

DEFORMATIONS OF COMPLEXES FOR FINITE DIMENSIONAL ALGEBRAS

FRAUKE M. BLEHER AND JOSÉ A. VÉLEZ-MARULANDA

ABSTRACT. Let k be a field and let Λ be a finite dimensional k -algebra. We prove that every bounded complex V^\bullet of finitely generated Λ -modules has a well-defined versal deformation ring $R(\Lambda, V^\bullet)$ which is a complete local commutative Noetherian k -algebra with residue field k . We also prove that nice two-sided tilting complexes between Λ and another finite dimensional k -algebra Γ preserve these versal deformation rings. Additionally, we investigate stable equivalences of Morita type between self-injective algebras in this context. We apply these results to the derived equivalence classes of the members of a particular family of algebras of dihedral type that were introduced by Erdmann and shown by Holm to be not derived equivalent to any block of a group algebra.

1. INTRODUCTION

The main objective of the theory of deformations of algebraic objects, such as modules or group representations, is to study the behavior of these objects under perturbations. Suppose \mathcal{O} is a complete local commutative Noetherian ring with residue field k , $\Lambda_{\mathcal{O}}$ is an \mathcal{O} -algebra and $\Lambda = k \otimes_{\mathcal{O}} \Lambda_{\mathcal{O}}$. If V is a Λ -module of finite k -dimension, the deformations of V are defined to be isomorphism classes of lifts of V over complete local commutative Noetherian \mathcal{O} -algebras R with residue field k . Here a lift of V over R is an $R \otimes_{\mathcal{O}} \Lambda_{\mathcal{O}}$ -module M that is free over R , together with a Λ -module isomorphism $\phi : k \otimes_R M \rightarrow V$.

Lifts and deformations of this kind were studied by Green in [18] in the 1950's in the case when \mathcal{O} is a ring of p -adic integers, for some prime number p , and $\Lambda_{\mathcal{O}}$ is the group algebra of a group G over \mathcal{O} . Green's work inspired Auslander, Ding and Solberg in [1] to consider more general \mathcal{O} -algebras $\Lambda_{\mathcal{O}}$ and more general lifting problems. In the 1970's Laudal developed a theory of formal moduli of algebraic structures, and he used Massey products to describe deformations of k -algebras and their modules over complete local commutative Artinian k -algebras with residue field k (see [23] and its references). In the 1980's Mazur developed a theory of deformations of group representations to systematically study p -adic lifts of representations of profinite Galois groups over finite fields of characteristic p (see [25, 26]). Both Laudal and Mazur used Schlessinger's criteria in [32] for the pro-representability of functors of Artinian rings. One advantage of Mazur's approach is that he uses a continuous deformation functor, which allows him to include the deformations over arbitrary complete local commutative Noetherian \mathcal{O} -algebras with residue field k directly in his functorial description and not just as inverse limits. In particular, Mazur proved in [25] that a finite dimensional Galois representation over a finite field always has a versal deformation ring, and in the case when the representation is absolutely irreducible that this versal deformation ring is universal. In [9], Mazur's approach was used by the authors to study deformation rings and deformations of modules for arbitrary finite dimensional k -algebras Λ when $\mathcal{O} = k$, and to provide additional structure theorems in the case when Λ is self-injective or Frobenius.

Let now k be a field of arbitrary characteristic, let $\mathcal{O} = k$, and let $\Lambda = \Lambda_{\mathcal{O}}$ be a finite dimensional k -algebra. Our first goal is to generalize the deformation theory for finitely generated Λ -modules

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in [9] to bounded complexes of finitely generated Λ -modules. We accomplish this goal in Section 2. Many of our techniques are based on the generalization of Mazur’s deformation theory to bounded complexes of Galois representations in [5, 6].

More precisely, let $D^-(\Lambda)$ be the derived category of bounded above cochain complexes of pseudocompact Λ -modules (see Section 2.1 for a review of pseudocompact rings and modules). The following is our first main result (see Theorem 2.1.12 for a more precise statement):

Theorem 1.1. *Let V^\bullet be an object of $D^-(\Lambda)$ such that V^\bullet only has finitely many non-zero cohomology groups, all of which have finite k -dimension. Then V^\bullet always has a versal deformation ring $R(\Lambda, V^\bullet)$ which is a complete local commutative Noetherian k -algebra with residue field k . Moreover, $R(\Lambda, V^\bullet)$ is universal if the endomorphism ring of V^\bullet in $D^-(\Lambda)$ is isomorphic to k .*

Additionally, we prove that the case of modules corresponds to the case when V^\bullet has precisely one non-zero cohomology group. The main challenge of the proof of Theorem 1.1 is to ensure that the arguments in [5, 6] can be modified to work for arbitrary finite dimensional k -algebras Λ that may be neither Frobenius nor self-injective.

Note that we provide more details concerning the continuity of our deformation functor than were provided for the deformation functor defined in [6]. This can also be used to better explain the arguments used to prove [6, Prop. 7.2]; see Remark 2.4.5.

There has been a lot of interest in classifying finite dimensional k -algebras up to derived or stable equivalences. One of the most famous conjectures in this context is Broué’s conjecture that blocks of group rings of finite groups G with an abelian defect group D are derived equivalent to blocks of the normalizer of D in G (see [10, 31] and their references). This conjecture has spurred a lot of work on derived equivalences for more general algebras. For example, Rickard proved in [29] that two finite dimensional k -algebras Λ and Γ are derived equivalent if and only if there is a derived equivalence between them that is given by the left derived tensor product functor with a so-called two-sided tilting complex. Such a derived equivalence is also called a standard derived equivalence. It is then a natural question to ask whether standard derived equivalences preserve versal deformation rings of complexes V^\bullet as in Theorem 1.1.

In [30], Rickard showed that if Λ and Γ are derived equivalent block algebras for finite groups then one can choose the two-sided tilting complex providing the standard derived equivalence to be particularly nice (namely, to be a split-endomorphism two-sided tilting complex). In [3], it was shown that such nice two-sided tilting complexes indeed preserve versal deformation rings when Λ and Γ are block algebras.

Our second goal, accomplished in Section 3, is to provide a variation on “niceness” of two-sided tilting complexes that works for arbitrary finite dimensional algebras (see Definition 3.1.1). This leads to our second main result (see Theorem 3.1.5 for a more precise statement):

Theorem 1.2. *Suppose Γ is another finite dimensional k -algebra such that Λ and Γ are derived equivalent. Then there exists a nice two-sided tilting complex P^\bullet of finitely generated Γ - Λ -bimodules such that if V^\bullet is a bounded complex of finitely generated Λ -modules and $V'^\bullet = P^\bullet \otimes_\Lambda^L V^\bullet$, then the versal deformation rings $R(\Lambda, V^\bullet)$ and $R(\Gamma, V'^\bullet)$ are isomorphic.*

To prove Theorem 1.2, the main challenge is again to modify the arguments in [3] so that they work for arbitrary finite dimensional k -algebras Λ that may be neither Frobenius nor self-injective.

In [29], Rickard proved that if Λ and Γ are self-injective derived equivalent k -algebras, then there is a stable equivalence of Morita type between them, as introduced by Broué in [11]. Such stable equivalences of Morita type provide especially well-behaved equivalences between the stable module categories of Λ and Γ . On the other hand, not every stable equivalence of Morita type between self-injective algebras is induced by a derived equivalence (see, for example, [13] and its references). Therefore, we prove in Proposition 3.2.6 that arbitrary stable equivalences of Morita type between self-injective algebras preserve versal deformation rings of modules.

In Section 4, we show how the main results from Sections 2 and 3 can be applied to the derived equivalence classes of the members of a particular family of algebras of dihedral type that was

introduced by Erdmann in [14] and denoted by $D(3\mathcal{R})$. Note that Holm showed in [19, Sect. 3.2] that none of the algebras in the family $D(3\mathcal{R})$ is derived equivalent to a block of a group algebra. Let Λ_0 be a particular algebra in the family $D(3\mathcal{R})$. Theorems 4.2.2, 4.2.4 and Proposition 4.2.5 demonstrate how the knowledge of the universal deformation rings of certain Λ_0 -modules can be used to determine the universal deformation rings of modules for another algebra Λ that is just known to be derived equivalent to Λ_0 .

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Convention 1.3. Throughout this paper “complex” means “cochain complex.” The degree n term of a complex C^\bullet is denoted by C^n and its degree n differential is denoted by $\delta^n = \delta_C^n : C^n \rightarrow C^{n+1}$.

Even if we apply contravariant functors to cochain complexes, we shall assume, without saying this explicitly, that the terms of the resulting complexes are renumbered in order to regain cochain complexes. Thus, for example, the k -dual $\text{Hom}_k(V^\bullet, k)$ of a bounded complex V^\bullet of Λ -modules will have degree n term given by $\text{Hom}_k(V^{-n}, k)$.

If C^\bullet is a complex and i is an integer, then $C^\bullet[i]$ is the complex obtained by “shifting C^\bullet to the left by i places.” More precisely, the degree n term of $C^\bullet[i]$ is C^{n+i} and the degree n differential of $C^\bullet[i]$ is $(-1)^i \delta_C^{n+i}$.

Our complexes are either bounded above, i.e. complexes C^\bullet with $C^n = 0$ for $n \gg 0$, or bounded, i.e. complexes C^\bullet with $C^n = 0$ for all but finitely many n .

2. VERSAL DEFORMATION RINGS FOR COMPLEXES OVER FINITE DIMENSIONAL ALGEBRAS

In [5, 6], it was proved that if k is a field of positive characteristic, G is a profinite group satisfying a certain finiteness condition and V^\bullet is quasi-isomorphic to a bounded complex of pseudocompact $[[kG]]$ -modules, then V^\bullet always has a versal deformation ring. Moreover, it was proved that if the endomorphism ring of V^\bullet in the derived category of bounded above complexes of pseudocompact $[[kG]]$ -modules is isomorphic to k , then this versal deformation ring is universal.

It is the goal of this section to prove an analogous result when k is an arbitrary field and $[[kG]]$ is replaced by an arbitrary finite dimensional k -algebra Λ . In Section 2.1, we will recall some results on pseudocompact rings and modules, define quasi-lifts and deformations of complexes, and state our main result of this section, Theorem 2.1.12. In Section 2.2, we analyze the structure of quasi-lifts over Artinian rings. In Section 2.3, we provide some more results on complexes and quasi-lifts. In Section 2.4, we prove Theorem 2.1.12. In Section 2.5, we consider quasi-lifts of one-term and two-term complexes, and completely split complexes.

2.1. Pseudocompact rings and modules, quasi-lifts, and deformations of complexes.

Recall from [12] that a pseudocompact ring Ω is a complete Hausdorff topological ring that admits a system of open neighborhoods of 0 consisting of two-sided ideals I for which Ω/I is an Artinian ring. In particular, every finite dimensional algebra Λ over any field k is a pseudocompact ring, by choosing all two-sided ideals of Λ as an open neighborhood basis of 0. In other words, Λ is a pseudocompact ring with the discrete topology. Moreover, since k is a commutative pseudocompact ring (again with the discrete topology), Λ is a pseudocompact k -algebra. A Λ -module M is called pseudocompact if M is a complete Hausdorff topological Λ -module that has a basis of open neighborhoods of 0 consisting of submodules N for which M/N has finite length as Λ -module. In particular, each finitely generated Λ -module M is pseudocompact, by choosing all submodules of M as an open neighborhood basis of 0 (i.e. by giving M the discrete topology). The category of pseudocompact Λ -modules is an abelian category; see below for more details. Let $D^-(\Lambda)$ be the derived category of bounded above complexes of pseudocompact Λ -modules.

Hypothesis 1. *Throughout this paper, we assume that k is an arbitrary field, Λ is a finite dimensional k -algebra, and V^\bullet is a complex in $D^-(\Lambda)$ that has only finitely many non-zero cohomology groups, all of which have finite k -dimension.*

Define $\hat{\mathcal{C}}$ to be the category of all complete local commutative Noetherian k -algebras with residue field k . The morphisms in $\hat{\mathcal{C}}$ are continuous k -algebra homomorphisms that induce the identity on k . Let \mathcal{C} be the full subcategory of Artinian rings in $\hat{\mathcal{C}}$.

For $R \in \text{Ob}(\hat{\mathcal{C}})$, define $R\Lambda = R \otimes_k \Lambda$. Then R is a commutative pseudocompact ring, and $R\Lambda$ is a pseudocompact R -algebra. Define $\text{PCMod}(R\Lambda)$ to be the category of pseudocompact $R\Lambda$ -modules.

Pseudocompact rings, algebras and modules have been studied, for example, in [16, 17] and [12]. For the convenience of the reader, we state some useful facts from these references.

Remark 2.1.1. Let $R \in \text{Ob}(\hat{\mathcal{C}})$.

- (i) The ring $R\Lambda$ is the inverse limit of Artinian quotient rings. An $R\Lambda$ -module is pseudocompact if and only if it is the inverse limit of $R\Lambda$ -modules of finite length. Moreover, an $R\Lambda$ -module has finite length if and only if it has finite length as an R -module. The category $\text{PCMod}(R\Lambda)$ is an abelian category with exact inverse limits.
- (ii) A pseudocompact $R\Lambda$ -module M is said to be topologically free on a set $X = \{x_i\}_{i \in I}$ if M is isomorphic to the product of a family $(R\Lambda_i)_{i \in I}$ where $R\Lambda_i = R\Lambda$ for all i . Every topologically free pseudocompact $R\Lambda$ -module is a projective object of $\text{PCMod}(R\Lambda)$, and every pseudocompact $R\Lambda$ -module is the quotient of a topologically free $R\Lambda$ -module. Hence $\text{PCMod}(R\Lambda)$ has enough projective objects.
- (iii) Suppose M and N are pseudocompact $R\Lambda$ -modules. Then we define the right derived functors $\text{Ext}_{R\Lambda}^n(M, N)$ by using a projective resolution of M in $\text{PCMod}(R\Lambda)$.

Remark 2.1.2. Let R be an object of $\hat{\mathcal{C}}$ with maximal ideal m_R . Suppose that $(R/m_R^i)X_i$ denotes an abstractly free (R/m_R^i) -module on the finite topological space X_i for all i , and that $\{X_i\}_i$ forms an inverse system. Define $X = \varprojlim_i X_i$ and $[[RX]] = \varprojlim_i (R/m_R^i)X_i$. Then $[[RX]]$ is a topologically free pseudocompact R -module on X .

Remark 2.1.3. Suppose $R \in \text{Ob}(\hat{\mathcal{C}})$, and $\Omega = R$ or $\Omega = R\Lambda$. Let M be a right (resp. left) pseudocompact Ω -module.

- (i) Let $\hat{\otimes}_\Omega$ denote the completed tensor product in the category $\text{PCMod}(\Omega)$ (see [12, Sect. 2]). Then $M\hat{\otimes}_\Omega -$ (resp. $-\hat{\otimes}_\Omega M$) is a right exact functor. Moreover, M is said to be topologically flat, if the functor $M\hat{\otimes}_\Omega -$ (resp. $-\hat{\otimes}_\Omega M$) is exact.
- (ii) By [12, Lemma 2.1] and [12, Prop. 3.1], M is topologically flat if and only if M is projective as a pseudocompact Ω -module.
- (iii) If M is finitely generated as a pseudocompact Ω -module, it follows from [12, Lemma 2.1(ii)] that the functors $M\hat{\otimes}_\Omega -$ and $M\hat{\otimes}_\Omega -$ (resp. $-\hat{\otimes}_\Omega M$ and $-\hat{\otimes}_\Omega M$) are naturally isomorphic.
- (iv) If $\Omega = R$ and M is a pseudocompact R -module, it follows from [17, Proof of Prop. 0.3.7] and [17, Cor. 0.3.8] that M is topologically flat if and only if M is topologically free if and only if M is abstractly flat. In particular, if R is Artinian, a pseudocompact R -module is topologically flat if and only if it is abstractly free.

For $R \in \text{Ob}(\hat{\mathcal{C}})$, let $C^-(R\Lambda)$ be the abelian category of complexes of pseudocompact $R\Lambda$ -modules that are bounded above, let $K^-(R\Lambda)$ be the homotopy category of $C^-(R\Lambda)$, and let $D^-(R\Lambda)$ be the derived category of $K^-(R\Lambda)$. Let $[1]$ denote the translation functor on $C^-(R\Lambda)$ (resp. $K^-(R\Lambda)$, resp. $D^-(R\Lambda)$), i.e. $[1]$ shifts complexes one place to the left and changes the signs of the differentials (see Convention 1.3). Recall that a homomorphism in $C^-(R\Lambda)$ is a quasi-isomorphism if and only if the induced homomorphisms on all the cohomology groups are bijective.

Remark 2.1.4. Let $R \in \text{Ob}(\hat{\mathcal{C}})$, and let \mathcal{P}_R be the additive subcategory of $\text{PCMod}(R\Lambda)$ of projective objects. By Remark 2.1.1(ii) and [34, Thm. 10.4.8], the natural functor $K^-(\mathcal{P}_R) \rightarrow D^-(R\Lambda)$ is an equivalence of triangulated categories. Let $\sigma_R : D^-(R\Lambda) \rightarrow K^-(\mathcal{P}_R)$ be a quasi-inverse.

Suppose S is a pseudocompact R -module. In the case when $S \in \text{Ob}(\hat{\mathcal{C}})$ and there exists a morphism $\alpha : R \rightarrow S$ in $\hat{\mathcal{C}}$ defining the R -module structure on S , let $R_S = S$. In all other cases, let

$R_S = R$. Consider the completed tensor product functor

$$S\hat{\otimes}_{R-} : K^-(R\Lambda) \rightarrow K^-(R_S\Lambda).$$

By [34, Thm. 10.5.6], its left derived functor $S\hat{\otimes}_{R-}^{\mathbf{L}} : D^-(R\Lambda) \rightarrow D^-(R_S\Lambda)$ is the following composition of functors of triangulated categories:

$$(2.1) \quad D^-(R\Lambda) \xrightarrow{\sigma_R} K^-(\mathcal{P}_R) \xrightarrow{S\hat{\otimes}_{R-}} K^-(R_S\Lambda) \xrightarrow{q_S} D^-(R_S\Lambda)$$

where $q_S : K^-(R_S\Lambda) \rightarrow D^-(R_S\Lambda)$ is the localization functor. In other words, if X^\bullet is in $\text{Ob}(K^-(R\Lambda))$, then there exists an isomorphism $\rho_X : X^\bullet \rightarrow \sigma_R(X^\bullet)$ in $D^-(R\Lambda)$ and

$$(2.2) \quad S\hat{\otimes}_{R-}^{\mathbf{L}} X^\bullet = S\hat{\otimes}_{R-} \sigma_R(X^\bullet).$$

The following definitions and remarks are adapted from [6, Sect. 2] to our situation. Note that we follow Illusie's definition of finite tor dimension as given in [20, Def. 5.2] (see also [15, Sect. 8.3.6]).

Definition 2.1.5. We will say that a complex M^\bullet in $K^-(R\Lambda)$ has *finite pseudocompact R -tor dimension*, if there exists an integer N such that for all pseudocompact R -modules S , and for all integers $i < N$, $H^i(S\hat{\otimes}_{R-}^{\mathbf{L}} M^\bullet) = 0$. If we want to emphasize the integer N in this definition, we say M^\bullet has *finite pseudocompact R -tor dimension at N* .

Remark 2.1.6. Suppose M^\bullet is a complex in $K^-(R\Lambda)$ of topologically flat, hence topologically free, pseudocompact R -modules that has finite pseudocompact R -tor dimension at N . Then the bounded complex M'^\bullet , which is obtained from M^\bullet by replacing M^N by $M'^N = M^N/\delta^{N-1}(M^{N-1})$ and by setting $M'^i = 0$ if $i < N$, is quasi-isomorphic to M^\bullet and has topologically free pseudocompact terms over R .

Definition 2.1.7. Let R be an object of $\hat{\mathcal{C}}$. A *quasi-lift* of V^\bullet over R is a pair (M^\bullet, ϕ) consisting of a complex M^\bullet in $D^-(R\Lambda)$ that has finite pseudocompact R -tor dimension together with an isomorphism $\phi : k\hat{\otimes}_{R-}^{\mathbf{L}} M^\bullet \rightarrow V^\bullet$ in $D^-(\Lambda)$. Two quasi-lifts (M^\bullet, ϕ) and (M'^\bullet, ϕ') of V^\bullet over R are *isomorphic* if there is an isomorphism $f : M^\bullet \rightarrow M'^\bullet$ in $D^-(R\Lambda)$ with $\phi' \circ (k\hat{\otimes}_{R-}^{\mathbf{L}} f) = \phi$. A *deformation* of V^\bullet over R is an isomorphism class of quasi-lifts of V^\bullet over R . We denote the deformation of V^\bullet over R represented by (M^\bullet, ϕ) by $[M^\bullet, \phi]$.

A *proflat quasi-lift* of V^\bullet over R is a quasi-lift (M^\bullet, ϕ) of V^\bullet over R whose cohomology groups are topologically flat, and hence topologically free, pseudocompact R -modules. A *proflat deformation* of V^\bullet over R is an isomorphism class of proflat quasi-lifts of V^\bullet over R .

Remark 2.1.8. There exist quasi-lifts that are not isomorphic to proflat quasi-lifts in $D^-(R\Lambda)$. For example, suppose $\Lambda = k$ and $V^\bullet = k \xrightarrow{0} k$ is the two-term complex concentrated in the degrees -1 and 0 with trivial differential. Then the two-term complex $M^\bullet = k[[t]] \xrightarrow{t} k[[t]]$ concentrated in the degrees -1 and 0 defines a quasi-lift of V^\bullet over $k[[t]]$. However, since M^\bullet is isomorphic to the one-term complex $k[[t]]/tk[[t]] \cong k$ concentrated in degree 0 , this quasi-lift is not isomorphic to a proflat quasi-lift of V^\bullet over $k[[t]]$.

Remark 2.1.9. The following two statements are proved in the same way as [6, Lemmas 2.9 and 2.11] by using Remark 2.1.1(ii) and the fact that a bounded above complex of topologically free pseudocompact modules whose cohomology groups are all topologically free splits completely.

- (i) Suppose $R \in \text{Ob}(\hat{\mathcal{C}})$ and (M^\bullet, ϕ) is a quasi-lift of V^\bullet over R . Then there exists a quasi-lift (P^\bullet, ψ) of V^\bullet over R that is isomorphic to (M^\bullet, ϕ) such that the terms of P^\bullet are topologically free pseudocompact $R\Lambda$ -modules.
- (ii) Suppose $R \in \text{Ob}(\hat{\mathcal{C}})$ and (M^\bullet, ϕ) is a proflat quasi-lift of V^\bullet over R . Then $H^n(M^\bullet)$ is an abstractly free R -module of rank $d_n = \dim_k H^n(V^\bullet)$ for all n . Moreover, for any $R' \in \text{Ob}(\hat{\mathcal{C}})$ and for any morphism $\alpha : R \rightarrow R'$ in $\hat{\mathcal{C}}$, there is a natural R' -linear isomorphism $R'\hat{\otimes}_R H^n(M^\bullet) \cong H^n(R'\hat{\otimes}_R^{\mathbf{L}} M^\bullet)$.

Definition 2.1.10. Let $\hat{F} = \hat{F}_{V^\bullet} : \hat{\mathcal{C}} \rightarrow \text{Sets}$ (resp. $\hat{F}^{\text{fl}} = \hat{F}_{V^\bullet}^{\text{fl}} : \hat{\mathcal{C}} \rightarrow \text{Sets}$) be the map that sends an object R of $\hat{\mathcal{C}}$ to the set $\hat{F}(R)$ (resp. $\hat{F}^{\text{fl}}(R)$) of all deformations (resp. all proflat deformations) of V^\bullet over R , and that sends a morphism $\alpha : R \rightarrow R'$ in $\hat{\mathcal{C}}$ to the set map $\hat{F}(R) \rightarrow \hat{F}(R')$ (resp. $\hat{F}^{\text{fl}}(R) \rightarrow \hat{F}^{\text{fl}}(R')$) given by $[M^\bullet, \phi] \mapsto [R' \hat{\otimes}_{R, \alpha}^{\mathbf{L}} M^\bullet, \phi_\alpha]$. Here ϕ_α denotes the composition $k \hat{\otimes}_{R'}^{\mathbf{L}} (R' \hat{\otimes}_{R, \alpha}^{\mathbf{L}} M^\bullet) \cong k \hat{\otimes}_R^{\mathbf{L}} M^\bullet \xrightarrow{\phi} V^\bullet$. Let $F = F_{V^\bullet}$ (resp. $F^{\text{fl}} = F_{V^\bullet}^{\text{fl}}$) be the restriction of \hat{F} (resp. \hat{F}^{fl}) to the subcategory \mathcal{C} of Artinian rings in $\hat{\mathcal{C}}$. In the following, we will use the subscript \mathcal{D} to denote the empty condition if we consider the map \hat{F} , and the condition of having topologically free cohomology groups if we consider the map \hat{F}^{fl} . In particular, the notation $\hat{F}_{\mathcal{D}}$ will be used to refer to both \hat{F} and \hat{F}^{fl} .

Let $k[\varepsilon]$, where $\varepsilon^2 = 0$, denote the ring of dual numbers over k . The set $F_{\mathcal{D}}(k[\varepsilon])$ is called the *tangent space* to $F_{\mathcal{D}}$, denoted by $t_{F_{\mathcal{D}}}$.

Remark 2.1.11. Let $F_{\mathcal{D}}$ and $\hat{F}_{\mathcal{D}}$ be as in Definition 2.1.10. Note that $\hat{F}_{\mathcal{D}}(R)$ is indeed a set for each object R of $\hat{\mathcal{C}}$, as can be seen, for example, by using the concepts of [24, Sect. 3A]. Using similar arguments as in the proof of [6, Prop. 2.12], it follows that the map $\hat{F}_{\mathcal{D}}$ is a functor $\hat{\mathcal{C}} \rightarrow \text{Sets}$. Moreover, \hat{F}^{fl} is a subfunctor of \hat{F} in the sense that there is a natural transformation $\hat{F}^{\text{fl}} \rightarrow \hat{F}$ that is injective. If V'^\bullet is a complex in $D^-(\Lambda)$ satisfying Hypothesis 1 such that there is an isomorphism $\nu : V^\bullet \rightarrow V'^\bullet$ in $D^-(\Lambda)$, then the natural transformation $\hat{F}_{\mathcal{D}, V^\bullet} \rightarrow \hat{F}_{\mathcal{D}, V'^\bullet}$ (resp. $F_{\mathcal{D}, V^\bullet} \rightarrow F_{\mathcal{D}, V'^\bullet}$) induced by $[M^\bullet, \phi] \mapsto [M^\bullet, \nu \circ \phi]$ is an isomorphism of functors.

The following theorem is the main result of Section 2.

Theorem 2.1.12. *Assume Hypothesis 1, and let $F_{\mathcal{D}}$ and $\hat{F}_{\mathcal{D}}$ be as in Definition 2.1.10.*

- (i) *The functor $F_{\mathcal{D}}$ has a pro-representable hull $R_{\mathcal{D}}(\Lambda, V^\bullet) \in \text{Ob}(\hat{\mathcal{C}})$ (c.f. [32, Def. 2.7] and [26, Sect. 1.2]), and the functor $\hat{F}_{\mathcal{D}}$ is continuous (c.f. [26]).*
- (ii) *If $F_{\mathcal{D}} = F$, then t_F is a vector space over k and there is a k -vector space isomorphism $h : t_F \rightarrow \text{Ext}_{D^-(\Lambda)}^1(V^\bullet, V^\bullet)$. If $F_{\mathcal{D}} = F^{\text{fl}}$, then the composition of the natural map $t_{F^{\text{fl}}} \rightarrow t_F$ and h induces an isomorphism between $t_{F^{\text{fl}}}$ and the kernel of the natural map $\text{Ext}_{D^-(\Lambda)}^1(V^\bullet, V^\bullet) \rightarrow \text{Ext}_{D^-(k)}^1(V^\bullet, V^\bullet)$ given by forgetting the Λ -action.*
- (iii) *If $\text{Hom}_{D^-(\Lambda)}(V^\bullet, V^\bullet) = k$, then $\hat{F}_{\mathcal{D}}$ is represented by $R_{\mathcal{D}}(\Lambda, V^\bullet)$.*

Remark 2.1.13. By Theorem 2.1.12(i), there exists a quasi-lift $(U_{\mathcal{D}}(\Lambda, V^\bullet), \phi_U)$ of V^\bullet over $R_{\mathcal{D}}(\Lambda, V^\bullet)$ with the following property. For each $R \in \text{Ob}(\hat{\mathcal{C}})$, the map $\text{Hom}_{\hat{\mathcal{C}}}(R_{\mathcal{D}}(\Lambda, V^\bullet), R) \rightarrow \hat{F}_{\mathcal{D}}(R)$ given by $\alpha \mapsto [R \hat{\otimes}_{R_{\mathcal{D}}(\Lambda, V^\bullet), \alpha}^{\mathbf{L}} U_{\mathcal{D}}(\Lambda, V^\bullet), \phi_{U, \alpha}]$ is surjective, and this map is bijective if R is the ring of dual numbers $k[\varepsilon]$ over k where $\varepsilon^2 = 0$.

In general, the isomorphism type of the pair consisting of the pro-representable hull $R_{\mathcal{D}}(\Lambda, V^\bullet)$ and the deformation $[U_{\mathcal{D}}(\Lambda, V^\bullet), \phi_U]$ of V^\bullet over $R_{\mathcal{D}}(\Lambda, V^\bullet)$ is unique up to a non-canonical isomorphism. If $R_{\mathcal{D}}(\Lambda, V^\bullet)$ represents $\hat{F}_{\mathcal{D}}$, the pair $(R_{\mathcal{D}}(\Lambda, V^\bullet), [U_{\mathcal{D}}(\Lambda, V^\bullet), \phi_U])$ is uniquely determined up to a canonical isomorphism.

Definition 2.1.14. Using the notation of Theorem 2.1.12 and Remark 2.1.13, if $\hat{F}_{\mathcal{D}} = \hat{F}$ then we call $R_{\mathcal{D}}(\Lambda, V^\bullet) = R(\Lambda, V^\bullet)$ the *versal deformation ring* of V^\bullet and the deformation $[U(\Lambda, V^\bullet), \phi_U]$ is called the *versal deformation* of V^\bullet . If $\hat{F}_{\mathcal{D}} = \hat{F}^{\text{fl}}$ then we call $R_{\mathcal{D}}(\Lambda, V^\bullet) = R^{\text{fl}}(\Lambda, V^\bullet)$ the *versal proflat deformation ring* of V^\bullet and the deformation $[U^{\text{fl}}(\Lambda, V^\bullet), \phi_U]$ is called the *versal proflat deformation* of V^\bullet .

If $R_{\mathcal{D}}(\Lambda, V^\bullet)$ represents $\hat{F}_{\mathcal{D}}$, then $R(\Lambda, V^\bullet)$ (resp. $R^{\text{fl}}(\Lambda, V^\bullet)$) will be called the *universal deformation ring* (resp. *the universal proflat deformation ring*) of V^\bullet , and the deformation $[U(\Lambda, V^\bullet), \phi_U]$ (resp. $[U^{\text{fl}}(\Lambda, V^\bullet), \phi_U]$) will be called the *universal deformation* (resp. *the universal proflat deformation*) of V^\bullet .

Remark 2.1.15.

- (i) By part (ii) of Theorem 2.1.12, the tangent space $t_{F^{\text{fl}}}$ consists of those elements

$$\gamma \in \text{Ext}_{D^-(\Lambda)}^1(V^\bullet, V^\bullet) = \text{Hom}_{D^-(\Lambda)}(V^\bullet, V^\bullet[1])$$

that induce the trivial map on cohomology. In other words, the k -vector space maps $\gamma^i : H^i(V^\bullet) \rightarrow H^{i+1}(V^\bullet)$ that are induced by γ have to be zero for all i .

- (ii) By part (i) of Theorem 2.1.12, there exists a non-canonical continuous k -algebra homomorphism $f_{\text{fl}} : R(\Lambda, V^\bullet) \rightarrow R^{\text{fl}}(\Lambda, V^\bullet)$. By part (ii) of Theorem 2.1.12, it follows that the induced map between the Zariski cotangent spaces of these rings is surjective, implying that f_{fl} is surjective.
- (iii) If V^\bullet consists of a single module V^0 in degree 0, the versal deformation ring $R(\Lambda, V^\bullet)$ is isomorphic to the versal deformation ring $R(\Lambda, V^0)$ studied in [9] (see Proposition 2.5.2).

To prove Theorem 2.1.12, we adapt the argumentation in [6] to our situation. Our first task is to adapt results from [6, Sects. 3, 4 and 14] which prove key properties of quasi-lifts of V^\bullet .

2.2. Properties of quasi-lifts of V^\bullet . In this subsection, we analyze the structure of quasi-lifts of V^\bullet over Artinian rings R in \mathcal{C} . The following full subcategories of $C^-(R\Lambda)$, $K^-(R\Lambda)$ and $D^-(R\Lambda)$ play an important role in this situation.

Definition 2.2.1. Let $R \in \text{Ob}(\mathcal{C})$ be Artinian. Define $C_{\text{fin}}^-(R\Lambda)$ (resp. $K_{\text{fin}}^-(R\Lambda)$, resp. $D_{\text{fin}}^-(R\Lambda)$) to be the full subcategory of $C^-(R\Lambda)$ (resp. $K^-(R\Lambda)$, resp. $D^-(R\Lambda)$) whose objects are those complexes M^\bullet of finite pseudocompact R -tor dimension having finitely many non-zero cohomology groups, all of which have finite R -length.

Remark 2.2.2. Suppose R is an Artinian ring in $\text{Ob}(\mathcal{C})$.

- (i) By Remark 2.1.1(i), an $R\Lambda$ -module has finite length if and only if it has finite length as an R -module. Since R is local Artinian, an R -module has finite R -length if and only if it has finite k -length.
- (ii) Suppose N^\bullet is a complex in $D_{\text{fin}}^-(R\Lambda)$ and X^\bullet is a complex in $D^-(R\Lambda)$ such that there is an isomorphism $\xi : X^\bullet \rightarrow N^\bullet$ in $D^-(R\Lambda)$. Then X^\bullet is an object of $D_{\text{fin}}^-(R\Lambda)$ and ξ is an isomorphism in $D_{\text{fin}}^-(R\Lambda)$. This follows since having finite pseudocompact R -tor dimension is an invariant of isomorphisms in $D^-(R\Lambda)$, and since such isomorphisms induce isomorphisms between the cohomology groups. In particular, if $A^\bullet \rightarrow B^\bullet$ is a quasi-isomorphism in $K^-(R\Lambda)$ and one of A^\bullet or B^\bullet is an object of $K_{\text{fin}}^-(R\Lambda)$, then so is the other.
- (iii) Let $N^\bullet, N_1^\bullet, N_2^\bullet$ be complexes in $D_{\text{fin}}^-(R\Lambda)$ such that *all their terms have finite k -length*, and let $g : N_1^\bullet \rightarrow N_2^\bullet$ be a morphism in $D_{\text{fin}}^-(R\Lambda)$. By Remark 2.1.1(ii), and since $R\Lambda$ is Artinian, there exist bounded above complexes M^\bullet, M_1^\bullet and M_2^\bullet of abstractly free finitely generated $R\Lambda$ -modules such that there are isomorphisms $\beta : N^\bullet \rightarrow M^\bullet$ and $\beta_i : N_i^\bullet \rightarrow M_i^\bullet$ in $D_{\text{fin}}^-(R\Lambda)$ ($i = 1, 2$). Then $f = \beta_2 \circ g \circ \beta_1^{-1}$ is a morphism $f : M_1^\bullet \rightarrow M_2^\bullet$ in $D_{\text{fin}}^-(R\Lambda)$. Let \mathcal{P}_R be the additive subcategory of $\text{PCMod}(R\Lambda)$ of projective objects. By [34, Thm. 10.4.8], the natural functor $K^-(\mathcal{P}_R) \rightarrow D^-(R\Lambda)$ is an equivalence of categories. Hence f can be taken to be a morphism in $K_{\text{fin}}^-(R\Lambda)$.

The following two results, Lemmas 2.2.3 and 2.2.5, establish key properties of objects and morphisms in $D_{\text{fin}}^-(R\Lambda)$. They replace [6, Cor. 3.6 and Lemma 3.8] in our situation. Since Λ is Artinian, some of the statements can be simplified.

Lemma 2.2.3. *Suppose $R \in \text{Ob}(\mathcal{C})$ is Artinian, and N^\bullet, N_1^\bullet and N_2^\bullet are objects in $D_{\text{fin}}^-(R\Lambda)$. Let $g : N_1^\bullet \rightarrow N_2^\bullet$ be a morphism in $D_{\text{fin}}^-(R\Lambda)$.*

- (i) *There exists a bounded above complex M^\bullet of abstractly free finitely generated $R\Lambda$ -modules, and an isomorphism $\beta : N^\bullet \rightarrow M^\bullet$ in $D_{\text{fin}}^-(R\Lambda)$.*
- (ii) *There exist bounded above complexes M_1^\bullet and M_2^\bullet of abstractly free finitely generated $R\Lambda$ -modules, a morphism $f : M_1^\bullet \rightarrow M_2^\bullet$ in $K_{\text{fin}}^-(R\Lambda)$, and isomorphisms $\beta_i : N_i^\bullet \rightarrow M_i^\bullet$ in $D_{\text{fin}}^-(R\Lambda)$ ($i = 1, 2$) such that $f = \beta_2 \circ g \circ \beta_1^{-1}$ as morphisms in $D_{\text{fin}}^-(R\Lambda)$.*

Proof. In view of Remark 2.2.2(iii), the main ingredient in the proof is the following claim, which is proved using similar arguments as in the proof of [6, Lemma 3.4(i)].

Claim 1. Suppose N^\bullet is an object of $D_{\text{fin}}^-(R\Lambda)$ satisfying $H^j(N^\bullet) = 0$ for $j < n$. Then there exists an exact sequence of complexes

$$(2.3) \quad 0 \rightarrow U^\bullet \xrightarrow{t} N^\bullet \rightarrow N'^\bullet \rightarrow 0$$

in $C_{\text{fin}}^-(R\Lambda)$ such that U^\bullet is acyclic, and such that the terms of N'^\bullet have finite k -length and satisfy $N'^j = 0$ for $j < n$.

Part (i) of Lemma 2.2.3 now follows from Claim 1 and Remark 2.2.2(iii). To prove part(ii) of Lemma 2.2.3, we use Claim 1 to see that there exist bounded complexes $N'_1{}^\bullet$ and $N'_2{}^\bullet$ such that all their terms have finite k -length, together with quasi-isomorphisms $\gamma_i : N_i^\bullet \rightarrow N'_i{}^\bullet$ in $C_{\text{fin}}^-(R\Lambda)$ ($i = 1, 2$) that are surjective on terms. Let $g' = \gamma_2 \circ g \circ \gamma_1^{-1} : N'_1{}^\bullet \rightarrow N'_2{}^\bullet$, so g' is a morphism in $D_{\text{fin}}^-(R\Lambda)$. Using Remark 2.2.2(iii), there exist bounded above complexes M_1^\bullet and M_2^\bullet of abstractly free finitely generated $R\Lambda$ -modules, a morphism $f : M_1^\bullet \rightarrow M_2^\bullet$ in $K_{\text{fin}}^-(R\Lambda)$, and isomorphisms $\beta'_i : N'_i{}^\bullet \rightarrow M_i^\bullet$ in $D_{\text{fin}}^-(R\Lambda)$ ($i = 1, 2$) such that $f = \beta'_2 \circ g' \circ \beta'_1{}^{-1}$ as morphisms in $D_{\text{fin}}^-(R\Lambda)$. Letting $\beta_i = \beta'_i \circ \gamma_i$ ($i = 1, 2$), part (ii) follows. \square

Definition 2.2.4. In the situation of Lemma 2.2.3(i), we say *we can replace N^\bullet by M^\bullet* . In the situation of Lemma 2.2.3(ii), we say *we can replace N_i^\bullet by M_i^\bullet ($i = 1, 2$), and g by f* .

Lemma 2.2.5. *Suppose $R \in \text{Ob}(\mathcal{C})$ is Artinian, and M^\bullet is an object of $D_{\text{fin}}^-(R\Lambda)$ such that $H^j(M^\bullet) = 0$ for $j < n$. Then M^\bullet has finite pseudocompact R -tor dimension at n .*

Proof. By Lemma 2.2.3(i), we may assume that M^\bullet is a bounded above complex of abstractly free finitely generated $R\Lambda$ -modules. Hence all terms of M^\bullet are abstractly free finitely generated R -modules. By Remark 2.1.6, there exists an integer $n_1 \leq n$ such that $M^{n_1}/\delta^{n_1-1}(M^{n_1-1})$ is a topologically free pseudocompact R -module. Since M^{n_1} is a finitely generated R -module, it follows that this is an abstractly free finitely generated R -module. To prove Lemma 2.2.5, it is enough to show that $M^n/\delta^{n-1}(M^{n-1})$ is an abstractly free finitely generated R -module. This is proved exactly in the same way as in the proof of [6, Lemma 3.8]. \square

Remark 2.2.6. Suppose $R \in \text{Ob}(\mathcal{C})$ is Artinian. Using Lemma 2.2.3(i) together with Remark 2.1.6 and Lemma 2.2.5, it follows that every object M^\bullet of $D_{\text{fin}}^-(R\Lambda)$ is isomorphic in $D_{\text{fin}}^-(R\Lambda)$ to a bounded complex M'^\bullet such that all terms of M'^\bullet are abstractly free finitely generated R -modules and such that all its terms, except possibly its leftmost non-zero term, are actually abstractly free finitely generated $R\Lambda$ -modules.

The following remark replaces [6, Cors. 3.10 and 3.11] in our situation. Note that since Λ is already Artinian, we do not need to deal with quotient algebras of Λ .

Remark 2.2.7. Suppose $R, S, T \in \text{Ob}(\mathcal{C})$ are Artinian rings with morphisms $R \xrightarrow{\alpha} T \xleftarrow{\beta} S$ in \mathcal{C} . Let X^\bullet be an object of $D_{\text{fin}}^-(R\Lambda)$, and let Z^\bullet be an object of $D_{\text{fin}}^-(S\Lambda)$. Suppose $\tau : T \hat{\otimes}_S^{\mathbf{L}} Z^\bullet \rightarrow T \hat{\otimes}_R^{\mathbf{L}} X^\bullet$ is a morphism in $D_{\text{fin}}^-(T\Lambda)$. By Remark 2.1.4,

$$T \hat{\otimes}_R^{\mathbf{L}} X^\bullet = T \hat{\otimes}_R \sigma_R(X^\bullet) \quad \text{and} \quad T \hat{\otimes}_S^{\mathbf{L}} Z^\bullet = T \hat{\otimes}_S \sigma_S(Z^\bullet)$$

where $\sigma_R(X^\bullet)$ (resp. $\sigma_S(Z^\bullet)$) is an object of $K^-(\mathcal{P}_R)$ (resp. $K^-(\mathcal{P}_S)$) and there exists an isomorphism $\rho_X : X^\bullet \rightarrow \sigma_R(X^\bullet)$ in $D^-(R\Lambda)$ (resp. $\rho_Z : Z^\bullet \rightarrow \sigma_S(Z^\bullet)$ in $D^-(S\Lambda)$). By Remark 2.2.2(ii), $\sigma_R(X^\bullet)$ and ρ_X (resp. $\sigma_S(Z^\bullet)$ and ρ_Z) are in $D_{\text{fin}}^-(R\Lambda)$ (resp. $D_{\text{fin}}^-(S\Lambda)$).

By Lemma 2.2.3(i), there exists a bounded above complex \tilde{X}^\bullet (resp. \tilde{Z}^\bullet) of abstractly free finitely generated $R\Lambda$ -modules (resp. $S\Lambda$ -modules) and an isomorphism $\tilde{\beta} : \tilde{X}^\bullet \rightarrow X^\bullet$ in $D_{\text{fin}}^-(R\Lambda)$ (resp. $\tilde{\gamma} : \tilde{Z}^\bullet \rightarrow Z^\bullet$ in $D_{\text{fin}}^-(S\Lambda)$). Let $\beta = \rho_X \circ \tilde{\beta}$ and $\gamma = \rho_Z \circ \tilde{\gamma}$. Then $\beta : \tilde{X}^\bullet \rightarrow \sigma_R(X^\bullet)$ (resp. $\gamma : \tilde{Z}^\bullet \rightarrow \sigma_S(Z^\bullet)$) is an isomorphism in $D_{\text{fin}}^-(R\Lambda)$ (resp. $D_{\text{fin}}^-(S\Lambda)$). Since $\tilde{X}^\bullet, \sigma_R(X^\bullet)$ (resp. $\tilde{Z}^\bullet, \sigma_S(Z^\bullet)$) are objects in $K^-(\mathcal{P}_R)$ (resp. $K^-(\mathcal{P}_S)$), β (resp. γ) can be taken to be a morphism in

$K_{\text{fin}}^-(R\Lambda)$ (resp. $K_{\text{fin}}^-(S\Lambda)$). Moreover, $T\hat{\otimes}_R\beta = T\hat{\otimes}_R^{\mathbf{L}}\beta$ (resp. $T\hat{\otimes}_S\gamma = T\hat{\otimes}_S^{\mathbf{L}}\gamma$) is an isomorphism in $K_{\text{fin}}^-(T\Lambda)$, and $\tau : T\hat{\otimes}_S\sigma_S(Z^\bullet) \rightarrow T\hat{\otimes}_R\sigma_R(X^\bullet)$ can be taken to be a morphism in $K_{\text{fin}}^-(T\Lambda)$. Define

$$(2.4) \quad \tilde{\tau} = (T\hat{\otimes}_R\beta)^{-1} \circ \tau \circ (T\hat{\otimes}_S\gamma) : T\hat{\otimes}_S\tilde{Z}^\bullet \rightarrow T\hat{\otimes}_R\tilde{X}^\bullet.$$

Then we can replace X^\bullet (resp. Z^\bullet) by \tilde{X}^\bullet (resp. \tilde{Z}^\bullet), in the sense of Definition 2.2.4, and we can replace τ by $\tilde{\tau}$. Note that β (resp. γ) only depends on \tilde{X}^\bullet and $\tilde{\beta}$ (resp. \tilde{Z}^\bullet and $\tilde{\gamma}$).

Using Remark 2.1.9(i), the following result is proved in a similar way to [6, Lemma 3.1].

Lemma 2.2.8. *Suppose (M^\bullet, ϕ) is a quasi-lift of V^\bullet over some Artinian ring $R \in \text{Ob}(\mathcal{C})$. Then M^\bullet is an object of $D_{\text{fin}}^-(R\Lambda)$. More precisely, $H^i(M^\bullet)$ is a subquotient of an abstractly free R -module of rank $d_i = \dim_k H^i(V^\bullet)$ for all i .*

The following result summarizes the main properties of quasi-lifts and proflat quasi-lifts of V^\bullet over arbitrary objects R in $\hat{\mathcal{C}}$. The proof is very similar to the proof of [7, Thm. 2.10], once we replace the results from [6] by the corresponding results stated above. For the convenience of the reader, we provide some of the details.

Theorem 2.2.9. *Suppose that $H^i(V^\bullet) = 0$ unless $n_1 \leq i \leq n_2$. Every quasi-lift of V^\bullet over an object R of $\hat{\mathcal{C}}$ is isomorphic to a quasi-lift (P^\bullet, ψ) for a complex P^\bullet with the following properties:*

- (i) *The terms of P^\bullet are topologically free $R\Lambda$ -modules.*
- (ii) *The cohomology group $H^i(P^\bullet)$ is finitely generated as an abstract R -module for all i , and $H^i(P^\bullet) = 0$ unless $n_1 \leq i \leq n_2$.*
- (iii) *One has $H^i(S\hat{\otimes}_R^{\mathbf{L}}P^\bullet) = 0$ for all pseudocompact R -modules S unless $n_1 \leq i \leq n_2$.*

Proof. Let R be an object of $\hat{\mathcal{C}}$, and let (M^\bullet, ϕ) be a quasi-lift of V^\bullet over R . By Remark 2.1.9(i), it follows that there exists a quasi-lift (P^\bullet, ψ) of V^\bullet over R that is isomorphic to the quasi-lift (M^\bullet, ϕ) such that P^\bullet satisfies property (i). It remains to verify properties (ii) and (iii). By (i), we can assume that the terms of P^\bullet are topologically free pseudocompact $R\Lambda$ -modules. In particular, the functors $-\hat{\otimes}_R^{\mathbf{L}}P^\bullet$ and $-\hat{\otimes}_R P^\bullet$ are naturally isomorphic. Let m_R denote the maximal ideal of R , and let n be an arbitrary positive integer. By Lemmas 2.2.5 and 2.2.8, $H^i((R/m_R^n)\hat{\otimes}_R P^\bullet) = 0$ for $i > n_2$ and $i < n_1$. Moreover, for $n_1 \leq i \leq n_2$, $H^i((R/m_R^n)\hat{\otimes}_R P^\bullet)$ is a subquotient of an abstractly free (R/m_R^n) -module of rank $d_i = \dim_k H^i(V^\bullet)$, and $(R/m_R^n)\hat{\otimes}_R P^\bullet$ has finite pseudocompact (R/m_R^n) -tor dimension at $N = n_1$. Since $P^\bullet \cong \varinjlim_n (R/m_R^n)\hat{\otimes}_R P^\bullet$ and since by Remark 2.1.1(i), the category $\text{PCMod}(R)$ has exact inverse limits, it follows that for all pseudocompact R -modules S

$$H^i(S\hat{\otimes}_R P^\bullet) = \varinjlim_n H^i\left((S/m_R^n S)\hat{\otimes}_{R/m_R^n}((R/m_R^n)\hat{\otimes}_R P^\bullet)\right)$$

for all i . Hence Theorem 2.2.9 follows. \square

2.3. More results on complexes and quasi-lifts. In this subsection, we first provide some results from Milne [27], which are adapted from [6, Sect. 14] to our situation. Note that in [27, Lemma VI.8.17] (resp. [27, Lemma VI.8.18]), the condition “ π is surjective on terms” (resp. “ ψ is surjective on terms”) is necessary in the statement.

Lemma 2.3.1. ([27, Lemma VI.8.17]) *Let $R \in \text{Ob}(\hat{\mathcal{C}})$, and let $M^\bullet \xrightarrow{\phi} L^\bullet \xleftarrow{\pi} N^\bullet$ be morphisms in $C^-(R\Lambda)$ such that π is a quasi-isomorphism that is surjective on terms. If M^\bullet is a complex of topologically free pseudocompact $R\Lambda$ -modules, there exists a morphism $\psi : M^\bullet \rightarrow N^\bullet$ in $C^-(R\Lambda)$ such that $\pi \circ \psi = \phi$.*

Remark 2.3.2. Suppose $R, R_0 \in \text{Ob}(\hat{\mathcal{C}})$ such that R_0 is a quotient ring of R . We write $X \rightarrow X_0$, $\phi \rightarrow \phi_0$ for the functor $R_0\hat{\otimes}_R-$. Suppose M, N are topologically free pseudocompact $R\Lambda$ -modules. Then every continuous $R_0\Lambda$ -module homomorphism $\pi : M_0 \rightarrow N_0$ can be lifted to a continuous $R\Lambda$ -module homomorphism $\phi : M \rightarrow N$ such that $\pi = \phi_0$.

Lemma 2.3.3. ([27, Sublemma VI.8.20]) *Let $R, R_0 \in \text{Ob}(\hat{\mathcal{C}})$ such that R_0 is a quotient ring of R . As in Remark 2.3.2, we write $X \rightarrow X_0, \phi \rightarrow \phi_0$ for the functor $R_0 \hat{\otimes}_R -$. Let $\phi : L^\bullet \rightarrow M^\bullet$ be a morphism in $C^-(R\Lambda)$ of complexes of topologically free pseudocompact $R\Lambda$ -modules. Then any morphism $L_0^\bullet \rightarrow M_0^\bullet$ in $C^-(R_0\Lambda)$ that is homotopic to ϕ_0 is of the form ψ_0 , where $\psi : L^\bullet \rightarrow M^\bullet$ is a morphism in $C^-(R\Lambda)$ that is homotopic to ϕ .*

Lemma 2.3.4. ([27, Lemma VI.8.18]) *Let $R, R_0 \in \text{Ob}(\mathcal{C})$ be Artinian such that R_0 is a quotient ring of R . As in Remark 2.3.2, we write $X \rightarrow X_0, \phi \rightarrow \phi_0$ for the functor $R_0 \hat{\otimes}_R -$. Let M^\bullet (resp. N^\bullet) be a bounded above complex of abstractly free finitely generated $R\Lambda$ -modules (resp. $R_0\Lambda$ -modules), and let ψ be a quasi-isomorphism $\psi : M_0^\bullet \rightarrow N^\bullet$ in $C^-(R_0\Lambda)$ that is surjective on terms. Then there exist a bounded above complex L^\bullet of abstractly free finitely generated $R\Lambda$ -modules, a quasi-isomorphism $\phi : M^\bullet \rightarrow L^\bullet$ in $C^-(R\Lambda)$, and an isomorphism $\rho : L_0^\bullet \rightarrow N^\bullet$ in $C^-(R_0\Lambda)$, such that $\rho \circ \phi_0 = \psi$.*

The following remark (which replaces [6, Remark 5.2] in our situation) shows how one can relate a morphism f in $C^-(R\Lambda)$ to a morphism g in $C^-(R\Lambda)$ that is surjective on terms.

Remark 2.3.5. Suppose $R \in \text{Ob}(\hat{\mathcal{C}})$. Let M^\bullet and N^\bullet be two bounded above complexes of pseudocompact $R\Lambda$ -modules, and let $f : M^\bullet \rightarrow N^\bullet$ be a morphism in $C^-(R\Lambda)$. Let P^\bullet be a bounded above complex of topologically free pseudocompact $R\Lambda$ -modules such that there is a quasi-isomorphism $P^\bullet \rightarrow N^\bullet$ in $C^-(R\Lambda)$ that is surjective on terms. Then the mapping cone C^\bullet of $P^\bullet[-1] \xrightarrow{\text{id}} P^\bullet[-1]$ is an acyclic complex, and there is a morphism $\pi : C^\bullet \rightarrow N^\bullet$ in $C^-(R\Lambda)$ that is surjective on terms. Define $g : M^\bullet \oplus C^\bullet \rightarrow N^\bullet$ by $g = (f, \pi)$, and define $s : M^\bullet \rightarrow M^\bullet \oplus C^\bullet$ by $s = \begin{pmatrix} \text{id}_{M^\bullet} \\ 0 \end{pmatrix}$. Then g is surjective on terms, s is a quasi-isomorphism and $g \circ s = f$.

Suppose there is a surjective morphism $R_1 \rightarrow R$ in $\hat{\mathcal{C}}$, and there is a bounded above complex X^\bullet of topologically free pseudocompact $R_1\Lambda$ -modules such that $M^\bullet = R \hat{\otimes}_{R_1} X^\bullet$. Since $R\Lambda = R \hat{\otimes}_{R_1} R_1\Lambda$, there exists a bounded above complex Q^\bullet of topologically free pseudocompact $R_1\Lambda$ -modules with $P^\bullet = R \hat{\otimes}_{R_1} Q^\bullet$. Hence $C^\bullet = R \hat{\otimes}_{R_1} D^\bullet$, where D^\bullet is the mapping cone of $Q^\bullet[-1] \xrightarrow{\text{id}} Q^\bullet[-1]$, and $M^\bullet \oplus C^\bullet = R \hat{\otimes}_{R_1} (X^\bullet \oplus Q^\bullet)$.

Next, we look at quasi-lifts of V^\bullet in the case when the endomorphism ring of V^\bullet in $D^-(\Lambda)$ is isomorphic to k . We need the following remark.

Remark 2.3.6. Define \hat{F}_1 (resp. F_1) to be the functor from $\hat{\mathcal{C}}$ (resp. \mathcal{C}) to the category Sets that sends R to the set $\hat{F}_1(R)$ (resp. $F_1(R)$) of isomorphism classes of quasi-lifts of V^\bullet over R that are represented by bounded above complexes of topologically free pseudocompact $R\Lambda$ -modules.

Using Remark 2.1.9(i), it follows as in the proof of [6, Lemma 4.2] that the natural transformation $\hat{F}_1 \rightarrow \hat{F}$ (resp. $F_1 \rightarrow F$) is an isomorphism of functors.

Using Remark 2.3.6 and Lemma 2.3.3, the following result is proved in a similar way to [6, Prop. 4.3].

Proposition 2.3.7. *Suppose $\text{Hom}_{D^-(\Lambda)}(V^\bullet, V^\bullet) = k$. Then $\text{Hom}_{D^-(R\Lambda)}(M^\bullet, M^\bullet) = R$ for every quasi-lift (M^\bullet, ϕ) of V^\bullet over an Artinian ring $R \in \text{Ob}(\mathcal{C})$.*

2.4. Proof of Theorem 2.1.12. In this subsection, we prove Theorem 2.1.12. We follow the argumentation in Sections 5 through 7 of [6] and explain how the key steps are adapted to our situation. As in [6], we use Schlessinger's criteria (H1) - (H4) for the pro-representability, respectively for the existence of a pro-representable hull, of a functor of Artinian rings. We refer to [32, Thm. 2.11] for a precise description of these criteria (H1) - (H4).

Proposition 2.4.1. *Schlessinger's criteria (H1) and (H2) are always satisfied for $F_{\mathcal{D}}$. In the case when $\text{Hom}_{D^-(\Lambda)}(V^\bullet, V^\bullet) = k$, (H4) is also satisfied.*

Proof. The proof of Proposition 2.4.1 closely follows the proof of [6, Prop. 5.1]. By Remark 2.1.11 and Lemma 2.2.3(i), we may assume, without loss of generality, that V^\bullet is a bounded above complex

of abstractly free finitely generated Λ -modules. Suppose A, B, C are Artinian rings in $\text{Ob}(\mathcal{C})$ and that we have a diagram in \mathcal{C}

$$\begin{array}{ccc} A & & B \\ & \searrow \alpha & \swarrow \beta \\ & C & \end{array}$$

Let D be the pullback $D = A \times_C B = \{(a, b) \in A \times B \mid \alpha(a) = \beta(b)\}$. Consider the natural map

$$\chi_{\mathcal{D}} : F_{\mathcal{D}}(D) \rightarrow F_{\mathcal{D}}(A) \times_{F_{\mathcal{D}}(C)} F_{\mathcal{D}}(B).$$

Claim 1. If β is surjective, then $\chi_{\mathcal{D}}$ is surjective. In particular, this implies that (H1) is always satisfied for $F_{\mathcal{D}}$.

To prove Claim 1, suppose $[X_A^{\bullet}, \xi_A] \in F_{\mathcal{D}}(A)$ and $[X_B^{\bullet}, \xi_B] \in F_{\mathcal{D}}(B)$ such that there exists an isomorphism

$$\tau : C \hat{\otimes}_B^{\mathbf{L}} X_B^{\bullet} \rightarrow C \hat{\otimes}_A^{\mathbf{L}} X_A^{\bullet}$$

in $D^-(CA)$ with $\xi_A \circ (k \hat{\otimes}_C^{\mathbf{L}} \tau) = \xi_B$. By Remark 2.2.7, we can assume the following. The complex X_A^{\bullet} (resp. X_B^{\bullet}) is a bounded above complex of abstractly free finitely generated $A\Lambda$ -modules (resp. $B\Lambda$ -modules), τ is given by a quasi-isomorphism in $C^-(CA)$, and ξ_A (resp. ξ_B) is given by a quasi-isomorphism in $C^-(\Lambda)$. By Remark 2.3.5, we can add to X_B^{\bullet} an acyclic complex of abstractly free finitely generated $B\Lambda$ -modules to be able to assume that τ is surjective on terms. We can now complete the proof of Claim 1 in a similar way to the proof of [6, Lemma 5.3].

Claim 2. If β is surjective, and either $\text{Hom}_{D^-(\Lambda)}(V^{\bullet}, V^{\bullet}) = k$ or $C = k$, then $\chi_{\mathcal{D}}$ is injective. In particular, this implies that (H2) is always satisfied for $F_{\mathcal{D}}$, and (H4) is satisfied if $\text{Hom}_{D^-(\Lambda)}(V^{\bullet}, V^{\bullet}) = k$.

Since F^{fl} is a subfunctor of F by Remark 2.1.11, it is enough to prove Claim 2 in the case when $F_{\mathcal{D}} = F$. Suppose $[X_D^{\bullet}, \xi]$ and $[Z_D^{\bullet}, \zeta]$ are two elements in $F(D)$ such that there is an isomorphism $\tau_A : A \hat{\otimes}_D^{\mathbf{L}} Z_D^{\bullet} \rightarrow A \hat{\otimes}_D^{\mathbf{L}} X_D^{\bullet}$ (resp. $\tau_B : B \hat{\otimes}_D^{\mathbf{L}} Z_D^{\bullet} \rightarrow B \hat{\otimes}_D^{\mathbf{L}} X_D^{\bullet}$) in $D^-(A\Lambda)$ (resp. $D^-(B\Lambda)$) with $\xi \circ (k \hat{\otimes}_A^{\mathbf{L}} \tau_A) = \zeta$ (resp. $\xi \circ (k \hat{\otimes}_B^{\mathbf{L}} \tau_B) = \zeta$) in $D^-(\Lambda)$. In other words $(A \hat{\otimes}_D^{\mathbf{L}} Z_D^{\bullet}, \zeta)$ and $(A \hat{\otimes}_D^{\mathbf{L}} X_D^{\bullet}, \xi)$ (resp. $(B \hat{\otimes}_D^{\mathbf{L}} Z_D^{\bullet}, \zeta)$ and $(B \hat{\otimes}_D^{\mathbf{L}} X_D^{\bullet}, \xi)$) are isomorphic as quasi-lifts of V^{\bullet} over A (resp. B). Consider $\varphi_C : C \hat{\otimes}_D^{\mathbf{L}} Z_D^{\bullet} \rightarrow C \hat{\otimes}_D^{\mathbf{L}} X_D^{\bullet}$ in $D^-(CA)$, defined by $\varphi_C = (C \hat{\otimes}_A^{\mathbf{L}} \tau_A)^{-1} \circ (C \hat{\otimes}_B^{\mathbf{L}} \tau_B)$. If $C = k$, then

$$\varphi_k = (k \hat{\otimes}_A^{\mathbf{L}} \tau_A)^{-1} \circ (k \hat{\otimes}_B^{\mathbf{L}} \tau_B) = (\zeta^{-1} \circ \xi) \circ (\xi^{-1} \circ \zeta) = \text{id}_{k \hat{\otimes}_D^{\mathbf{L}} Z_D^{\bullet}}$$

in $D^-(\Lambda)$. If $\text{Hom}_{D^-(\Lambda)}(V^{\bullet}, V^{\bullet}) = k$, then, by Proposition 2.3.7,

$$\text{Hom}_{D^-(CA)}(C \hat{\otimes}_D^{\mathbf{L}} Z_D^{\bullet}, C \hat{\otimes}_D^{\mathbf{L}} X_D^{\bullet}) = C.$$

Hence, in either case there exists a unit $\alpha_C \in C$, with image 1 in k , such that φ_C is multiplication by α_C in $D^-(CA)$. By Remark 2.2.7, we can assume the following.

- (i) The complexes X_D^{\bullet} and Z_D^{\bullet} are bounded above complexes of abstractly free finitely generated $D\Lambda$ -modules,
- (ii) The morphisms ξ and ζ are given by quasi-isomorphisms $\xi : k \otimes_D X_D^{\bullet} \rightarrow V^{\bullet}$ and $\zeta : k \otimes_D Z_D^{\bullet} \rightarrow V^{\bullet}$ in $C^-(\Lambda)$.
- (iii) The morphism τ_A (resp. τ_B) is given by a quasi-isomorphism $\tau_A : A \otimes_D Z_D^{\bullet} \rightarrow A \otimes_D X_D^{\bullet}$ (resp. $\tau_B : B \otimes_D Z_D^{\bullet} \rightarrow B \otimes_D X_D^{\bullet}$) in $C^-(A\Lambda)$ (resp. in $C^-(B\Lambda)$) such that $\xi \circ (k \otimes_A \tau_A) = \zeta$ (resp. $\xi \circ (k \otimes_B \tau_B) = \zeta$) in $K^-(\Lambda)$.

Since β is surjective, $D \rightarrow A$ is also surjective. By Remark 2.3.5, we can add to Z_D^{\bullet} an acyclic complex of abstractly free finitely generated $D\Lambda$ -modules to be able to assume that τ_A is surjective on terms. We can now complete the proof of Claim 2 in a similar way to the proof of [6, Lemma 5.4]. \square

Proposition 2.4.2. *Let $k[\varepsilon]$ be the ring of dual numbers over k where $\varepsilon^2 = 0$.*

(i) *The tangent space t_F is a vector space over k , and there is a k -vector space isomorphism*

$$h : t_F = F(k[\varepsilon]) \longrightarrow \mathrm{Hom}_{D^-(\Lambda)}(V^\bullet, V^\bullet[1]) = \mathrm{Ext}_{D^-(\Lambda)}^1(V^\bullet, V^\bullet).$$

(ii) *Composing the natural injection $t_{F^\natural} \rightarrow t_F$ with h , we obtain an isomorphism between t_{F^\natural} and the kernel of the natural forgetful map*

$$(2.5) \quad \mathrm{Ext}_{D^-(\Lambda)}^1(V^\bullet, V^\bullet) \rightarrow \mathrm{Ext}_{D^-(k)}^1(V^\bullet, V^\bullet).$$

Proof. The proof of part (i) closely follows the proof of [6, Lemma 6.1]. By Remark 2.1.11, we may assume that V^\bullet is a bounded above complex of topologically free pseudocompact Λ -modules. Suppose (M^\bullet, ϕ) is a quasi-lift of V^\bullet over $k[\varepsilon]$. By Remark 2.3.6, we can assume that M^\bullet is a bounded above complex of topologically free pseudocompact $k[\varepsilon]\Lambda$ -modules. We have a short exact sequence

$$0 \rightarrow \varepsilon M^\bullet \xrightarrow{\iota} M^\bullet \xrightarrow{\pi} M^\bullet / \varepsilon M^\bullet \rightarrow 0$$

in $C^-(k[\varepsilon]\Lambda)$. The mapping cone of ι is $C(\iota)^\bullet = \varepsilon M^\bullet[1] \oplus M^\bullet$ with i -th differential

$$\delta_{C(\iota)}^i = \begin{pmatrix} -\delta_M^{i+1} & 0 \\ \iota^{i+1} & \delta_M^i \end{pmatrix}.$$

We obtain a distinguished triangle in $K^-(k[\varepsilon]\Lambda)$

$$(2.6) \quad \varepsilon M^\bullet \xrightarrow{\iota} M^\bullet \xrightarrow{g} C(\iota)^\bullet \xrightarrow{f} \varepsilon M^\bullet[1]$$

where $g^i(b) = (0, b)$ and $f^i(a, b) = -a$. We define two morphisms in $C^-(k[\varepsilon]\Lambda)$

$$(2.7) \quad \begin{aligned} (0, \pi) & : C(\iota)^\bullet = \varepsilon M^\bullet[1] \oplus M^\bullet \rightarrow M^\bullet / \varepsilon M^\bullet \\ \psi & : \varepsilon M^\bullet \rightarrow M^\bullet / \varepsilon M^\bullet \end{aligned}$$

by $(0, \pi)^i(a, b) = \pi^i(b)$ and $\psi^i(\varepsilon x) = \pi^i(x)$. The kernel of $(0, \pi)$ is the mapping cone of $\varepsilon M^\bullet \xrightarrow{\mathrm{id}} \varepsilon M^\bullet$ which is acyclic; hence $(0, \pi)$ is a quasi-isomorphism. The morphism ψ is an isomorphism of complexes with inverse ψ^{-1} given by $(\psi^{-1})^i(\pi^i(x)) = \varepsilon x$. Let $C(f)^\bullet$ be the mapping cone of f from (2.6). By the triangle axioms (TR2) and (TR3) (see, for example, [34, Def. 10.2.1]) and by the 5-lemma for distinguished triangles (see, for example, [34, Ex. 10.2.2]), there exists an isomorphism $\rho : C(f)^\bullet \rightarrow M^\bullet[1]$ in $K^-(k[\varepsilon]\Lambda)$, which is represented by a quasi-isomorphism in $C^-(k[\varepsilon]\Lambda)$. Therefore, we get a distinguished triangle in $K^-(k[\varepsilon]\Lambda)$

$$(2.8) \quad \begin{array}{ccccccc} C(\iota)^\bullet & \xrightarrow{f} & \varepsilon M^\bullet[1] & \rightarrow & C(f)^\bullet & \longrightarrow & C(\iota)^\bullet[1] \\ (0, \pi) \downarrow & & \parallel & & \downarrow \rho & & \downarrow (0, \pi)[1] \\ M^\bullet / \varepsilon M^\bullet & & \varepsilon M^\bullet[1] & & M^\bullet[1] & & (M^\bullet / \varepsilon M^\bullet)[1] \end{array}$$

where the downward arrows are quasi-isomorphisms in $C^-(k[\varepsilon]\Lambda)$. Hence the diagram

$$(2.9) \quad \begin{array}{ccc} & C(\iota)^\bullet & \\ (0, \pi) \swarrow & & \searrow f \\ M^\bullet / \varepsilon M^\bullet & & \varepsilon M^\bullet[1] \end{array}$$

defines a morphism $\hat{f} : M^\bullet / \varepsilon M^\bullet \rightarrow \varepsilon M^\bullet[1]$ in $D^-(k[\varepsilon]\Lambda)$. Because of (2.8), we obtain a distinguished triangle

$$M^\bullet / \varepsilon M^\bullet \xrightarrow{\hat{f}} \varepsilon M^\bullet[1] \rightarrow M^\bullet[1] \rightarrow (M^\bullet / \varepsilon M^\bullet)[1]$$

in $D^-(k[\varepsilon]\Lambda)$. Using the isomorphism $\phi : M^\bullet / \varepsilon M^\bullet \rightarrow V^\bullet$ in $D^-(\Lambda)$, we obtain a morphism

$$\hat{f}_1 \in \mathrm{Hom}_{D^-(k[\varepsilon]\Lambda)}(V^\bullet, V^\bullet[1])$$

associated to \hat{f} , namely $\hat{f}_1 = \phi' \circ \hat{f} \circ \phi^{-1}$, where $\phi' = \phi[1] \circ \psi[1]$ and ψ is as in (2.7). We define a map

$$(2.10) \quad \begin{aligned} \hat{h} : F(k[\varepsilon]) &\rightarrow \text{Hom}_{D^-(k[\varepsilon]\Lambda)}(V^\bullet, V^\bullet[1]), \\ [M^\bullet, \phi] &\mapsto \hat{f}_1. \end{aligned}$$

As in the proof of [6, Lemma 6.1], it follows that \hat{h} is a well-defined injective set map. Moreover, it follows that if (X^\bullet, ξ) is given by $X^\bullet = k[\varepsilon] \hat{\otimes}_k V^\bullet$ and $\xi : k \hat{\otimes}_{k[\varepsilon]} X^\bullet \cong V^\bullet \xrightarrow{\text{id}} V^\bullet$, i.e. $[X^\bullet, \xi]$ is the trivial deformation of V^\bullet over $k[\varepsilon]$, then

$$\hat{f}_1 = \hat{h}([X^\bullet, \xi]) + \text{Inf}_\Lambda^{k[\varepsilon]\Lambda} \text{Res}_{k[\varepsilon]\Lambda}^\Lambda(\hat{f}_1)$$

in $\text{Hom}_{D^-(k[\varepsilon]\Lambda)}(V^\bullet, V^\bullet[1])$. Because the map \hat{h} in (2.10) is injective and the inflation map

$$\text{Inf}_\Lambda^{k[\varepsilon]\Lambda} : \text{Hom}_{D^-(\Lambda)}(V^\bullet, V^\bullet[1]) \rightarrow \text{Hom}_{D^-(k[\varepsilon]\Lambda)}(V^\bullet, V^\bullet[1])$$

is injective, we obtain an injective map

$$\begin{aligned} h : F(k[\varepsilon]) &\rightarrow \text{Hom}_{D^-(\Lambda)}(V^\bullet, V^\bullet[1]), \\ [M^\bullet, \phi] &\mapsto \text{Res}_{k[\varepsilon]\Lambda}^\Lambda(\hat{f}_1). \end{aligned}$$

Since Schlessinger's criterion (H2) is valid by Proposition 2.4.1, it follows from [32, Lemma 2.10] that $t_F = F(k[\varepsilon])$ has a vector space structure. Hence we obtain as in the proof of [6, Lemma 6.1] that the map h is k -linear and surjective. More precisely, an element $\alpha \in \text{Hom}_{D^-(\Lambda)}(V^\bullet, V^\bullet[1])$ defines a quasi-lift, and hence a deformation, of V^\bullet over $k[\varepsilon]$ as follows. Since V^\bullet is assumed to be a bounded above complex of topologically free pseudocompact Λ -modules, $\alpha : V^\bullet \rightarrow V^\bullet[1]$ can be represented by a morphism in $C^-(\Lambda)$. Define an ε -action on the mapping cone $C(\alpha[-1])^\bullet$ by $\varepsilon(a, b) = (0, a)$ for all $(a, b) \in C(\alpha[-1])^i = V^i \oplus V^i$. Then the complex $M^\bullet = C(\alpha[-1])^\bullet$ defines a quasi-lift (M^\bullet, ϕ) of V^\bullet over $k[\varepsilon]$ whose deformation is sent to α under h . This proves part (i) of Proposition 2.4.2.

Part (ii) is proved similarly to [6, Lemma 6.3]. Namely, by part (i), t_{F^fl} is isomorphic to the subspace of $\text{Ext}_{D^-(\Lambda)}^1(V^\bullet, V^\bullet)$ consisting of those elements that define proflat deformations of V^\bullet over $k[\varepsilon]$. Let (M^\bullet, ϕ) be a quasi-lift of V^\bullet over $k[\varepsilon]$. Then (M^\bullet, ϕ) is a proflat quasi-lift if the cohomology groups of M^\bullet are topologically free pseudocompact, and hence abstractly free, $k[\varepsilon]$ -modules. In this case M^\bullet is isomorphic in $D^-(k[\varepsilon])$ to a bounded complex with trivial differentials whose term in degree n is a free $k[\varepsilon]$ -module of rank $\dim_k H^n(V^\bullet)$ for all integers n . Therefore, all proflat quasi-lifts (M^\bullet, ϕ) of V^\bullet over $k[\varepsilon]$ are isomorphic in $D^-(k[\varepsilon])$, when we forget the Λ -action. This means that the tangent space t_{F^fl} is mapped to $\{0\}$ under the forgetful map (2.5). Suppose now that $\alpha \in \text{Ext}_{D^-(\Lambda)}^1(V^\bullet, V^\bullet)$ is mapped to the zero morphism under (2.5). Then it follows from the proof of part (i) that the deformation $[M^\bullet, \phi]$ of V^\bullet over $k[\varepsilon]$ corresponding to α is equal to the trivial deformation $[X^\bullet, \xi]$ of V^\bullet over $k[\varepsilon]$ where $X^\bullet = k[\varepsilon] \hat{\otimes}_k V^\bullet$ and $\xi : k \hat{\otimes}_{k[\varepsilon]} X^\bullet \cong V^\bullet \xrightarrow{\text{id}} V^\bullet$. Since V^\bullet is completely split in $D^-(k)$, it follows that the cohomology groups of X^\bullet are abstractly free, and hence topologically free, over $k[\varepsilon]$. Thus $[M^\bullet, \phi]$ is a proflat deformation of V^\bullet over $k[\varepsilon]$. This completes the proof of part (ii), and hence of Proposition 2.4.2. \square

Proposition 2.4.3. *Schlessinger's criterion (H3) is satisfied, i.e. the k -dimension of the tangent space t_{F_D} is finite.*

Proof. The proof of Proposition 2.4.3 closely follows the proof of [6, Prop. 6.4]. By Proposition 2.4.2 it is enough to find an upper bound for the k -dimension of $\text{Ext}_{D^-(\Lambda)}^1(V^\bullet, V^\bullet)$. By truncating and shifting, we can assume that V^\bullet has the form

$$V^\bullet : \dots \rightarrow V^{-n} \rightarrow V^{-n+1} \rightarrow \dots \rightarrow V^0 \rightarrow 0 \dots$$

We first prove the following claim, which replaces the assumption in [6] that G has finite pseudocompact cohomology in our situation.

Claim 1. If M_1, M_2 are pseudocompact Λ -modules that are finite dimensional over k , then $\text{Ext}_\Lambda^j(M_1, M_2)$ is finite dimensional over k for all integers j .

To prove Claim 1, we note that by Remark 2.1.1(iii), $\text{Ext}_\Lambda^j(M_1, M_2)$ is computed by using a projective resolution of M_1 in $\text{PCMod}(\Lambda)$. Since $\dim_k M_1$ is finite, there exists a resolution of M_1 in $\text{PCMod}(\Lambda)$ consisting of abstractly free finitely generated Λ -modules. In other words, $\text{Ext}_\Lambda^j(M_1, M_2)$ can be identified with the corresponding j -th Ext group in the category of finitely generated Λ -modules. The latter group is known to be a finite dimensional k -vector space, which proves Claim 1.

Claim 2. Suppose n is a non-negative integer, and L_1^\bullet (resp. L_2^\bullet) is a complex of pseudocompact Λ -modules whose terms are concentrated between the degrees $-n_1$ and $-n_1 + n$ (resp. between $-n_2$ and $-n_2 + n$), for integers n_1 and n_2 . Then for all integers j , $\text{Ext}_{D^-(\Lambda)}^j(L_1^\bullet, L_2^\bullet)$ has finite k -dimension, if all cohomology groups of L_1^\bullet and of L_2^\bullet have finite k -dimension.

Claim 2 is proved by induction on n . If $n = 0$, then L_1^\bullet (resp. L_2^\bullet) is a module in degree $-n_1$ (resp. $-n_2$). Hence

$$\text{Ext}_{D^-(\Lambda)}^j(L_1^\bullet, L_2^\bullet) \cong \text{Ext}_\Lambda^j(\mathbb{H}^{-n_1}(L_1^\bullet)[n_1], \mathbb{H}^{-n_2}(L_2^\bullet)[n_2]) \cong \text{Ext}_\Lambda^{j+n_2-n_1}(L_1^{-n_1}, L_2^{-n_2})$$

which has finite k -dimension by Claim 1. We can now complete the proof of Claim 2 in a similar way to the proof of [6, Prop. 6.4].

By setting $L_1^\bullet = V^\bullet = L_2^\bullet$ and $j = 1$, Proposition 2.4.3 follows from Claim 2. \square

Proposition 2.4.4. *The functor $\hat{F}_{\mathcal{D}} : \hat{\mathcal{C}} \rightarrow \text{Sets}$ is continuous. In other words, for all objects R in $\hat{\mathcal{C}}$ with maximal ideal m_R we have*

$$\hat{F}_{\mathcal{D}}(R) = \varprojlim_i \hat{F}_{\mathcal{D}}(R/m_R^i).$$

Proof. The proof of Proposition 2.4.4 closely follows the proof of [6, Prop. 7.2]. By Remark 2.1.11 and Lemma 2.2.3(i), we may assume, without loss of generality, that V^\bullet is a bounded above complex of abstractly free finitely generated Λ -modules. Let R be an object of $\hat{\mathcal{C}}$ with maximal ideal m_R . Consider the natural map

$$(2.11) \quad \Xi_{\mathcal{D}} : \hat{F}_{\mathcal{D}}(R) \rightarrow \varprojlim_i \hat{F}_{\mathcal{D}}(R/m_R^i)$$

defined by $\Xi_{\mathcal{D}}([M^\bullet, \phi]) = \{\hat{F}_{\mathcal{D}}(\pi_i)([M^\bullet, \phi])\}_{i=1}^\infty = \{(R/m_R^i) \hat{\otimes}_R^{\mathbf{L}} M^\bullet, \phi_{\pi_i}\}_{i=1}^\infty$ when $\pi_i : R \rightarrow R/m_R^i$ is the natural surjection for all i .

We first show that $\Xi_{\mathcal{D}}$ is surjective. Suppose we have a sequence of deformations $\{[M_i^\bullet, \phi_i]\}_{i=1}^\infty$ with $[M_i^\bullet, \phi_i] \in \hat{F}_{\mathcal{D}}(R/m_R^i) = F_{\mathcal{D}}(R/m_R^i)$ for all i such that there is an isomorphism

$$\alpha_i : (R/m_R^i) \hat{\otimes}_{R/m_R^{i+1}}^{\mathbf{L}} M_{i+1}^\bullet \rightarrow M_i^\bullet$$

in $D^-((R/m_R^i)\Lambda)$ with $\phi_i \circ (k \hat{\otimes}_{R/m_R^i}^{\mathbf{L}} \alpha_i) = \phi_{i+1}$. We need to construct a quasi-lift (M^\bullet, ϕ) with $[M^\bullet, \phi] \in \hat{F}_{\mathcal{D}}(R)$ such that, for all i , there is an isomorphism $\beta_i : (R/m_R^i) \hat{\otimes}_R^{\mathbf{L}} M^\bullet \rightarrow M_i^\bullet$ in $D^-((R/m_R^i)\Lambda)$ with $\alpha_i \circ ((R/m_R^i) \hat{\otimes}_{R/m_R^{i+1}}^{\mathbf{L}} \beta_{i+1}) = \beta_i$ and $\phi_i \circ (k \hat{\otimes}_{R/m_R^i}^{\mathbf{L}} \beta_i) = \phi$. We now construct (M^\bullet, ϕ) inductively.

By Remark 2.2.7, we can replace M_i^\bullet by a bounded above complex N_i^\bullet of abstractly free finitely generated $(R/m_R^i)\Lambda$ -modules, and we can replace ϕ_i by a quasi-isomorphism $\psi_i : k \hat{\otimes}_{R/m_R^i}^{\mathbf{L}} N_i^\bullet \rightarrow V^\bullet$ in $C^-(\Lambda)$. Suppose $\gamma_i : N_i^\bullet \rightarrow M_i^\bullet$ is an isomorphism in $D^-((R/m_R^i)\Lambda)$ associated to these

replacements. Then the diagram

$$(2.12) \quad (R/m_R^i) \hat{\otimes}_{R/m_R^{i+1}} N_{i+1}^\bullet \xrightarrow{(R/m_R^i) \hat{\otimes}^{\mathbf{L}} \gamma_{i+1}} (R/m_R^i) \hat{\otimes}_{R/m_R^{i+1}}^{\mathbf{L}} M_{i+1}^\bullet$$

$$\begin{array}{c} \downarrow \alpha_i \\ M_i^\bullet \\ \downarrow \gamma_i^{-1} \\ N_i^\bullet \end{array}$$

defines an isomorphism $\tilde{\alpha}_i : (R/m_R^i) \hat{\otimes}_{R/m_R^{i+1}} N_{i+1}^\bullet \rightarrow N_i^\bullet$ in $D^-((R/m_R^i)\Lambda)$, and $\tilde{\alpha}_i$ replaces α_i . Moreover, $\psi_i = \phi_i \circ (k \hat{\otimes}_{R/m_R^i}^{\mathbf{L}} \gamma_i)$ satisfies

$$\psi_i \circ (k \hat{\otimes}_{R/m_R^i} \tilde{\alpha}_i) = \psi_{i+1}.$$

Since N_{i+1}^\bullet is a bounded above complex of abstractly free finitely generated $(R/m_R^{i+1})\Lambda$ -modules, we can assume that $\tilde{\alpha}_i$ is represented by a quasi-isomorphism in $C^-((R/m_R^i)\Lambda)$. By Remark 2.3.5, we can add to N_{i+1}^\bullet a suitable acyclic bounded above complex of abstractly free finitely generated $(R/m_R^{i+1})\Lambda$ -modules to be able to assume that $\tilde{\alpha}_i$ is surjective on terms. Continuing this process inductively, we obtain an inverse system of quasi-lifts

$$\{(N_i^\bullet, \psi_i)\}_{i=1}^\infty \quad \text{where} \quad [N_i^\bullet, \psi_i] \in F_{\mathcal{D}}(R/m_R^i).$$

Further, N_i^\bullet is a complex of abstractly free finitely generated $(R/m_R^i)\Lambda$ -modules such that in the diagram

$$(2.13) \quad \begin{array}{ccc} & N_{i+1}^\bullet & \\ & \downarrow & \\ (R/m_R^i) \hat{\otimes}_{R/m_R^{i+1}} N_{i+1}^\bullet & \xrightarrow{\tilde{\alpha}_i} & N_i^\bullet \end{array}$$

all arrows are morphisms in $C^-((R/m_R^{i+1})\Lambda)$ that are surjective on terms. Define

$$M^\bullet = \varprojlim_i N_i^\bullet \quad \text{and} \quad \phi = \varprojlim_i \psi_i.$$

Then we obtain in a similar way as in the proof of [6, Prop. 7.2] that all terms of M^\bullet are topologically free pseudocompact R -modules and that $[M^\bullet, \phi] \in \hat{F}_{\mathcal{D}}(R)$. Letting for each i

$$\tilde{\beta}_i : (R/m_R^i) \hat{\otimes}_R M^\bullet \longrightarrow N_i^\bullet$$

be the natural isomorphism in $C^-((R/m_R^i)\Lambda)$, it follows that $\tilde{\alpha}_i \circ ((R/m_R^i) \hat{\otimes}_{R/m_R^{i+1}} \tilde{\beta}_{i+1}) = \tilde{\beta}_i$, where $\tilde{\alpha}_i$ is as defined in (2.12), and $\psi_i \circ (k \hat{\otimes}_{R/m_R^i} \tilde{\beta}_i) = \phi$. Hence, defining $\beta_i = \gamma_i \circ \tilde{\beta}_i$, we have $\alpha_i \circ ((R/m_R^i) \hat{\otimes}_{R/m_R^{i+1}}^{\mathbf{L}} \beta_{i+1}) = \beta_i$ and $\phi_i \circ (k \hat{\otimes}_{R/m_R^i}^{\mathbf{L}} \beta_i) = \phi$. Therefore, the map $\Xi_{\mathcal{D}}$ in (2.11) is surjective.

We now show that $\Xi_{\mathcal{D}}$ is injective. Since \hat{F}^{fl} is a subfunctor of \hat{F} by Remark 2.1.11, it is enough to show that $\Xi_{\mathcal{D}}$ is injective in the case when $\hat{F}_{\mathcal{D}} = \hat{F}$. We notice that in the proof of [6, Prop. 7.2] an assumption was made to arrive at a morphism f_i as in [6, Eq. (7.5)]. This needs more explanation, which we will now provide in our situation. We first prove three claims.

Claim 1. Let $[M^\bullet, \phi] \in \hat{F}(R)$. For all positive integers i , define

$$Z_i = \{\zeta_i \in \text{End}_{D^-((R/m_R^i)\Lambda)}((R/m_R^i) \hat{\otimes}_R^{\mathbf{L}} M^\bullet) \mid k \hat{\otimes}_{R/m_R^i}^{\mathbf{L}} \zeta_i = 0 \text{ in } D^-(\Lambda)\}.$$

Then Z_i is a finitely generated nilpotent (R/m_R^i) -module.

To prove Claim 1, we first use Remarks 2.1.4 and 2.3.6 to assume without loss of generality that M^\bullet is a bounded above complex of topologically free pseudocompact $R\Lambda$ -modules. Hence

$$(2.14) \quad Z_i = \{\zeta_i \in \text{End}_{K^-((R/m_R^i)\Lambda)}((R/m_R^i) \hat{\otimes}_R M^\bullet) \mid k \hat{\otimes}_{R/m_R^i} \zeta_i = 0 \text{ in } K^-(\Lambda)\}.$$

It is obvious that Z_i is an (R/m_R^i) -module. Suppose $H^j(V^\bullet) = 0$ for $j < n_1$ and $j > n_2$. By Theorem 2.2.9, it follows that also $H^j((R/m_R^i)\hat{\otimes}_R M^\bullet) = 0$ for $j < n_1$ and $j > n_2$. By Lemma 2.2.3(i), there exists a bounded above complex N_i^\bullet of abstractly free finitely generated $(R/m_R^i)\Lambda$ -modules such that there is an isomorphism $(R/m_R^i)\hat{\otimes}_R M^\bullet \rightarrow N_i^\bullet$ in $K_{\text{fin}}^-((R/m_R^i)\Lambda)$. Hence we can truncate N_i^\bullet at n_1 and n_2 to be able to assume that $N_i^j = 0$ for $j < n_1$ and $j > n_2$ and that N_i^j is an abstractly free finitely generated (R/m_R^i) -module for $n_1 \leq j \leq n_2$. We obtain that Z_i is an (R/m_R^i) -submodule of

$$\text{End}_{K^-(R/m_R^i)}((R/m_R^i)\hat{\otimes}_R M^\bullet) \cong \text{End}_{K^-(R/m_R^i)}(N_i^\bullet)$$

which is isomorphic to a quotient module of $\text{End}_{C^-(R/m_R^i)}(N_i^\bullet)$. This, in turn, is a submodule of $\prod_{j=n_1}^{n_2} \text{End}_{R/m_R^i}(N_i^j)$, which is a free (R/m_R^i) -module of finite rank. Since R/m_R^i is Noetherian, it follows that Z_i is a finitely generated (R/m_R^i) -module. Since $(\zeta_i)^i = 0$ in $K^-((R/m_R^i)\Lambda)$ for all $\zeta_i \in Z_i$, Claim 1 follows.

Claim 2. Let $[M^\bullet, \phi]$ and Z_i be as in Claim 1. For all positive integers i and for all $j \geq i$, define $\tau_i^j : Z_j \rightarrow Z_i$ to be the map that sends $\zeta_j \in Z_j$ to $(R/m_R^i)\hat{\otimes}_{R/m_R^j}^{\mathbf{L}} \zeta_j$. Then there exists $N \geq i$ such that for all $j \geq N$, $\tau_i^j(Z_j) = \tau_i^N(Z_N)$. In other words, the inverse system $\{Z_i, \tau_i^j\}$ satisfies the Mittag-Leffler condition.

Claim 2 follows almost immediately from Claim 1. Namely, by Claim 1, Z_i is a finitely generated module over the Artinian ring R/m_R^i . In particular, Z_i is Artinian. Consider the descending chain of submodules

$$Z_i = \tau_i^i(Z_i) \supseteq \tau_i^{i+1}(Z_{i+1}) \supseteq \tau_i^{i+2}(Z_{i+2}) \supseteq \dots$$

Since Z_i is Artinian, this must stabilize after finitely many steps, proving Claim 2.

Claim 3. Suppose $\hat{F}_{\mathcal{D}} = \hat{F}$ and $[M^\bullet, \phi], [\widetilde{M}^\bullet, \widetilde{\phi}] \in \hat{F}(R)$ are such that $\Xi_{\mathcal{D}}([M^\bullet, \phi]) = \Xi_{\mathcal{D}}([\widetilde{M}^\bullet, \widetilde{\phi}])$. Then for all i , there exist isomorphisms $f_i : (R/m_R^i)\hat{\otimes}_R^{\mathbf{L}} M^\bullet \rightarrow (R/m_R^i)\hat{\otimes}_R^{\mathbf{L}} \widetilde{M}^\bullet$ in $D^-((R/m_R^i)\Lambda)$ such that $(R/m_R^i)\hat{\otimes}_{R/m_R^{i+1}}^{\mathbf{L}} f_{i+1} = f_i$ in $D^-((R/m_R^i)\Lambda)$ and $\widetilde{\phi} \circ (k\hat{\otimes}_{R/m_R^i}^{\mathbf{L}} f_i) = \phi$ in $D^-(\Lambda)$.

To prove Claim 3, we can assume, as in the proof of Claim 1, that M^\bullet and \widetilde{M}^\bullet are bounded above complexes of topologically free pseudocompact $R\Lambda$ -modules. In particular, Z_i is given as in (2.14). For each positive integer i , define

$$S_i = \bigcap_{j \geq i} \tau_i^j(Z_j)$$

which is a subset of Z_i , and define $\sigma_i : S_{i+1} \rightarrow S_i$ by the restriction of τ_i^{i+1} to S_i . Then σ_i is a well-defined surjective map for all $i \geq 1$. By Claim 2, for each $i \geq 1$ there exists a positive integer $N_i \geq i$ such that $S_i = \tau_i^{N_i}(Z_{N_i})$. Assume $N_i \geq i$ is chosen to be smallest with this property. Note that by construction, $N_i \leq N_{i+1}$ for all i .

By our assumptions for Claim 3, $\Xi_{\mathcal{D}}([M^\bullet, \phi]) = \Xi_{\mathcal{D}}([\widetilde{M}^\bullet, \widetilde{\phi}])$, which means that for each positive integer i , there exists an isomorphism

$$\gamma_i : (R/m_R^i)\hat{\otimes}_R M^\bullet \rightarrow (R/m_R^i)\hat{\otimes}_R \widetilde{M}^\bullet$$

in $K^-((R/m_R^i)\Lambda)$ such that $\widetilde{\phi} \circ (k\hat{\otimes}_{R/m_R^i}^{\mathbf{L}} \gamma_i) = \phi$ in $K^-(\Lambda)$.

For each positive integer i , let Id_i be the identity cochain map of $(R/m_R^i)\hat{\otimes}_R M^\bullet$ and define

$$(2.15) \quad \widetilde{f}_i = (R/m_R^i)\hat{\otimes}_{R/m_R^{N_i}}^{\mathbf{L}} \gamma_{N_i}$$

and

$$(2.16) \quad \zeta_i = \widetilde{f}_i^{-1} \circ ((R/m_R^i)\hat{\otimes}_{R/m_R^{i+1}}^{\mathbf{L}} \widetilde{f}_{i+1}) - \text{Id}_i.$$

Then

$$\zeta_i = (R/m_R^i)\hat{\otimes}_{R/m_R^{N_i}}^{\mathbf{L}} \left(\gamma_{N_i}^{-1} \circ \left((R/m_R^{N_i})\hat{\otimes}_{R/m_R^{N_{i+1}}}^{\mathbf{L}} \gamma_{N_{i+1}} \right) - \text{Id}_{N_i} \right)$$

is an element of $\tau_i^{N_i}(Z_{N_i}) = S_i$, since $(k\hat{\otimes}_{R/m_R^i}\gamma_i) = \tilde{\phi}^{-1} \circ \phi$ in $K^-(\Lambda)$ for all $i \geq 1$.

We define $f_1 = \tilde{f}_1$ and $f_2 = \tilde{f}_2$. Then

$$(R/m_R)\hat{\otimes}_{R/m_R^2}f_2 = k\hat{\otimes}_{R/m_R^{N_2}}\gamma_{N_2} = \tilde{\phi}^{-1} \circ \phi = k\hat{\otimes}_{R/m_R^{N_1}}\gamma_{N_1} = f_1.$$

For all $j \geq 2$, define $\zeta_j^{(j)} = \zeta_j$ to be the element in S_j defined in (2.16). Inductively, for all $\ell \geq 1$ and all $j \geq \ell + 2$, use that $\sigma_{j-1} : S_j \rightarrow S_{j-1}$ is surjective to choose an element $\zeta_{j-\ell}^{(j)} \in S_j$ with $\sigma_{j-1}(\zeta_{j-\ell}^{(j)}) = \zeta_{j-\ell}^{(j-1)}$. In particular, this means that for all $j \geq 3$ and all $2 \leq i < j$, the element $\zeta_i^{(j)} \in S_j$ is such that for all t with $i \leq t < j$, $(\sigma_t \circ \sigma_{t+1} \circ \cdots \circ \sigma_{j-1})(\zeta_i^{(j)}) = \zeta_i^{(t)}$. For all $j \geq 3$, we define

$$f_j = \tilde{f}_j \circ \left(\text{Id}_j + \zeta_{j-1}^{(j)}\right)^{-1} \circ \left(\text{Id}_j + \zeta_{j-2}^{(j)}\right)^{-1} \circ \cdots \circ \left(\text{Id}_j + \zeta_2^{(j)}\right)^{-1}$$

where we use Claim 1 to see that, for all $2 \leq i \leq j$, $\zeta_i^{(j)} \in S_j \subseteq Z_j$ is nilpotent, and hence $(\text{Id}_j + \zeta_i^{(j)})$ is invertible in $K^-((R/m_R^j)\Lambda)$. We obtain

$$\begin{aligned} (R/m_R^{j-1})\hat{\otimes}_{R/m_R^j}f_j &= \left((R/m_R^{j-1})\hat{\otimes}_{R/m_R^j}\tilde{f}_j\right) \circ \left(\text{Id}_{j-1} + \zeta_{j-1}^{(j-1)}\right)^{-1} \circ \\ &\quad \left(\text{Id}_{j-1} + \zeta_{j-2}^{(j-1)}\right)^{-1} \circ \cdots \circ \left(\text{Id}_{j-1} + \zeta_2^{(j-1)}\right)^{-1} \\ &= \tilde{f}_{j-1} \circ \left(\text{Id}_{j-1} + \zeta_{j-2}^{(j-1)}\right)^{-1} \circ \cdots \circ \left(\text{Id}_{j-1} + \zeta_2^{(j-1)}\right)^{-1} \\ &= f_{j-1} \end{aligned}$$

in $K^-((R/m_R^{j-1})\Lambda)$, where the second to last equality follows from (2.16). Moreover, we have

$$\begin{aligned} \tilde{\phi} \circ (k\hat{\otimes}_{R/m_R^j}f_j) &= \tilde{\phi} \circ (k\hat{\otimes}_{R/m_R^j}\tilde{f}_j) \circ \left(\text{Id}_1 + k\hat{\otimes}_{R/m_R^{j-1}}\zeta_{j-1}\right)^{-1} \circ \cdots \circ \left(\text{Id}_1 + k\hat{\otimes}_{R/m_R^2}\zeta_2\right)^{-1} \\ &= \tilde{\phi} \circ (k\hat{\otimes}_{R/m_R^j}\tilde{f}_j) = \tilde{\phi} \circ (k\hat{\otimes}_{R/m_R^{N_j}}\gamma_{N_j}) = \phi \end{aligned}$$

in $K^-(\Lambda)$, where the second equation follows since all $\zeta_i \in S_i \subseteq Z_i$. This proves Claim 3.

We are now ready to prove that $\Xi_{\mathcal{D}}$ in (2.11) is injective in the case when $\hat{F}_{\mathcal{D}} = \hat{F}$. Suppose $[M^\bullet, \phi], [\tilde{M}^\bullet, \tilde{\phi}]$ in $\hat{F}(R)$ satisfy $\Xi_{\mathcal{D}}([M^\bullet, \phi]) = \Xi_{\mathcal{D}}([\tilde{M}^\bullet, \tilde{\phi}])$. As in the proof of Claim 1, we can assume without loss of generality that M^\bullet and \tilde{M}^\bullet are bounded above complexes of topologically free pseudocompact RA -modules. Hence it follows from Claim 3 that for all $i \geq 1$, there exist isomorphisms

$$f_i : (R/m_R^i)\hat{\otimes}_R M^\bullet \rightarrow (R/m_R^i)\hat{\otimes}_R \tilde{M}^\bullet$$

in $K^-((R/m_R^i)\Lambda)$ such that $(R/m_R^i)\hat{\otimes}_{R/m_R^{i+1}}f_{i+1} = f_i$ in $K^-((R/m_R^i)\Lambda)$ and $\tilde{\phi} \circ (k\hat{\otimes}_{R/m_R^i}f_i) = \phi$ in $K^-(\Lambda)$. Suppose that for all i , f_i is represented by a quasi-isomorphism in $C^-((R/m_R^i)\Lambda)$. By Remark 2.3.5, we can add to M^\bullet a suitable acyclic bounded above complex of topologically free pseudocompact RA -modules to be able to assume that all f_i are surjective on terms. Define $h_1 = f_1$. As in the proof of [6, Prop. 7.2], we can construct inductively, for all i , a quasi-isomorphism

$$h_i : (R/m_R^i)\hat{\otimes}_R M^\bullet \rightarrow (R/m_R^i)\hat{\otimes}_R \tilde{M}^\bullet$$

in $C^-((R/m_R^i)\Lambda)$ that is surjective on terms such that h_i is homotopic to f_i and such that

$$(R/m_R^j)\hat{\otimes}_{R/m_R^i}h_i = h_j$$

for all $j < i$. It follows that

$$h = \varprojlim_i h_i : M^\bullet = \varprojlim_i ((R/m_R^i)\hat{\otimes}_R M^\bullet) \longrightarrow \varprojlim_i ((R/m_R^i)\hat{\otimes}_R \tilde{M}^\bullet) = \tilde{M}^\bullet$$

is an isomorphism in $K^-(RA)$ with $\tilde{\phi} \circ (k\hat{\otimes}_R h) = \phi$ in $K^-(\Lambda)$. This completes the proof of Proposition 2.4.4. \square

By [32, Sect. 2], Theorem 2.1.12 now follows from Propositions 2.4.1 through 2.4.4.

Remark 2.4.5. Note that we can prove claims that are similar to Claims 1 - 3 in the proof of Proposition 2.4.4 to also provide more details in the proof of the continuity of the deformation functor defined in [6]. As mentioned above, the main point is that in the proof of the injectivity of the map $\Gamma_{\mathcal{D}}$ defined in the proof of [6, Prop. 7.2] an assumption was made to arrive at a morphism f_i as in [6, Eq. (7.5)] that needs more explanation. We now briefly sketch the necessary arguments.

Suppose G is a profinite group with finite pseudocompact cohomology, as defined in [6, Def. 2.13]. Make the assumptions on k and $\hat{\mathcal{C}}$ as in [6]. Assume V^\bullet is a complex in $D^-([[kG]])$ that has only finitely many non-zero cohomology groups, all of which have finite k -dimension. Let $\hat{F}_{\mathcal{D}} = \hat{F}_{\mathcal{D}, V^\bullet} : \hat{\mathcal{C}} \rightarrow \text{Sets}$ be the deformation functor defined in [6, Def. 2.10]. We want to show that $\hat{F}_{\mathcal{D}}$ is continuous.

By [6, Prop. 2.12 and Cor. 3.6(i)] we may assume, without loss of generality, that there is a closed normal subgroup Δ of finite index in G such that V^\bullet is a bounded above complex of abstractly free finitely generated $[k(G/\Delta)]$ -modules. Let R be an object of $\hat{\mathcal{C}}$ with maximal ideal m_R , and consider the natural map

$$\Gamma_{\mathcal{D}} : \hat{F}_{\mathcal{D}}(R) \rightarrow \varprojlim_i \hat{F}_{\mathcal{D}}(R/m_R^i)$$

defined by $\Gamma_{\mathcal{D}}([M^\bullet, \phi]) = \{[(R/m_R^i) \hat{\otimes}_R^{\mathbf{L}} M^\bullet, \phi_{\pi_i}]\}_{i=1}^\infty$ when $\pi_i : R \rightarrow R/m_R^i$ is the natural surjection for all i . The proof that $\Gamma_{\mathcal{D}}$ is surjective is the same as in the proof of [6, Prop. 7.2]. For the proof that $\Gamma_{\mathcal{D}}$ is injective, we use that \hat{F}^{fl} is a subfunctor of \hat{F} by [6, Prop. 2.12]. Hence it suffices to show that $\Gamma_{\mathcal{D}}$ is injective in the case when $\hat{F}_{\mathcal{D}} = \hat{F}$.

To prove this, we modify Claims 1 - 3 in the proof of Proposition 2.4.4 to arrive at the three claims Claims $G.1$ - $G.3$ below. Using f_i from Claim $G.3$, instead of the morphism f_i in [6, Eq. (7.5)], the remainder of the proof of the injectivity of $\Gamma_{\mathcal{D}}$ follows then as in the proof of [6, Prop. 7.2].

Claim G.1. Let $[M^\bullet, \phi] \in \hat{F}(R)$. For all positive integers i , define

$$Z_i = \{\zeta_i \in \text{End}_{D^-([[(R/m_R^i)G]])}((R/m_R^i) \hat{\otimes}_R^{\mathbf{L}} M^\bullet) \mid k \hat{\otimes}_{R/m_R^i}^{\mathbf{L}} \zeta_i = 0 \text{ in } D^-([kG])\}.$$

Then Z_i is a finitely generated nilpotent (R/m_R^i) -module.

Claim G.2. Let $[M^\bullet, \phi]$ and Z_i be as in Claim $G.1$. For all positive integers i and for all $j \geq i$, let $\tau_i^j : Z_j \rightarrow Z_i$ be the map that sends ζ_j to $(R/m_R^i) \hat{\otimes}_{R/m_R^j}^{\mathbf{L}} \zeta_j$. Then there exists $N \geq i$ such that for all $j \geq N$, $\tau_i^j(Z_j) = \tau_i^N(Z_N)$. In other words, the inverse system $\{Z_i, \tau_i^j\}$ satisfies the Mittag-Leffler condition.

Claim G.3. Suppose $\hat{F}_{\mathcal{D}} = \hat{F}$ and $[M^\bullet, \phi], [\tilde{M}^\bullet, \tilde{\phi}] \in \hat{F}(R)$ are such that $\Gamma_{\mathcal{D}}([M^\bullet, \phi]) = \Gamma_{\mathcal{D}}([\tilde{M}^\bullet, \tilde{\phi}])$. Then for all i , there are isomorphisms $f_i : (R/m_R^i) \hat{\otimes}_R^{\mathbf{L}} M^\bullet \rightarrow (R/m_R^i) \hat{\otimes}_R^{\mathbf{L}} \tilde{M}^\bullet$ in $D^-([[(R/m_R^i)G]])$ such that $(R/m_R^i) \hat{\otimes}_{R/m_R^i}^{\mathbf{L}} f_{i+1} = f_i$ in $D^-([[(R/m_R^i)G]])$ and $\tilde{\phi} \circ (k \hat{\otimes}_{R/m_R^i}^{\mathbf{L}} f_i) = \phi$ in $D^-([kG])$.

The proofs of Claims $G.1$ - $G.3$ are rather similar to the proofs of Claims 1 - 3 in the proof of Proposition 2.4.4. We use [6, Lemma 4.2] to be able to assume that M^\bullet (and in Claim $G.3$ also \tilde{M}^\bullet) is a bounded above complex of topologically free pseudocompact $[[RG]]$ -modules. If $H^j(V^\bullet) = 0$ for $j < n_1$ and $j > n_2$, then we use [7, Thm. 2.10] to argue that $H^j((R/m_R^i) \hat{\otimes}_R^{\mathbf{L}} M^\bullet) = 0$ for $j < n_1$ and $j > n_2$. Moreover, we use [6, Cor. 3.6(i)] to see that there exists a closed normal subgroup Δ_i of finite index in G and a bounded above complex N_i^\bullet of abstractly free finitely generated $[(R/m_R^i)(G/\Delta_i)]$ -modules such that there is an isomorphism $(R/m_R^i) \hat{\otimes}_R^{\mathbf{L}} M^\bullet \rightarrow \text{Inf}_{G/\Delta_i}^G(N_i^\bullet)$ in $K_{\text{fin}}^-([[(R/m_R^i)G]])$. In particular, $\prod_{j=n_1}^{n_2} \text{End}_{R/m_R^i}(N_i^j)$ is then a free (R/m_R^i) -module of finite rank. The arguments of the remainder of the proofs of Claims $G.1$ - $G.3$ are basically the same as in the proof of Proposition 2.4.4.

We now return to the situation in the present paper and, in particular, to the assumptions put forward in Section 2.1.

2.5. Special complexes V^\bullet . In this section we consider several cases of more special complexes V^\bullet . Namely, we consider one-term and two-term complexes, and completely split complexes. The results are adapted from [6, Sects. 9 and 11].

We first recall the deformation functor associated to a finitely generated Λ -module that was studied in [9].

Remark 2.5.1. Let C be a finitely generated Λ -module, and let R be an object of $\hat{\mathcal{C}}$. According to [9, Sect. 2], a lift of C over R is a pair (L, λ) consisting of a finitely generated $R\Lambda$ -module L that is abstractly free as an R -module together with a Λ -module isomorphism $\lambda : k \otimes_R M \rightarrow V$. Two lifts (L, λ) and (L', λ') of C over R are said to be isomorphic if there is an $R\Lambda$ -module isomorphism $f : L \rightarrow L'$ with $\lambda' \circ (k \otimes_R f) = \lambda$. A deformation of C over R is an isomorphism class of lifts of C over R . We denote the deformation of C over R represented by (L, λ) by $[L, \lambda]$. The deformation functor $\hat{F}_C : \hat{\mathcal{C}} \rightarrow \text{Sets}$ is defined to be the covariant functor that sends an object R of $\hat{\mathcal{C}}$ to the set $\hat{F}_C(R)$ of all deformations of C over R , and that sends a morphism $\alpha : R \rightarrow R'$ in $\hat{\mathcal{C}}$ to the set map $\hat{F}_C(R) \rightarrow \hat{F}_C(R')$ given by $[L, \lambda] \mapsto [R' \otimes_{R, \alpha} L, \lambda_\alpha]$ where λ_α is the composition $k \otimes_{R'} (R' \otimes_{R, \alpha} L) \cong k \otimes_R L \xrightarrow{\lambda} C$. Let $F_C : \mathcal{C} \rightarrow \text{Sets}$ be the restriction of \hat{F}_C to the full subcategory \mathcal{C} of Artinian rings in $\hat{\mathcal{C}}$.

It was shown in [9, Prop. 2.1] that F_C always has a pro-representable hull $R(\Lambda, C)$ in $\hat{\mathcal{C}}$, called the versal deformation ring of C , and that \hat{F}_C is continuous. Moreover, if $\text{End}_\Lambda(V) = k$, it was shown that \hat{F}_C is represented by $R(\Lambda, C)$, in which case $R(\Lambda, C)$ is called the universal deformation ring of C .

Proposition 2.5.2. *Assume Hypothesis 1, and let $F_{\mathcal{D}}$ and $\hat{F}_{\mathcal{D}}$ be as in Definition 2.1.10. Suppose V^\bullet has exactly one non-zero cohomology group C , which has finite k -dimension. Let \hat{F}_C be the deformation functor considered in [9] (see Remark 2.5.1). Then $\hat{F}_{\mathcal{D}}$ and \hat{F}_C are naturally isomorphic, and $R(\Lambda, V^\bullet)$ is isomorphic to the versal deformation ring $R(\Lambda, C)$ of C . In particular, $R(\Lambda, V^\bullet) = R^{\text{fl}}(\Lambda, V^\bullet)$. The groups $\text{Hom}_{\mathcal{D}^-(\Lambda)}(V^\bullet, V^\bullet)$ and $\text{Hom}_\Lambda(C, C)$ are isomorphic.*

Proof. The proof of Proposition 2.5.2 closely follows the proof of [6, Prop. 9.1]. It will suffice to show that the functor $\hat{F} : \hat{\mathcal{C}} \rightarrow \text{Sets}$ defined by the isomorphism classes of quasi-lifts of V^\bullet is naturally isomorphic to the functor \hat{F}_C from [9] (see Remark 2.5.1). Because \hat{F} and \hat{F}_C are continuous (see Proposition 2.4.4 and [9, Prop. 2.1]), it will suffice to show that the restrictions F and F_C of these functors to \mathcal{C} are naturally isomorphic. Without loss of generality we can assume $C = H^0(V^\bullet)$ and V^\bullet is the one-term complex with $V^0 = C$ and $V^i = 0$ for $i \neq 0$ (see Remark 2.1.11).

If R is an Artinian ring in $\text{Ob}(\mathcal{C})$ and (M^\bullet, ϕ) is a quasi-lift of V^\bullet over R , then M^\bullet has non-zero cohomology only in degree 0 by Lemma 2.2.8. Hence (M^\bullet, ϕ) is isomorphic to a quasi-lift (M'^\bullet, ϕ') where $M'^0 = H^0(M^\bullet)$ and $M'^i = 0$ otherwise. Since, by Lemma 2.2.5, M^\bullet has finite pseudocompact R -tor dimension at 0, it follows by Remark 2.1.6 and Lemma 2.2.8 that M'^0 has projective dimension 0 as abstract R -module. Hence M'^0 is an abstractly free R -module because R is local Artinian. Therefore $k \otimes_R M'^0 = k \hat{\otimes}_R M'^0$ is isomorphic to $C = H^0(V^\bullet)$ since $k \hat{\otimes}_R M^\bullet$ is isomorphic to V^\bullet in the derived category. We have now shown that (M'^0, ϕ'^0) is a lift of C over R , in the sense of [9], and the isomorphism class of (M'^0, ϕ'^0) as a lift of C over R determines the isomorphism class of (M^\bullet, ϕ) as a quasi-lift of V^\bullet over R . Conversely, suppose (L, λ) is a lift of C over R , in the sense of [9]. Then L is an abstractly free R -module of rank equal to $\dim_k C$. Since R is Artinian, this implies that L is a discrete $R\Lambda$ -module of finite length, and hence a pseudocompact $R\Lambda$ -module. Thus the complex L^\bullet with $L^0 = L$ and $L^i = 0$ for $i \neq 0$ together with the morphism $\psi : k \otimes_R L^\bullet = k \hat{\otimes}_R L^\bullet \rightarrow V^\bullet$ in $C^-(\Lambda)$ given by $\psi^0 = \lambda$ and $\psi^i = 0$ for $i \neq 0$ defines a quasi-lift (L^\bullet, ψ) of V^\bullet over R . This shows F and F_C are naturally isomorphic functors. \square

Remark 2.5.3. Suppose V^\bullet has precisely two non-zero cohomology groups. Without loss of generality, we can assume these groups are $U_0 = H^0(V^\bullet)$ and $U_{-n} = H^{-n}(V^\bullet)$ for some $n > 0$, both of finite k -dimension by Hypothesis 1. We will also regard U_0, U_{-n} as complexes concentrated in

degree 0. In particular, we obtain a distinguished triangle in $D^-(\Lambda)$ of the form

$$(2.17) \quad U_{-n}[n] \xrightarrow{\iota} V^\bullet \xrightarrow{\pi} U_0 \xrightarrow{\beta} U_{-n}[n+1].$$

By the triangle axioms, there exists a complex $C(\beta)^\bullet$ in $D^-(\Lambda)$, which is unique up to isomorphism, such that

$$U_0 \xrightarrow{\beta} U_{-n}[n+1] \rightarrow C(\beta)^\bullet \rightarrow U_0[1]$$

is a distinguished triangle in $D^-(\Lambda)$. In other words, $V^\bullet \cong C(\beta)^\bullet[-1]$ in $D^-(\Lambda)$.

The statements and proofs of [6, Sect. 9] can be adapted to our situation. Here is a summary of some of the main results:

- (i) If the endomorphism rings of U_0 and U_{-n} are both given by scalars, then $\mathrm{Hom}_{D^-(\Lambda)}(V^\bullet, V^\bullet) = k$ if and only if
 - (a) $\mathrm{Ext}_\Lambda^n(U_0, U_{-n}) = 0$, and
 - (b) there exists a nontrivial element $\beta \in \mathrm{Ext}_\Lambda^{n+1}(U_0, U_{-n}) \cong \mathrm{Hom}_{D^-(\Lambda)}(U_0, U_{-n}[n+1])$ such that $V^\bullet \cong C(\beta)^\bullet[-1]$ in $D^-(\Lambda)$.
- (ii) We have the following description of the tangent space of the proflat deformation functor \hat{F}^{fl} using Remark 2.1.15(i):
 - If $n \geq 2$, then $t_{F^{\mathrm{fl}}} = t_F$. If $n = 1$, then $t_{F^{\mathrm{fl}}}$ is the subspace of $t_F = \mathrm{Ext}_{D^-(\Lambda)}^1(V^\bullet, V^\bullet) = \mathrm{Hom}_{D^-(\Lambda)}(V^\bullet, V^\bullet[1])$ consisting of those elements $f \in t_F$ satisfying $\pi[1] \circ f \circ \iota = 0$ in $\mathrm{Hom}_{D^-(\Lambda)}(U_{-1}[1], U_0[1])$, where ι and π are as in (2.17) for $n = 1$.
- (iii) Suppose that U_{-n} and U_0 have universal deformation rings R_{-n} and R_0 and universal deformations $[X_{-n}, \psi_{-n}]$ and $[X_0, \psi_0]$, respectively, in the sense of [9], and that

$$\dim_k \mathrm{Ext}_\Lambda^1(U_{-n}, U_{-n}) + \dim_k \mathrm{Ext}_\Lambda^1(U_0, U_0) = \dim_k t_{F^{\mathrm{fl}}}.$$

Suppose furthermore that there exists a proflat quasi-lift (M^\bullet, ϕ) of V^\bullet over $R_{-n} \hat{\otimes}_k R_0$ such that

$$H^{-n}(M^\bullet) \cong (R_{-n} \hat{\otimes}_k R_0) \hat{\otimes}_{R_{-n}} X_{-n} \quad \text{and} \quad H^0(M^\bullet) \cong (R_{-n} \hat{\otimes}_k R_0) \hat{\otimes}_{R_0} X_0.$$

Then the versal proflat deformation ring $R^{\mathrm{fl}}(\Lambda, V^\bullet)$ is universal and isomorphic to $R_{-n} \hat{\otimes}_k R_0$.

Remark 2.5.4. Suppose V^\bullet is isomorphic in $D^-(\Lambda)$ to a complex whose differentials are trivial. Thus in $D^-(\Lambda)$, V^\bullet is isomorphic to the direct sum $\bigoplus_i H^i(V^\bullet)[-i]$, where there are only finitely many non-zero terms in this sum and all terms have finite k -dimension by Hypothesis 1. Without loss of generality, we can assume that all the differentials of V^\bullet are trivial, so $H^i(V^\bullet) = V^i$ for all i .

In this situation, a *split quasi-lift* of V^\bullet over an object R of $\hat{\mathcal{C}}$ is a proflat quasi-lift (M^\bullet, ϕ) of V^\bullet over R such that M^\bullet is isomorphic in $D^-(R\Lambda)$ to a complex whose differentials are trivial. A split deformation is the isomorphism class of a split quasi-lift. Let $\hat{F}^{\mathrm{sp}} = \hat{F}_{V^\bullet}^{\mathrm{sp}} : \hat{\mathcal{C}} \rightarrow \mathrm{Sets}$ be the functor that sends each object R of $\hat{\mathcal{C}}$ to the set $\hat{F}^{\mathrm{sp}}(R)$ of all split deformations of V^\bullet over R .

The statements and proofs of [6, Sect. 11] can be adapted to our situation. Here is a summary of some of the main results:

- (i) The functor \hat{F}^{sp} is isomorphic to the product of the functors on $\hat{\mathcal{C}}$ associated to deformations, in the sense of [9], of the non-zero cohomology groups of V^\bullet considered as Λ -modules. Moreover, the functor \hat{F}^{sp} is naturally isomorphic to the functor \hat{F}^{fl} .
- (ii) The versal split deformation ring $R^{\mathrm{sp}}(\Lambda, V^\bullet)$ associated to the split deformation functor \hat{F}^{sp} is the tensor product $\hat{\otimes}_i R(\Lambda, V^i)$ over k of the versal deformation rings, in the sense of [9], of the non-zero cohomology groups of V^\bullet . Suppose $[U(\Lambda, V^i), \phi_i]$ is the versal deformation of V^i , in the sense of [9], when $V^i = H^i(V^\bullet) \neq \{0\}$. A versal split deformation of V^\bullet is represented by the direct sum

$$U^{\mathrm{sp}}(\Lambda, V^\bullet) = \bigoplus_i R^{\mathrm{sp}}(\Lambda, V^\bullet) \hat{\otimes}_{R(\Lambda, V^i)} U(\Lambda, V^i)[-i]$$

where i runs over those integers for which $H^i(V^\bullet) \neq \{0\}$, together with the morphism $\phi_U : k \otimes_{R^{\text{sp}}(\Lambda, V^\bullet)} U^{\text{sp}}(\Lambda, V^\bullet) \rightarrow V^\bullet$ in $C^-(\Lambda)$ given by $\phi_U^i = \phi_i$ for all i with $H^i(V^\bullet) \neq \{0\}$ and $\phi_U^i = 0$ for all other i .

- (iii) The natural map on tangent spaces $\tau : F^{\text{sp}}(k[\varepsilon]) \rightarrow F(k[\varepsilon])$ may be identified with the natural inclusion

$$\iota : \bigoplus_i \text{Ext}_\Lambda^1(V^i, V^i) \rightarrow \text{Ext}_{D^-(\Lambda)}^1(V^\bullet, V^\bullet) = \bigoplus_{i,j} \text{Ext}_\Lambda^{1+i-j}(V^i, V^j).$$

- (iv) We have a non-canonical surjective continuous k -algebra homomorphism $f_{\text{sp}} : R(\Lambda, V^\bullet) \rightarrow R^{\text{sp}}(\Lambda, V^\bullet)$. If the versal deformation ring $R(\Lambda, V^i)$ is universal for all i , then $R^{\text{sp}}(\Lambda, V^\bullet)$ is a universal split deformation ring. If in addition f_{sp} is an isomorphism, then $R(\Lambda, V^\bullet)$ is a universal deformation ring for V^\bullet .

3. DERIVED EQUIVALENCES AND STABLE EQUIVALENCES OF MORITA TYPE

In [3], the first author proved that if k is a field of positive characteristic, and A and B are block algebras of finite groups over a complete local commutative Noetherian ring with residue field k , then a split-endomorphism two-sided tilting complex (as introduced by Rickard [30]) for the derived categories of bounded complexes of finitely generated modules over A , resp. B , preserves the versal deformation rings of bounded complexes of finitely generated modules over kA , resp. kB .

It is the goal of Section 3.1 to prove an analogous result, Theorem 3.1.5, when k is an arbitrary field and A and B are replaced by arbitrary finite dimensional k -algebras. In Section 3.2, we will then study the connection to stable equivalences of Morita type for self-injective algebras; see Propositions 3.2.5 and 3.2.6. We assume the notation of Section 2.

If S is a ring, $S\text{-mod}$ denotes the category of finitely generated left S -modules. Let $C^b(S\text{-mod})$ be the category of bounded complexes in $S\text{-mod}$, let $K^b(S\text{-mod})$ be the homotopy category of $C^b(S\text{-mod})$, and let $D^b(S\text{-mod})$ be the derived category of $K^b(S\text{-mod})$.

3.1. Deformations of complexes and derived equivalences. Recall from Section 2 that we view the finite dimensional k -algebra Λ as a pseudocompact k -algebra with the discrete topology, and every finitely generated Λ -module as a pseudocompact Λ -module with the discrete topology. In particular, $D^b(\Lambda\text{-mod})$ can be identified with a full subcategory of the derived category $D^-(\Lambda)$ of bounded above complexes of pseudocompact Λ -modules.

Suppose Γ is another finite dimensional k -algebra, and $R \in \text{Ob}(\hat{\mathcal{C}})$ is arbitrary. Then $R\Lambda$ and $R\Gamma$ are free R -modules of finite rank. Rickard proved in [29] that the derived categories $D^b(R\Lambda\text{-mod})$ and $D^b(R\Gamma\text{-mod})$ are equivalent as triangulated categories if and only if there is a bounded complex P_R^\bullet of finitely generated $R\Gamma$ - RA -bimodules and a bounded complex Q_R^\bullet of finitely generated RA - $R\Gamma$ -bimodules such that

$$(3.1) \quad \begin{aligned} Q_R^\bullet \otimes_{R\Gamma}^{\mathbf{L}} P_R^\bullet &\cong RA \quad \text{in } D^b((RA \otimes_R RA^{op})\text{-mod}), \text{ and} \\ P_R^\bullet \otimes_{RA}^{\mathbf{L}} Q_R^\bullet &\cong R\Gamma \quad \text{in } D^b((R\Gamma \otimes_R R\Gamma^{op})\text{-mod}). \end{aligned}$$

If P_R^\bullet and Q_R^\bullet exist, then the functors

$$(3.2) \quad \begin{aligned} P_R^\bullet \otimes_{RA}^{\mathbf{L}} - &: D^b(R\Lambda\text{-mod}) \rightarrow D^b(R\Gamma\text{-mod}) \quad \text{and} \\ Q_R^\bullet \otimes_{R\Gamma}^{\mathbf{L}} - &: D^b(R\Gamma\text{-mod}) \rightarrow D^b(R\Lambda\text{-mod}) \end{aligned}$$

are equivalences of derived categories, and Q_R^\bullet is isomorphic to $\mathbf{R}\text{Hom}_{R\Gamma}(P_R^\bullet, R\Gamma)$ in the derived category of RA - $R\Gamma$ -bimodules. The complexes P_R^\bullet and Q_R^\bullet are called *two-sided tilting complexes* (see [29, Def. 4.2]).

By [29, Prop. 3.1], P_R^\bullet is isomorphic in $D^b(R\Gamma\text{-mod})$ (resp. in $D^b(RA^{op}\text{-mod})$) to a bounded complex X^\bullet of finitely generated projective left $R\Gamma$ -modules (resp. a bounded complex Y^\bullet of finitely generated projective right RA -modules). Since $R\Gamma$ and RA are free R -modules, projective bimodules for these algebras are projective as left and right modules. Let

$$T^\bullet : \quad \dots \rightarrow T^{n-1} \xrightarrow{\delta_T^{n-1}} T^n \xrightarrow{\delta_T^n} T^{n+1} \rightarrow \dots$$

be a projective $R\Gamma$ - RA -bimodule resolution of P_R^\bullet such that all terms of T^\bullet are finitely generated projective $R\Gamma$ - RA -bimodules. By adding an acyclic complex of finitely generated projective $R\Gamma$ - RA -bimodules to T^\bullet if necessary, we can find quasi-isomorphisms

$$(3.3) \quad \begin{aligned} f : T^\bullet &\rightarrow X^\bullet \quad \text{in } C^-(R\Gamma\text{-mod}) \text{ and} \\ g : T^\bullet &\rightarrow Y^\bullet \quad \text{in } C^-(RA^{op}\text{-mod}) \end{aligned}$$

that are surjective on terms. More precisely, we construct the acyclic complex of finitely generated projective $R\Gamma$ - RA -bimodules to be added to T^\bullet inductively, starting from the right. Without loss of generality, assume $T^i = 0 = X^i = Y^i$ for $i > 0$. Let F^{-1} be a finitely generated free $R\Gamma$ - RA -bimodule such that there exists a surjective RA -module homomorphism $\phi_X^{-1} : F^{-1} \rightarrow X^{-1}$ and a surjective $R\Gamma$ -module homomorphism $\phi_Y^{-1} : F^{-1} \rightarrow Y^{-1}$. Let $\psi_X^0 : F^{-1} \rightarrow X^0$ be the composition $\delta_X^{-1} \circ \phi_X^{-1}$, and let $\psi_Y^0 : F^{-1} \rightarrow Y^0$ be the composition $\delta_Y^{-1} \circ \phi_Y^{-1}$. Since $f^0 : T^0 \rightarrow X^0$ and $g^0 : T^0 \rightarrow Y^0$ induce isomorphisms on the zero-th cohomology groups, it follows that $\tilde{f}^0 = (\psi_X^0, f^0) : F^{-1} \oplus T^0 \rightarrow X^0$ and $\tilde{g}^0 = (\psi_Y^0, g^0) : F^{-1} \oplus T^0 \rightarrow Y^0$ are surjective. In other words, we have added the acyclic complex $F^{-1} \xrightarrow{\text{id}} F^{-1}$, concentrated in the degrees -1 and 0 , to T^\bullet to ensure that the resulting homomorphisms \tilde{f}^0 and \tilde{g}^0 are surjective. Inductively, we now work our way to the left and add acyclic complexes of finitely generated free $R\Gamma$ - RA -bimodules of the form $F^{i-1} \xrightarrow{\text{id}} F^{i-1}$, concentrated in the degrees $i-1$ and i , to T^\bullet , for $i \leq -1$ such that there exists a surjective RA -module homomorphism $\phi_X^{i-1} : F^{i-1} \rightarrow X^{i-1}$ and a surjective $R\Gamma$ -module homomorphism $\phi_Y^{i-1} : F^{i-1} \rightarrow Y^{i-1}$. We also let $\psi_X^i : F^{i-1} \rightarrow X^i$ be the composition $\delta_X^{i-1} \circ \phi_X^{i-1}$, and we let $\psi_Y^i : F^{i-1} \rightarrow Y^i$ be the composition $\delta_Y^{i-1} \circ \phi_Y^{i-1}$, to receive surjective $R\Gamma$ -module homomorphisms $\tilde{f}^i = (\psi_X^i, \phi_X^i, f^i) : F^{i-1} \oplus F^i \oplus T^i \rightarrow X^i$ and surjective RA -module homomorphisms $\tilde{g}^i = (\psi_Y^i, \phi_Y^i, g^i) : F^{i-1} \oplus F^i \oplus T^i \rightarrow Y^i$ for all $i \leq -1$. Replacing f by \tilde{f} and g by \tilde{g} , if necessary, we arrive at quasi-isomorphisms f, g as in (3.3) that are surjective on terms.

Since $\text{Ker}(f)$ (resp. $\text{Ker}(g)$) is an acyclic complex of projective left $R\Gamma$ -modules (resp. projective right RA -modules), it splits completely. If $X^i = 0$ and $Y^i = 0$ for all $i \leq n$, it follows that $\text{Ker}(\delta_T^n)$ is isomorphic to the kernel of the n -th differential of $\text{Ker}(f)$ (resp. $\text{Ker}(g)$) as a left $R\Gamma$ -module (resp. right RA -module). But the latter kernel is projective as a left $R\Gamma$ -module (resp. right RA -module). Hence we can truncate T^\bullet at n :

$$\cdots \rightarrow 0 \rightarrow \text{Ker}(\delta_T^n) \rightarrow T^n \xrightarrow{\delta_T^n} T^{n+1} \rightarrow \cdots$$

to produce a bounded complex that is isomorphic to P_R^\bullet in $D^b((R\Gamma \otimes_R RA^{op})\text{-mod})$ with the additional property that all the terms of this complex are projective as left and as right modules and that all terms, except possibly the leftmost non-zero term, are actually projective as bimodules. Similarly, we can assume that all terms of Q_R^\bullet are projective as left RA -modules and as right $R\Gamma$ -modules and that all terms, except possibly the leftmost non-zero term, are actually projective as RA - $R\Gamma$ -bimodules. In particular, we can take Q_R^\bullet to be the $R\Gamma$ -dual

$$(3.4) \quad \tilde{P}_R^\bullet = \text{Hom}_{R\Gamma}(P_R^\bullet, R\Gamma).$$

In this situation, (3.1) is equivalent to

$$(3.5) \quad \begin{aligned} \tilde{P}_R^\bullet \otimes_{R\Gamma} P_R^\bullet &\cong RA \quad \text{in } D^b((RA \otimes_R RA^{op})\text{-mod}), \text{ and} \\ P_R^\bullet \otimes_{RA} \tilde{P}_R^\bullet &\cong R\Gamma \quad \text{in } D^b((R\Gamma \otimes_R R\Gamma^{op})\text{-mod}). \end{aligned}$$

Definition 3.1.1. Let $R \in \text{Ob}(\hat{\mathcal{C}})$. Suppose P_R^\bullet is a bounded complex of finitely generated $R\Gamma$ - RA -bimodules.

- (a) We call P_R^\bullet a *nice two-sided tilting complex*, if all terms of P_R^\bullet are projective as left $R\Gamma$ -modules and as right RA -modules and (3.5) is satisfied for \tilde{P}_R^\bullet as in (3.4).
- (b) If $R = k$, then we write $P^\bullet = P_R^\bullet$ and $\tilde{P}^\bullet = \text{Hom}_\Gamma(P^\bullet, \Gamma)$. In other words, P^\bullet is a *nice two-sided tilting complex* if P^\bullet is a bounded complex of finitely generated Γ - Λ -bimodules such that all terms of P^\bullet are projective as left Γ -modules and as right Λ -modules and such that $\tilde{P}^\bullet \otimes_\Gamma P^\bullet \cong \Lambda$ in $D^b((\Lambda \otimes_k \Lambda^{op})\text{-mod})$, and $P^\bullet \otimes_\Lambda \tilde{P}^\bullet \cong \Gamma$ in $D^b((\Gamma \otimes_k \Gamma^{op})\text{-mod})$.

Remark 3.1.2. If Λ and Γ are symmetric k -algebras, then $R\Lambda$ and $R\Gamma$ are symmetric R -algebras for all $R \in \text{Ob}(\hat{\mathcal{C}})$, in the sense that $R\Lambda$ (resp. $R\Gamma$), considered as a bimodule over itself, is isomorphic to its R -linear dual $\text{Hom}_R(R\Lambda, R)$ (resp. $\text{Hom}_R(R\Gamma, R)$). In particular, the functors $\text{Hom}_{R\Gamma}(-, R\Gamma)$ and $\text{Hom}_R(-, R)$ are then naturally isomorphic. Therefore, under the assumptions made prior to (3.4), we may take Q_R^\bullet to be the R -dual

$$(3.6) \quad \check{P}_R^\bullet = \text{Hom}_R(P_R^\bullet, R).$$

Rickard calls a bounded complex P_R^\bullet of finitely generated $R\Gamma$ - $R\Lambda$ -bimodules a *split-endoromorphism two-sided tilting complex*, if all terms of P_R^\bullet are projective as left $R\Gamma$ -modules and as right $R\Lambda$ -modules and (3.5) is satisfied with \check{P}_R^\bullet replaced by \tilde{P}_R^\bullet (see [30, p. 336]).

Lemma 3.1.3. *Suppose Λ and Γ are finite dimensional k -algebras, P^\bullet is a nice two-sided tilting complex in $D^b((\Gamma \otimes_k \Lambda^{op})\text{-mod})$ as in Definition 3.1.1, and $R \in \text{Ob}(\hat{\mathcal{C}})$. Then $P_R^\bullet = R \otimes_k P^\bullet$ is a nice two-sided tilting complex in $D^b((R\Gamma \otimes_R R\Lambda^{op})\text{-mod})$. Moreover,*

$$(3.7) \quad \begin{aligned} \tilde{P}_R^\bullet \hat{\otimes}_{R\Gamma} P_R^\bullet &\cong R\Lambda && \text{in } D^-(R\Lambda \otimes_R R\Lambda^{op}), \text{ and} \\ P_R^\bullet \hat{\otimes}_{R\Lambda} \tilde{P}_R^\bullet &\cong R\Gamma && \text{in } D^-(R\Gamma \otimes_R R\Gamma^{op}). \end{aligned}$$

In particular, the functors $P_R^\bullet \hat{\otimes}_{R\Lambda} -$ and $\tilde{P}_R^\bullet \hat{\otimes}_{R\Gamma} -$ provide quasi-inverse equivalences:

$$(3.8) \quad \begin{aligned} P_R^\bullet \hat{\otimes}_{R\Lambda} - &: D^-(R\Lambda) \rightarrow D^-(R\Gamma), \quad \text{and} \\ \tilde{P}_R^\bullet \hat{\otimes}_{R\Gamma} - &: D^-(R\Gamma) \rightarrow D^-(R\Lambda). \end{aligned}$$

Proof. We have

$$R \otimes_k (\Lambda \otimes_k \Lambda^{op}) \cong R\Lambda \otimes_R R\Lambda^{op}$$

and

$$R \otimes_k (\Gamma \otimes_k \Gamma^{op}) \cong R\Gamma \otimes_R R\Gamma^{op}$$

as pseudocompact R -algebras. Therefore, it follows from [29, Lemma 4.3] that $P_R^\bullet = R \otimes_k P^\bullet$ is a two-sided tilting complex in $D^b((R\Gamma \otimes_R R\Lambda^{op})\text{-mod})$. Note that $R \otimes_k \tilde{P}^\bullet = R \otimes_k \text{Hom}_\Gamma(P^\bullet, \Gamma) \cong \text{Hom}_{R\Gamma}(P_R^\bullet, R\Gamma) = \tilde{P}_R^\bullet$ as complexes of $R\Lambda$ - $R\Gamma$ -bimodules. Since all terms of P^\bullet are projective as left Γ -modules and as right Λ -modules, it follows that all terms of P_R^\bullet are projective as left $R\Gamma$ -modules and as right $R\Lambda$ -modules. Since $R \otimes_k (\tilde{P}^\bullet \otimes_\Gamma P^\bullet) \cong \tilde{P}_R^\bullet \otimes_{R\Gamma} P_R^\bullet$ in $C^b((R\Lambda \otimes_R R\Lambda^{op})\text{-mod})$ and since $R \otimes_k (P^\bullet \otimes_\Lambda \tilde{P}^\bullet) \cong P_R^\bullet \otimes_{R\Lambda} \tilde{P}_R^\bullet$ in $C^b((R\Gamma \otimes_R R\Gamma^{op})\text{-mod})$, we obtain (3.5). In other words, P_R^\bullet is a nice two-sided tilting complex in $D^b((R\Gamma \otimes_R R\Lambda^{op})\text{-mod})$.

To prove the remaining statements of Lemma 3.1.3, we consider the isomorphisms in (3.5) more closely. First, we note that since the terms of P_R^\bullet are finitely generated projective left $R\Gamma$ -modules and finitely generated projective right $R\Lambda$ -modules and since the terms of \tilde{P}_R^\bullet are finitely generated projective left $R\Lambda$ -modules and finitely generated projective right $R\Gamma$ -modules, we can replace the tensor products by completed tensor products. Moreover, since $\tilde{P}^\bullet \otimes_\Gamma P^\bullet \cong \Lambda$ in $D^b((\Lambda \otimes_k \Lambda^{op})\text{-mod})$, it follows that this is also true in the derived category $D^-(\Lambda \otimes_k \Lambda^{op})$ of bounded above pseudocompact Λ - Λ -bimodules. Similarly, $P^\bullet \otimes_\Lambda \tilde{P}^\bullet \cong \Gamma$ in the derived category $D^-(\Gamma \otimes_k \Gamma^{op})$ of bounded above pseudocompact Γ - Γ -bimodules. Therefore, we obtain

$$(3.9) \quad \begin{aligned} R \otimes_k (\tilde{P}^\bullet \otimes_\Gamma P^\bullet) &\cong R \otimes_k \Lambda = R\Lambda && \text{in } D^-(R\Lambda \otimes_R R\Lambda^{op}), \text{ and} \\ R \otimes_k (P^\bullet \otimes_\Lambda \tilde{P}^\bullet) &\cong R \otimes_k \Gamma = R\Gamma && \text{in } D^-(R\Gamma \otimes_R R\Gamma^{op}). \end{aligned}$$

Since

$$R \otimes_k (\tilde{P}^\bullet \otimes_\Gamma P^\bullet) \cong \tilde{P}_R^\bullet \otimes_{R\Gamma} P_R^\bullet \cong \tilde{P}_R^\bullet \hat{\otimes}_{R\Gamma} P_R^\bullet$$

in $C^-(R\Lambda \otimes_R R\Lambda^{op})$ and since

$$R \otimes_k (P^\bullet \otimes_\Lambda \tilde{P}^\bullet) \cong P_R^\bullet \otimes_{R\Lambda} \tilde{P}_R^\bullet \cong P_R^\bullet \hat{\otimes}_{R\Lambda} \tilde{P}_R^\bullet$$

in $C^-(R\Gamma \otimes_R R\Gamma^{op})$, the isomorphisms in (3.7) follow. This proves that the functors in (3.8) are quasi-inverses between the derived categories of bounded above complexes of pseudocompact $R\Lambda$ - and $R\Gamma$ -modules, completing the proof of Lemma 3.1.3. \square

Remark 3.1.4. Suppose Λ , Γ , P^\bullet and R are as in Lemma 3.1.3. Then Lemma 3.1.3 implies, in particular, that P_R^\bullet is a bounded complex of finitely generated $R\Gamma$ - $R\Lambda$ -bimodules whose terms are all projective as abstract right $R\Lambda$ -modules. Hence we can use Remark 2.1.3 to see that all terms of P_R^\bullet are projective objects in the category of pseudocompact right $R\Lambda$ -modules. Therefore, the functor $P_R^\bullet \hat{\otimes}_{R\Lambda} - : K^-(R\Lambda) \rightarrow K^-(R\Gamma)$ sends acyclic complexes to acyclic complexes (see, for example, [34, Acyclic Assembly Lemma 2.7.3]). This implies that $P_R^\bullet \hat{\otimes}_{R\Lambda} -$ is its own left derived functor, i.e.

$$P_R^\bullet \hat{\otimes}_{R\Lambda}^{\mathbf{L}} - = P_R^\bullet \hat{\otimes}_{R\Lambda} - : D^-(R\Lambda) \rightarrow D^-(R\Gamma)$$

(see, for example, [34, Ex. 10.5.5]). Similarly, we see that

$$\tilde{P}_R^\bullet \hat{\otimes}_{R\Gamma}^{\mathbf{L}} - = \tilde{P}_R^\bullet \hat{\otimes}_{R\Gamma} - : D^-(R\Gamma) \rightarrow D^-(R\Lambda).$$

In particular, we can identify the functors $P^\bullet \hat{\otimes}_\Lambda^{\mathbf{L}} -$, $P^\bullet \hat{\otimes}_\Lambda -$ and $P^\bullet \otimes_\Lambda -$ as functors from $D^-(\Lambda)$ to $D^-(\Gamma)$.

Theorem 3.1.5. *Suppose Λ and Γ are finite dimensional k -algebras, and P^\bullet is a nice two-sided tilting complex in $D^b((\Gamma \otimes_k \Lambda^{op})\text{-mod})$ as in Definition 3.1.1. Let V^\bullet be a complex in $D^-(\Lambda)$ satisfying Hypothesis 1, and let $V'^\bullet = P^\bullet \hat{\otimes}_\Lambda^{\mathbf{L}} V^\bullet = P^\bullet \otimes_\Lambda V^\bullet$. Then the deformation functors \hat{F}_{V^\bullet} and $\hat{F}_{V'^\bullet}$ are naturally isomorphic. In particular, the versal deformation rings $R(\Lambda, V^\bullet)$ and $R(\Gamma, V'^\bullet)$ are isomorphic in $\hat{\mathcal{C}}$, and $R(\Lambda, V^\bullet)$ is a universal deformation ring of V^\bullet if and only if $R(\Gamma, V'^\bullet)$ is a universal deformation ring of V'^\bullet .*

Proof. By Remarks 2.1.11 and 3.1.4, we may assume, without loss of generality, that V^\bullet is a bounded above complex of topologically free pseudocompact Λ -modules. We use the notation from Lemma 3.1.3. Let $R \in \text{Ob}(\hat{\mathcal{C}})$ and let (M^\bullet, ϕ) be a quasi-lift of V^\bullet over R . By Remark 2.1.9(i), we can assume that the terms of M^\bullet are topologically free pseudocompact $R\Lambda$ -modules. Since M^\bullet has finite pseudocompact R -tor dimension, we can truncate M^\bullet to obtain a complex N^\bullet that is isomorphic to M^\bullet in $D^-(R\Gamma)$ such that N^\bullet is a bounded complex of $R\Lambda$ -modules, all of which are topologically free as R -modules.

Define $M'^\bullet = P_R^\bullet \hat{\otimes}_{R\Lambda} M^\bullet$ and $N'^\bullet = P_R^\bullet \hat{\otimes}_{R\Lambda} N^\bullet$, so M'^\bullet and N'^\bullet are isomorphic objects of $D^-(R\Gamma)$. Since P_R^\bullet is a bounded complex of finitely generated projective right $R\Lambda$ -modules and since N^\bullet is a bounded complex of topologically free R -modules, it follows that N'^\bullet is also a bounded complex of topologically free R -modules. But this means that there exists an integer n such that for all pseudocompact R -modules S and all integers $i < n$ we have $H^i(S \hat{\otimes}_R N'^\bullet) = H^i(S \hat{\otimes}_R^{\mathbf{L}} N'^\bullet) = 0$. Therefore N'^\bullet , and hence M'^\bullet , both have finite pseudocompact R -tor dimension.

Next we note that we can view $K^-(\Lambda)$ as the full subcategory of $K^-(R\Lambda)$ consisting of bounded above complexes of pseudocompact $R\Lambda$ -modules on which the maximal ideal m_R of R acts trivially. Moreover, on $K^-(\Lambda)$ the functor $P_R^\bullet \hat{\otimes}_{R\Lambda} -$ coincides with the functor $P^\bullet \hat{\otimes}_\Lambda - : K^-(\Lambda) \rightarrow K^-(\Gamma)$. Viewing both functors as functors on $K^-(\Lambda)$, their corresponding left derived functors $L^-(P_R^\bullet \hat{\otimes}_{R\Lambda} -)$ and $L^-(P^\bullet \hat{\otimes}_\Lambda -) = P^\bullet \hat{\otimes}_\Lambda^{\mathbf{L}} - = P^\bullet \hat{\otimes}_\Lambda -$ coincide as functors from $D^-(\Lambda)$ to $D^-(\Gamma)$. Define $\phi' = L^-(P_R^\bullet \hat{\otimes}_{R\Lambda} -)(\phi)$, so ϕ' is an isomorphism in $D^-(\Gamma)$. Since M^\bullet is a bounded above complex of topologically free pseudocompact $R\Lambda$ -modules and since we assumed that V^\bullet is a bounded above complex of topologically free pseudocompact Λ -modules, we obtain $\phi' = P_R^\bullet \hat{\otimes}_{R\Lambda} \phi$. Therefore,

$$M'^\bullet \hat{\otimes}_R^{\mathbf{L}} k = M'^\bullet \hat{\otimes}_R k = (P_R^\bullet \hat{\otimes}_{R\Lambda} M^\bullet) \hat{\otimes}_R k = P_R^\bullet \hat{\otimes}_{R\Lambda} (M^\bullet \hat{\otimes}_R k) \xrightarrow{\phi'} P_R^\bullet \hat{\otimes}_{R\Lambda} V^\bullet = V'^\bullet$$

which means (M'^\bullet, ϕ') is a quasi-lift of V'^\bullet . It follows that for each $R \in \text{Ob}(\hat{\mathcal{C}})$, the functor $P_R^\bullet \hat{\otimes}_{R\Lambda} -$ induces a bijection τ_R from the set of deformations of V^\bullet over R onto the set of deformations of V'^\bullet over R .

It remains to show that the maps τ_R are natural with respect to morphisms $\alpha : R \rightarrow R'$ in $\hat{\mathcal{C}}$. Considering (M^\bullet, ϕ) and (M'^\bullet, ϕ') as above, it suffices to show that $(R' \hat{\otimes}_{R, \alpha} M'^\bullet, (\phi')_\alpha)$ and $(P_{R'}^\bullet \hat{\otimes}_{R'\Lambda} (R' \hat{\otimes}_{R, \alpha} M^\bullet), P_{R'}^\bullet \hat{\otimes}_{R'\Lambda} (\phi_\alpha))$ are isomorphic as quasi-lifts of V'^\bullet over R' . Since all the

terms of P_R^\bullet are finitely generated projective right $R\Lambda$ -modules and since all the terms of M^\bullet are topologically free as R -modules, it follows that there is a natural isomorphism

$$f : M'^\bullet \hat{\otimes}_{R,\alpha} R' = (P_R^\bullet \hat{\otimes}_{R\Lambda} M^\bullet) \hat{\otimes}_{R,\alpha} R' \rightarrow P_{R'}^\bullet \hat{\otimes}_{R'\Lambda} (M^\bullet \hat{\otimes}_{R,\alpha} R')$$

in $D^-(R'\Gamma)$ (in fact in $C^-(R'\Gamma)$). Moreover, the diagram

$$\begin{array}{ccc} (M'^\bullet \hat{\otimes}_{R,\alpha} R') \hat{\otimes}_{R'} k & \xrightarrow{f \hat{\otimes}_{R'} k} & (P_{R'}^\bullet \hat{\otimes}_{R'\Lambda} (M^\bullet \hat{\otimes}_{R,\alpha} R')) \hat{\otimes}_{R'} k \\ \cong \downarrow & & \downarrow \cong \\ M'^\bullet \hat{\otimes}_R k & & P_{R'}^\bullet \hat{\otimes}_{R'\Lambda} (M^\bullet \hat{\otimes}_R k) \\ & \searrow \phi' & \swarrow P_{R'}^\bullet \hat{\otimes}_{R'\Lambda} \phi \\ & & V'^\bullet \end{array}$$

commutes in $D^-(\Gamma)$. Hence $(R' \hat{\otimes}_{R,\alpha} M'^\bullet, (\phi')_\alpha) \cong (P_{R'}^\bullet \hat{\otimes}_{R'\Lambda} (R' \hat{\otimes}_{R,\alpha} M^\bullet), P_{R'}^\bullet \hat{\otimes}_{R'\Lambda} (\phi_\alpha))$.

This means that the functors \hat{F}_{V^\bullet} and $\hat{F}_{V'^\bullet}$ are naturally isomorphic, which implies Theorem 3.1.5. \square

3.2. Stable equivalences of Morita type for self-injective algebras. We use the notation introduced in Section 3.1. Moreover, we assume throughout this section that both Λ and Γ are self-injective finite dimensional k -algebras.

If V is any finitely generated Λ -module or Γ -module, let $\hat{F}_V : \hat{\mathcal{C}} \rightarrow \text{Sets}$ be the deformation functor considered in [9] (see also Remark 2.5.1), and let F_V be its restriction to the full subcategory \mathcal{C} of $\hat{\mathcal{C}}$ consisting of Artinian rings.

We first collect some useful facts, which were proved as Claims 1, 2 and 6 in the proof of [9, Thm. 2.6], only using the assumption that Λ is a self-injective finite dimensional k -algebra. Note that we need to add the assumption that M (resp. M_0) is free over R (resp. R_0) in Claims 1 and 2 in the proof of [9, Thm. 2.6]. This makes no difference in the overall proof of [9, Thm. 2.6] since these claims were only used under this assumption.

Remark 3.2.1. Suppose Λ is a self-injective finite dimensional k -algebra. Let R, R_0 be Artinian rings in \mathcal{C} , and let $\pi : R \rightarrow R_0$ be a surjection in \mathcal{C} . Let M, Q (resp. M_0, Q_0) be finitely generated $R\Lambda$ -modules (resp. $R_0\Lambda$ -modules) that are free over R (resp. R_0), and assume that Q (resp. Q_0) is projective. Suppose there are $R_0\Lambda$ -module isomorphisms $g : R_0 \otimes_{R,\pi} M \rightarrow M_0$, $h : R_0 \otimes_{R,\pi} Q \rightarrow Q_0$.

- (i) If $\nu_0 \in \text{Hom}_{R_0\Lambda}(M_0, Q_0)$, then there exists $\nu \in \text{Hom}_{R\Lambda}(M, Q)$ with $\nu_0 = h \circ (R_0 \otimes_{R,\pi} \nu) \circ g^{-1}$.
- (ii) If $\sigma_0 \in \text{End}_\Lambda(M_0)$ factors through a projective $R_0\Lambda$ -module, then there exists $\sigma \in \text{End}_{R\Lambda}(M)$ such that σ factors through a projective $R\Lambda$ -module and $\sigma_0 = g \circ (R_0 \otimes_{R,\pi} \sigma) \circ g^{-1}$.

Let P be a finitely generated projective Λ -module. Let $\iota_R : k \rightarrow R$ be the unique morphism in \mathcal{C} endowing R with a k -algebra structure, and let $\pi_R : R \rightarrow k$ be the morphism from R to its residue field k in \mathcal{C} . Then $\pi_R \circ \iota_R$ is the identity on k , and $P_R = R \otimes_{k,\iota_R} P$ is a projective $R\Lambda$ -module cover of P , which is unique up to isomorphism. In particular, $(P_R, \pi_{R,P})$ is a lift of P over R , where $\pi_{R,P}$ is the natural isomorphism $k \otimes_{R,\pi_R} (R \otimes_{k,\iota_R} P) \rightarrow P$ of Λ -modules.

(iii) Suppose there is a commutative diagram of finitely generated $R\Lambda$ -modules

$$(3.10) \quad \begin{array}{ccccccc} 0 & \longrightarrow & P_R & \xrightarrow{g} & T & \xrightarrow{h} & C \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & P & \xrightarrow{\bar{g}} & k \otimes_R T & \xrightarrow{\bar{h}} & k \otimes_R C \longrightarrow 0 \end{array}$$

in which T and C are free over R and the bottom row arises by tensoring the top row with k over R and using the Λ -module isomorphism $\pi_{R,P} : k \otimes_{R,\pi_R} P_R \rightarrow P$. Then the top row of (3.10) splits as a sequence of $R\Lambda$ -modules.

We need the following result, which is a generalization of [9, Thm. 2.6(iii)].

Lemma 3.2.2. *Suppose Λ is a self-injective finite dimensional k -algebra and that V and P are finitely generated non-zero Λ -modules and P is projective. Then P has a universal deformation ring $R(\Lambda, P)$ and $R(\Lambda, P) \cong k$. The deformation functors \hat{F}_V and $\hat{F}_{V \oplus P}$ are naturally isomorphic. In particular, the versal deformation rings $R(\Lambda, V)$ and $R(\Lambda, V \oplus P)$ are isomorphic in $\hat{\mathcal{C}}$, and $R(\Lambda, V)$ is universal if and only if $R(\Lambda, V \oplus P)$ is universal.*

Proof. Since P is a projective Λ -module, it follows that $\text{Ext}_\Lambda^1(P, P) = 0$, which implies by [9, Prop. 2.1] that the versal deformation ring of P is isomorphic to k . For each $R \in \text{Ob}(\hat{\mathcal{C}})$, let $\iota_R : k \rightarrow R$ be the unique morphism in $\hat{\mathcal{C}}$ endowing R with a k -algebra structure, and let $\pi_R : R \rightarrow k$ be the morphism from R to its residue field k in $\hat{\mathcal{C}}$. Then $\pi_R \circ \iota_R$ is the identity morphism of k . This implies that k is the universal deformation ring of P .

Let $R \in \text{Ob}(\mathcal{C})$ be Artinian. Define a map

$$(3.11) \quad \begin{aligned} F_V(R) &\rightarrow F_{V \oplus P}(R), \\ [M, \phi] &\mapsto [M \oplus P_R, \phi \oplus \pi_{R,P}] \end{aligned}$$

where $P_R = R \otimes_{k, \iota_R} P$ and $\pi_{R,P} : k \otimes_{R, \pi_R} P_R \rightarrow P$ are as in Remark 3.2.1. Then (3.11) is a well-defined map that is natural with respect to morphisms $\alpha : R \rightarrow R'$ in \mathcal{C} , since $\alpha \circ \iota_R = \iota_{R'}$ and $\pi_{R'} \circ \alpha = \pi_R$. Since the deformation functors \hat{F}_V and $\hat{F}_{V \oplus P}$ are continuous, it suffices to show that the map (3.11) is bijective for all $R \in \text{Ob}(\mathcal{C})$ to complete the proof of Lemma 3.2.2.

Suppose first that (M, ϕ) and (M', ϕ') are two lifts of V over R such that there exists an $R\Lambda$ -module isomorphism $f = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} : M \oplus P_R \rightarrow M' \oplus P_R$ with $(\phi' \oplus \pi_{R,P}) \circ (k \otimes f) = \phi \oplus \pi_{R,P}$. In particular, $\phi' \circ (k \otimes f_{11}) = \phi$ and $k \otimes f_{22}$ is the identity morphism on P . By Nakayama's Lemma, this implies that f_{11} and f_{22} are $R\Lambda$ -module isomorphisms, since M and M' are free R -modules of the same finite rank. Therefore, $[M, \phi] = [M', \phi']$ and the map (3.11) is injective.

We now show that the map (3.11) is surjective. Let (T, τ) be a lift of $V \oplus P$ over R . Since P_R is a projective $R\Lambda$ -module, there exists an $R\Lambda$ -module homomorphism g that makes the following diagram commute

$$(3.12) \quad \begin{array}{ccc} P_R & \xrightarrow{g} & T \\ \left(\begin{array}{c} 0 \\ \text{pr} \end{array} \right) \downarrow & & \downarrow \\ V \oplus P & \xrightarrow{\tau^{-1}} & k \otimes_R T \end{array}$$

where pr is obtained by first tensoring P_R with k over R and then using the Λ -module isomorphism $\pi_{R,P} : k \otimes_{R, \pi_R} P_R \rightarrow P$. Then g induces an injective Λ -module homomorphism g' and a commutative diagram of Λ -modules

$$(3.13) \quad \begin{array}{ccc} P_R/m_R P_R & \xrightarrow{g'} & T/m_R T \\ \pi_{R,P} \downarrow & & \parallel \\ P & & \\ \iota_P \downarrow & & \\ V \oplus P & \xleftarrow{\tau} & k \otimes_R T \end{array}$$

where $\iota_P : P \rightarrow V \oplus P$ is the natural injection and, as before, m_R denotes the maximal ideal of R . Using Nakayama's Lemma, it follows that g is injective. Letting C be the cokernel of g , we obtain a commutative diagram (3.10). Since P_R and T are free R -modules and since g and g' (and hence $\bar{g} = g' \circ \pi_{R,P}^{-1}$) are injective, an elementary Nakayama's Lemma argument shows that C is also free as an R -module. Therefore, by Remark 3.2.1(iii), the top row of (3.10) splits as a sequence

of $R\Lambda$ -modules. Let $j : C \rightarrow T$ be an $R\Lambda$ -module splitting of h . By tensoring with k over R , we obtain a Λ -module splitting $\bar{j} : k \otimes_R C \rightarrow k \otimes_R T$ of \bar{h} . Consider the $R\Lambda$ -module isomorphism

$$(j, g) : C \oplus P_R \rightarrow T.$$

Since $\tau \circ \bar{g} = \tau \circ g' \circ \pi_{R,P}^{-1} = \iota_P$ by (3.13), there exists a Λ -module isomorphism $\xi : k \otimes_R C \rightarrow V$ such that $p_V \circ \tau = \xi \circ \bar{h}$, where $p_V : V \oplus P \rightarrow V$ is the natural projection onto V . Letting $p_P : V \oplus P \rightarrow P$ be the natural projection onto P , we have

$$(3.14) \quad \begin{aligned} p_P \circ \tau \circ (k \otimes_R g) &= p_P \circ \tau \circ g' = p_P \circ \iota_P \circ \pi_{R,P} = \pi_{R,P}, \\ p_V \circ \tau \circ (k \otimes_R g) &= p_V \circ \tau \circ g' = p_V \circ \iota_P \circ \pi_{R,P} = 0, \\ p_V \circ \tau \circ (k \otimes_R j) &= p_V \circ \tau \circ \bar{j} = \xi \circ \bar{h} \circ \bar{j} = \xi, \text{ and} \\ p_P \circ \tau \circ (k \otimes_R j) &= p_P \circ \tau \circ \bar{j}. \end{aligned}$$

By Remark 3.2.1(i), there exists an $R\Lambda$ -module homomorphism $\lambda : C \rightarrow P_R$ such that

$$k \otimes_R \lambda = \pi_{R,P}^{-1} \circ p_P \circ \tau \circ \bar{j}.$$

Letting $j_1 = j - g \circ \lambda$ shows that j_1 is also an $R\Lambda$ -module splitting of h and $\bar{j}_1 = k \otimes_R j_1$ is also a Λ -module splitting of \bar{h} . Hence, on replacing j by j_1 and using the equations (3.14), we see that $p_V \circ \tau \circ (k \otimes_R j_1) = \xi$ and

$$p_P \circ \tau \circ (k \otimes_R j_1) = p_P \circ \tau \circ \bar{j} - p_P \circ \tau \circ (k \otimes_R g) \circ (k \otimes_R \lambda) = 0.$$

This means that $(j_1, g) : C \oplus P_R \rightarrow T$ provides an isomorphism between the lifts $(C \oplus P_R, \xi \oplus \pi_{R,P})$ and (T, τ) of $V \oplus P$ over R . Hence the map (3.11) is bijective for all $R \in \text{Ob}(\mathcal{C})$, which completes the proof of Lemma 3.2.2. \square

The following definition of stable equivalence of Morita type goes back to Broué [11].

Definition 3.2.3. Suppose Λ and Γ are self-injective finite dimensional k -algebras. Let X be a Γ - Λ -bimodule and let Y be a Λ - Γ -bimodule. We say X and Y induce a *stable equivalence of Morita type* between Λ and Γ , if X and Y are projective both as left and as right modules, and if

$$(3.15) \quad \begin{aligned} Y \otimes_\Gamma X &\cong \Lambda \oplus P \quad \text{as } \Lambda\text{-}\Lambda\text{-bimodules, and} \\ X \otimes_\Lambda Y &\cong \Gamma \oplus Q \quad \text{as } \Gamma\text{-}\Gamma\text{-bimodules,} \end{aligned}$$

where P is a projective Λ - Λ -bimodule, and Q is a projective Γ - Γ -bimodule. In particular, $X \otimes_\Lambda -$ and $Y \otimes_\Gamma -$ induce mutually inverse equivalences between the stable module categories $\Lambda\text{-}\underline{\text{mod}}$ and $\Gamma\text{-}\underline{\text{mod}}$.

Remark 3.2.4. It follows from a result by Rickard (see [29, Cor. 5.5] and [21, Prop. 6.3.8]) that a derived equivalence between $D^b(\Lambda\text{-mod})$ and $D^b(\Gamma\text{-mod})$ induces a stable equivalence of Morita type between Λ and Γ .

More precisely, let $K^b(\Lambda\text{-proj})$ be the full subcategory of $D^b(\Lambda\text{-mod})$ consisting of all objects isomorphic to bounded complexes of finitely generated projective Λ -modules. Then $K^b(\Lambda\text{-proj})$ is a thick subcategory of $D^b(\Lambda\text{-mod})$ and we can build the Verdier quotient

$$D^b(\Lambda\text{-mod})/K^b(\Lambda\text{-proj})$$

(see, for example, [22, Sect. 4.6] for the construction of this quotient). Rickard proved in [28, Thm. 2.1] that this Verdier quotient is equivalent as a triangulated category to the stable module category $\Lambda\text{-}\underline{\text{mod}}$.

Suppose now that there is a derived equivalence between $D^b(\Lambda\text{-mod})$ and $D^b(\Gamma\text{-mod})$. As in Section 3.1, there exists a nice two-sided tilting complex P^\bullet (see Definition 3.1.1(b) in the case where $R = k$). Following the proof of [29, Cor. 5.5], let T^\bullet be a projective Γ - Λ -bimodule resolution of P^\bullet such that all terms of T^\bullet are finitely generated projective Γ - Λ -bimodules. For large $n > 0$, we can truncate T^\bullet to obtain a bounded complex

$$S^\bullet : \dots \rightarrow 0 \rightarrow S^{-n} \rightarrow T^{-n+1} \rightarrow T^{-n+2} \rightarrow \dots$$

that is isomorphic to P^\bullet in $D^b((\Gamma \otimes_k \Lambda^{op})\text{-mod})$, where all terms but S^{-n} are projective Γ - Λ -bimodules and S^{-n} is projective as a left Γ -module and as a right Λ -module. If we let $X = \Omega_{\Gamma\Lambda}^{-n}(S^{-n})$, the $(-n)$ -th syzygy as a Γ - Λ -bimodule, then S^\bullet is isomorphic to the one-term complex X concentrated in degree 0 in $D^b((\Gamma \otimes_k \Lambda^{op})\text{-mod})/K^b(\Gamma \otimes_k \Lambda^{op}\text{-proj})$. Moreover, we have that $\tilde{S}^\bullet = \text{Hom}_\Gamma(S^\bullet, \Gamma)$ has the form

$$\tilde{S}^\bullet : \quad \dots \rightarrow \tilde{T}^{n-2} \rightarrow \tilde{T}^{n-1} \rightarrow \tilde{S}^n \rightarrow 0 \rightarrow \dots$$

where $\tilde{T}^i = \text{Hom}_\Gamma(T^{-i}, \Gamma)$ for $i < n$ and $\tilde{S}^n = \text{Hom}_\Gamma(S^{-n}, \Gamma)$. If we let $Y = \Omega_{\Lambda\Gamma}^n(\tilde{S}^n)$, the n -th syzygy as a Λ - Γ -bimodule, then \tilde{S}^\bullet is isomorphic to the one-term complex Y concentrated in degree 0 in $D^b((\Lambda \otimes_k \Gamma^{op})\text{-mod})/K^b(\Lambda \otimes_k \Gamma^{op}\text{-proj})$. Using (3.5), we obtain

$$\begin{aligned} Y \otimes_\Gamma X &\cong \tilde{S}^\bullet \otimes_\Gamma S^\bullet \cong \tilde{P}^\bullet \otimes_\Gamma P^\bullet \cong \Lambda && \text{in } D^b((\Lambda \otimes_k \Lambda^{op})\text{-mod})/K^b((\Lambda \otimes_k \Lambda^{op})\text{-proj}), \text{ and} \\ X \otimes_\Lambda Y &\cong S^\bullet \otimes_\Gamma \tilde{S}^\bullet \cong P^\bullet \otimes_\Lambda \tilde{P}^\bullet \cong \Gamma && \text{in } D^b((\Gamma \otimes_k \Gamma^{op})\text{-mod})/K^b((\Gamma \otimes_k \Gamma^{op})\text{-proj}). \end{aligned}$$

Using that $D^b((\Lambda \otimes_k \Lambda^{op})\text{-mod})/K^b((\Lambda \otimes_k \Lambda^{op})\text{-proj})$ is equivalent as a triangulated category to $(\Lambda \otimes_k \Lambda^{op})\text{-mod}$ and that $D^b((\Gamma \otimes_k \Gamma^{op})\text{-mod})/K^b((\Gamma \otimes_k \Gamma^{op})\text{-proj})$ is equivalent as a triangulated category to $(\Gamma \otimes_k \Gamma^{op})\text{-mod}$, we obtain that X and Y satisfy (3.15). In other words, X and Y induce a stable equivalence of Morita type between Λ and Γ .

The following result is proved using Theorem 3.1.5 and Lemma 3.2.2.

Proposition 3.2.5. *Suppose Λ and Γ are finite dimensional self-injective k -algebras. Let P^\bullet be a nice two-sided tilting complex in $D^b((\Gamma \otimes_k \Lambda^{op})\text{-mod})$ such that P^\bullet is isomorphic in $D^b((\Gamma \otimes_k \Lambda^{op})\text{-mod})/K^b((\Gamma \otimes_k \Lambda^{op})\text{-proj})$ to a one-term complex X concentrated in degree 0, as in Remark 3.2.4. Let V be a finitely generated Λ -module, and let $V' = X \otimes_\Lambda V$, so V' is a finitely generated Γ -module. Then $R(\Lambda, V)$ and $R(\Gamma, V')$ are isomorphic in $\hat{\mathcal{C}}$.*

Proof. If we view V as a one-term complex concentrated in degree 0, then it follows from Proposition 2.5.2 and Theorem 3.1.5 that $R(\Lambda, V) \cong R(\Gamma, P^\bullet \otimes_\Lambda V)$ in $\hat{\mathcal{C}}$. We have that

$$P^\bullet \otimes_\Lambda V \cong S^\bullet \otimes_\Lambda V \cong X \otimes_\Lambda V$$

in $D^b(\Gamma\text{-mod})/K^b(\Gamma\text{-proj})$, where S^\bullet is as in Remark 3.2.4. By [28, Thm. 2.1],

$$D^b(\Gamma\text{-mod})/K^b(\Gamma\text{-proj}) \cong \Gamma\text{-mod}$$

as triangulated categories. By Definition 3.1.1(b) and by (3.15), we have

$$\begin{aligned} \tilde{P}^\bullet \otimes_\Gamma (P^\bullet \otimes_\Lambda V) &\cong V && \text{in } D^-(\Lambda), \text{ and} \\ Y \otimes_\Gamma (X \otimes_\Lambda V) &\cong V \oplus (P \otimes_\Lambda V) && \text{in } \Lambda\text{-mod}. \end{aligned}$$

Since moreover $R(\Lambda, V) \cong R(\Lambda, V \oplus (P \otimes_\Lambda V))$ by Lemma 3.2.2, we obtain that

$$R(\Gamma, P^\bullet \otimes_\Lambda V) \cong R(\Gamma, X \otimes_\Lambda V)$$

in $\hat{\mathcal{C}}$. Therefore, $R(\Lambda, V) \cong R(\Gamma, P^\bullet \otimes_\Lambda V) \cong R(\Gamma, X \otimes_\Lambda V)$ in $\hat{\mathcal{C}}$. \square

Note that not every stable equivalence of Morita type between self-injective algebras is induced by a derived equivalence (see, for example, [13] and its references). Therefore, we next show that an arbitrary stable equivalence of Morita type between self-injective algebras preserves versal deformation rings. The arguments are very similar to those used in [4, Sect. 2.2].

Proposition 3.2.6. *Suppose Λ and Γ are self-injective finite dimensional k -algebras. Suppose X is a Γ - Λ -bimodule and Y is a Λ - Γ -bimodule that induce a stable equivalence of Morita type between Λ and Γ . Let V be a finitely generated Λ -module, and define $V' = X \otimes_\Lambda V$. Then the deformation functors \hat{F}_V and $\hat{F}_{V'}$ are naturally isomorphic. In particular, the versal deformation rings $R(\Lambda, V)$ and $R(\Gamma, V')$ are isomorphic in $\hat{\mathcal{C}}$, and $R(\Lambda, V)$ is a universal deformation ring of V if and only if $R(\Gamma, V')$ is a universal deformation ring of V' .*

Proof. Let $R \in \text{Ob}(\mathcal{C})$ be Artinian. Then $X_R = R \otimes_k X$ is projective as left $R\Gamma$ -module and as right $R\Lambda$ -module, and $Y_R = R \otimes_k Y$ is projective as left $R\Lambda$ -module and as right $R\Gamma$ -module. Since $X_R \otimes_{R\Lambda} (Y_R) \cong R \otimes_k (X \otimes_\Lambda Y)$, we have, using (3.15),

$$\begin{aligned} Y_R \otimes_{R\Gamma} X_R &\cong R\Lambda \oplus P_R \quad \text{as } R\Lambda\text{-}R\Lambda\text{-bimodules, and} \\ X_R \otimes_{R\Lambda} Y_R &\cong R\Gamma \oplus Q_R \quad \text{as } R\Gamma\text{-}R\Gamma\text{-bimodules,} \end{aligned}$$

where $P_R = R \otimes_k P$ is a projective $R\Lambda$ - $R\Lambda$ -bimodule and $Q_R = R \otimes_k Q$ is a projective $R\Gamma$ - $R\Gamma$ -bimodule.

Since P is a projective Λ - Λ -bimodule, it follows that $P \otimes_\Lambda V$ is a projective left Λ -module. By Lemma 3.2.2 it follows that $P \otimes_\Lambda V$ has a universal deformation ring, which is isomorphic to k . In particular, every lift of $P \otimes_\Lambda V$ over R is isomorphic to $(R \otimes_k (P \otimes_\Lambda V), \pi_{R, P \otimes_\Lambda V})$, where $\pi_{R, P \otimes_\Lambda V} : k \otimes_R (R \otimes_k (P \otimes_\Lambda V)) \rightarrow P \otimes_\Lambda V$ is the natural isomorphism of Λ -modules.

Let now (M, ϕ) be a lift of V over R . Then M is a finitely generated $R\Lambda$ -module. Define $M' = X_R \otimes_{R\Lambda} M$. Since X_R is a finitely generated projective right $R\Lambda$ -module and since M is a finitely generated abstractly free R -module, it follows that M' is a finitely generated projective, and hence abstractly free, R -module.

Next we note that we can view $\Lambda\text{-mod}$ as the full subcategory of $R\Lambda\text{-mod}$ consisting of all finitely generated $R\Lambda$ -modules on which the maximal ideal m_R of R acts trivially. Moreover, on $\Lambda\text{-mod}$ the functor $X_R \otimes_{R\Lambda} -$ coincides with the functor $X \otimes_\Lambda -$. Define $\phi' = X_R \otimes_{R\Lambda} \phi$, so ϕ' is a Γ -module isomorphism, since X_R is projective as a right $R\Lambda$ -module. Then

$$(3.16) \quad M' \otimes_R k = (X_R \otimes_{R\Lambda} M) \otimes_R k = X_R \otimes_{R\Lambda} (M \otimes_R k) \xrightarrow{\phi'} X_R \otimes_{R\Lambda} V = V'$$

which means (M', ϕ') is a lift of V' over R . We therefore obtain for all $R \in \text{Ob}(\mathcal{C})$ a well-defined map

$$\begin{aligned} \tau_R : F_V(R) &\rightarrow F_{V'}(R), \\ [M, \phi] &\mapsto [M', \phi'] = [X_R \otimes_{R\Lambda} M, X_R \otimes_{R\Lambda} \phi]. \end{aligned}$$

We need to show that τ_R is bijective. Arguing as in (3.16), we see that $(Y_R \otimes_{R\Gamma} M', Y_R \otimes_{R\Gamma} \phi')$ is a lift of $Y \otimes_\Gamma V' \cong V \oplus (P \otimes_\Lambda V)$ over R . Moreover,

$$(3.17) \quad \begin{aligned} (Y_R \otimes_{R\Gamma} M', Y_R \otimes_{R\Gamma} \phi') &\cong ((R\Lambda \oplus P_R) \otimes_{R\Lambda} M, (R\Lambda \oplus P_R) \otimes_{R\Lambda} \phi) \\ &\cong (M \oplus (P_R \otimes_{R\Lambda} M), \phi \oplus (P_R \otimes_{R\Lambda} \phi)). \end{aligned}$$

Since $(P_R \otimes_{R\Lambda} M, P_R \otimes_{R\Lambda} \phi)$ is a lift of the projective Λ -module $P \otimes_\Lambda V$ over R , it follows from Lemma 3.2.2 that τ_R is injective.

Now let (L, ψ) be a lift of $V' = X \otimes_\Lambda V$ over R . Then $(L', \psi') = (Y_R \otimes_{R\Gamma} L, Y_R \otimes_{R\Gamma} \psi)$ is a lift of $V'' = Y \otimes_\Gamma V' \cong V \oplus (P \otimes_\Lambda V)$ over R . By Lemma 3.2.2, there exists a lift (M, ϕ) of V over R such that (L', ψ') is isomorphic to the lift $(M \oplus (R \otimes_k (P \otimes_\Lambda V)), \phi \oplus \pi_{R, P \otimes_\Lambda V})$. Arguing similarly as in (3.17), we then have that (L', ψ') is isomorphic to $(M'', \phi'') = (Y_R \otimes_{R\Gamma} M', Y_R \otimes_{R\Gamma} \phi')$ where $(M', \phi') = (X_R \otimes_{R\Lambda} M, X_R \otimes_{R\Lambda} \phi)$. Therefore, $(X_R \otimes_{R\Lambda} L', X_R \otimes_{R\Lambda} \psi') \cong (X_R \otimes_{R\Lambda} M'', X_R \otimes_{R\Lambda} \phi'')$. Arguing again similarly as in (3.17) and using Lemma 3.2.2, we have

$$\begin{aligned} (X_R \otimes_{R\Lambda} L', X_R \otimes_{R\Lambda} \psi') &\cong (L \oplus (R \otimes_k (Q \otimes_\Gamma V')), \psi \oplus \pi_{R, Q \otimes_\Gamma V'}), \text{ and} \\ (X_R \otimes_{R\Lambda} M'', X_R \otimes_{R\Lambda} \phi'') &\cong (M' \oplus (R \otimes_k (Q \otimes_\Gamma V')), \phi' \oplus \pi_{R, Q \otimes_\Gamma V'}). \end{aligned}$$

Thus by Lemma 3.2.2, it follows that $(L, \psi) \cong (M', \phi')$, i.e. τ_R is surjective.

To show that the maps τ_R are natural with respect to morphisms $\alpha : R \rightarrow R'$ in \mathcal{C} , consider (M, ϕ) and (M', ϕ') as above. Since X_R is a projective right $R\Lambda$ -module and M is a free R -module, there exists a natural isomorphism

$$f : R' \otimes_{R, \alpha} M' = R' \otimes_{R, \alpha} (X_R \otimes_{R\Lambda} M) \rightarrow X_{R'} \otimes_{R'\Lambda} (R' \otimes_{R, \alpha} M)$$

of $R'\Gamma$ -modules. It is straightforward to see that f provides an isomorphism between the lifts $(R' \otimes_{R, \alpha} M', (\phi')_\alpha)$ and $(X_{R'} \otimes_{R'\Lambda} (R' \otimes_{R, \alpha} M), X_{R'} \otimes_{R'\Lambda} (\phi_\alpha))$ of V' over R' .

Since the deformation functors \hat{F}_V and $\hat{F}_{V'}$ are continuous, this implies that they are naturally isomorphic. Hence the versal deformation rings $R(\Lambda, V)$ and $R(\Gamma, V')$ are isomorphic in $\hat{\mathcal{C}}$. Moreover, $R(\Lambda, V)$ is universal if and only if $R(\Gamma, V')$ is universal. \square

The following two remarks will be important in Section 4.

Remark 3.2.7. Using the notation of Proposition 3.2.6, suppose that the stable endomorphism ring $\underline{\text{End}}_\Lambda(V)$ is isomorphic to k . Then it follows that $\underline{\text{End}}_\Gamma(V')$ is also isomorphic to k . Because Λ and Γ are self-injective, this implies by [9, Thm. 2.6] that the versal deformation rings $R(\Lambda, V)$ and $R(\Gamma, V')$ are in fact universal. Since $\underline{\text{End}}_\Gamma(V') = k$, there exists a non-projective indecomposable Γ -module V'_0 (unique up to isomorphism) that is a direct summand of V' with $\underline{\text{End}}_\Gamma(V'_0) = k$ and $R(\Gamma, V') \cong R(\Gamma, V'_0)$ (see Lemma 3.2.2). It follows that $R(\Lambda, V) \cong R(\Gamma, V'_0)$.

Remark 3.2.8. Using the notation of Proposition 3.2.6, let $G : \Lambda\text{-mod} \rightarrow \Gamma\text{-mod}$ denote the stable equivalence of Morita type induced by $X \otimes_\Lambda -$. Denote by $\text{mod}_{\mathcal{P}}(\Lambda)$ (resp. $\text{mod}_{\mathcal{P}}(\Gamma)$) the full subcategory of $\Lambda\text{-mod}$ (resp. $\Gamma\text{-mod}$) whose objects are the modules that have no non-zero projective summands, and denote the correspondence between $\text{mod}_{\mathcal{P}}(\Lambda)$ and $\text{mod}_{\mathcal{P}}(\Gamma)$ that is induced by G again by G . Let

$$0 \rightarrow A \xrightarrow{\begin{pmatrix} f \\ s \end{pmatrix}} B \oplus P \xrightarrow{(g, t)} C \rightarrow 0$$

be an almost split sequence in $\Lambda\text{-mod}$ where A, B, C are in $\text{mod}_{\mathcal{P}}(\Lambda)$, B is non-zero and P is projective. Then, by [2, Prop. X.1.6], for any morphism $g' : G(B) \rightarrow G(C)$ with $G(g) = g'$ in $\Gamma\text{-mod}$, there is an almost split sequence

$$0 \rightarrow G(A) \xrightarrow{\begin{pmatrix} f' \\ u \end{pmatrix}} G(B) \oplus P' \xrightarrow{(g', v)} G(C) \rightarrow 0$$

in $\Gamma\text{-mod}$ where P' is projective and $G(f) = f'$ in $\Gamma\text{-mod}$.

Suppose now additionally that Λ and Γ are symmetric algebras with no blocks of Loewy length 2. Then it follows by [2, Cor. X.1.9 and Prop. X.1.12] that the stable Auslander-Reiten quivers of Λ and Γ are isomorphic stable translation quivers, and G commutes with the syzygy functor Ω .

If V and V' are as in Proposition 3.2.6, this means, in particular, that the stable Auslander-Reiten quiver components of V and V' match up, including the relative positions of V and V' in these components. We will see in Section 4 how to use this to transfer results about universal deformation rings of Λ -modules V with $\underline{\text{End}}_\Lambda(V) = k$ to results about the universal deformation rings of the corresponding Γ -modules V' .

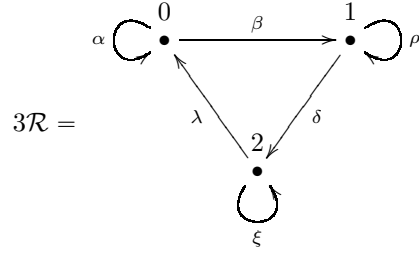
4. A FAMILY OF EXAMPLES

We use the notation from Section 3. Moreover, we assume that k is an algebraically closed field. In this section, we consider the derived equivalence classes of the family of symmetric k -algebras $D(3\mathcal{R})$ introduced by Erdmann in [14]. In Section 4.1, we describe these derived equivalence classes, which were obtained by Holm in [19, Sect. 3.2]. In Section 4.2, we apply the results from Section 3 together with the results in [9] and [33] to obtain universal deformation rings of Λ -modules for other algebras Λ of dihedral type that are derived equivalent to members of the family $D(3\mathcal{R})$.

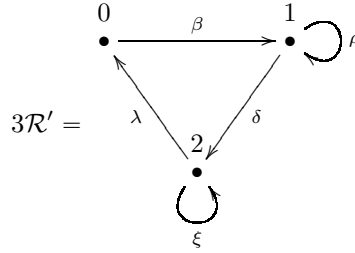
4.1. The derived equivalence classes of the algebras in the family $D(3\mathcal{R})$. In [14], Erdmann introduced the symmetric k -algebras of dihedral type $D(3\mathcal{R})^{a,b,c,d}$, for integers $a \geq 1$, $b, c, d \geq 2$, see Figure 1.

In [19, Sect. 3.2], Holm determined all algebras of dihedral type with precisely three isomorphism classes of simple modules that are derived equivalent to $D(3\mathcal{R})^{a,b,c,d}$ for various $a, b \geq 1$; $c, d \geq 2$. Note that for $a \geq 1$; $c, d \geq 2$, we define the algebra $D(3\mathcal{R})^{a,1,c,d}$ as in Figure 2.

Holm showed in [19, Thm. 3.4] that no block of a group algebra with dihedral defect groups is derived equivalent to $D(3\mathcal{R})^{a,b,c,d}$ for any $a, b \geq 1$; $c, d \geq 2$. Note that up to derived equivalence, we can order $1 \leq a \leq b \leq c \leq d$; $2 \leq c$ in $D(3\mathcal{R})^{a,b,c,d}$. By [19, Sect. 3.2], there are precisely five additional families of Morita equivalence classes of algebras of dihedral type that are derived

FIGURE 1. The family $D(3\mathcal{R})^{a,b,c,d} = k[3\mathcal{R}]/I_{3\mathcal{R},a,b,c,d}$ for $a \geq 1; b, c, d \geq 2$.


$$I_{3\mathcal{R},a,b,c,d} = \langle \alpha\lambda, \lambda\xi, \xi\delta, \delta\rho, \rho\beta, \beta\alpha, \alpha^b - (\lambda\delta\beta)^a, \rho^c - (\beta\lambda\delta)^a, \xi^d - (\delta\beta\lambda)^a \rangle.$$

 FIGURE 2. The family $D(3\mathcal{R}')^{a,1,c,d} = k[3\mathcal{R}']/I_{3\mathcal{R}',a,c,d}$ for $a \geq 1; c, d \geq 2$.


$$I_{3\mathcal{R}',a,c,d} = \langle \lambda\xi, \xi\delta, \delta\rho, \rho\beta, \rho^c - (\beta\lambda\delta)^a, \xi^d - (\delta\beta\lambda)^a \rangle.$$

equivalent to the algebras in the family $D(3\mathcal{R})^{a,b,c,d}$, $a, b \geq 1; c, d \geq 2$. We list these Morita equivalence classes in Figure 3.

4.2. Universal deformation rings for algebras in the derived equivalence classes of members of $D(3\mathcal{R})$. In [8], [9] and [33], the universal deformation rings of certain modules of $D(3\mathcal{R})^{a,b,c,d}$ were determined for various $a, b \geq 1; c, d \geq 2$. Recall that it follows from [9, Thm. 2.6] (see also Remark 3.2.7) that if Λ is a self-injective finite dimensional k -algebra and V is a finitely generated Λ -module whose stable endomorphism ring is isomorphic to k then the versal deformation ring $R(\Lambda, V)$ is universal.

Remark 4.2.1. In [8], the first author and S. Talbott studied the case $D(3\mathcal{R})^{1,1,2,2}$, which is of polynomial growth, and determined all indecomposable $D(3\mathcal{R})^{1,1,2,2}$ -modules whose stable endomorphism rings are isomorphic to k , together with their universal deformation rings.

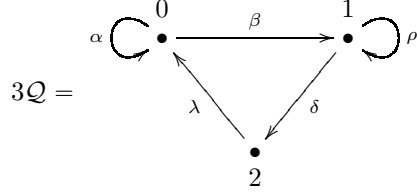
Using Figure 3, we obtain the following list of Morita equivalence classes of algebras of dihedral type that are derived equivalent to $D(3\mathcal{R})^{1,1,2,2}$:

$$D(3\mathcal{R})^{1,1,2,2}, D(3\mathcal{Q})^{1,2,2}, D(3\mathcal{L})^{2,2}, D(3\mathcal{A})_2^{2,2}, D(3\mathcal{B})_2^{1,2,2}, D(3\mathcal{B})_2^{2,1,2}, D(3\mathcal{D})_2^{1,1,2,2}.$$

By Proposition 3.2.5, it follows that if Λ is any of the algebras in this derived equivalence class and V is an indecomposable Λ -module with $\underline{\text{End}}_{\Lambda}(V) = k$, then $R(\Lambda, V) \cong R(D(3\mathcal{R})^{1,1,2,2}, V')$ if V corresponds to V' under the stable equivalence of Morita type between Λ and $D(3\mathcal{R})^{1,1,2,2}$ that is induced by the derived equivalence. By Remark 3.2.8, we see additionally that the stable Auslander-Reiten quiver components of V and V' match up, including the relative positions of V

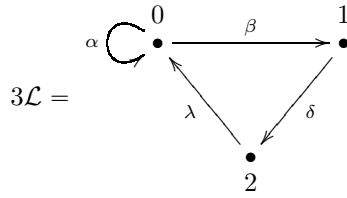
FIGURE 3. Morita equivalence classes of algebras of dihedral type that are derived equivalent to $D(3\mathcal{R})^{a,b,c,d}$ for various $a, b \geq 1$; $c, d \geq 2$ (see [19, Sect. 3.2]).

- (A) The family $D(3\mathcal{Q})^{b,c,d} = k[3\mathcal{Q}]/I_{3\mathcal{Q},b,c,d}$, $b \geq 1$; $c, d \geq 2$, which is derived equivalent to $D(3\mathcal{R})^{1,b,c,d}$:



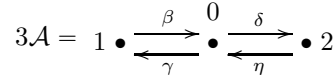
$$I_{3\mathcal{Q},b,c,d} = \langle \alpha\lambda, \delta\rho, \rho\beta, \beta\alpha, \alpha^c - (\lambda\delta\beta)^b, \rho^d - (\beta\lambda\delta)^b \rangle.$$

- (B) The family $D(3\mathcal{L})^{c,d} = k[3\mathcal{L}]/I_{3\mathcal{L},c,d}$, $c, d \geq 2$, which is derived equivalent to $D(3\mathcal{R})^{1,1,c,d}$:



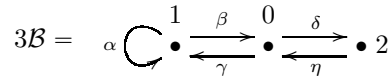
$$I_{3\mathcal{L},c,d} = \langle \alpha\lambda, \beta\alpha, \alpha^d - (\lambda\delta\beta)^c, \delta(\beta\lambda\delta)^c \rangle.$$

- (C) The family $D(3\mathcal{A})_2^{c,d} = k[3\mathcal{A}]/I_{(3\mathcal{A})_2,c,d}$, $c \geq d \geq 2$, which is derived equivalent to $D(3\mathcal{R})^{1,1,c,d}$:



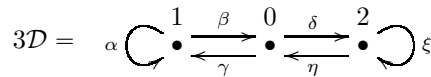
$$I_{(3\mathcal{A})_2,c,d} = \langle \gamma\eta, \delta\beta, (\beta\gamma)^c - (\eta\delta)^d \rangle.$$

- (D) The family $D(3\mathcal{B})_2^{b,c,d} = k[3\mathcal{B}]/I_{(3\mathcal{B})_2,b,c,d}$, $b, c \geq 1$ ($b + c > 2$); $d \geq 2$ which is derived equivalent to $D(3\mathcal{R})^{1,b,c,d}$ if $c \geq 2$ and to $D(3\mathcal{R})^{1,c,b,d}$ if $c = 1$ (and hence $b \geq 2$):



$$I_{(3\mathcal{B})_2,b,c,d} = \langle \alpha\gamma, \beta\alpha, \gamma\eta, \delta\beta, \alpha^d - (\gamma\beta)^b, (\beta\gamma)^b - (\eta\delta)^c \rangle.$$

- (E) The family $D(3\mathcal{D})_2^{a,b,c,d} = k[3\mathcal{D}]/I_{(3\mathcal{D})_2,a,b,c,d}$, $a, b \geq 1$; $c, d \geq 2$ which is derived equivalent to $D(3\mathcal{R})^{a,b,c,d}$:



$$I_{(3\mathcal{D})_2,a,b,c,d} = \langle \alpha\gamma, \beta\alpha, \gamma\eta, \delta\beta, \eta\xi, \xi\delta, \alpha^c - (\gamma\beta)^a, (\beta\gamma)^a - (\eta\delta)^b, \xi^d - (\delta\eta)^b \rangle.$$

and V' in these components. This reaffirms the results in [8, Thm. 1.1] (see also [8, Props. 3.1-3.3]) for these algebras.

In [9, Sect. 3], the authors studied the algebra $D(3\mathcal{R})^{1,2,2,2}$ and determined all indecomposable $D(3\mathcal{R})^{1,2,2,2}$ -modules whose stable endomorphism rings are isomorphic to k , together with their universal deformation rings.

Using Proposition 3.2.5, Remark 3.2.8 and [9, Thm. 1.2] (see also [9, Thm. 3.8, Props. 3.9-3.11, Thm. 3.16, Prop. 3.17]), we obtain the following result for all algebras of dihedral type in the derived equivalence class of $D(3\mathcal{R})^{1,2,2,2}$.

Theorem 4.2.2. *Let Λ be one of the following algebras:*

$$(4.1) \quad D(3\mathcal{R})^{1,2,2,2}, D(3\mathcal{Q})^{2,2,2,2}, D(3\mathcal{B})_2^{2,2,2,2}, D(3\mathcal{D})_2^{1,2,2,2}.$$

Suppose \mathfrak{C} is a component of the stable Auslander-Reiten quiver $\Gamma_s(\Lambda)$.

- (i) *If \mathfrak{C} is one of the two 3-tubes, then $\Omega(\mathfrak{C})$ is the other 3-tube. There are exactly three Ω^2 -orbits of modules in \mathfrak{C} whose stable endomorphism rings are isomorphic to k . If U_0 is a module that belongs to the boundary of \mathfrak{C} , then these three Ω^2 -orbits are represented by U_0 , by a successor U_1 of U_0 , and by a successor U_2 of U_1 that does not lie in the Ω^2 -orbit of U_0 . The universal deformation rings are*

$$R(\Lambda, U_0) \cong R(\Lambda, U_1) \cong k, \quad R(\Lambda, U_2) \cong k[[t]].$$

- (ii) *There are infinitely many components of $\Gamma_s(\Lambda)$ of type $\mathbb{Z}A_\infty^\infty$ that each contain a module whose stable endomorphism ring is isomorphic to k . If \mathfrak{C} is such a component, then $\mathfrak{C} = \Omega(\mathfrak{C})$ and there are exactly six Ω^2 -orbits (resp. exactly three Ω -orbits) of modules in \mathfrak{C} whose stable endomorphism rings are isomorphic to k . These three Ω -orbits are represented by a module V_0 , by a successor V_1 of V_0 that does not lie in the Ω -orbit of V_0 , and by a successor V_2 of V_1 that does not lie in the Ω^2 -orbit of V_0 . The universal deformation rings are*

$$R(\Lambda, V_0) \cong k[[t]]/(t^2), \quad R(\Lambda, V_1) \cong k, \quad R(\Lambda, V_2) \cong k[[t]].$$

- (iii) *There are infinitely many 1-tubes of $\Gamma_s(\Lambda)$ that each contain a module whose stable endomorphism ring is isomorphic to k . If \mathfrak{C} is such a component, then there is exactly one Ω^2 -orbit of modules in \mathfrak{C} whose stable endomorphism ring is isomorphic to k , represented by a module W_0 belonging to the boundary of \mathfrak{C} . The universal deformation ring of W_0 is*

$$R(\Lambda, W_0) \cong k[[t]].$$

Remark 4.2.3. Suppose Λ is one of the algebras in (4.1), and suppose V is a Λ -module with $\text{End}_\Lambda(V) = k$.

- (i) If $\Lambda = D(3\mathcal{R})^{1,2,2,2}$, then V is one of the modules

$$S_0, S_1, S_2, \begin{array}{c} 0 \quad 1 \quad 2 \\ 1 \quad , \quad 2 \quad , \quad 0 \quad , \quad 1 \quad , \quad 2 \quad , \quad 0 \quad . \\ 2 \quad 0 \quad 1 \end{array}.$$

Moreover, $R(\Lambda, V) \cong k$ if V has composition series length 2 or 3, and $R(\Lambda, V) \cong k[[t]]/(t^2)$ if V is simple.

- (ii) If $\Lambda = D(3\mathcal{Q})^{2,2,2,2}$, then V is one of the modules

$$S_0, S_1, S_2, \begin{array}{c} 0 \quad 1 \quad 2 \\ 1 \quad , \quad 2 \quad , \quad 0 \quad , \quad 1 \quad , \quad 2 \quad , \quad 0 \quad . \\ 2 \quad 0 \quad 1 \end{array}.$$

Moreover, $R(\Lambda, V) \cong k$ if $V = S_2$ or V has composition series length 2, and $R(\Lambda, V) \cong k[[t]]/(t^2)$ if $V \in \{S_0, S_1\}$ or V has composition series length 3.

- (iii) If $\Lambda = D(3\mathcal{B})_2^{2,2,2,2}$, then V is one of the modules

$$S_0, S_1, S_2, \begin{array}{c} 0 \quad 1 \quad 0 \quad 2 \quad 0 \quad 1 \quad 2 \\ 1 \quad , \quad 0 \quad , \quad 2 \quad , \quad 0 \quad , \quad 1 \quad 2 \quad , \quad 0 \quad . \end{array}$$

Moreover, $R(\Lambda, V) \cong k$ if $V \in \{S_0, S_2\}$ or V has composition series length 3, and $R(\Lambda, V) \cong k[[t]]/(t^2)$ if $V = S_1$ or V has composition series length 2.

(iv) If $\Lambda = D(3\mathcal{D})_2^{1,2,2,2}$, then V is one of the modules

$$S_0, S_1, S_2, \begin{array}{cccccc} 0 & 1 & 0 & 2 & 0 & 1 & 2 \\ 1 & 0 & 2 & 0 & 1 & 2 & 0 \end{array}.$$

Moreover, $R(\Lambda, V) \cong k$ if $V \in \left\{ S_0, \begin{array}{c} 0 \\ 1 \end{array}, \begin{array}{c} 1 \\ 0 \end{array} \right\}$ or V has composition series length 3, and

$$R(\Lambda, V) \cong k[[t]]/(t^2) \text{ if } V \in \left\{ S_1, S_2, \begin{array}{c} 0 \\ 2 \end{array}, \begin{array}{c} 2 \\ 0 \end{array} \right\}.$$

In [33], the second author considered all possible $a \geq 1, b, c, d \geq 2$ and determined all indecomposable $D(3\mathcal{R})^{a,b,c,d}$ -modules whose (usual) endomorphism rings are isomorphic to k . Moreover, he looked at their components in the stable Auslander-Reiten quiver of $D(3\mathcal{R})^{a,b,c,d}$ and determined all modules in these components whose stable endomorphism rings are isomorphic to k , together with their universal deformation rings.

We use [33, Thm. 1.1(iv)] to obtain a result concerning 3-tubes for all allowed parameters a, b, c, d . By using similar arguments as in the proof of [33, Prop. 4.4], we can include the case of $D(3\mathcal{R})^{a,1,c,d}$ for $a \geq 1$ and $c, d \geq 2$.

Theorem 4.2.4. *Let Λ be one of the following algebras:*

$$(4.2) \quad D(3\mathcal{R})^{a,b,c,d}, D(3\mathcal{Q})^{b,c,d}, D(3\mathcal{L})^{c,d}, D(3\mathcal{A})_2^{c,d}, D(3\mathcal{B})_2^{b,c,d}, D(3\mathcal{B})_2^{c,b,d}, D(3\mathcal{D})_2^{a,b,c,d},$$

where $a, b \geq 1, c, d \geq 2$ are allowed parameters according to Figure 3 such that Λ is not of polynomial growth. Suppose \mathfrak{T} is one of the two 3-tubes of the stable Auslander-Reiten quiver $\Gamma_s(\Lambda)$. Then $\Omega(\mathfrak{T})$ is the other 3-tube, and there are precisely three Ω -orbits of modules in $\mathfrak{T} \cup \Omega(\mathfrak{T})$ whose stable endomorphism rings are isomorphic to k . If U_0 is a module that belongs to the boundary of \mathfrak{T} , then these three Ω -orbits are represented by U_0 , by a successor U_1 of U_0 , and by a successor U_2 of U_1 that does not lie in the Ω^2 -orbit of U_0 . The universal deformation rings are

$$R(\Lambda, U_0) \cong R(\Lambda, U_1) \cong k, \quad R(\Lambda, U_2) \cong k[[t]].$$

Since in [33] only those components of the stable Auslander-Reiten quiver of $D(3\mathcal{R})^{a,b,c,d}$, for all $a \geq 1, b, c, d \geq 2$, were studied that contain modules whose (usual) endomorphism rings are isomorphic to k , we can only say something about finitely many components of the stable Auslander-Reiten quiver of type $\mathbb{Z}A_\infty^\infty$, as far as universal deformation rings are concerned. Using [33, Thm. 1.1(i)-(iii)], we obtain the following result for all allowed parameters a, b, c, d . By using similar arguments as in the proof of [33, Props. 4.1-4.3], we can include the case of $D(3\mathcal{R})^{a,1,c,d}$ for $a \geq 1$ and $c, d \geq 2$.

Proposition 4.2.5. *Let Λ be one of the algebras in (4.2), where $a, b \geq 1, c, d \geq 2$ are allowed parameters according to Figure 3 such that Λ is not of polynomial growth. If the quiver of Λ is $3\mathcal{Q}$ or $3\mathcal{B}$ we set $a = 1$, and if the quiver of Λ is $3\mathcal{L}$ or $3\mathcal{A}$ we set $a = b = 1$. Let \mathfrak{C} be a component of $\Gamma_s(\Lambda)$ of type $\mathbb{Z}A_\infty^\infty$ containing a module whose stable endomorphism ring is k .*

- (i) *Suppose $a = 1 = b$. Then there is at least one component \mathfrak{C} such that the following is true: There are precisely three Ω -orbits of modules in $\mathfrak{C} \cup \Omega(\mathfrak{C})$ whose stable endomorphism rings are isomorphic to k , represented by V_0, V_1, V_2 such that V_1 is a successor of V_0 that does not lie in the Ω -orbit of V_0 , V_2 is a successor of V_1 that does not lie in the Ω^2 -orbit of V_0 , and*

$$R(\Lambda, V_0) \cong k[[t]]/(t^c), \quad R(\Lambda, V_1) \cong k, \quad R(\Lambda, V_2) \cong k[[t]]/(t^d).$$

Moreover, $\mathfrak{C} = \Omega(\mathfrak{C})$ if and only if $c = 2$ or $d = 2$.

- (ii) *Suppose $a = 1$ and $b \geq 2$. Then there are at least three components $\mathfrak{C} = \mathfrak{C}_{i,j}$ for $(i, j) \in \{(1, b), (2, c), (3, d)\}$ such that the following is true: There are precisely three Ω -orbits of modules in $\mathfrak{C}_{i,j} \cup \Omega(\mathfrak{C}_{i,j})$ whose stable endomorphism rings are isomorphic to k , represented by $V_{i,j,0}, V_{i,j,1}, V_{i,j,2}$ such that $V_{i,j,1}$ is a successor of $V_{i,j,0}$ that does not lie in the Ω -orbit of $V_{i,j,0}$, $V_{i,j,2}$ is a successor of $V_{i,j,1}$ that does not lie in the Ω^2 -orbit of $V_{i,j,0}$, and*

$$R(\Lambda, V_{i,j,0}) \cong k[[t]]/(t^j), \quad R(\Lambda, V_{i,j,1}) \cong k, \quad R(\Lambda, V_{i,j,2}) \cong k[[t]].$$

Moreover, $\mathfrak{C}_{i,j} = \Omega(\mathfrak{C}_{i,j})$ if and only if $j = 2$.

- (iii) Suppose $a \geq 2$ and $b = 1$. Then there are at least four components $\mathfrak{C} = \mathfrak{C}_{i,j}$ for $(i, j) \in \{(1, a), (2, a), (3, c), (4, d)\}$ such that the following is true: There are precisely three Ω -orbits of modules in $\mathfrak{C}_{i,j} \cup \Omega(\mathfrak{C}_{i,j})$ whose stable endomorphism rings are isomorphic to k , represented by $V_{i,j,0}, V_{i,j,1}, V_{i,j,2}$ such that $V_{i,j,1}$ is a successor of $V_{i,j,0}$ that does not lie in the Ω -orbit of $V_{i,j,0}$, $V_{i,j,2}$ is a successor of $V_{i,j,1}$ that does not lie in the Ω^2 -orbit of $V_{i,j,0}$, and

$$R(\Lambda, V_{i,j,0}) \cong k[[t]]/(t^j), \quad R(\Lambda, V_{i,j,1}) \cong k, \quad R(\Lambda, V_{i,j,2}) \cong k[[t]].$$

Moreover, $\mathfrak{C}_{i,j} = \Omega(\mathfrak{C}_{i,j})$ if and only if $j = 2$.

- (iv) Suppose $a \geq 2$ and $b \geq 2$. Then there are at least nine components $\mathfrak{C} = \mathfrak{C}_{i,j}$ for $(i, j) \in \{(1, a), (2, a), (3, a), (4, b), (5, c), (6, d), (7, \infty), (8, \infty), (9, \infty)\}$ such that the following is true: There are precisely three Ω -orbits of modules in $\mathfrak{C}_{i,j} \cup \Omega(\mathfrak{C}_{i,j})$ whose stable endomorphism rings are isomorphic to k , represented by $V_{i,j,0}, V_{i,j,1}, V_{i,j,2}$ such that $V_{i,j,1}$ is a successor of $V_{i,j,0}$ that does not lie in the Ω -orbit of $V_{i,j,0}$, $V_{i,j,2}$ is a successor of $V_{i,j,1}$ that does not lie in the Ω^2 -orbit of $V_{i,j,0}$, and

$$R(\Lambda, V_{i,j,0}) \cong \begin{cases} k[[t]]/(t^j) & : j \neq \infty \\ k[[t]] & : j = \infty \end{cases}, \quad R(\Lambda, V_{i,j,1}) \cong k, \quad R(\Lambda, V_{i,j,2}) \cong k[[t]].$$

Moreover, $\mathfrak{C}_{i,j} = \Omega(\mathfrak{C}_{i,j})$ if and only if $j = 2$.

Remark 4.2.6. Suppose Λ is one of the algebras in (4.2), where $a, b \geq 1$, $c, d \geq 2$ are allowed parameters according to Figure 3. Moreover assume Λ is not of polynomial growth. Note that by [19, Lemma 3.15], $D(3\mathcal{R})^{a,b,c,d}$ and $D(3\mathcal{R})^{b,a,c,d}$ are derived equivalent for all $a, b \geq 1$; $c, d \geq 2$. In view of Theorem 4.2.2(ii), it seems plausible that there are usually infinitely many components of the stable Auslander-Reiten quiver of Λ of type $\mathbb{Z}A_\infty^\infty$ that contain modules whose stable endomorphism rings are isomorphic to k .

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F.B.: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF IOWA, 14 MACLEAN HALL, IOWA CITY, IA 52242-1419, U.S.A.

E-mail address: frauke-bleher@uiowa.edu

J.V.: DEPARTMENT OF MATHEMATICS & COMPUTER SCIENCE, VALDOSTA STATE UNIVERSITY, 2072 NEVINS HALL, VALDOSTA, GA 31698-0040, U.S.A.

E-mail address: javelezmarulanda@valdosta.edu