Spacetimes of Weyl and Ricci type N in higher dimensions

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March 12, 2021

Abstract

We study the geometrical properties of null congruences generated by an aligned null direction of the Weyl tensor (WAND) in spacetimes of the Weyl and Ricci type N (possibly with a non-vanishing cosmological constant) in an arbitrary dimension. We prove that a type N Ricci tensor and a type III or N Weyl tensor have to be aligned. In such spacetimes, the multiple WAND has to be geodetic. For spacetimes with type N aligned Weyl and Ricci tensors, the canonical form of the optical matrix in the twisting and non-twisting case is derived and the dependence of the Weyl and the Ricci tensors and Ricci rotation coefficients on the affine parameter of the geodetic null congruence generated by the WAND is obtained.

1 Introduction and summary

In general, the Einstein equations and their various generalizations in higher dimensions are notoriously difficult to solve. Therefore, when studying various classes of spacetimes one has to resort to certain assumptions which lead to a considerable simplification of the field equations. One possibility is to make simplifying assumptions on the curvature of the spacetime. In particular, assuming an algebraically special spacetime (i.e. a spacetime that admits a multiple Weyl aligned null direction (mWAND)) in the Weyl alignment classification [1,2] (see also [3] for a recent review) leads to a substantial simplification of the field equations. Moreover, it is usually also assumed that the spacetime is Einstein (the Ricci tensor is proportional to the metric). Under these simplifying assumptions, various results of interest have been already obtained. For example, a generalization of the (necessary part of) the Goldberg-Sachs theorem was obtained in five dimension [4], which subsequently led to the determination of all the algebraically special vacuum solutions of the Einstein equations in five dimensions (including new solutions) [5], [6] and [7].

In this work, we want to go beyond the Einstein spaces and we set out to study Ricci type N[1] spacetimes (possibly with a non-vanishing cosmological constant added)[2], i.e. spacetimes with the Ricci tensor of the form

$$R_{ab} = \lambda g_{ab} + \eta k_a k_b, \tag{1.1}$$

where k_a is a null vector, η is a non-vanishing scalar ('radiation density') and λ is proportional to the cosmological constant.[3] Four-dimensional solutions of the Einstein equations with the Ricci tensor of the form (1.1) with $\lambda = 0$, representing spacetimes containing pure radiation (null dust, e.g. null Maxwell field), have already been studied in detail in the literature (see e.g. [8] and [9] and references therein). In the present paper, we study the geometrical properties of Weyl type N (and partly also of more general type III) spacetimes with the Ricci tensor of the form (1.1) in higher dimensions.

Our assumptions are geometrical and thus our results, apart from Einstein gravity, also hold in the context of generalized gravities, such as e.g. quadratic gravity. However, note that the matter content of spacetimes with the Ricci tensor (1.1) may be theory-dependent. Indeed, a Weyl and Ricci type N

[1] In this paper, algebraic types are considered to be genuine unless stated otherwise, thus, e.g. by 'type N' we mean type N not including type O.

[2] Thus, more precisely, in our case the traceless Ricci tensor is of type N.

[3] In fact, $\lambda = 2\Lambda/(d-2)$, where Λ is the cosmological constant. Note that the 'λ' term in the Ricci tensor (1.1) does not affect the Bianchi equations.

metric representing a pure radiation spacetime in Einstein gravity may represent a vacuum spacetime in quadratic gravity [10].

For Weyl type N and III spacetimes, there is a unique preferred null direction - a multiple Weyl aligned null direction (mWAND) $\boldsymbol{\ell}$. Since, in principle, the Ricci tensor (1.1) defines another null direction $\boldsymbol{k}$, there may be two distinct prefered null directions (i.e., the Weyl and the Ricci tensor may be non-aligned). However, in section 3, we show that *for Weyl type III and N spacetimes, the Ricci tensor (1.1) is necessarily aligned with the Weyl tensor*[4] (i.e. $\boldsymbol{k} \propto \boldsymbol{\ell}$,[5] see theorem 3.1), thus generalizing an earlier result by Wils [13] which was obtained in four dimensions.

We then proceed with studying the geodeticity of $\boldsymbol{\ell}$, arriving at the conclusion that *for a Weyl type III and N spacetime with the Ricci tensor of the form* (1.1), *the common aligned null direction of the Weyl and the Ricci tensor is geodetic* (theorem 3.2).

A next natural step, which we take in sections 4, 5, is to study geometrical properties of this preferred geodetic null direction, which are encoded in the optical matrix ρ_{ij} (introduced in section 2.2). However, note that here we limit ourselves to the type N case since the type III case has not yet been completely resolved even in the simpler case of Einstein spacetimes (see [14, 15] and [3] for partial results).

For Einstein type N spacetimes, the canonical form of the optical matrix is [14–16]

$$\boldsymbol{\rho} = s \left(\begin{array}{cc|c} 1 & a & \multirow{2}{*}{$\mathbf{0}$} \\ -a & 1 & \\ \hline \multicolumn{2}{c|}{\mathbf{0}} & \mathbf{0} \end{array} \right). \tag{1.2}$$

In this case, the optical matrix obeys the so-called optical constraint $\rho_{ik}\rho_{jk} \propto \rho_{(ij)}$. Note that for Einstein spacetimes, the optical constraint also holds in much more general situations (e.g. for Kerr-Schild metrics [10, 17], non-degenerate geodetic double WANDs in asymptotically flat type II vacuum spacetimes [18], 'general' non-twisting type II Einstein spacetimes [19], etc). This result was used in [17] and [20] to integrate certain Bianchi identities to determine the r-dependence (where r is an affine parameter of the null congruence corresponding to the geodetic multiple WAND) of the curvature and to study asymptotic properties of such spacetimes.

We show that in contrast to the type N Einstein case, the optical constraint does not hold if the the term $\eta k_a k_b$ is present in the Ricci tensor (unless, trivially, the spacetime is Kundt, i.e. $\boldsymbol{\rho} = \mathbf{0}$). Nevertheless, the symmetric and skew-symmetric parts of the optical matrix ρ_{ij} still commute and thus ρ_{ij} is a normal[6] matrix. Similarly to the Einstein case, we find that the rank of ρ_{ij} is at most 2, however, the 2×2 block is now shearing and thus the canonical form of ρ_{ij} is (theorems 4.1 and 5.4)

$$\boldsymbol{\rho} = s \left(\begin{array}{cc|c} 1 & a & \multirow{2}{*}{$\mathbf{0}$} \\ -a & b & \\ \hline \multicolumn{2}{c|}{\mathbf{0}} & \mathbf{0} \end{array} \right). \tag{1.3}$$

For the conformally flat case, the optical matrix vanishes completely (corollary 5.3). Let us summarize possible subcases in table 1.

rank of ρ_{ij}	conditions	spacetime
0	$s = 0$	Kundt
1	$s \neq 0,\ b = 0$	non-twisting
2	$s \neq 0,\ b = 1$	Einstein
2	$s \neq 0,\ b \neq 0, 1,\ a = 0$	non-twisting
2	$s \neq 0,\ b \neq 0, 1,\ a \neq 0$	twisting

Table 1: Possible subcases of the optical matrix (1.3) for spacetimes of the Weyl type N and the Ricci tensor of the form (1.1). Note that the first ($s = 0$) case also allows the Weyl type O.

For a normal optical matrix ρ_{ij}, the Sachs equation (see (6.14)), which determines the evolution of the optical matrix along the multiple WAND $\boldsymbol{\ell}$, can be integrated [20]. Thus, in section 6, the r-dependence

[4]For brevity, whenever the Weyl and the traceless Ricci tensors are (non)aligned, we say that the spacetime is (non)aligned.

[5]Thus, for Weyl and Ricci type N spacetimes, there is always a frame where both the Weyl and the Ricci tensors admit only boost weight (-2) components. This implies that all rank-2 tensors constructed from the Weyl and the Ricci tensor that are at least quadratic in the Weyl/Ricci tensor vanish. This implies that apart from the Einstein equations, these metrics also solve the field equations of various generalizations of the Einstein theory, such as Gauss-Bonnet or Lovelock gravity, with a type N stress energy tensor ($T_{ab} \propto \ell_a \ell_b$), see e.g. [11] and [12] for higher-dimensional Weyl and Ricci type N spacetimes as solutions of generalized gravity theories.

[6] Real normal matrix A satisfies $A^T A = A A^T$.

(where r is the affine parameter along $\boldsymbol{\ell}$) of the 'radiation density' and the Weyl tensor, as well as some Ricci rotation coefficients, is determined.

Finally, we briefly discuss a direct product of an algebraically special aligned Ricci type N spacetime and a Euclidean space. For the resulting spacetime, we show that it is again an algebraically special spacetime of Ricci type N with the same WAND as that of the original spacetime. It turns out that the resulting spacetime does not change its primary Weyl type, unless it is conformally flat. We also show the connection between the geometry of the congruence generated by the multiple WAND in the original and the resulting spacetime. In particular, the resulting spacetime is necessarily shearing, unless the original spacetime is Kundt. The four-dimensional pure radiation metrics given in table 3 can be used as seed metrics to construct higher-dimensional spacetimes of Weyl and Ricci type N with various optical properties (sec. 7).

2 Preliminaries

2.1 Null frames and Lorentz transformations

Let d denote the dimension of a spacetime. In a d-dimensional spacetime, we shall consider a *null frame* $\{\mathbf{e}_{(a)}\}$, i.e. a real frame

$$\{\boldsymbol{\ell} \equiv \mathbf{e}_{(0)} = \mathbf{e}^{(1)},\ \boldsymbol{n} \equiv \mathbf{e}_{(1)} = \mathbf{e}^{(0)},\ \boldsymbol{m}_{(i)} \equiv \mathbf{e}_{(i)} = \mathbf{e}^{(i)}\} \tag{2.1}$$

with two null vector fields $\boldsymbol{\ell}$ and $\boldsymbol{n}$ and $d-2$ spacelike vector fields $\boldsymbol{m}_{(i)}$ such that they satisfy the following relations

$$\ell_a \ell^a = n_a n^a = \ell_a m^a_{(i)} = n_a m^a_{(i)} = 0, \qquad \ell_a n^a = 1, \qquad m^{(i)}_a m^a_{(j)} = \delta^i_j. \tag{2.2}$$

The metric expressed in terms of dual null frame vectors thus reads

$$g_{ab} = 2\ell_{(a} n_{b)} + \delta_{ij} m^{(i)}_a m^{(j)}_b. \tag{2.3}$$

Throughout the paper, we make use of two types of indices: indices $a, b, \ldots$ take values $0, \ldots, d-1$, while indices $i, j, \ldots$ take values $2, \ldots, d-1$, unless stated otherwise. We employ the Einstein summation convention for both types of indices, however, since the frame indices $i, j, \ldots$ are raised and lowered by δ_{ij}, we do not distinguish between the covariant and contravariant null frame tensor components corresponding to indices $i, j, \ldots$. When we want to emphasize that there is no summation over repeated indices, we put them both in brackets.

Relations (2.2) are preserved under local Lorentz transformations acting on a tangent space. Every proper orthochronous Lorentz transformation can be decomposed into *boost* with a positive function λ

$$\boldsymbol{\ell} \mapsto \lambda \boldsymbol{\ell}, \qquad \boldsymbol{n} \mapsto \lambda^{-1} \boldsymbol{n}, \qquad \boldsymbol{m}_{(i)} \mapsto \boldsymbol{m}_{(i)}, \tag{2.4}$$

spin determined by $X_{ij} \in SO(d-2)$ at a given point of the spacetime

$$\boldsymbol{\ell} \mapsto \boldsymbol{\ell}, \qquad \boldsymbol{n} \mapsto \boldsymbol{n}, \qquad \boldsymbol{m}_{(i)} \mapsto X_{ij} \boldsymbol{m}_{(j)}, \tag{2.5}$$

and *null rotations* about vectors $\boldsymbol{\ell}$ or $\boldsymbol{n}$ determined by a set of (real) functions z_i

$$\begin{aligned} &\boldsymbol{\ell} \mapsto \boldsymbol{\ell}, \qquad \boldsymbol{n} \mapsto \boldsymbol{n} + z^i \boldsymbol{m}_{(i)} - \frac{1}{2} z^i z_i \boldsymbol{\ell}, \qquad \boldsymbol{m}_{(i)} \mapsto \boldsymbol{m}_{(i)} - z_i \boldsymbol{\ell}, \\ &\boldsymbol{n} \mapsto \boldsymbol{n}, \qquad \boldsymbol{\ell} \mapsto \boldsymbol{\ell} + z^i \boldsymbol{m}_{(i)} - \frac{1}{2} z^i z_i \boldsymbol{n}, \qquad \boldsymbol{m}_{(i)} \mapsto \boldsymbol{m}_{(i)} - z_i \boldsymbol{n}, \end{aligned} \tag{2.6}$$

respectively.

2.2 Optical matrix and optical scalars

Consider a covariant derivative $L_{ab} \equiv \nabla_b \ell_a$ of the null frame vector $\boldsymbol{\ell}$. Projecting L_{ab} on the frame vectors $\mathbf{e}_{(a)}$, one obtains its null frame components

$$L_{(a)(b)} \equiv L_{cd} e^c_{(a)} e^d_{(b)}. \tag{2.7}$$

The vector field $\boldsymbol{\ell}$ is tangent to a geodetic null congruence if and only if $\kappa_i \equiv L_{(i)(0)} = 0$. Then, such geodetic congruence can be always affinely parametrized, i.e. $L_{(1)(0)} = 0$.

Similarly, one can project covariant derivatives of the remaining frame vectors,

$$N_{ab} = \nabla_b n_a, \qquad \overset{(i)}{M}_{ab} = \nabla_b m^{(i)}_a, \tag{2.8}$$

into the basis (2.1) to obtain the scalars $N_{(a)(b)}$, $\overset{(i)}{M}_{(a)(b)}$. From (2.2), these scalars satisfy the identities

$$N_{(0)(a)} + L_{(1)(a)} = 0, \quad \overset{(i)}{M}_{(0)(a)} + L_{(i)(a)} = 0, \quad \overset{(i)}{M}_{(1)(a)} + N_{(i)(a)} = 0, \quad \overset{(i)}{M}_{(j)(a)} + \overset{(j)}{M}_{(i)(a)} = 0, \tag{2.9}$$

and

$$L_{(0)(a)} = N_{(1)(a)} = \overset{(i)}{M}_{(i)(a)} = 0. \tag{2.10}$$

For a geodetic null field $\boldsymbol{\ell}$, the so called optical matrix $\rho_{ij} \equiv L_{(i)(j)}$ contains information about certain geometric properties of the null congruence, namely, one can define the *optical scalars* σ (*shear*), θ (*expansion*) and A (*twist*) of $\boldsymbol{\rho}$ in the following way:

$$\sigma^2 \equiv \sigma_{ij}\sigma^{ij}, \qquad \theta \equiv \frac{1}{d-2}\rho^i{}_i, \qquad A^2 \equiv A_{ij}A^{ij}, \tag{2.11}$$

where $\sigma_{ij} \equiv S_{ij} - \theta\delta_{ij}$ is the traceless part of the symmetric part $\boldsymbol{S}$ of $\boldsymbol{\rho}$ and $\boldsymbol{A}$ is the skew-symmetric part of $\boldsymbol{\rho}$. We say that a spacetime is non-twisting or twisting if $A_{ij} = 0$ or $A_{ij} \neq 0$, respectively.

2.3 GHP formalism

Throughout the paper, we use some of the features of the higher-dimensional GerochHeldPenrose (GHP) formalism that was developed in [16]. The basic notion of GHP formalism is a *GHP scalar*. We say that an object $T_{i_1\ldots i_s}$ is a GHP scalar of spin weight s and boost weight b if it transforms as

$$T_{i_1\ldots i_s} \mapsto X_{i_1 j_1} \ldots X_{i_s j_s} T_{j_1 \ldots j_s} \tag{2.12}$$

under spins X_{ij} and as

$$T_{i_1\ldots i_s} \mapsto \lambda^b T_{i_1\ldots i_s} \tag{2.13}$$

under boosts λ. Note that, for example, the frame components

$$\omega' \equiv R_{(1)(1)} \equiv R_{ab} n^a n^b, \tag{2.14}$$

$$\omega \equiv R_{(0)(0)} \equiv R_{ab} \ell^a \ell^b, \tag{2.15}$$

of the Ricci tensor R_{ab} are GHP scalars of spin and boost weights 0, (-2) and 0, $(+2)$, respectively, while the matrix

$$\Omega'_{ij} \equiv C_{(1)(i)(1)(j)} \equiv C_{abcd} n^a m^b_{(i)} n^c m^d_{(j)} \tag{2.16}$$

of frame components of the Weyl tensor is a GHP scalar of spin weight 2 and boost weight (-2).

For GHP scalars,[7] we introduce the notation following [16], shown in table 2.

Quantity	Notation	Boost weight b	Spin s	Interpretation
L_{ij}	ρ_{ij}	1	2	expansion, shear and twist of $\boldsymbol{\ell}$
L_{ii}	$\rho = \rho_{ii}$	1	0	expansion of $\boldsymbol{\ell}$
L_{i0}	κ_i	2	1	non-geodesity of $\boldsymbol{\ell}$
L_{i1}	τ_i	0	1	transport of $\boldsymbol{\ell}$ along $\boldsymbol{n}$
N_{ij}	ρ'_{ij}	-1	2	expansion, shear and twist of $\boldsymbol{n}$
N_{ii}	$\rho' = \rho'_{ii}$	-1	0	expansion of $\boldsymbol{n}$
N_{i1}	κ'_i	-2	1	non-geodesity of $\boldsymbol{n}$
N_{i0}	τ'_i	0	1	transport of $\boldsymbol{n}$ along $\boldsymbol{\ell}$

Table 2: A higher-dimensional generalization of GHP scalars corresponding to the Ricci rotation coefficients [16].

[7] For brevity, we omit brackets in the scalar indices.

In type III spacetimes, there is a frame such that the type III Weyl tensor has the form[8]

$$C_{abcd} = 8\Psi'_i \ell_{\{a} n_b \ell_c m^{(i)}_{d\}} + 4\Psi'_{ijk} \ell_{\{a} m^{(i)}_b m^{(j)}_c m^{(k)}_{d\}} + 4\Omega'_{ij} \ell_{\{a} m^{(i)}_b \ell_c m^{(j)}_{d\}}. \tag{2.17}$$

where $\Psi'_{ijk} = -\Psi'_{ikj}$, $\Psi'_i = \Psi'_{jij}$ and $\Psi'_{[ijk]} = 0$. For the type N Weyl tensor, only terms with Ω'_{ij} are non-vanishing, i.e.

$$C_{abcd} = \Omega'_{ij} \ell_{\{a} m^{(i)}_b \ell_c m^{(j)}_{d\}}, \tag{2.18}$$

with Ω'_{ij} being symmetric and traceless.

Further, one can define a *GHP derivative operators* $\text{þ}, \text{þ}'$ and $\eth_i$ acting on a GHP scalar $\boldsymbol{T}$ of boost weight b and spin s and creating again a GHP scalar (see (2.15)–(2.17) in [16]):

$$\text{þ} T_{i_1 i_2 \dots i_s} \equiv D T_{i_1 i_2 \dots i_s} - b L_{(1)(0)} T_{i_1 i_2 \dots i_s} + \sum_{r=1}^{s} \overset{(k)}{M}_{(i_r)(0)} T_{i_1 \dots i_{r-1} k i_{r+1} \dots i_s}, \tag{2.19}$$

$$\text{þ}' T_{i_1 i_2 \dots i_s} \equiv \Delta T_{i_1 i_2 \dots i_s} - b L_{(1)(1)} T_{i_1 i_2 \dots i_s} + \sum_{r=1}^{s} \overset{(k)}{M}_{(i_r)(1)} T_{i_1 \dots i_{r-1} k i_{r+1} \dots i_s}, \tag{2.20}$$

$$\eth_i T_{j_1 j_2 \dots j_s} \equiv \delta_i T_{j_1 j_2 \dots j_s} - b L_{(1)(i)} T_{j_1 j_2 \dots j_s} + \sum_{r=1}^{s} \overset{(k)}{M}_{(j_r)(i)} T_{j_1 \dots j_{r-1} k j_{r+1} \dots j_s}, \tag{2.21}$$

where the derivatives along the frame vectors are defined as

$$D \equiv \ell^a \nabla_a, \qquad \triangle \equiv n^a \nabla_a, \qquad \delta_i \equiv m^a_{(i)} \nabla_a. \tag{2.22}$$

For more details and for other features of the higher-dimensional GHP formalism, see [16].

In this paper, projections of the Bianchi identity

$$\nabla_{[a} R_{bc]de} = 0 \tag{2.23}$$

onto the null frame (2.1) ([16], see also [14]) are used. The complete set of all the independent null frame components of the Bianchi identity for algebraically special Einstein spacetimes using the GHP notation was given in [16]. The Bianchi equations with the non-trivial Ricci tensor are obtained by simple replacement (2.43)–(2.48) in [16].

Now, one has to distinguish between an *aligned* and a *non-aligned* case. In the aligned case, one has $\langle \boldsymbol{\ell} \rangle = \langle \boldsymbol{k} \rangle$[9], i.e. the Weyl and the Ricci tensors have a common aligned null direction. Otherwise, we say that the Weyl and the Ricci tensors are non-aligned. If this is the case, one can further choose the null frame vector $\boldsymbol{n}$ such that $\boldsymbol{n} \propto \boldsymbol{k}$.

For the *non-aligned* Ricci tensor (1.1), where $\boldsymbol{n} = \boldsymbol{k}$, the only necessary replacement in the Bianchi equations is (2.43) in [16], i.e.

$$\Omega_{ij} \to \Omega_{ij} + \tilde{\omega} \delta_{ij}. \tag{2.24}$$

Here, $\tilde{\omega}$ is defined as

$$\tilde{\omega} \equiv \frac{\eta}{d-2}. \tag{2.25}$$

The Bianchi equations (B2), (B3) and (B4) in [16] then read

$$\text{þ}' \tilde{\omega} \delta_{ij} = -(\Psi'_j \delta_{ik} - \Psi'_{jik}) \kappa_k - \tilde{\omega} \delta_{ik} \rho'_{kj}, \tag{2.26}$$

$$\Psi'_{[i|kl|} \kappa_{j]} + \Psi'_{[k|ij|} \kappa_{l]} + \tilde{\omega} \delta_{i[k} \rho'_{|j|l]} - \tilde{\omega} \delta_{j[k} \rho'_{|i|l]} = 0, \tag{2.27}$$

$$\tilde{\omega} \delta_{i[j} \rho'_{kl]} = 0, \tag{2.28}$$

respectively. Apart from the Bianchi equations, we will also employ equations (2.50) and (2.51) of [16], which for the Ricci tensor of the form (1.1) reduce to

$$\text{þ}' \omega = -\rho' \omega, \tag{2.29}$$

$$\kappa'_i \omega = 0, \tag{2.30}$$

where ρ' denotes the trace of ρ'_{ij} and $\omega = \eta$.

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[8] The operation {} is defined as $2T_{\{abcd\}} \equiv T_{[ab][cd]} + T_{[cd][ab]}$.

[9] Here, $\langle \boldsymbol{\ell} \rangle$ denotes the equivalence class of vectors having the same direction as $\boldsymbol{\ell}$, i.e. $\langle \boldsymbol{\ell} \rangle = \langle \boldsymbol{k} \rangle$ iff $\boldsymbol{\ell} \propto \boldsymbol{k}$.

For the *aligned* Ricci (1.1) and Weyl (2.18) tensors, where $\boldsymbol{\ell} = \boldsymbol{k}$, the only necessary replacement in the Bianchi equations is (2.43') in [16], i.e.

$$\Omega'_{ij} \to \Omega'_{ij} + \tilde{\omega}\delta_{ij}. \tag{2.31}$$

We will employ equations (B2'), (B3') and (B4') of [16]

$$\text{þ}\Omega'_{ij} + \text{þ}\tilde{\omega}\delta_{ij} = -\Omega'_{ik}\rho_{kj} - \tilde{\omega}\rho_{ij}, \tag{2.32}$$

$$\Omega'_{i[k}\rho_{|j|l]} + \tilde{\omega}\delta_{i[k}\rho_{|j|l]} = \Omega'_{j[k}\rho_{|i|l]} + \tilde{\omega}\delta_{j[k}\rho_{|i|l]}, \tag{2.33}$$

$$\Omega'_{i[j}\rho_{kl]} + \tilde{\omega}\delta_{i[j}\rho_{kl]} = 0 \tag{2.34}$$

and (2.50') and (2.51) in [16]

$$\text{þ}\omega' = -\rho\omega', \tag{2.35}$$

$$\kappa_i\omega' = 0, \tag{2.36}$$

where $\omega' = \eta$.

3 Alignment and geodeticity for Weyl type III and N, Ricci type N spacetimes

In [13], it is proven that non-aligned pure radiation spacetimes of the Weyl type III and N do not exist in four dimensions (see [13], Theorem 3). It turns out that this is also true in arbitrary dimension.

Proposition 3.1. *Weyl type III and N spacetimes with the Ricci tensor of the form* (1.1) *are necessarily aligned.*

Proof. Thanks to Theorem 3 of [13], it is sufficient to prove the assertion for $d > 4$. Let $\boldsymbol{\ell}$ be a multiple WAND of a spacetime. Suppose that $\boldsymbol{k}$ is the aligned null direction of the Ricci tensor such that $\langle \boldsymbol{k} \rangle \neq \langle \boldsymbol{\ell} \rangle$. Then, one can consider a null frame $\{\mathbf{e}_{(a)}\}$ with $\mathbf{e}_{(0)} \equiv \boldsymbol{\ell}$ and $\mathbf{e}_{(1)} \equiv \boldsymbol{k}$. In this null frame, the Ricci tensor has the form (1.1) (with non-vanishing η), while the Weyl tensor takes the form (2.17).

First, by contracting equation (2.28) with respect to i and j, one obtains

$$(d-4)\rho'_{[kl]} = 0, \tag{3.1}$$

i.e. $\boldsymbol{k}$ is non-twisting for every $d > 4$ (and thus also geodetic).[10] Now, we will prove that the optical matrix ρ'_{ij} associated with $\boldsymbol{k}$ in fact vanishes. Substituting (2.29) into (2.26), one has that

$$-\rho'\tilde{\omega}\delta_{ij} = -\Psi'_j\kappa_i + \Psi'_{jik}\kappa^k - \tilde{\omega}\rho'_{ij}. \tag{3.2}$$

Then, taking the trace of (3.2), one arrives at

$$2\Psi'_i\kappa^i = (d-3)\rho'\tilde{\omega}, \tag{3.3}$$

while the contraction of (3.2) with $\kappa^i\kappa^j$ gives

$$\rho'\tilde{\omega}\kappa_i\kappa^i = \Psi'_j\kappa^j\kappa_i\kappa^i - \Psi'_{jik}\kappa^i\kappa^j\kappa^k + \tilde{\omega}\rho'_{ij}\kappa^i\kappa^j. \tag{3.4}$$

Using (3.3) and $\Psi'_{ijk} = -\Psi'_{ikj}$, equation (3.4) simplifies to

$$2\rho'_{ij}\kappa^i\kappa^j = -(d-5)\rho'\kappa_i\kappa^i. \tag{3.5}$$

Tracing equation (2.27) in i and k leads to

$$2\Psi'_{(l}\kappa_{j)} + 2\Psi'_{(j|k|l)}\kappa^k - (d-4)\tilde{\omega}\rho'_{jl} - \tilde{\omega}\rho'\delta_{jl} = 0. \tag{3.6}$$

Contracting (3.6) with $\kappa^j\kappa^l$, one obtains

$$2\Psi'_l\kappa^l\kappa_j\kappa^j - (d-4)\tilde{\omega}\rho'_{jl}\kappa^j\kappa^l - \tilde{\omega}\rho'\kappa_j\kappa^j = 0. \tag{3.7}$$

[10] For a non-twisting null congruence $\boldsymbol{k}$, one can introduce a foliation u =const., such that $k_a = u_{,a}$. Then $k_{a;b}k^b = u_{;ab}k^b = u_{;ba}k^b = k_{b;a}k^b = 0$. In the non-aligned case, this can be also seen directly from (2.30).

Finally, substituting (3.3) and (3.5) into (3.7), one obtains

$$(d-3)(d-4)\tilde{\omega}\rho'\kappa_i\kappa^i = 0, \tag{3.8}$$

thus, either $\rho' = 0$, i.e. $\boldsymbol{k}$ is non-expanding, and Proposition 1 in [21] implies $\rho'_{ij} = 0$, or $\kappa_i = 0$, i.e. $\boldsymbol{\ell}$ is geodetic, and equation (3.6) reduces to

$$(d-4)\tilde{\omega}\rho'_{jl} + \tilde{\omega}\rho'\delta_{jl} = 0. \tag{3.9}$$

Its trace implies $\rho' = 0$. Thus, we again arrive at $\rho'_{ij} = 0$. Since assumptions of Proposition 2 of [21] are met, $\boldsymbol{k}$ must be a multiple WAND. However, since we assume that the spacetime is of Weyl type III or N with (the only) multiple WAND $\boldsymbol{\ell}$, one has that necessarily $\langle \boldsymbol{k} \rangle = \langle \boldsymbol{\ell} \rangle$, which is a contradiction. ■

Thus, the type III or N Weyl tensor and the Ricci tensor (1.1) are always aligned, and, without loss of generality, we can set $\boldsymbol{k} = \boldsymbol{\ell}$. The boost weight (-2) component (2.14) of the Ricci tensor (1.1) is then $\omega' = \eta$.

Let us study geodeticity of the aligned direction $\boldsymbol{\ell}$. Equation (2.36) immediately implies $\kappa_i = 0$ and thus $\boldsymbol{\ell}$ is geodetic. Therefore

Proposition 3.2. *In a Weyl type III or N spacetime with the Ricci tensor of the form* (1.1)*, the common aligned null direction of the Weyl and the Ricci tensors is geodetic.*

In the rest of the paper, we focus solely on Weyl type N spacetimes with the Ricci tensor of the form (1.1).

Recall that $\boldsymbol{S}$ and $\boldsymbol{A}$ denote the symmetric and skew-symmetric parts of $\boldsymbol{\rho}$, respectively. Before we proceed any further, we give the following characterization of the non-twisting spacetimes of the Weyl N with the Ricci tensor of the form (1.1).

Lemma 3.3. *A spacetime of the Weyl type N with the Ricci tensor of the form* (1.1) *is non-twisting if and only if its* $\boldsymbol{\Omega}'$ *and* $\boldsymbol{S}$ *commute.*

Proof. By taking the trace of both sides of equation (2.34) in i and k, we obtain the matrix relation

$$\boldsymbol{\Omega}'\boldsymbol{A} + \boldsymbol{A}\boldsymbol{\Omega}' = (d-4)\tilde{\omega}\boldsymbol{A}. \tag{3.10}$$

The skew-symmetric part of equation (2.32) reads

$$0 = [\boldsymbol{\Omega}', \boldsymbol{S}] + \boldsymbol{\Omega}'\boldsymbol{A} + \boldsymbol{A}\boldsymbol{\Omega}' + 2\tilde{\omega}\boldsymbol{A}. \tag{3.11}$$

Using (3.10), this equation further simplifies to

$$\omega'\boldsymbol{A} = -[\boldsymbol{\Omega}', \boldsymbol{S}]. \tag{3.12}$$

Since ω' is non-vanishing, this proves our assertion. ■

We see that, in contrast to the Einstein spacetimes of Weyl type N, where $\boldsymbol{\Omega}'$ and $\boldsymbol{S}$ always commute (see [16]), in spacetimes of Weyl N with the Ricci tensor of the form (1.1), $\boldsymbol{\Omega}'$ and $\boldsymbol{S}$ can be simultaneously diagonalized by Lorentz spins only in the non-twisting case. Thus, it is convenient to treat the twisting and the non-twisting case separately.

4 Twist-free spacetimes

In this section, we consider spacetimes with a non-twisting multiple WAND. In [19], the canonical form of the optical matrix corresponding to a non-twisting WAND was obtained for algebraically special Einstein spacetimes of dimension $d \geq 5$ (for results on five dimensions, see also [4] and for type III and N, see [14]). It turned out that for these spacetimes, the optical matrix has at least one double eigenvalue, unless they are conformally flat. However, we observe that for non-twisting spacetimes of Weyl type N and with the Ricci tensor of the form (1.1), this is not the case.

Proposition 4.1. *Let* $\boldsymbol{\ell}$ *be a multiple WAND of a twist-free spacetime of the Weyl type N with the Ricci tensor of the form* (1.1)*. Then the canonical form of the corresponding optical matrix is*

$$\boldsymbol{\rho} = s \left(\begin{array}{cc|c} 1 & 0 & \multirow{2}{*}{$\boldsymbol{0}$} \\ 0 & b & \\ \hline \multicolumn{2}{c|}{\boldsymbol{0}} & \boldsymbol{0} \end{array} \right). \tag{4.1}$$

where $b \neq 1$*, otherwise the spacetime is Einstein,*[11] *for* $s = 0$*, it is Kundt.*

[11] It follows from (4.9) that $b = 1$ iff $\tilde{\omega} = 0 = R_{11}$ and thus an Einstein spacetime corresponds to $b = 1$.

Note that $\boldsymbol{\rho}$ (4.1) is a normal matrix, not satisfying the optical constraint.

Proof. Let $d > 4$. From lemma 3.3 we know that $\boldsymbol{\Omega}'$ and $\boldsymbol{S}$ commute. Thus, the null frame $\{\mathbf{e}_{(a)}\}$ can be transformed by Lorentz spin (2.5) to a null frame $\{\hat{\mathbf{e}}_{(a)}\}$ such that the matrices $\hat{\boldsymbol{\Omega}}'$, $\hat{\boldsymbol{S}}$ of the null frame components in $\{\hat{\mathbf{e}}_{(a)}\}$ are simultaneously diagonal. For the sake of clarity, let us denote $\hat{\boldsymbol{\Omega}}'$ and $\hat{\boldsymbol{S}}$ again by $\boldsymbol{\Omega}'$ and $\boldsymbol{S}$, respectively. Thus $\boldsymbol{\Omega}'$ and $\boldsymbol{S}$ take the form $\boldsymbol{\Omega}' = \mathrm{diag}(\Omega'_2, \dots, \Omega'_{d-1})$ and $\boldsymbol{S} = \mathrm{diag}(S_2, \dots, S_{d-1})$. Taking the trace of both sides of (2.33) in i and k, we obtain

$$\rho\left(\boldsymbol{\Omega}' + \tilde{\omega}\boldsymbol{I}\right) = \boldsymbol{\Omega}'\boldsymbol{\rho} + \boldsymbol{\rho}\boldsymbol{\Omega}' - (d-4)\tilde{\omega}\boldsymbol{\rho}. \tag{4.2}$$

Decomposing $\boldsymbol{\rho}$ into its symmetric and skew-symmetric part, $\boldsymbol{\rho} = \boldsymbol{S} + \boldsymbol{A}$, and using (3.10) one obtains

$$\rho\left(\boldsymbol{\Omega}' + \tilde{\omega}\boldsymbol{I}\right) = \boldsymbol{\Omega}'\boldsymbol{S} + \boldsymbol{S}\boldsymbol{\Omega}' - (d-4)\tilde{\omega}\boldsymbol{S}. \tag{4.3}$$

Hence from (4.3) we obtain the following relation between S_i and Ω'_i:

$$S_i\left(2\Omega'_i - (d-4)\tilde{\omega}\right) = \rho(\Omega'_i + \tilde{\omega}). \tag{4.4}$$

Firstly, we will prove that $2\Omega'_i \neq (d-4)\tilde{\omega}$ for all i, unless $\boldsymbol{\rho} = \mathbf{0}$. Now, assume that there exists i such that $2\Omega'_i = (d-4)\tilde{\omega}$. Then, using (2.35), we obtain

$$\text{þ}\Omega'_i = -\frac{d-4}{2}\tilde{\omega}\rho. \tag{4.5}$$

However, equation (4.4) implies that $\rho = 0$ and thus both þΩ'_i and þ$\tilde{\omega}$ vanish. Hence the Bianchi equation (2.32) reduces to (no summation)

$$\Omega'_{(i)}S_{(i)} + \tilde{\omega}S_i = 0, \tag{4.6}$$

which, using the relation for Ω'_i, immediately implies that $S_i = 0$ for such i. Now, considering equation (2.33) for the choice of indices $i = k$, $i \neq j$, $i \neq l$ and $j = l$, one has

$$(\Omega'_k + \tilde{\omega})S_j = -(\Omega'_j + \tilde{\omega})S_k \tag{4.7}$$

for all $j \neq i$. Thus for $k = i$, one obtains that $(\Omega'_i + \tilde{\omega})S_j = 0$ for all $j \neq i$. This implies that $S_j = 0$ for all $j \neq i$ and consequently $\boldsymbol{\rho} = \mathbf{0}$, i.e. the spacetime belongs to the Kundt class.

Let us discuss the case $2\Omega'_i \neq (d-4)\tilde{\omega}$ for all i. In this case, it is possible to express S_i using relation (4.4) as

$$S_i = \rho\frac{\Omega'_i + \tilde{\omega}}{2\Omega'_i - (d-4)\tilde{\omega}}. \tag{4.8}$$

Now, one can distinguish between four possible scenarios which lead to three qualitatively different canonical forms of $\boldsymbol{\rho}$. Namely, these are:

(i) $\Omega'_i = -\tilde{\omega}$ for all i;

(ii) There is exactly one i such that $\Omega'_i \neq -\tilde{\omega}$;

(iii) There are exactly two i such that $\Omega'_i \neq -\tilde{\omega}$;

(iv) There are at least three i such that $\Omega'_i \neq -\tilde{\omega}$;

One immediately obtains that scenario (i) leads to $\Omega'_i = 0 = \tilde{\omega}$ thanks to $\Omega'_{ii} = 0$ and this contradicts our assumption of non-vanishing $\tilde{\omega}$.

Scenario (ii) leads to one possible non-vanishing component S_i of $\boldsymbol{S}$ and since $S_j = 0$ for all $j \neq i$, we obtain that necessarily $S_i = \rho$ (with $\Omega'_i = (d-3)\tilde{\omega}$).

The third scenario leads to the possibility of two non-vanishing components, say S_2 and S_3, of $\boldsymbol{S}$. In addition, one can use expression (4.8) to verify that $S_2 \neq S_3$, otherwise $\boldsymbol{\rho} = \mathbf{0}$. In fact, from (4.8) (also using $\Omega'_{ii} = 0$) one can determine that

$$S_3 = S_2\frac{\Omega'_2 - (d-3)\tilde{\omega}}{\Omega'_2 + \tilde{\omega}}. \tag{4.9}$$

It remains to show that scenario (iv) leads to the vanishing optical matrix $\boldsymbol{\rho}$. Denote I the set of all indices i such that $\Omega'_i \neq -\tilde{\omega}$. Then, using (4.8), equation (4.7) implies that

$$\rho\left[2\Omega'_i - (d-4)\tilde{\omega}\right] = -\rho\left[2\Omega'_j - (d-4)\tilde{\omega}\right] \tag{4.10}$$

for all $i, j \in I,\ i \neq j$. Since $\rho = 0$ leads to $\boldsymbol{\rho} = \mathbf{0}$, assume that ρ is non-vanishing. Then

$$\Omega'_i + \Omega'_j = (d-4)\tilde{\omega} \tag{4.11}$$

for all $i, j \in I,\ i \neq j$. This immediately implies that $2\Omega'_i = (d-4)\tilde{\omega}$ for all $i \in I$, which contradicts our assumption. Thus $\rho = 0$ and consequently $\boldsymbol{\rho} = \mathbf{0}$. This completes the proof for the case $d > 4$. The proof of the case $d = 4$ is similar and one obtains that $S_3 = S_2(\Omega'_2 - \tilde{\omega})/(\Omega'_2 + \tilde{\omega})$ or $\rho_{ij} = 0$. ■

Also note that for spacetimes with $\boldsymbol{\Omega}' = \mathbf{0}$ and non-vanishing $\tilde{\omega}$, equation (4.3) implies that $\rho = 0$ and Bianchi equation (2.32) then gives $\boldsymbol{\rho} = \mathbf{0}$. Thus we conclude with the following remark.

Remark 4.2. A non-twisting conformally flat spacetime with the Ricci tensor of the form (1.1) is necessarily a Kundt spacetime (see [22] for a discussion of such spacetimes).

5 Twisting spacetimes

In this section, we consider spacetimes with a twisting multiple WAND. In [21], it was proven that in spacetimes of odd dimensions, a twisting geodetic WAND must be also shearing. We show that for spacetimes of Weyl type N with the Ricci tensor of the form (1.1), this result extends to even dimensions.

Proposition 5.1. *Twisting, shear-free spacetimes of Weyl type N with the Ricci tensor of the form* (1.1) *do not exist.*

Proof. Let $d > 4$ and let $\{\hat{\mathbf{e}}_{(a)}\}$ be the null frame, in which $\boldsymbol{S}$ is diagonal, and let us employ the notation used in the proof of proposition 4.1. Consider equations (2.33) and (2.34) for the choice of indices $i = k$, $i \neq j$, $i \neq l$ and $j \neq l$ to obtain

$$\left(\Omega'_i + \tilde{\omega}\right) A_{jl} + \Omega'_{l(i)} A_{(i)j} - \Omega'_{j(i)} A_{(i)l} = -\Omega'_{jl} S_i \tag{5.1}$$

and

$$\Omega'_{(i)j} A_{(i)l} + \Omega'_{(i)l} A_{j(i)} - \left(\Omega'_i + \tilde{\omega}\right) A_{jl} = 0. \tag{5.2}$$

Combining (5.1) and (5.2), one immediately obtains that

$$S_i \Omega'_{jl} = 0 \tag{5.3}$$

for all $i \neq j$, $i \neq l$, $j \neq l$. Rewriting the equation (4.3) as

$$\rho \left(\boldsymbol{\Omega}' + \tilde{\omega} \boldsymbol{I}\right) = [\boldsymbol{\Omega}', \boldsymbol{S}] + 2\boldsymbol{S}\boldsymbol{\Omega}' - (d-4)\tilde{\omega}\boldsymbol{S} \tag{5.4}$$

and using (3.12), we obtain

$$\rho \left(\boldsymbol{\Omega}' + \tilde{\omega} \boldsymbol{I}\right) = 2\boldsymbol{S}\boldsymbol{\Omega}' - \omega' \boldsymbol{A} - (d-4)\tilde{\omega}\boldsymbol{S}. \tag{5.5}$$

Then, considering (5.5) for the choice of the indices $i \neq j$, one obtains the relation for the components of $\boldsymbol{A}$:

$$A_{ij} = \frac{2S_{(i)} - \rho}{\omega'} \Omega'_{(i)j}. \tag{5.6}$$

Clearly, for spacetimes to be twisting, necessarily $\boldsymbol{S} \neq \mathbf{0}$ and thus rank $\boldsymbol{S} > 0$. At the same time, there has to be at least one non-vanishing off-diagonal component of $\boldsymbol{\Omega}'$. Hence from (5.3) we have that rank $\boldsymbol{S} < d-2$. Therefore, $0 <$ rank $\boldsymbol{S} < d-2$ and the shear scalar σ^2 is necessarily non-vanishing, which proves the assertion for $d > 4$. Using (5.6), one immediately obtains the result also for $d = 4$, i.e. S_2 and S_3 have to be distinct for A_{23} to be non-vanishing. ■

From (5.6), we see that, in the frame in which $\boldsymbol{S}$ takes the diagonal form, the number of non-vanishing components of $\boldsymbol{A}$ is less than or equal to the number of non-vanishing off-diagonal components of $\boldsymbol{\Omega}'$. Thus, we conclude:

Remark 5.2. Twisting, conformally flat spacetimes with the Ricci tensor of the form (1.1) do not exist.

Combining remarks 4.2 and 5.2, we obtain that the following statement holds in arbitrary dimension.

Corollary 5.3. *A conformally flat spacetime with the Ricci tensor of the form* (1.1) *is Kundt.*

In the following proposition, we show an allowed canonical form of the optical matrix for a twisting multiple WAND.

Proposition 5.4. *Let $\boldsymbol{\ell}$ be a multiple WAND of a twisting spacetime of Weyl type N with the Ricci tensor of the form* (1.1). *Then the canonical form of the corresponding optical matrix is*

$$\boldsymbol{\rho} = s \left(\begin{array}{cc|c} 1 & a & \\ -a & b & \mathbf{0} \\ \hline \multicolumn{2}{c|}{\mathbf{0}} & \mathbf{0} \end{array} \right), \tag{5.7}$$

where $b \neq 1$ (or otherwise the spacetime is Einstein)[12] *and a and b satisfy*

$$a = \frac{\Omega'_{23}}{\Omega'_2 + \tilde{\omega}}, \qquad b = \frac{\Omega'_2 - (d-3)\tilde{\omega}}{\Omega'_2 + \tilde{\omega}} = 1 - \frac{\omega'}{\Omega'_2 + \tilde{\omega}}. \tag{5.8}$$

For $s = 0$, it is Kundt.

The Weyl tensor frame components are

$$\Omega'_{ij} = \left(\begin{array}{cc|c} \Omega'_2 & \Omega'_{23} & \\ \Omega'_{23} & (d-4)\tilde{\omega} - \Omega'_2 & \mathbf{0} \\ \hline \multicolumn{2}{c|}{\mathbf{0}} & -\tilde{\omega}\boldsymbol{I} \end{array} \right). \tag{5.9}$$

Note that $\boldsymbol{\rho}$ (5.7) is again a normal matrix, which does not satisfy the optical constraint. Note also that relations (5.7)–(5.9) hold in the non-twisting case as well, where we just substitute $a = 0 = \Omega'_{23}$.

Proof. We prove again only the case $d > 4$. Let $\{\hat{\mathbf{e}}_{(a)}\}$ be the null frame, in which $\boldsymbol{S}$ takes a diagonal form and let us employ the notation used in the previous sections again. Based on the above discussion, there are two possible scenarios:

(i) There is exactly one non-vanishing Ω'_{ij} for some $i < j$;

(ii) There are at least two non-vanishing Ω'_{ij} for some $i < j$.

Let us start with the case (ii). Equation (5.3) implies that there is at most one non-vanishing diagonal component S_i of $\boldsymbol{S}$ and the only non-vanishing off-diagonal components of $\boldsymbol{\Omega}'$ can be those that appear in the same row or column as S_i. Otherwise, $\boldsymbol{S} = \mathbf{0}$ and thus $\boldsymbol{A} = \mathbf{0}$, which is a contradiction. Therefore, let us assume that S_i is non-vanishing. Let us also rearrange $\{\boldsymbol{m}_{(i)}\}$ so that $\boldsymbol{S} = \mathrm{diag}(S_2, 0, \dots, 0)$. Since the rest of the diagonal of $\boldsymbol{S}$ vanishes, we have that $S_2 = \rho$, and thus from (5.3) only the off-diagonal components Ω'_{2i}, $i > 2$ (and thanks to (5.6), also A_{2i}) can be non-vanishing. From (4.7), we have $S_2(\Omega'_j + \tilde{\omega}) = 0$ for all $j > 2$. Hence $\Omega'_j = -\tilde{\omega}$ for all $j > 2$. From the condition $\mathrm{Tr}\,\boldsymbol{\Omega}' = 0$, we conclude that $\Omega'_2 = (d-3)\tilde{\omega}$. Using these relations for the components $\{\Omega'_i\}$, from (2.35), we obtain

$$\text{þ}\Omega'_2 = -(d-3)\tilde{\omega}\rho, \tag{5.10}$$

while for $i > 2$ one has

$$\text{þ}\Omega'_i = \tilde{\omega}\rho. \tag{5.11}$$

Considering (2.32) for the choice of indices $j = i$ thus gives

$$\Omega'_{(i)k}\rho_{k(i)} = 0. \tag{5.12}$$

Since $\rho_{ki} = 0$ for all $i, k > 2$, equation (5.12) reduces to

$$\Omega'_{(i)2}A_{2(i)} = 0 \tag{5.13}$$

for all $i > 2$. This implies that $\boldsymbol{A} = \mathbf{0}$, which again contradicts our assumption that the spacetime is twisting. Thus scenario (ii) is excluded.

Now, let us discuss case (i). According to (5.3), the only possible non-vanishing components S_i of $\boldsymbol{S}$ are those which appear in the same row or column as the only non-vanishing off-diagonal components Ω'_{ij} and Ω'_{ji} of $\boldsymbol{\Omega}'$. Again, we rearrange the spacelike vectors $\{\boldsymbol{m}_{(i)}\}$ of the null frame such that $\boldsymbol{S} = \mathrm{diag}(S_2, S_3, 0, \dots, 0)$ and the non-vanishing off-diagonal components of $\boldsymbol{\Omega}'$ are Ω'_{23} and Ω'_{32}. We have that $S_2 + S_3 = \rho$ and, similar to the discussion of scenario (ii), from (4.7) we have $\Omega'_j = -\tilde{\omega}$ for all $j > 3$.

[12] As in the non-twisting case, it follows from (4.9) or (5.8) that $b = 1$ iff $\tilde{\omega} = 0 = R_{11}$ and thus an Einstein spacetime corresponds to $b = 1$.

There are exactly $d-4$ such components Ω'_j, thus the tracelessness of $\boldsymbol{\Omega}'$ implies that $\Omega'_2+\Omega'_3=(d-4)\tilde{\omega}$. Using these relations, we conclude that neither of S_2, S_3 vanishes and $S_2\neq S_3$, otherwise (2.32) and (3.12), respectively, again implies that $\boldsymbol{\rho}$ is twist-free.

The relations (5.8) for parameters a and b can be derived using (4.7) or (4.9) and (5.6).[13] ■

Putting propositions 4.1 and 5.4 together, we observe that in both cases (twisting and the non-twisting), the optical matrix $\boldsymbol{\rho}$ takes the form (5.7), with $b\neq 1$. In the case of twisting $\boldsymbol{\ell}$, all functions s, a and b must be non-vanishing, while in the non-twisting case ($a=0$), b or s may vanish. Observe that for type N and generic type III *Einstein* spacetimes, as studied in [14], the optical matrix takes the form (5.7) with $b=1$.

6 The Ricci and Bianchi equations and r-dependence

Let us present all Ricci and Bianchi equations and commutators [16, 21] for spacetimes of aligned Weyl type N ($\boldsymbol{\Omega}'\neq 0$) with the Ricci tensor of the form (1.1) ($\omega'=\eta$, $\phi=\lambda$, $\phi_{ij}=\lambda\delta_{ij}$). Note that the multiple WAND $\boldsymbol{\ell}$ is geodetic ($\kappa_i=0$). For simplicity, we choose a parallelly propagated frame ($\tau'_i=0$, $\overset{i}{M}_{j0}=0$).

6.1 Derivatives and commutators

In a parallelly propagated frame, the *GHP derivative operators* þ, þ$'$, ð$_i$ (2.19)–(2.21) act on a GHP scalar $\boldsymbol{T}$ of boost weight b and spin s as

$$þT_{i_1i_2\dots i_s} \equiv DT_{i_1i_2\dots i_s}, \tag{6.1}$$

$$þ'T_{i_1i_2\dots i_s} \equiv \Delta T_{i_1i_2\dots i_s} - bL_{11}T_{i_1i_2\dots i_s} + \sum_{r=1}^{s}\overset{k}{M}_{i_r1}T_{i_1\dots i_{r-1}ki_{r+1}\dots i_s}, \tag{6.2}$$

$$ð_iT_{j_1j_2\dots j_s} \equiv \delta_iT_{j_1j_2\dots j_s} - bL_{1i}T_{j_1j_2\dots j_s} + \sum_{r=1}^{s}\overset{k}{M}_{j_ri}T_{j_1\dots j_{r-1}kj_{r+1}\dots j_s}, \tag{6.3}$$

where the derivatives along the frame vectors are defined in (2.22).

The commutators (C1)–(C3) and (C2') in [16] read:

$$[þ,þ']T_{i_1\dots i_s} = \left(-\tau_jð_j - b\frac{\lambda}{d-1}\right)T_{i_1\dots i_s}, \tag{6.4}$$

$$[þ,ð_i]T_{k_1\dots k_s} = -\rho_{ji}ð_jT_{k_1\dots k_s}, \tag{6.5}$$

$$\begin{aligned}[ð_i,ð_j]T_{k_1\dots k_s} &= 2\left(\rho_{[ij]}þ' + \rho'_{[ij]}þ + b\rho_{l[i|}\rho'_{l|j]}\right)T_{k_1\dots k_s}\\ &\quad +2\sum_{r=1}^{s}\left[\rho_{k_r[i|}\rho'_{l|j]} + \rho'_{k_r[i|}\rho_{l|j]} + \frac{\lambda\delta_{[i|k_r}\delta_{|j]l}}{(d-1)}\right]T_{k_1\dots l\dots k_s},\end{aligned} \tag{6.6}$$

$$\begin{aligned}[þ',ð_i]T_{k_1\dots k_s} &= \left[-(\kappa'_iþ + \tau_iþ' + \rho'_{ji}ð_j) + b\left(-\tau_j\rho'_{ji} + \kappa'_j\rho_{ji}\right)\right]T_{k_1\dots k_s}\\ &\quad + \sum_{r=1}^{s}\left[\kappa'_{k_r}\rho_{li} - \rho'_{k_ri}\tau_l + \tau_{k_r}\rho'_{li} - \rho_{k_ri}\kappa'_l\right]T_{k_1\dots l\dots k_s}.\end{aligned} \tag{6.7}$$

6.2 Bianchi equations

The Bianchi equations (B2'), (B3'), (B4')=(B3')$_{i[jkl]}$ and (B1') in [16] using (2.43') read (the remaining Bianchi equations are satisfied identically):

$$þ(\Omega'_{ij}+\tilde{\omega}\delta_{ij}) = -(\Omega'_{ik}+\tilde{\omega}\delta_{ik})\rho_{kj}, \tag{6.8}$$

$$0 = -(\Omega'_{i[k|}+\tilde{\omega}\delta_{i[k|})\rho_{j|l]} + (\Omega'_{j[k|}+\tilde{\omega}\delta_{j[k|})\rho_{i|l]}, \tag{6.9}$$

$$0 = -(\Omega'_{i[j}+\tilde{\omega}\delta_{i[j})\rho_{kl]}, \tag{6.10}$$

$$-ð_{[j}(\Omega'_{k]i}+\tilde{\omega}\delta_{k]i}) = (\Omega'_{i[j}+\tilde{\omega}\delta_{i[j})\tau_{k]}, \tag{6.11}$$

[13]Note that thanks to (4.8), the denominator $\Omega'_2+\tilde{\omega}$ is always non-vanishing for $S_2\neq 0$.

The contracted Bianchi identity $\nabla^a R_{ab} = \frac{1}{2}\nabla_b R$ ((2.50') and (2.51) in [16]) reads

$$
\begin{aligned}
\text{þ}\omega' &= -\rho\omega', && (6.12)\\
0 &= \kappa_i\omega'. && (6.13)
\end{aligned}
$$

6.3 Ricci equations

The Ricci equations (i.e. projections of $v_{a;bc} - v_{a;cb} = R_{sabc}v^s$ into the frame (2.1)) for GHP scalars (equations (NP1)–(NP4) and (NP1')–(NP4') in [16]) read

$$
\begin{aligned}
\text{þ}\rho_{ij} &= -\rho_{ik}\rho_{kj}, && (6.14)\\
\text{þ}\tau_i &= -\rho_{ij}\tau_j, && (6.15)
\end{aligned}
$$

$$
\begin{aligned}
\eth_{[j|}\rho_{i|k]} &= \tau_i\rho_{[jk]}, && (6.16)\\
\text{þ}'\rho_{ij} - \eth_j\tau_i &= -\tau_i\tau_j - \rho_{ik}\rho'_{kj} - \frac{\lambda}{d-1}\delta_{ij}, && (6.17)\\
\text{þ}\rho'_{ij} &= -\rho'_{ik}\rho_{kj} - \frac{\lambda}{d-1}\delta_{ij}, && (6.18)\\
-\text{þ}\kappa'_i &= \rho'_{ij}\tau_j, && (6.19)
\end{aligned}
$$

$$
\begin{aligned}
\eth_{[j|}\rho'_{i|k]} &= \kappa'_i\rho_{[jk]}, && (6.20)\\
\text{þ}'\rho'_{ij} - \eth_j\kappa'_i &= -\rho'_{ik}\rho'_{kj} - \kappa'_i\tau_j - \Omega'_{ij} - \tilde{\omega}\delta_{ij}. && (6.21)
\end{aligned}
$$

From (6.4)–(6.6), we obtain the Ricci equations for "non-GHP" scalars (i.e. NP scalars that do not transform covariantly as GHP scalars), see (11b), (11n), (11m), (11a), (11d), (11p), (11c) and (11o) in [21]:

$$
\begin{aligned}
DL_{1i} &= -L_{1j}\rho_{ji}, && (6.22)\\
D\overset{i}{M}_{jk} &= -\overset{i}{M}_{jl}\rho_{lk}, && (6.23)\\
D\overset{i}{M}_{j1} &= -\overset{i}{M}_{jk}\tau_k, && (6.24)\\
DL_{11} &= -L_{1i}\tau_i + \tfrac{\lambda}{d-1}, && (6.25)
\end{aligned}
$$

$$
\begin{aligned}
\delta_{[j|}L_{1|i]} &= -L_{11}\rho_{[ij]} - L_{1k}\overset{k}{M}_{[ij]} - \rho_{k[j|}\rho'_{k|i]}, && (6.26)\\
\delta_{[k|}\overset{i}{M}_{j|l]} &= \rho'_{i[l|}\rho_{j|k]} + \rho_{i[l|}\rho'_{j|k]} + \rho_{[kl]}\overset{i}{M}_{j1} + \overset{i}{M}_{p[k|}\overset{p}{M}_{j|l]} + \overset{i}{M}_{jp}\overset{p}{M}_{[kl]} - \tfrac{\lambda}{d-1}\delta_{i[k}\delta_{l]j}, && (6.27)\\
\triangle L_{1i} - \delta_i L_{11} &= L_{11}(L_{1i} - \tau_i) - \tau_j\rho'_{ji} + \rho_{ji}\kappa'_j - L_{1j}(\rho'_{ji} + \overset{j}{M}_{i1}), && (6.28)\\
\triangle\overset{i}{M}_{jk} - \delta_k\overset{i}{M}_{j1} &= 2\kappa'_{[j}\rho_{i]k} + 2\tau_{[j}\rho'_{i]k} + \overset{i}{M}_{j1}(L_{1k} - \tau_k) + 2\overset{i}{M}_{l[1|}\overset{l}{M}_{j|k]} - \overset{i}{M}_{jl}(\rho'_{lk} + \overset{l}{M}_{k1}). && (6.29)
\end{aligned}
$$

6.4 The r-dependence

To integrate some of these equations, let r denote an affine parameter along null geodesics generated by the multiple WAND $\boldsymbol{\ell}$ and let us work in a parallelly propagated frame.[14] Then the operator þ (2.19) reduces to the derivative along the integral curves of $\boldsymbol{\ell}$, i.e. þ$f = Df$ for a function f. The Sachs equation (6.14) for the optical matrix of the form (5.7) can be integrated (see [20], where the r-dependence of the optical matrix of a block form has been derived):

$$
s_p = \mathcal{R}\left[s_{0p} + r(s_{02}s_{03} + a_0^2)\right], \qquad p = 2, 3, \tag{6.30}
$$

$$
A_{23} = \mathcal{R}a_0, \quad \text{where} \quad \mathcal{R} = \frac{1}{1 + r\big(s_{02} + s_{03}\big) + r^2\big(s_{02}s_{03} + a_0^2\big)}, \tag{6.31}
$$

where s_{0p} and a_0 do not depend on r. Note that $s_2 = s$, $s_3 = sb$ and $A_{23} = sa$, where s, b and a are defined in (5.8).

[14] The form (5.7) of the optical matrix is compatible with the parallel transport [20].

Integrating (2.35)=(6.12) and (2.32)=(6.8), and using (6.30, 6.31), we may further obtain the r-dependence of the non-trivial null frame components of the Ricci (1.1) and Weyl tensors (5.9), respectively:

$$\tilde{\omega} = \tilde{\omega}_0 \mathcal{R}, \tag{6.32}$$

$$\begin{aligned}
\Omega'_{22} &= (o_{02} r + o_{01})\mathcal{R}, \\
\Omega'_{23} &= -\frac{\mathcal{R}}{a_0^2}\Big\{ r\left[(\tilde{\omega}_0 + o_{01})(s_{02}s_{03} + a_0^2) - o_{02}s_{02}\right] + (\tilde{\omega}_0 + o_{01})s_{03} - o_{02}\Big\}, \\
\Omega'_{33} &= -\frac{\mathcal{R}}{a_0^2}\Big\{ r\left[(\tilde{\omega}_0 + o_{01})(s_{02}s_{03} + a_0^2)(s_{02} - s_{03}) + o_{02}(a_0^2 + s_{02}s_{03} - s_{02}^2)\right] \\
&\qquad + (\tilde{\omega}_0 + o_{01})(a_0^2 + s_{02}s_{03} - s_{03}^2) + \tilde{\omega}_0 a_0^2 + o_{02}(s_{03} - s_{02})\Big\}, \\
\Omega'_{(w)(w)} &= -\tilde{\omega}_0 \mathcal{R},
\end{aligned} \tag{6.33}$$

where $\tilde{\omega}_0$, o_{01} and o_{02} are independent of r and $w = 4, \dots, d-1$.

As in the Einstein spacetime case, curvature singularities will appear (see equation (29) in [20] and the discussion below).

One can further integrate (6.15) and (6.18):

$$\tau_2 = (\tau_2^{(1)} r + \tau_2^{(0)})\mathcal{R}, \tag{6.34}$$

$$\tau_3 = -\frac{1}{a_0}\left\{ r\left[\tau_2^{(1)} s_{02} - \tau_2^{(0)}(s_{02}s_{03} + a_0^2)\right] + \tau_2^{(1)} - \tau_2^{(0)} s_{03}\right\}\mathcal{R}, \tag{6.35}$$

$$\tau_w = \tau_w^{(0)}, \tag{6.36}$$

$$\rho'_{22} = ({\rho'_{22}}^{(1)} r + {\rho'_{22}}^{(0)})\mathcal{R} - \frac{\lambda r}{2(d-1)}\left[1 + (r s_{03} + 1)\mathcal{R}\right], \tag{6.37}$$

$$\rho'_{23} = \frac{1}{a_0}\left\{ r\left[{\rho'_{22}}^{(1)} s_{02} - {\rho'_{22}}^{(0)}(s_{02}s_{03} + a_0^2)\right] + {\rho'_{22}}^{(1)} - {\rho'_{22}}^{(0)} s_{03} + \frac{\lambda a_0^2 r^2}{2(d-1)}\right\}\mathcal{R}, \tag{6.38}$$

$$\rho'_{33} = ({\rho'_{33}}^{(1)} r + {\rho'_{33}}^{(0)})\mathcal{R} - \frac{\lambda r}{2(d-1)}\left[1 + (r s_{02} + 1)\mathcal{R}\right], \tag{6.39}$$

$$\rho'_{32} = -\frac{1}{a_0}\left\{ r\left[{\rho'_{33}}^{(1)} s_{03} - {\rho'_{33}}^{(0)}(s_{02}s_{03} + a_0^2)\right] + {\rho'_{33}}^{(1)} - {\rho'_{33}}^{(0)} s_{02} + \frac{\lambda a_0^2 r^2}{2(d-1)}\right\}\mathcal{R}, \tag{6.40}$$

$$\rho'_{v2} = ({\rho'_{v2}}^{(1)} r + {\rho'_{v2}}^{(0)})\mathcal{R}, \tag{6.41}$$

$$\rho'_{v3} = \frac{1}{a_0}\left\{ r\left[{\rho'_{v2}}^{(1)} s_{02} - {\rho'_{v2}}^{(0)}(s_{02}s_{03} + a_0^2)\right] + {\rho'_{v2}}^{(1)} - {\rho'_{v2}}^{(0)} s_{03}\right\}\mathcal{R}, \tag{6.42}$$

$$\rho'_{iw} = -\frac{\lambda \delta_{iw}}{d-1} r + {\rho'_{iw}}^{(0)}, \tag{6.43}$$

where functions with the superscript (0) or (1) do not depend on r and $v, w = 4, \dots, d-1$.

Equations (6.22) and (6.23) can be also integrated:

$$L_{12} = (l_{12}^{(1)} r + l_{12}^{(0)})\mathcal{R}, \tag{6.44}$$

$$L_{13} = \frac{1}{a_0}\left\{ r\left[l_{12}^{(1)} s_{02} - l_{12}^{(0)}(s_{02}s_{03} + a_0^2)\right] + l_{12}^{(1)} - l_{12}^{(0)} s_{03}\right\}\mathcal{R}, \tag{6.45}$$

$$L_{1w} = l_{1w}^{(0)}, \tag{6.46}$$

$$\overset{i}{M}_{j2} = (\overset{i}{m}_{j2}{}^{(1)} r + \overset{i}{m}_{j2}{}^{(0)})\mathcal{R}, \tag{6.47}$$

$$\overset{i}{M}_{j3} = \frac{1}{a_0}\left\{ r\left[\overset{i}{m}_{j2}{}^{(1)} s_{02} - \overset{i}{m}_{j2}{}^{(0)}(s_{02}s_{03} + a_0^2)\right] + \overset{i}{m}_{j2}{}^{(1)} - \overset{i}{m}_{j2}{}^{(0)} s_{03}\right\}\mathcal{R}, \tag{6.48}$$

$$\overset{i}{M}_{jw} = \overset{i}{m}_{jw}{}^{(0)}. \tag{6.49}$$

The r-dependence of κ'_i, $\overset{i}{M}_{j1}$ and L_{11} follows from equations (6.19), (6.24) and (6.25), respectively, and is given by

$$\kappa'_i = -\int \rho'_{ij}\tau_j \mathrm{d}r, \tag{6.50}$$

$$\overset{i}{M}_{j1} = -\int \overset{i}{M}_{jk}\tau_k \mathrm{d}r, \tag{6.51}$$

$$L_{11} = -\int L_{1i}\tau_i \mathrm{d}r + \tfrac{\lambda r}{d-1}. \tag{6.52}$$

To investigate the consequences of the remaining equations, one would need to introduce the remaining coordinates and the rest of the frame, i.e. the frame vectors $\boldsymbol{n}$, $\boldsymbol{m}_{(i)}$, similarly as was done in [5], [6] and [7], and this is is not in the scope of this paper.

7 Examples: Direct products of aligned Ricci type N spacetimes and $\mathbb{E}^n$

Higher-dimensional examples of aligned Weyl type N, Ricci type N spacetimes can be easily constructed as direct products of four-dimensional Weyl type N, pure radiation spacetimes with flat Euclidean spaces.

For a direct product $M = \tilde{M} \times N$ of two arbitrary manifolds (Lorentzian $\tilde{M}$ and Riemannian N), the metric, as well as the Christoffel symbols and the Ricci and Riemann tensors, are decomposable. Then, obviously, for any tensor $\tilde{\boldsymbol{T}}$ from $\tilde{M}$ lifted to M, the covariant derivative in M reduces to the covariant derivative in $\tilde{M}$, i.e. $\nabla \boldsymbol{T} = \tilde{\nabla} \tilde{\boldsymbol{T}}$.

Applying this result to a direct product of an aligned algebraically special Ricci type N spacetime of dimension m with the Euclidean space $\mathbb{E}^n$ for some $n \in \mathbb{N}$, we obtain that $\nabla \boldsymbol{\ell}$ reduces to $\tilde{\nabla} \tilde{\boldsymbol{\ell}}$ and the form of the optical matrix (7.4) follows (where frames from $\tilde{M}$ and N lifted to M are used). Optical scalars thus read[15]

$$\theta = \frac{m-2}{d-2}\tilde{\theta}, \qquad \sigma^2 = \tilde{\sigma}^2 + \frac{(m-2)(d-m)}{d-2}\tilde{\theta}^2, \qquad A^2 = \tilde{A}^2. \tag{7.1}$$

Note that $\boldsymbol{\ell}$ is always shearing, unless the original spacetimes is Kundt. Moreover, $\boldsymbol{\ell}$ is expanding and twisting if and only if $\tilde{\boldsymbol{\ell}}$ is expanding and twisting, respectively, in the original spacetime. In addition, the resulting spacetime is Kundt if and only if the original spacetimes is Kundt.

The alignment type of the Weyl tensor

$$C_{abcd} = R_{abcd} - \frac{2}{d-2}(g_{a[c}R_{d]b} - g_{b[c}R_{d]a}) + \frac{2}{(d-1)(d-2)}R g_{a[c}g_{d]b} \tag{7.2}$$

is determined by the maximal boost weight components of both Weyl tensors and both Ricci tensors (see (12)–(14) in [23]). For non-flat N, the resulting Weyl type would in general be II. For flat Euclidean $\mathbb{E}^n$ and non-conformally flat $\tilde{M}$, the resulting Weyl type is the same as that of $\tilde{\boldsymbol{C}}$ and the Ricci tensor remains as

$$R_{ab} = \omega' \ell_a \ell_b, \tag{7.3}$$

i.e. a direct product of Weyl type N, Ricci type N spacetime with a Euclidean space is always of Weyl type N.

So, let us summarize:

Proposition 7.1. *Consider a direct product of an aligned algebraically special Ricci type N spacetime and $\mathbb{E}^n$. The following statements hold:*

(i) The Weyl type does not change, unless the original spacetime is conformally flat. If this is the case, the Weyl type changes from O to N. The mWAND $\boldsymbol{\ell}$ is the lift of the mWAND $\tilde{\boldsymbol{\ell}}$ in the original spacetime.

(ii) The Ricci tensor in the resulting spacetime is of the type N and it is aligned with the Weyl tensor.

(iii) The optical matrix $\boldsymbol{\rho}$ corresponding to $\boldsymbol{\ell}$ admits the form

$$\boldsymbol{\rho} = \left(\begin{array}{c|c} \tilde{\boldsymbol{\rho}} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array} \right), \tag{7.4}$$

where $\tilde{\boldsymbol{\rho}}$ is the optical matrix corresponding to $\tilde{\boldsymbol{\ell}}$ in the original spacetime.

Let us conclude with references to various classes of four-dimensional Weyl type N, Ricci type N spacetimes in table 3.

Acknowledgments

The authors are thankful to M. Ortaggio, V. Pravda and both referees for useful comments on the draft.

[15] Note that it is a special form of (17) in [23] for f =const.

$\boldsymbol{\rho}$	$\tilde{M}$
$\boldsymbol{\rho} = \mathbf{0}$, i.e. Kundt case	[9, (18.8)-(18.9)], [24] (The solution found in [25] is not included.)
$A_{ij} = 0$, i.e. non-twisting case	[8, (26.11a,b)]
$A_{ij} \neq 0$, i.e. twisting case	[26, (64)-(67)], [27]

Table 3: Four-dimensional Weyl type N and Ricci type N or O spacetimes.

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