ON STATISTICAL APPROXIMATION PROPERTIES OF (p,q)-BLEIMANN-BUTZER-HAHN OPERATORS

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Abstract. The aim of this paper is to introduce a generalization of the (p,q)-Bleimann-Butzer-Hahn operators based on (p,q)-integers and obtain Korovkin's type statistical approximation theorem for these operators. Also, we establish the rate of convergence of these operators using the modulus of continuity. Furthermore, we introduce (p,q)-Bleimann-Butzer-Hahn bivariate operators.

1. Introduction and Preliminaries

In order to approximate continuous functions defined on the positive half axis, Bleimann, Butzer and Hahn (BBH) introduced, in 1980, the following linear positive operators in [3];

$$L_n(f;x) = \frac{1}{(1+x)^n} \sum_{k=0}^{n} f\left(\frac{k}{n-k+1}\right) \left[\begin{array}{c} n \\ k \end{array}\right] x^k, x \geq 0 \tag{1.1}$$

The advent of q-calculus created a new venue of research in approximation theory. Lupaş [11] introduced the first q-analogue of the Bernstein polynomials in 1987. Phillips [17] presented another modification of Bernstein polynomials in 1997. He also established results for the convergence and the Voronovskaja's type asymptotic expansion for these operators.

The q-analogue of the BBH-type operators is defined as

$$L_n^q(f;x) = \frac{1}{\ell_n(x)} \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n-k+1]_q q^k}\right) q^{\frac{k(k-1)}{2}} \left[\begin{array}{c} n \\ k \end{array}\right]_q x^k \tag{1.2}$$

where $\ell_n(x) = \prod_{k=0}^{n-1}(1+q^s x)$.
In recent decades, the concept of (p,q)-calculus has also been introduced. Many researchers have used (p,q)-calculus to establish new and interesting results in approximation theory. Recently, Mursaleen et al [12] introduced the first (p,q)-analogue of Bernstein operators and (p,q)-analogue of Bernstein-Stancu operators [13]. They have investigated the approximation properties and convergence properties of these operators.

Let us give rudiments of (p,q)-calculus.

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The (p,q) integers $[n]_{p,q}$ are defined by

$$[n]_{p,q}=\frac{p^n-q^n}{p-q},\ \ n=0,1,2,\cdots,\ \ 0<q<p\leq 1.$$

whereas q-integers are given by

$$[n]_q=\frac{1-q^n}{1-q},\ \ n=0,1,2,\cdots,\ \ 0<q<1.$$

It is very clear that the two concepts are different but the former is a generalization of the later.

Also, we have (p,q)-binomial expansion as follows

$$(ax+by)^n_{p,q}:=\sum_{k=0}^{n}p^{\frac{(n-k)(n-k-1)}{2}}q^{\frac{k(k-1)}{2}}\left[\begin{array}{c}n\\k\end{array}\right]_{p,q}a^{n-k}b^kx^{n-k}y^k,$$

$$(x+y)^n_{p,q}=(x+y)(px+qy)(p^2x+q^2y)\cdots(p^{n-1}x+q^{n-1}y),$$

$$(1-x)^n_{p,q}=(1-x)(p-qx)(p^2-q^2x)\cdots(p^{n-1}-q^{n-1}x)$$

and the (p,q)-binomial coefficients are defined by

$$\left[\begin{array}{c}n\\k\end{array}\right]_{p,q}=\frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}.$$

By some simple calculation, we have the following relation

$$q^k[n-k+1]_{p,q}=[n+1]_{p,q}-p^{n-k+1}[k]_{p,q}.$$

For details on q-calculus and (p,q)-calculus, one is referred to [21] and [9, 18, 19] respectively.

The concept of statistical convergence was introduced by Fast [7] in the circa 1950 and in recent times it has become an active area of research. The concept of the limit of a sequence has been generalized to a statistical limit through the natural density of a set K of positive integers, defined as

$$\delta(K)=\lim_{n\to\infty}\frac{1}{n}\{k\leqslant n\ for\ k\in K\}$$

provided this limit exists [16]. We say that the sequence $x=(x_n)$ statistically converges to a number l, if for each $\varepsilon>0$, the density of the set $\{k:|x_k-l|\geqslant\varepsilon\}$ is zero. We denote it by $st-\lim_k x_x=l$. It is easily seen that every convergent sequence is statistically convergent but not inversely.

The main purpose of this paper is to introduce a modification of the operators defined by Mursaleen et al. [15] and investigate statistical approximation properties of the operators with the aid of Korovkin type theorem and estimate the rate of their statistical convergence.

Now based on (p,q)-integers, we construct (p,q)-analogue of BBH operators, and we call them as (p,q)-Bleimann-Butzer-Hahn Operators and investigate their

Korovokin's type statistical approximation properties by using the test functions $\left(\frac{t}{1+t}\right)^{\nu}$ for $\nu=0,1,2$. Also for a space of generalized Lipschitz-type maximal functions we give a pointwise estimation.

Let $C_B(\mathbb{R}_+)$ be the set of all bounded and continuous functions on $\mathbb{R}_+$, then $C_B(\mathbb{R}_+)$ is linear normed space with

$$\| f \|_{C_B} = \sup_{x\geq 0} | f(x) | .$$

Let ω denotes modulus of continuity satisfying the following condition:

(1) ω is a non-negative increasing function on $\mathbb{R}_+$
(2) $\omega(\delta_1+\delta_2)\leq\omega(\delta_1)+\omega(\delta_2)$
(3) $\lim_{\delta\to 0}\omega(\delta)=0$.

Let H_ω be the space of all real-valued functions f defined on the semiaxis $\mathbb{R}_+$ satisfying the condition

$$| f(x)-f(y) | \leq \omega\left(\left|\frac{x}{1+x}-\frac{y}{1+y}\right|\right),$$

for any $x,y\in\mathbb{R}_+$.

Theorem 1.1. [8] *Let $\{A_n\}$ be the sequence of positive linear operators from H_ω into $C_B(\mathbb{R}_+)$, satisfying the conditions*

$$\lim_{n\to\infty} \| A_n\left(\left(\frac{t}{1+t}\right)^{\nu};x\right)-\left(\frac{x}{1+x}\right)^{\nu} \|_{C_B},$$

for $\nu=0,1,2$. Then for any function $f\in H_\omega$

$$\lim_{n\to\infty} \| A_n(f)-f \|_{C_B}=0.$$

Now we introduce (p,q)-Bleimann-Butzer-Hahn type operators based on (p,q)-integers as follows:

$$L_n^{p,q}(f;x)=\frac{pq}{\ell_n^{p,q}(x)}\sum_{k=0}^{n} f\left(\frac{p^{n-k+1}[k]_{p,q}}{[n-k+1]_{p,q}q^k}\right)p^{\frac{(n-k)(n-k-1)}{2}}q^{\frac{k(k-1)}{2}}\left[\begin{array}{c} n\\ k\end{array}\right]_{p,q} x^k \quad (1.3)$$

where, $x\geq 0,\ 0<q<p\leq 1$

$$\ell_n^{p,q}(x)=\prod_{s=0}^{n-1}(p^s+q^s x)$$

and f is defined on semiaxis $\mathbb{R}_+$.
And also by induction, we construct the Euler identity based on (p,q)-analogue defined as follows:

$$\prod_{s=0}^{n-1}(p^s+q^s x)=\sum_{k=0}^{n} p^{\frac{(n-k)(n-k-1)}{2}}q^{\frac{k(k-1)}{2}}\left[\begin{array}{c} n\\ k\end{array}\right]_{p,q} x^k \quad (1.4)$$

If we put $p=1$, then we obtain q-BBH operators. In (1.3), if we take $f\left(\frac{[k]_{p,q}}{[n-k+1]_{p,q}}\right)$ in place of $f\left(\frac{p^{n-k+1}[k]_{p,q}}{[n-k+1]_{p,q}q^k}\right)$, then we obtain the usual generalization of Bleimann, Butzer and Hahn operators based on (p,q)-integers and then it is not possible to obtain explicit expressions for the monomials t^ν and $\left(\frac{t}{1+t}\right)^{\nu}$ for

$\nu = 1, 2$. Explicit formulas for the monomials $\left(\frac{t}{1+t}\right)^{\nu}$ for $\nu = 0, 1, 2$ are obtainable only if we define the Bleimann, Butzer and Hahn operators as in (1.3). It is to note that these operators are more flexible than the classical BBH operators and q-analogue of BBH operators. That is depending on the selection of (p, q)-integers, the rate of convergence of (p, q)-BBH operators is as good as the classical one atleast.

2. Main results

Lemma 2.1. *Let $L_n^{p,q}(f; x)$ be given by (1.3), then for any $x \geq 0$ and $0 < q < p \leq 1$ we have the following identities*

(1) $L_n^{p,q}(1; x) = pq$,

(2) $L_n^{p,q}(\frac{t}{1+t}; x) = \frac{p^2 q[n]_{p,q}}{[n+1]_{p,q}} \left(\frac{x}{1+x}\right)$,

(3) $L_n^{p,q}\left((\frac{t}{1+t})^2; x\right) = \frac{p^2 q^3 [n]_{p,q}[n-1]_{p,q}}{[n+1]_{p,q}^2} \frac{x^2}{(1+x)(p+qx)} + \frac{p^{n+2} q[n]_{p,q}}{[n+1]_{p,q}^2} \left(\frac{x}{1+x}\right)$.

Proof. (1) $L_n^{p,q}(1; x) = \frac{pq}{\ell_n^{p,q}(x)} \sum_{k=1}^{n} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} x^k$

but for $0 < q < p \leq 1$, we have

$$\sum_{k=0}^{n} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} x^k = \prod_{s=0}^{n-1} (p^s + q^s x) = \ell_n^{p,q}(x),$$

so

$$L_n^{p,q}(1; x) = \frac{pq}{l_n^{p,q}(x)} \times l_n^{p,q}(x) = pq.$$

This proves (1).

(2) Let $t = \frac{p^{n-k+1}[k]_{p,q}}{[n-k+1]_{p,q} q^k}$, then $\frac{t}{t+1} = \frac{[k]_{p,q} p^{n+1-k}}{[n+1]_{p,q}}$

$$\begin{aligned}
L_n^{p,q}\left(\frac{t}{1+t}; x\right) &= \frac{pq}{\ell_n^{p,q}(x)} \sum_{k=1}^{n} \frac{[k]_{p,q} p^{n-k+1}}{[n+1]_{p,q}} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} x^k \\
&= \frac{pq}{\ell_n^{p,q}(x)} \sum_{k=1}^{n} \frac{[n]_{p,q} p^{n-k+1}}{[n+1]_{p,q}} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \left[\begin{array}{c} n-1 \\ k-1 \end{array}\right]_{p,q} x^k \\
&= x \left(\frac{pq}{\ell_n^{p,q}(x)} \cdot \frac{[n]_{p,q}}{[n+1]_{p,q}} p\right) \sum_{k=0}^{n-1} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \left[\begin{array}{c} n-1 \\ k \end{array}\right]_{p,q} (qx)^k \\
&= p^2 q \frac{[n]_{p,q}}{[n+1]_{p,q}} \frac{x}{1+x}.
\end{aligned}$$

This completes the proof of (2).

(3) $L_n^{p,q}\left(\frac{t^2}{(1+t)^2}; x\right) = \frac{pq}{\ell_n^{p,q}(x)} \sum_{k=1}^{n} \frac{[k]_{p,q}^2 p^{2(n-k+1)}}{[n+1]_{p,q}^2} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} x^k.$

Now we have,

$[k]_{p,q} = p^{k-1} + q[k-1]_{p,q}$, and $[k]^2_{p,q} = q[k]_{p,q}[k-1]_{p,q} + p^{k-1}[k]_{p,q}$,

using it in above, we get

$$L_n^{p,q}\left(\frac{t^2}{(1+t)^2};x\right) = \frac{pq}{\ell_n^{p,q}(x)}\sum_{k=2}^{n}\frac{q[k]_{p,q}[k-1]_{p,q}p^{2n-2k+2}}{[n+1]^2_{p,q}}p^{\frac{(n-k)(n-k-1)}{2}}q^{\frac{k(k-1)}{2}}\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q}x^k$$

$$+\frac{pq}{\ell_n^{p,q}(x)}\sum_{k=1}^{n}p^{k-1}\frac{[k]_{p,q}p^{2n-2k+2}}{[n+1]^2_{p,q}}p^{\frac{(n-k)(n-k-1)}{2}}q^{\frac{k(k-1)}{2}}\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q}x^k$$

$$= \frac{pq}{\ell_n^{p,q}(x)}\frac{q[n]_{p,q}[n-1]_{p,q}}{[n+1]^2_{p,q}}\sum_{k=2}^{n}p^{\left((2n-2k+2)+\frac{(n-k)(n-k-1)}{2}\right)}q^{\frac{k(k-1)}{2}}\left[\begin{array}{c} n-2 \\ k-2 \end{array}\right]_{p,q}x^k$$

$$+\frac{pq}{\ell_n^{p,q}(x)}\frac{[n]_{p,q}}{[n+1]^2_{p,q}}\sum_{k=1}^{n}p^{\left((k-1)+(2n-2k+2)+\frac{(n-k)(n-k-1)}{2}\right)}q^{\frac{k(k-1)}{2}}\left[\begin{array}{c} n-1 \\ k-1 \end{array}\right]_{p,q}x^k$$

$$= x^2\frac{pq}{\ell_n^{p,q}(x)}\frac{q[n]_{p,q}[n-1]_{p,q}}{[n+1]^2_{p,q}}\sum_{k=0}^{n-2}p^{\left((2n-2k-2)+\frac{(n-k-2)(n-k-3)}{2}\right)}q^{\frac{(k+1)(k+2)}{2}}\left[\begin{array}{c} n-2 \\ k \end{array}\right]_{p,q}x^k$$

$$+x\frac{pq}{\ell_n^{p,q}(x)}\frac{[n]_{p,q}}{[n+1]^2_{p,q}}\sum_{k=0}^{n-1}p^{\left(k+(2n-2k)+\frac{(n-k-1)(n-k-2)}{2}\right)}q^{\frac{k(k+1)}{2}}\left[\begin{array}{c} n-1 \\ k \end{array}\right]_{p,q}x^k$$

$$= x^2\frac{pq}{\ell_n^{p,q}(x)}\frac{pq^2[n]_{p,q}[n-1]_{p,q}}{[n+1]^2_{p,q}}\sum_{k=0}^{n-2}p^{\frac{(n-k)(n-k-1)}{2}}q^{\frac{k(k-1)}{2}}\left[\begin{array}{c} n-2 \\ k \end{array}\right]_{p,q}(q^2x)^k$$

$$+x\frac{pq}{\ell_n^{p,q}(x)}\frac{p^{n+1}[n]_{p,q}}{[n+1]^2_{p,q}}\sum_{k=0}^{n-1}p^{\frac{(n-k)(n-k-1)}{2}}q^{\frac{k(k-1)}{2}}\left[\begin{array}{c} n-1 \\ k \end{array}\right]_{p,q}(qx)^k$$

$$= \frac{p^2q^3[n]_{p,q}[n-1]_{p,q}}{[n+1]^2_{p,q}}\frac{x^2}{(1+x)(p+qx)} + \frac{p^{n+2}q[n]_{p,q}}{[n+1]^2_{p,q}}\left(\frac{x}{1+x}\right).$$

This proves (3).

□

Korovkin's type approximation properties

In this section, we obtain the Korovkin's type statistical approximation properties for the operators defined by (1.3), using Theorem 1.1.

In order to obtain the convergence results for the operators $L_n^{p,q}$, we take $q = q_n$, $p = p_n$ where $q_n \in (0,1)$ and $p_n \in (q_n, 1]$ satisfy

$$\lim_n p_n = 1, \ \lim_n q_n = 1 \tag{2.1}$$

Theorem 2.2. *Let $p = p_n andq = q_n$ satisfy (2.1) for $0 < q_n < p_n \leq 1$, and if $L_n^{p_n,q_n}$ is defined by (1.3), then for any function $f \in H_\omega$,*

$$st - \lim_n \parallel L_n^{p_n,q_n}(f;x) - f \parallel_{C_B} = 0.$$

Proof. In the light of Theorem 1.1, it is sufficient to prove the followings:

$$st-\lim_{n\to\infty}\parallel L_n^{p_n,q_n}\left(\left(\frac{t}{1+t}\right)^{\nu};x\right)-\left(\frac{x}{1+x}\right)^{\nu}\parallel_{C_B}=0,\ \ for\ \nu=0,1,2 \quad (2.2)$$

From Lemma 2.1, the first condition of (2.2) is easily obtained for $\nu=0$. Also, we can easily see from (2) of Lemma 2.1 that

$$\begin{aligned}\parallel L_n^{p_n,q_n}\left(\left(\frac{t}{1+t}\right)^{\nu};x\right)-\left(\frac{x}{1+x}\right)^{\nu}\parallel_{C_B} &\le \left|\frac{p_nq_n[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}}-1\right| \\ &= 1-p_nq_n\frac{[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}}.\end{aligned}$$

Now for a given $\varepsilon>0$, we define the following sets

$$U=\left\{n:\|L_n^{p_n,q_n}\left(\frac{t}{1+t};x\right)-\frac{x}{1+x}\|\geqslant\varepsilon\right\},$$

$$U_1=\left\{n:1-p_nq_n\frac{[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}}\ge\varepsilon\right\}.$$

It is obvious that $U\subset U_1$, so we have

$$\delta\{k\le n:\|L_n^{p_n,q_n}\left(\frac{t}{1+t};x\right)-\frac{x}{1+x}\|\ge\varepsilon\}\leqslant\delta\{k\le n:1-p_nq_n\frac{[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}}\ge\varepsilon\}.$$

Now using (2.1) it is clear that

$$st-\lim_n\left(1-p_nq_n\frac{[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}}\right)=0,$$

so

$$\delta\{k\le n:1-p_nq_n\frac{[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}}\ge\varepsilon\}=0,$$

then

$$st-\lim_n\left\|L_n^{p_n,q_n}\left(\frac{t}{1+t};x\right)-\frac{x}{1+x}\right\|_{C_B}=0.$$

which proves that the condition (2.2) holds for $\nu=1$. To verify this condition for $\nu=2$, consider (3) of Lemma 2.1. Then, we see that

$$\left\|L_n^{p_n,q_n}\left(\left(\tfrac{t}{1+t}\right)^2;x\right)-\left(\tfrac{x}{1+x}\right)^2\right\|_{C_B}$$
$$=\sup_{x\ge0}\left\{\tfrac{x^2}{(1+x)^2}\left(\tfrac{p_n^2q_n^3[n]_{p_n,q_n}[n-1]_{p_n,q_n}}{[n+1]^2_{p_n,q_n}}.\tfrac{1+x}{p_n+q_nx}-1\right)+\tfrac{p_n^{n+2}q[n]_{p_n,q_n}}{[n+1]^2_{p_n,q_n}}\cdot\tfrac{x}{1+x}\right\}.$$

After some calculations, we get

$$\frac{[n]_{p_n,q_n}[n-1]_{p_n,q_n}}{[n+1]^2_{p_n,q_n}}=\frac{1}{q_n^3}\left\{1-p_n^n\left(2+\frac{q_n}{p_n}\right)\frac{1}{[n+1]_{p_n,q_n}}+(p_n^n)^2\left(1+\frac{q_n}{p_n}\right)\frac{1}{[n+1]^2_{p_n,q_n}}\right\},$$

and

$$\frac{[n]_{p_n,q_n}}{[n+1]^2_{p_n,q_n}}=\frac{1}{q_n}\left(\frac{1}{[n+1]_{p_n,q_n}}-p_n^n\frac{1}{[n+1]^2_{p_n,q_n}}\right).$$

Then we are led to

$$\left\|L_n^{p_n,q_n}\left(\left(\frac{t}{1+t}\right)^2;x\right)-\left(\frac{x}{1+x}\right)^2\right\|_{C_B}$$

$$\leq |(p_n^2-1)+p_n^{n+2}\left(\frac{-1}{[n+1]_{p_n,q_n}}+\frac{p_n^n}{[n+1]_{p_n,q_n}^2}\right)+p_n^{n+1}\left(\frac{-q_n}{[n+1]_{p_n,q_n}}+\frac{p_n^2}{[n+1]_{p_n,q_n}}+\frac{p_n^nq_n}{[n+1]_{p_n,q_n}}\right)|.$$

$$=1-p_n^2+p_n^{n+2}\left(\frac{1}{[n+1]_{p_n,q_n}}-\frac{p_n^n}{[n+1]_{p_n,q_n}^2}\right)+p_n^{n+1}\left(\frac{q_n}{[n+1]_{p_n,q_n}}-\frac{p_n^2+p_n^nq_n}{[n+1]_{p_n,q_n}^2}\right).$$

Now if we denote $1-p_n^2$, $p_n^{n+2}\left(\frac{1}{[n+1]_{p_n,q_n}}-\frac{p_n^n}{[n+1]_{p_n,q_n}^2}\right)$ and $p_n^{n+1}\left(\frac{q_n}{[n+1]_{p_n,q_n}}-\frac{p_n^2+p_n^nq_n}{[n+1]_{p_n,q_n}^2}\right)$ by α_n, β_n and γ_n respectively, then by using (2.1), we find that

$$st-\lim_n \alpha_n=0,\, st-\lim_n \beta_n=0 and st-\lim_n \gamma_n=0. \tag{2.3}$$

Now for a given $\varepsilon>0$, we define the following sets

$$U=\left\{n:\left\|L_n^{p_n,q_n}\left(\left(\frac{t}{1+t}\right)^2;x\right)-\left(\frac{x}{1+x}\right)^2\right\|_{C_B}\geq\varepsilon\right\},$$

$U_1=\left\{n:\alpha_n\geq\frac{\varepsilon}{3}\right\}$, $U_2=\left\{n:\beta_n\geq\frac{\varepsilon}{3}\right\}$ and $U_3=\left\{n:\gamma_n\geq\frac{\varepsilon}{3}\right\}$. It is obvious that $U\subseteq U_1\cup U_2\cup U_3$. So we have

$$\delta\left\{k\leq n:\left\|L_n^{p_n,q_n}\left(\left(\frac{t}{1+t}\right)^2;x\right)-\left(\frac{x}{1+x}\right)^2\right\|_{C_B}\geq\varepsilon\right\}$$

$$\leqslant\delta\left\{k\leq n:\alpha_n\geq\frac{\varepsilon}{3}\right\}+\delta\left\{k\leq n:\beta_n\geq\frac{\varepsilon}{3}\right\}+\delta\left\{k\leq n:\gamma_n\geq\frac{\varepsilon}{3}\right\}.$$

Now by virtue of (2.3), the right hand side of the above inequality is trivial, so we get

$$st-\lim_n\left\|L_n^{p_n,q_n}\left(\left(\frac{t}{1+t}\right)^2;x\right)-\left(\frac{x}{1+x}\right)^2\right\|_{C_B}=0.$$

Hence the proof of the theorem is complete. □

3. Rate of Convergence

In this section, we calculate the rate of convergence of the operators (1.3) by means of modulus of continuity and Lipschitz type maximal functions.

The modulus of continuity for $f\in H_\omega$ is defined by

$$\widetilde{\omega}(f;\delta)=\sum_{\substack{|\frac{t}{1+t}-\frac{x}{1+x}|\leq\delta,\\ x,t\geq 0}} |\ f(t)-f(x)\ |$$

where $\widetilde{\omega}(f;\delta)$ satisfies the following conditions. For all $f\in H_\omega(\mathbb{R}_+)$

(1) $\lim_{\delta\to 0}\widetilde{\omega}(f;\delta)=0$
(2) $|\ f(t)-f(x)\ |\leq\widetilde{\omega}(f;\delta)\left(\frac{|\frac{t}{1+t}-\frac{x}{1+x}|}{\delta}+1\right)$

Theorem 3.1. *Let $p = p_n andq = q_n$ satisfy (2.1), for $0 < q_n < p_n \leq 1$, and let $L_n^{p_n,q_n}$ be defined by (1.3). Then for each $x \geq 0$ and for any function $f \in H_\omega$, we have*

$$| L_n^{p_n,q_n}(f;x) - f | \leq 2\widetilde{\omega}(f;\sqrt{\delta_n(x)}),$$

where

$$\delta_n(x) = \frac{x^2}{(1+x)^2}\left(\frac{p_n^2 q_n^3 [n]_{p_n,q_n}[n-1]_{p_n,q_n}}{[n+1]^2_{p_n,q_n}}\frac{1+x}{p_n+q_n x} - 2\frac{p_n q_n [n]_{p_n,q_n}}{[n+1]_{p_n,q_n}} + 1\right) + \frac{p_n^{n+2} q_n [n]_{p_n,q_n}}{[n+1]^2_{p_n,q_n}}\frac{x}{1+x}.$$

Proof.

$$\begin{aligned} | L_n^{p_n,q_n}(f;x) - f | &\leq L_n^{p_n,q_n}(| f(t) - f(x) |;x) \\ &\leq \widetilde{\omega}(f;\delta)\left\{1 + \frac{1}{\delta}L_n^{p_n,q_n}\left(\left|\frac{t}{1+t} - \frac{x}{1+x}\right|;x\right)\right\}. \end{aligned}$$

Now by using the Cauchy-Schwarz inequality, we have

$$| L_n^{p_n,q_n}(f;x) - f | \leq \widetilde{\omega}(f;\delta_n)\left\{1 + \frac{1}{\delta_n}\left[\left(L_n^{p_n,q_n}\left(\left(\frac{t}{1+t} - \frac{x}{1+x}\right)^2;x\right)\right)\right]^{\frac{1}{2}}(L_n^{p_n,q_n}(1;x))^{\frac{1}{2}}\right\}$$

$$\leq \widetilde{\omega}(f;\delta_n)\left\{1 + \tfrac{1}{\delta_n}\left[\tfrac{x^2}{(1+x)^2}\left(\tfrac{p_n^2 q_n^3 [n]_{p_n,q_n}[n-1]_{p_n,q_n}}{[n+1]^2_{p_n,q_n}}\tfrac{1+x}{p_n+q_n x} - 2\tfrac{p_n q_n [n]_{p_n,q_n}}{[n+1]_{p_n,q_n}} + 1\right) + \tfrac{p_n^{n+2} q_n [n]_{p_n,q_n}}{[n+1]^2_{p_n,q_n}}\tfrac{x}{1+x}\right]^{\frac{1}{2}}\right\}.$$

This completes the proof. □

Now we will give an estimate concerning the rate of convergence by means of Lipschitz type maximal functions. In [2], the Lipschitz type maximal function space on $E \subset \mathbb{R}_+$ is defined as

$$\widetilde{W}_{\alpha,E} = \{f : \sup(1+x)^\alpha \widetilde{f}_\alpha(x) \leq M\frac{1}{(1+y)^\alpha} : x \leq 0, \text{ and } y \in E\} \tag{3.1}$$

where f is bounded and continuous function on $\mathbb{R}_+$, M is a positive constant and $0 < \alpha \leq 1$.

In [10], B. Lenze introduced a Lipschitz type maximal function f_α as follows:

$$f_\alpha(x,t) = \sum_{\substack{t>0 \\ t\neq x}} \frac{| f(t) - f(x) |}{| x - t |^\alpha}. \tag{3.2}$$

We denote by $d(x,E)$, the distance between x and E, that is

$$d(x,E) = \inf\{| x - y |; y \in E\}.$$

Theorem 3.2. *For all $f \in \widetilde{W}_{\alpha,E}$, we have*

$$| L_n^{p_n,q_n}(f;x) - f(x) | \leq M\left(\delta_n^{\frac{\alpha}{2}}(x) + 2\left(d(x,E)\right)^\alpha\right) \tag{3.3}$$

where $\delta_n(x)$ is defined as in Theorem 3.1.

Proof. Let $\overline{E}$ denote the closure of the set E. Then there exits a $x_0 \in \overline{E}$ such that $| x - x_0 |= d(x,E)$, where $x \in \mathbb{R}_+$. Thus we can write

$$| f - f(x) | \leq | f - f(x_0) | + | f(x_0) - f(x) | .$$

Since $L_n^{p_n,q_n}$ are positive linear operators, so for $f\in \widetilde{W}_{\alpha,E}$, by using the previous inequality, we have

$$\begin{aligned}&\mid L_n^{p_n,q_n}(f;x)-f(x)\mid\leq\mid L_n^{p_n,q_n}(\mid f-f(x_0)\mid;x)+\mid f(x_0)-f(x)\mid L_n^{p_n,q_n}(1;x)\\&\leq M\left(L_n^{p_n,q_n}\left(\left|\frac{t}{1+t}-\frac{x_0}{1+x_0}\right|^\alpha;x\right)+\frac{\mid x-x_0\mid^\alpha}{(1+x)^\alpha(1+x_0)^\alpha}L_n^{p_n,q_n}(1;x)\right).\end{aligned}$$

Now $(a+b)^\alpha\leq a^\alpha+b^\alpha$ consequently implies that

$$L_n^{p_n,q_n}\left(\left|\frac{t}{1+t}-\frac{x_0}{1+x_0}\right|^\alpha;x\right)\leq L_n^{p_n,q_n}\left(\left|\frac{t}{1+t}-\frac{x}{1+x}\right|^\alpha;x\right)+L_n^{p_n,q_n}\left(\left|\frac{x}{1+x}-\frac{x_0}{1+x_0}\right|^\alpha;x\right)$$

$$L_n^{p_n,q_n}\left(\left|\frac{t}{1+t}-\frac{x_0}{1+x_0}\right|^\alpha;x\right)\leq L_n^{p_n,q_n}\left(\left|\frac{t}{1+t}-\frac{x}{1+x}\right|^\alpha;x\right)+\frac{\mid x-x_0\mid^\alpha}{(1+x)^\alpha(1+x_0)^\alpha}L_n^{p_n,q_n}(1;x).$$

By using the Hölder's inequality with $p=\frac{2}{\alpha}$ and $q=\frac{2}{2-\alpha}$, we have

$$\begin{aligned}L_n^{p_n,q_n}\left(\left|\tfrac{t}{1+t}-\tfrac{x_0}{1+x_0}\right|^\alpha;x\right)&\leq L_n^{p_n,q_n}\left(\left(\tfrac{t}{1+t}-\tfrac{x}{1+x}\right)^2;x\right)^{\frac{\alpha}{2}}\left(L_n^{p_n,q_n}(1;x)\right)^{\frac{2-\alpha}{2}}\\&+\frac{\mid x-x_0\mid^\alpha}{(1+x)^\alpha(1+x_0)^\alpha}L_n^{p_n,q_n}(1;x)\\&=\delta_n^{\frac{\alpha}{2}}(x)+\frac{\mid x-x_0\mid^\alpha}{(1+x)^\alpha(1+x_0)^\alpha}.\end{aligned}$$

This completes the proof. □

Corollary 3.3. *If we take $E=\mathbb{R}_+$ as a particular case of Theorem 3.2, then for all $f\in\widetilde{W}_{\alpha,\mathbb{R}_+}$, we have*

$$\mid L_n^{p_n,q_n}(f;x)-f(x)\mid\leq M\delta_n^{\frac{\alpha}{2}}(x),$$

where $\delta_n(x)$ is defined as in Theorem 3.1.

Theorem 3.4. *If $x\in(0,\infty)\backslash\left\{p^{n-k+1}\frac{[k]_{p,q}}{[n-k+1]_{p,q}q^k}\middle|k=0,1,2,\cdots,n\right\}$, then*

$$L_n^{p,q}(f;x)-f\left(\frac{px}{q}\right)=-\frac{x^{n+1}}{\ell_n^{p,q}(x)}\left[\frac{px}{q};\frac{p[n]_{p,q}}{q^n};f\right]pq^{\frac{n(n-1)}{2}-n}$$

$$+\frac{x}{\ell_n^{p,q}(x)}\sum_{k=0}^{n-1}\left[\frac{px}{q};p^{n-k+1}\frac{[k]_{p,q}}{[n-k+1]_{p,q}q^k};f\right]\frac{1}{[n-k]_{p,q}}p^{\frac{(n-k)(n-k-1)}{2}-(k-n)-1}q^{\frac{k(k-1)}{2}-k}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}x^k. \tag{3.4}$$

Proof. By using (1.3), we have

$$L_n^{p,q}(f;x)-f\left(\tfrac{px}{q}\right)=\tfrac{pq}{\ell_n^{p,q}(x)}\sum_{k=0}^n\left\{f\left(\tfrac{p^{n-k+1}[k]_{p,q}}{[n-k+1]_{p,q}q^k}\right)-f\left(\tfrac{px}{q}\right)\right\}p^{\frac{(n-k)(n-k-1)}{2}}q^{\frac{k(k-1)}{2}}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}x^k$$

$$=-\tfrac{1}{\ell_n^{p,q}(x)}\sum_{k=0}^n\left(\tfrac{px}{q}-\tfrac{p^{n-k+1}[k]_{p,q}}{[n-k+1]_{p,q}q^k}\right)\left[\tfrac{px}{q};\tfrac{p^{n-k+1}[k]_{p,q}}{[n-k+1]_{p,q}q^k};f\right]p^{\frac{(n-k)(n-k-1)}{2}+1}q^{\frac{k(k-1)}{2}+1}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}x^k$$

By using $\frac{[k]_{p,q}}{[n-k+1]_{p,q}}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}=\begin{bmatrix}n\\k-1\end{bmatrix}_{p,q}$, we have

$$L_n^{p,q}(f;x)-f\left(\frac{px}{q}\right)=-\frac{x}{\ell_n^{p,q}(x)}\sum_{k=0}^{n}\left[\frac{px}{q};\frac{p^{n-k+1}[k]_{p,q}}{[n-k+1]_{p,q}q^k};f\right]p^{\frac{(n-k)(n-k-1)}{2}+2}q^{\frac{k(k-1)}{2}}\left[\begin{array}{c}n\\k\end{array}\right]_{p,q}x^k.$$

$$+\frac{1}{\ell_n^{p,q}(x)}\sum_{k=1}^{n}\left[\frac{px}{q};\frac{p^{n-k+1}[k]_{p,q}}{[n-k+1]_{p,q}q^k};f\right]p^{\frac{(n-k)(n-k-1)}{2}-(k-n-1)-1}q^{\frac{k(k-1)}{2}-(k-1)}\left[\begin{array}{c}n\\k-1\end{array}\right]_{p,q}x^k$$

$$=-\frac{x}{\ell_n^{p,q}(x)}\sum_{k=0}^{n}\left[\frac{px}{q};\frac{p^{n-k+1}[k]_{p,q}}{[n-k+1]_{p,q}q^k};f\right]p^{\frac{(n-k)(n-k-1)}{2}+2}q^{\frac{k(k-1)}{2}}\left[\begin{array}{c}n\\k\end{array}\right]_{p,q}x^k.$$

$$+\frac{x}{\ell_n^{p,q}(x)}\sum_{k=0}^{n-1}\left[\frac{px}{q};\frac{p^{n-k}[k+1]_{p,q}}{[n-k]_{p,q}q^{k+1}};f\right]p^{\frac{(n-k)(n-k-1)}{2}-(k-n-1)-2}q^{\frac{k(k-1)}{2}-k}\left[\begin{array}{c}n\\k\end{array}\right]_{p,q}x^k$$

$$=-\tfrac{x^{n+1}}{\ell_n^{p,q}(x)}\left[\tfrac{px}{q};\tfrac{p[n]_{p,q}}{q^n};f\right]pq^{\frac{n(n-1)}{2}-n}$$

$$+\tfrac{x}{\ell_n^{p,q}(x)}\textstyle\sum_{k=0}^{n-1}\left\{\left[\tfrac{px}{q};\tfrac{p^{n-k}[k+1]_{p,q}}{[n-k]_{p,q}q^{k+1}};f\right]-\left[\tfrac{px}{q};\tfrac{p^{n-k+1}[k]_{p,q}}{[n-k+1]_{p,q}q^k};f\right]\right\}p^{\frac{(n-k)(n-k-1)}{2}-(k-n-1)-2}q^{\frac{k(k-1)}{2}-k}\left[\begin{array}{c}n\\k\end{array}\right]_{p,q}x^k.$$

Now by using the results

$$\left[\tfrac{px}{q};\tfrac{p^{n-k}[k+1]_{p,q}}{[n-k]_{p,q}q^{k+1}};f\right]-\left[\tfrac{px}{q};\tfrac{p^{n-k+1}[k]_{p,q}}{[n-k+1]_{p,q}q^k};f\right]$$
$$=\left(\frac{p^{n-k}[k+1]_{p,q}}{[n-k]_{p,q}q^{k+1}}-\frac{p^{n-k+1}[k]_{p,q}}{[n-k+1]_{p,q}q^k}\right)\left\{\frac{px}{q};\frac{p^{n-k+1}[k]_{p,q}}{[n-k+1]_{p,q}q^k};\frac{p^{n-k}[k+1]_{p,q}}{[n-k]_{p,q}q^{k+1}};f\right\}$$

and

$$\frac{p^{n-k}[k+1]_{p,q}}{[n-k]_{p,q}q^{k+1}}-\frac{p^{n-k+1}[k]_{p,q}}{[n-k+1]_{p,q}q^k}=[n+1]_{p,q},$$

we have

$$L_n^{p,q}(f;x)-f\left(\tfrac{px}{q}\right)=-\tfrac{x^{n+1}}{\ell_n^{p,q}(x)}\left[\tfrac{px}{q};\tfrac{p[n]_{p,q}}{q^n};f\right]pq^{\frac{n(n-1)}{2}-n}$$
$$+\tfrac{x}{\ell_n^{p,q}(x)}\textstyle\sum_{k=0}^{n-1}\left\{\left[\tfrac{px}{q};\tfrac{p^{n-k+1}[k]_{p,q}}{[n-k+1]_{p,q}q^k};f\right]\tfrac{p^{n-k}[n+1]_{p,q}}{[n-k]_{p,q}[n-k+1]_{p,q}q^{k+1}}\right\}p^{\frac{(n-k)(n-k-1)}{2}-(k-n)-1}q^{\frac{k(k-1)}{2}-k}\left[\begin{array}{c}n\\k\end{array}\right]_{p,q}x^k.$$

which completes the proof. □

4. Some Generalizations of $L_n^{p,q}$

In this section, we present some generalizations of the operators $L_n^{p,q}$ based on (p,q)-integers similar to work done in [4, 2].

We consider a sequence of linear positive operators based on (p,q)-integers as follows:

$$L_n^{(p,q),\gamma}(f;x)=\frac{pq}{\ell_n^{p,q}(x)}\sum_{k=0}^{n}f\left(\frac{p^{n-k+1}[k]_{p,q}+\gamma}{b_{n,k}}\right)p^{\frac{(n-k)(n-k-1)}{2}}q^{\frac{k(k-1)}{2}}\left[\begin{array}{c}n\\k\end{array}\right]_{p,q}x^k,\ (\gamma\in\mathbb{R})\tag{4.1}$$

where $b_{n,k}$ satisfy the following conditions:

$$p^{n-k+1}[k]_{p,q}+b_{n,k}=c_n \text{ and } \frac{[n]_{p,q}}{c_n}\to 1 \text{ for } n\to\infty.$$

It is easy to check that if $b_{n,k}=q^k[n-k+1]_{p,q}+\beta$ for any n,k and $0<q<p\leq 1$, then $c_n=[n+1]_{p,q}+\beta$. If we choose $p=1$, then the operators reduce to the generalization of q-BBH opeartors defined in [2], and which turn out to be D. D. Stancu-type generalization of Bleimann, Butzer, and Hahn operators based on q-integers [20]. If we choose $\gamma=0,\ q=1$ as in [2] for $p=1$, then the operators become the special case of the Balzs-type generalization of the q-BBH operators [2] given in [4].

Theorem 4.1. *Let $p=p_n$ and $q=q_n$ satisfy (2.1) for $0<q_n<p_n\leq 1$ and let $L_n^{(p_n,q_n),\gamma}$ be defined by (4.1). Then for any function $f\in\widetilde{W}_{\alpha,[0,\infty)}$, we have*

$$\lim_n \parallel L_n^{(p_n,q_n),\gamma}(f;x)-f(x)\parallel_{C_B}\leq 3M$$

$$\times\max\left\{\left(\frac{[n]_{p_n,q_n}}{c_n+\gamma}\right)^{\alpha}\left(\frac{\gamma}{[n]_{p_n,q_n}}\right)^{\alpha},\left|1-\frac{[n+1]_{p_n,q_n}}{c_n+\gamma}\right|^{\alpha}\left(\frac{p_nq_n[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}}\right)^{\alpha},1-2\frac{p_nq_n[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}}+\frac{p_nq_n[n]_{p_n,q_n}[n-1]_{p_n,q_n}}{[n+1]^2_{p_n,q_n}}\right\}.$$

Proof. Using (1.3) and (4.1), we have

$$|\ L_n^{(p,q),\gamma}(f;x)-f(x)\ |$$

$$\leq\frac{pq}{\ell_n^{p_n,q_n}(x)}\sum_{k=0}^n\left|f\left(\frac{p_n^{n-k+1}[k]_{p_n,q_n}+\gamma}{b_{n,k}}\right)-f\left(\frac{p_n^{n-k+1}[k]_{p_n,q_n}}{\gamma+b_{n,k}}\right)\right|p_n^{\frac{(n-k)(n-k-1)}{2}}q_n^{\frac{k(k-1)}{2}}\left[\begin{array}{c}n\\k\end{array}\right]_{p_n,q_n}x^k$$

$$+\frac{pq}{\ell_n^{p_n,q_n}(x)}\sum_{k=0}^n\left|f\left(\frac{p_n^{n-k+1}[k]_{p_n,q_n}}{\gamma+b_{n,k}}\right)-f\left(\frac{p_n^{n-k+1}[k]_{p_n,q_n}}{[n-k+1]_{p_n,q_n}q_n^k}\right)\right|p_n^{\frac{(n-k)(n-k-1)}{2}}q_n^{\frac{k(k-1)}{2}}\left[\begin{array}{c}n\\k\end{array}\right]_{p_n,q_n}x^k,$$

we have

$$|\ L_n^{(p,q),\gamma}(f;x)-f(x)\ |$$

$$\leq\frac{pq}{\ell_n^{p_n,q_n}(x)}\sum_{k=0}^n\left|f\left(\frac{p_n^{n-k+1}[k]_{p_n,q_n}+\gamma}{b_{n,k}}\right)-f\left(\frac{p_n^{n-k+1}[k]_{p_n,q_n}}{\gamma+b_{n,k}}\right)\right|p_n^{\frac{(n-k)(n-k-1)}{2}+1}q_n^{\frac{k(k-1)}{2}+1}\left[\begin{array}{c}n\\k\end{array}\right]_{p_n,q_n}x^k$$

$$+\frac{pq}{\ell_n^{p_n,q_n}(x)}\sum_{k=0}^n\left|f\left(\frac{p_n^{n-k+1}[k]_{p_n,q_n}}{\gamma+b_{n,k}}\right)-f\left(\frac{p_n^{n-k+1}[k]_{p_n,q_n}}{[n-k+1]_{p_n,q_n}q_n^k}\right)\right|p_n^{\frac{(n-k)(n-k-1)}{2}+1}q_n^{\frac{k(k-1)}{2}+1}\left[\begin{array}{c}n\\k\end{array}\right]_{p_n,q_n}x^k$$

$$+\ |\ L_n^{p_n,q_n}(f;x)-f(x)\ |\ .$$

Now for $f\in\widetilde{W}_{\alpha,[0,\infty)}$, by using the Corollary 3.3, we can write

$$|\ L_n^{(p,q),\gamma}(f;x)-f(x)\ |$$

$$\leq\frac{M}{\ell_n^{p_n,q_n}(x)}\sum_{k=0}^n\left|\frac{p_n^{n-k+1}[k]_{p_n,q_n}+\gamma}{p_n^{n-k+1}[k]_{p_n,q_n}+\gamma+b_{n,k}}-\frac{p_n^{n-k+1}[k]_{p_n,q_n}}{\gamma+p_n^{n-k+1}[k]_{p_n,q_n}+b_{n,k}}\right|^{\alpha}p_n^{\frac{(n-k)(n-k-1)}{2}+1}q_n^{\frac{k(k-1)}{2}+1}\left[\begin{array}{c}n\\k\end{array}\right]_{p_n,q_n}x^k$$

$+\frac{M}{\ell_n^{p_n,q_n}(x)}\sum_{k=0}^{n}\left|\frac{p_n^{n-k+1}[k]_{p_n,q_n}}{p_n^{n-k+1}[k]_{p_n,q_n}+\gamma+b_{n,k}}-\frac{p_n^{n-k+1}[k]_{p_n,q_n}}{p_n^{n-k+1}[k]_{p_n,q_n}+[n-k+1]_{p_n,q_n}q_n^k}\right|$
$\times p_n^{\frac{(n-k)(n-k-1)}{2}+1}q_n^{\frac{k(k-1)}{2}+1}\left[\begin{array}{c} n \\ k \end{array}\right]_{p_n,q_n}x^k+M\delta_n^{\frac{\alpha}{2}}(x).$

This implies that

$$\mid L_n^{(p,q),\gamma}(f;x)-f(x)\mid\leq M\left(\frac{[n]_{p_n,q_n}}{c_n+\gamma}\right)^{\alpha}\left(\frac{\gamma}{[n]_{p_n,q_n}}\right)^{\alpha}$$
$$+\frac{M}{\ell_n^{p_n,q_n}(x)}\left|1-\frac{[n+1]_{p_n,q_n}}{c_n+\gamma}\right|^{\alpha}\sum_{k=0}^{n}\left(\frac{p_n^{n-k+1}[k]_{p_n,q_n}}{[n+1]_{p_n,q_n}}\right)^{\alpha}p_n^{\frac{(n-k)(n-k-1)}{2}+1}q_n^{\frac{k(k-1)}{2}+1}\left[\begin{array}{c} n \\ k \end{array}\right]_{p_n,q_n}x^k+M\delta_n^{\frac{\alpha}{2}}(x)$$
$$=M\left(\frac{[n]_{p_n,q_n}}{c_n+\gamma}\right)^{\alpha}\left(\frac{\gamma}{[n]_{p_n,q_n}}\right)^{\alpha}+M\left|1-\frac{[n+1]_{p_n,q_n}}{c_n+\gamma}\right|^{\alpha}L_n^{p_n,q_n}\left(\left(\frac{t}{1+t}\right)^{\alpha};x\right)+M\delta_n^{\frac{\alpha}{2}}(x).$$

Using the Hölder's inequality for $p=\frac{1}{\alpha},\ q=\frac{1}{1-\alpha}$, we get

$$\mid L_n^{(p,q),\gamma}(f;x)-f(x)\mid$$
$$\leq M\left(\frac{[n]_{p_n,q_n}}{c_n+\gamma}\right)^{\alpha}\left(\frac{\gamma}{[n]_{p_n,q_n}}\right)^{\alpha}+M\left|1-\frac{[n+1]_{p_n,q_n}}{c_n+\gamma}\right|^{\alpha}L_n^{p_n,q_n}\left(\frac{t}{1+t};x\right)^{\alpha}\left(L_n^{p_n,q_n}(1;x)\right)^{1-\alpha}+$$
$$M\delta_n^{\frac{\alpha}{2}}(x)$$
$$\leq M\left(\frac{[n]_{p_n,q_n}}{c_n+\gamma}\right)^{\alpha}\left(\frac{\gamma}{[n]_{p_n,q_n}}\right)^{\alpha}+M\left|1-\frac{[n+1]_{p_n,q_n}}{c_n+\gamma}\right|^{\alpha}\left(\frac{p_nq_n[n]_{p_n,q_n}}{[n+1]_{p_n,q_n}}\frac{x}{1+x}\right)^{\alpha}+M\delta_n^{\frac{\alpha}{2}}(x),$$

which completes the proof. □

5. Construction of the bivariate operators

In what follows we construct the bivariate extension of the operators (1.3). We will introduce the statistical convergence of the operators to a function f and investigate the statistical rate of convergence of these operators.

Let $\mathbb{R}_+^2=[0,\infty)\times[0,\infty)$, $f:\mathbb{R}_+^2\to\mathbb{R}$ and $0<p_{n_1},q_{n_1};p_{n_2},q_{n_2}\leq 1$. Then we define the bivariate companion of the operators (1.3) as follows:

$$L_{n_1,n_2}(f;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y)=\frac{p_{n_1}p_{n_2}q_{n_1}q_{n_2}}{l_{n_1}^{p_{n_1},q_{n_1}}\times l_{n_2}^{p_{n_2},q_{n_2}}}\sum_{k_1=0}^{n_1}\sum_{k_2=0}^{n_2}f\left(\frac{p_{n_1}^{n_1-k_1+1}[k_1]_{p_{n_1},q_{n_1}}}{[n_1-k_1+1]_{p_{n_1},q_{n_1}}q_{n_1}^{k_1}},\right.$$
$$\left.\frac{p_{n_2}^{n_2-k_2+1}[k_2]_{p_{n_2},q_{n_2}}}{[n_2-k_2+1]_{p_{n_2},q_{n_2}}q_{n_2}^{k_2}}\right)p_{n_1}^{\frac{(n_1-k_1)(n_1-k_1-1)}{2}}p_{n_2}^{\frac{(n_2-k_2)(n_2-k_2-1)}{2}}q_{n_1}^{\frac{k_1(k_1-1)}{2}}q_{n_2}^{\frac{k_2(k_2-1)}{2}}\left[\begin{array}{c} n_1 \\ k_1 \end{array}\right]_{p_{n_1},q_{n_1}}$$
$$\times\left[\begin{array}{c} n_2 \\ k_2 \end{array}\right]_{p_{n_2},q_{n_2}}x^{k_1}y^{k_2},$$

where $l_{n_1}^{p_{n_1},q_{n_1}}=\prod_{s=0}^{n_1-1}(p_{n_1}^s+q_{n_1}^s x)$ and $l_{n_2}^{p_{n_2},q_{n_2}}=\prod_{s=0}^{n_2-1}(p_{n_2}^s+q_{n_2}^s y)$.

For $K=[0,\infty)\times[0,\infty)$, the modulus of continuity for the bivariate case is defined as

$$\omega_2(g;\delta_1,\delta_2)=sup\{|g(u_1,v_1)-g(u_2,v_2)|:(u_1,v_1),(u_2,v_2)\in K\ and\ |u_1-u_2|\leq\delta_1,|v_1-v_2|\leq\delta_2\},$$

where, for each $g \in H_{\omega_2}$, $\omega_2(g;\delta_1,\delta_2)$ satisfies

$$|g(u_1,v_1)-g(u_2,v_2)| \leq \omega_2\left(g\left|\frac{u_1}{1+u_1}-\frac{u_2}{1+u_2}\right|,\left|\frac{v_1}{1+v_1}-\frac{v_2}{1+v_2}\right|\right).$$

For detailed study of modulus of continuity for the bivariate analogue one is referred to [1].

The first Korovkin type theorem for the statistical approximation for the bivariate analogue of linear positive operators defined in the space H_{ω_2} was obtained by Erkus and Duman [5] which is as follows.

Theorem 5.1. *Let $\{L_n\}$ be a sequence of positive linear operators from H_{ω_2} into $C_B(K)$. Then, for each $g \in H_{\omega_2}$,*

$$st-\lim_n \parallel L_n(g)-g \parallel = 0,$$

holds if the following is satisfied

$$st-\lim_n \parallel L_n(g_j)-g_j \parallel = 0, \;\; for \;\; j=0,1,2,3$$

where

$$g_0(u,v)=0,\;\; g_1(u,v)=\frac{u}{1+u},\;\; g_2(u,v)=\frac{v}{1+v},\;\; g_3(u,v)=(\frac{u}{1+u})^2+(\frac{v}{1+v})^2. \tag{5.1}$$

To study the statistical convergence of the bivariate operators, the following lemma is essential.

Lemma 5.2. *The bivariate operators defined above satisfy the followings:*

(1) $L_{n_1,n_2}(f_0;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y)=p_{n_1}p_{n_2}q_{n_1}q_{n_2}$,

(2) $L_{n_1,n_2}(f_1;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y)=p_{n_1}p_{n_2}q_{n_1}q_{n_2}\frac{[n_1]_{p_{n_1},q_{n_1}}}{[n_1+1]_{p_{n_1},q_{n_1}}}\frac{x}{1+x}$,

(3) $L_{n_1,n_2}(f_2;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y)=p_{n_1}p_{n_2}q_{n_1}q_{n_2}\frac{[n_2]_{p_{n_2},q_{n_2}}}{[n_2+1]_{p_{n_2},q_{n_2}}}\frac{y}{1+y}$

(4) $L_{n_1,n_2}(f_3;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y)=p^3_{n_1}p_{n_2}q^3_{n_1}q_{n_2}\frac{[n_1]_{p_{n_1},q_{n_1}}[n_1-1]_{p_{n_1},q_{n_1}}}{[n_1+1]^2_{p_{n_1},q_{n_1}}}\frac{x^2}{(1+x)(p_{n_1}+q_{n_1}x)}+$
$p_{n_1}p_{n_2}q_{n_1}q_{n_2}\frac{[n_1]_{p_{n_1},q_{n_1}}}{[n_1+1]^2_{p_{n_1},q_{n_1}}}\frac{x}{1+x}+p_{n_1}p^3_{n_2}q_{n_1}q^3_{n_2}\frac{[n_2]_{p_{n_2},q_{n_2}}[n_2-1]_{p_{n_2},q_{n_2}}}{[n_2+1]^2_{p_{n_2},q_{n_2}}}\frac{y^2}{(1+y)(p_{n_2}+q_{n_2}y)}+$
$p_{n_1}p_{n_2}q_{n_1}q_{n_2}\frac{[n_2]_{p_{n_2},q_{n_2}}}{[n_2+1]_{p_{n_2},q_{n_2}}}\frac{y}{1+y}$.

Proof. Exploiting the proofs for the bivariate operators in [6], the above can be easily established. So we skip the proof. □

Now let the sequences

$$p=(p_{n_1}),p=(p_{n_2}),q=(q_{n_1}),q=(q_{n_2})$$

be statistically convergent to unity but not convergent in usual sense, so we can write them for $0<p_{n_1},q_{n_1},p_{n_2},q_{n_2}\leq 1$ as

$$st-\lim_{n_1}p_{n_1}=st-\lim_{n_1}q_{n_1}=st-\lim_{n_2}p_{n_2}=st-\lim_{n_2}q_{n_2}=1. \tag{5.2}$$

Now making use of the proof of Theorem (2.2) and conditions (5.2), we establish the statistical convergence of the bivariate operators introduced above.

Theorem 5.3. *Let* $p = (p_{n_1}),\ p = (p_{n_2}),\ q = (q_{n_1})$ *and* $q = (q_{n_2})$ *be the sequences subject to conditions (2.10) and let* L_{n_1,n_2} *be the sequence of linear positive operators from* $H_{\omega_2}(R^2_+)$ *into* $C_B(R_+)$*. Then for each* $g \in H_{\omega_2}$,

$$st - \lim_{n_1,n_2} \|L_{n_1,n_2}(g) - g\| = 0$$

Proof. With the aid of the Lemma (5.2), a proof similar to the proof of the Theorem (2.2) can be easily obtained. So we shall omit the proof. □

Rates of convergence of the bivariate operators

For any $g \in H_{\omega_2}(R^2_+)$, the modulus of continuity of the bivariate analogue is defined as:

$$\tilde{\omega}(g;\delta_1,\delta_2) = \sup_{x_1,x_2\geq 0}\left\{|g(x_1,y_1)-g(x_2,y_2)| : \left|\frac{x_1}{1+x_1}-\frac{x_2}{1+x_2}\right| \leq \delta_1, \left|\frac{y_1}{1+y_1}-\frac{y_2}{1+y_2}\right| \leq \delta_2, \right.$$
$$\left. (x_1,y_1),(x_2,y_2) \in H_{\omega_2}(R^2_+)\right\}$$

For details of this sort of modulus, one is referred to [1].

Two chief properties of $\tilde{\omega}(g;\delta_1,\delta_2)$ are

(1) $\tilde{\omega}(g;\delta_1,\delta_2) \to 0$ as $\delta_1 \to 0$ and $\delta_2 \to 0$ and

(2) $|g(x_1,y_1)-g(x_2,y_2)| \leq \tilde{\omega}(g;\delta_1,\delta_2)\left(1+\frac{\left|\frac{x_1}{1+x_1}-\frac{x_2}{1+x_2}\right|}{\delta_1}\right)\left(1+\frac{\left|\frac{y_1}{1+y_1}-\frac{y_2}{1+y_2}\right|}{\delta_2}\right)$.

Now in the following theorem we study the rate of statistical convergence of the bivariate operators through modulus of continuity in H_{ω_2}.

Theorem 5.4. *Let* $p = (p_{n_1}), p = (p_{n_2}), q = (q_{n_1}), q = (q_{n_2})$ *be four sequences obeying conditions of (5.2). Then we have*

$$|L_{n_1,n_2}(f;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y)-f(x,y)| \leq 4p^2_{n_1}p^2_{n_2}q^2_{n_1}q^2_{n_2}\ \omega(f;\sqrt{\delta_{n_1}(x)},\sqrt{\delta_{n_2}(x)}),$$

where

$$\delta_{n_1}(x) = \frac{x^2}{(1+x)^2}\left(p^2_{n_1}q^2_{n_1}\frac{(1+x)}{p_{n_1}+q_{n_1}x}\frac{[n_1]_{p_{n_1},q_{n_1}}[n_1-1]_{p_{n_1},q_{n_1}}}{[n_1+1]^2_{p_{n_1},q_{n_1}}} - 2\frac{[n_1]_{p_{n_1},q_{n_1}}}{[n_1+1]_{p_{n_1},q_{n_1}}}+1\right)$$
$$+\frac{x}{1+x}\frac{[n_1]_{p_{n_1},q_{n_1}}}{[n_1+1]^2_{p_{n_1},q_{n_1}}},$$

$$\delta_{n_2}(y) = \frac{y^2}{(1+y)^2}\left(p^2_{n_2}q^2_{n_2}\frac{(1+y)}{p_{n_2}+q_{n_1}y}\frac{[n_2]_{p_{n_2},q_{n_2}}[n_2-1]_{p_{n_2},q_{n_2}}}{[n_1+1]^2_{p_{n_2},q_{n_2}}} - 2\frac{[n_2]_{p_{n_2},q_{n_2}}}{[n_2+1]_{p_{n_2},q_{n_2}}}+1\right)$$
$$+\frac{y}{1+y}\frac{[n_2]_{p_{n_2},q_{n_2}}}{[n_2+1]^2_{p_{n_2},q_{n_2}}}.$$

Proof. Using the property of the modulus above, we have

$$|L_{n_1,n_2}(f;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y)-f(x,y)|\leq \omega(f:\delta_{n_1},\delta_{n_2})\{L_{n_1,n_2}(f_0;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y)$$
$$+\frac{1}{\delta_{n_1}}L_{n_1,n_2}(|\frac{t}{1+t}-\frac{x}{1+x}|;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y)\}\{L_{n_1,n_2}(f_0;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y)$$
$$+\frac{1}{\delta_{n_2}}L_{n_1,n_2}(|\frac{s}{1+s}-\frac{y}{1+y}|;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y)\}.$$

Applying the Cauchy-Schwarz inequality, we get

$$L_{n_1,n_2}(|\frac{t}{1+t}-\frac{x}{1+x}|;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y)\leq\left(L_{n_1,n_2}\left(\left(\frac{t}{1+t}-\frac{x}{1+x}\right)^2;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y\right)\right)^{\frac{1}{2}}$$
$$\times\left(L_{n_1,n_2}(f_0;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y)\right)^{\frac{1}{2}}.$$

On substituting this in the above inequality, we get the proof of the theorem. □

Now we shall study the statistical convergence of the bivariate operators using Lipschitz type maximal functions. The Lipschitz type maximal function space on $E\times E\subset\mathbb{R}_+\times\mathbb{R}_+$ is defined as follows

$$\tilde{W}_{\alpha_1,\alpha_2,E^2}=\left\{f:sup(1+t)^{\alpha_1}(1+s)^{\alpha_2}\tilde{f}_{\alpha_1,\alpha_2}(x,y)\leq M\frac{1}{(1+x)^{\alpha_1}}\frac{1}{(1+y)^{\alpha_2}};\ x,y\geq 0,(t,s)\in E^2\right\}. \tag{5.3}$$

Where f is a bounded and continuous function on $\mathbb{R}_+$, M is a positive constant and $0\leq\alpha_1,\alpha_2\leq 1$ and $\tilde{f}_{\alpha_1,\alpha_2}(x,y)$ is defined as follows:

$$\tilde{f}_{\alpha_1,\alpha_2}(x,y)=\sup_{t,s\geq 0}\frac{|f(t,s)-f(x,y)|}{|t-x|^{\alpha_1}|s-y|^{\alpha_2}}.$$

Theorem 5.5. *Let $p=(p_{n_1}),p=(p_{n_2}),q=(q_{n_1}),q=(q_{n_2})$ be four sequences satisfying the conditions of (5.2). Then we have*

$$|L_{n_1,n_2}(f;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y)-f(x,y)|\leq Mp_{n_1}p_{n_2}q_{n_1}q_{n_2}\{\delta_{n_1}(x)^{\frac{\alpha_1}{2}}\delta_{n_2}(y)^{\frac{\alpha_2}{2}}(p_{n_1}p_{n_2}q_{n_1}q_{n_2})$$
$$+\delta_{n_1}(x)^{\frac{\alpha_1}{2}}d(y,E)^{\alpha_2}+\delta_{n_2}(y)^{\frac{alpha_2}{2}}d(x,E)^{\alpha_1}+2d(x,E)^{\alpha_1}d(y,E)^{\alpha_2}\},$$

where $0\leq\alpha_1,\alpha_2\leq 1$ and $\delta_{n_1}(x)$, $\delta_{n_2}(y)$ are defined as in Theorem (2.11) and $d(x,E)=inf\{|x-y|:y\in E\}$.

Proof. For $x,y\geq 0$ and $(x_1,y_1)\in E\times E$, we can write

$$|f(t,s)-f(x,y)|\leq|f(t,s)-f(x_1,y_1)|+|f(x_1,y_1)-f(x,y)|.$$

Applying the operator L_{n_1,n_2} to both sides of the above inequality and making use of (2.11), we have

$$|L_{n_1,n_2}(f;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y)-f(x,y)|\leq L_{n_1,n_2}(|f(t,s)-f(x_1,y_1)|;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y)$$
$$+|f(x_1,y_1)-f(x,y)|L_{n_1,n_2}(f_0;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y)$$
$$\leq ML_{n_1,n_2}\left(|\frac{y_2}{1+y_2}-\frac{x_1}{1+x_1}|^{\alpha_1}|\frac{x_2}{1+x_2}-\frac{y_1}{1+y_1}|^{\alpha_2};p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y)\right)$$
$$+M|\frac{x}{1+x}-\frac{x_1}{1+x_1}|^{\alpha_1}|\frac{y}{1+y}-\frac{y_1}{1+y_1}|^{\alpha_2}L_{n_1,n_2}(f_0;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y).$$

Now for $0 \leq p \leq 1$, using $(a+b)^p \leq a^p + b^q$, we can write

$$\Big|\frac{y_2}{1+y_2} - \frac{x_1}{1+x_1}\Big|^{\alpha_1} \leq \Big|\frac{y_2}{1+y_2} - \frac{x}{1+x}\Big|^{\alpha_1} + \Big|\frac{x}{1+x} - \frac{x_1}{1+x_1}\Big|^{\alpha_1}$$

and

$$\Big|\frac{x_2}{1+x_2} - \frac{y_1}{1+y_1}\Big|^{\alpha_2} \leq \Big|\frac{x_2}{1+x_2} - \frac{y}{1+y}\Big|^{\alpha_2} + \Big|\frac{y}{1+y} - \frac{y_1}{1+y_1}\Big|^{\alpha_2}.$$

Using these inequalities in the above, we get

$$|L_{n_1,n_2}(f;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y) - f(x,y)| \leq L_{n_1,n_2}(\Big|\frac{y_2}{1+y_2} - \frac{x}{1+x}\Big|^{\alpha_1}\Big|\frac{x_2}{1+x_2} - \frac{y}{1+y}\Big|^{\alpha_2}$$
$$;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y) + \Big|\frac{y}{1+y} - \frac{y_1}{1+y_1}\Big|^{\alpha_2} L_{n_1,n_2}(\Big|\frac{y_2}{1+y_2} - \frac{x}{1+x}\Big|^{\alpha_1};p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y)$$
$$+\Big|\frac{x}{1+x} - \frac{x_1}{1+x_1}\Big|^{\alpha_1} L_{n_1,n_2}(\Big|\frac{x_2}{1+x_2} - \frac{y}{1+y}\Big|^{\alpha_2};p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y) + \Big|\frac{x}{1+x} - \frac{x_1}{1+x_1}\Big|^{\alpha_1}$$
$$\times\Big|\frac{y}{1+y} - \frac{y_1}{1+y_1}\Big|^{\alpha_2} L_{n_1,n_2}(f_0;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y).$$

Now using the Hölder's inequality for $p_1 = \frac{2}{\alpha_1}, p_2 = \frac{2}{\alpha_2}, q_1 = \frac{2}{2-\alpha_1} q_2 = \frac{2}{2-\alpha_2}$, we get

$$L_{n_1,n_2}(\Big|\frac{y_2}{1+y_2} - \frac{x}{1+x}\Big|^{\alpha_1}\Big|\frac{x_2}{1+x_2} - \frac{y}{1+y}\Big|^{\alpha_2};p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y) = L_{n_1,n_2}(\Big|\frac{y_2}{1+y_2} - \frac{x}{1+x}\Big|^{\alpha_1}$$
$$;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y) L_{n_1,n_2}(\Big|\frac{x_2}{1+x_2} - \frac{y}{1+y}\Big|^{\alpha_2};p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y) \leq (L_{n_1,n_2}(\frac{y_2}{1+y_2}$$
$$-\frac{x}{1+x})^2;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y)^{\frac{\alpha_1}{2}} (L_{n_1,n_2}(f_0;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y))^{\frac{2-\alpha_1}{2}} (L_{n_1,n_2}(\frac{x_2}{1+x_2}$$
$$-\frac{y}{1+y})^2;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y)^{\frac{\alpha_2}{2}} (L_{n_1,n_2}(f_0;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y))^{\frac{2-\alpha_2}{2}}.$$

This consequently gives the desired result. Therefore the proof is complete. □

Remark 5.6. For $E = [0,\infty)$, we see that $d(x,E) = 0$ and $d(y,E) = 0$, so that we have $|L_{n_1,n_2}(f;p_{n_1},p_{n_2};q_{n_1},q_{n_2};x,y) - f(x,y)| \leq M(p_{n_1}p_{n_2}q_{n_1}q_{n_2})^{4-\frac{\alpha_1+\alpha_2}{2}} \delta_{n_1}(x)^{\frac{\alpha_1}{2}} \delta_{n_2}(y)^{\frac{\alpha_2}{2}}$.

Remark 5.7. By means of (2.10), it can be easily seen that $st-\lim_{n_1} \delta_{n_1} = 0$ and $st-\lim_{n_2} \delta_{n_2} = 0$. So we can estimate the order of statistical approximation of our bivariate operators by means of Lipschitz type maximal functions using this result.

Also as

$$\sup_{x\geq 0} \delta_{n_1}(x) \leq \frac{p_{n_1}^{2n_1} q_{n_1}^{2n_1}}{[n_1+1]^2_{p_{n_1},q_{n_1}}}$$

and

$$p_{n_1}^{n_1} q_{n_1}^{n_1}(n_1+1) \leq \left(\frac{1}{p_{n_1}^{n_1} q_{n_1}^{n_1}} + \dots + \frac{1}{p_{n_1}q_{n_1}} + 1\right) p_{n_1}^{n_1} q_{n_1}^{n_1}$$

So for $0 \leq p_{n_1}, q_{n_1} \leq 1$, we get

$$\frac{p_{n_1}^{2n_1} q_{n_1}^{2n_1}}{[n_1+1]^2_{p_{n_1},q_{n_1}}} \leq \frac{1}{(n_1+1)^2}.$$

In a similar fashon we can obtain it for $\delta_{n_2}(y)$. So we have the following concluding remark.

Remark 5.8. This chapter has two main features:

(1) δ_{n_1} and δ_{n_2} approach to zero in statistical sense however they may not tend to zero in the usual sense.
(2) In our case δ_{n_1} and δ_{n_2} approach to zero faster than that of the classical BBH operators.

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