac{\alpha(y - y_c)}{\sinh \alpha(y - y_c)} dy,$$

from which, it follows that

$$\|\Theta_{2}(c)\|_{L^{\infty}} \leq C\alpha \int_{0}^{1} \frac{\int_{y_{c}}^{y} \|g\|_{L^{\infty}} dz}{y - y_{c}} \frac{\alpha(y - y_{c})}{\sinh \alpha(y - y_{c})} dy$$

$$\leq C\alpha \|g\|_{L^{\infty}} \int_{0}^{1} \frac{\alpha(y - y_{c})}{\sinh \alpha(y - y_{c})} dy$$

$$\leq C\|g\|_{L^{\infty}} \|\frac{z}{\sinh z}\|_{L^{1}} \leq C\|g\|_{L^{\infty}},$$

and

$$\begin{split} \|\Theta_{2}(c)\|_{L^{1}} &\leq C \int_{0}^{1} \alpha \int_{0}^{1} \frac{\int_{y_{c}}^{y} |g(u(z))| dz}{y - y_{c}} \frac{\alpha(y - y_{c})}{\sinh \alpha(y - y_{c})} dy dy_{c} \\ &\leq C \int_{0}^{1} |g(u(z))| dz \Big\| \int_{0}^{z} \int_{z}^{1} \frac{\alpha^{2}}{\sinh \alpha(y_{c} - y)} dy_{c} dy \Big\|_{L^{\infty}} \leq C \|g\|_{L^{1}}. \end{split}$$

By the interpolation, we get

$$\|\Theta_2(c)\|_{L^p} \le C\|g\|_{L^p}.$$

Thus, we obtain

$$\|[\mathcal{L}_1,\rho](g')\|_{L^p} \le \|\Theta_1(c)\|_{L^p} + \|\Theta_2(c)\|_{L^p} + \|\mathcal{L}_1(\rho'g)\|_{L^p} \le C\|g\|_{L^p}.$$

This completes the proof of the lemma.

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