CHARACTERISTIC INITIAL VALUE PROBLEM FOR SPHERICALLY SYMMETRIC BAROTROPIC FLOW

ANDRÉ LISIBACH

ABSTRACT. We study the equations of motion for a barotropic fluid in spherical symmetric flow. Making use of the Riemann invariants we consider the characteristic form of these equations. In a first part, we show that the resulting constraint equations along characteristics can be solved globally away from the center of symmetry. In a second part, given data on two intersecting characteristics, we show existence and uniqueness of a smooth solution in a neighborhood in the future of these characteristics.

1. INTRODUCTION

The equations of motion describing a compressible inviscid fluid are of hyperbolic type. For such equations, along a characteristic hypersurface not all of the unknown functions can be prescribed freely. The ones that can will be denoted in the following as *free data*. When restricting the equations of motion to a characteristic hypersurface they become the so called *constraint equations*. Given free data these are equations for the remaining unknowns which will be called *derived data*. This situation is in contrast to the Cauchy problem where all of the unknown functions can be prescribed at t = 0. The system of constraint equations is nonlinear, therefore a solution does not exist in general. We study the constraint equations for the Euler equations in the case of a barotropic fluid, i.e. under the assumption that $p = f(\rho)$, where p, ρ are the pressure and density of the fluid respectively. In addition we assume that the flow is spherically symmetric, hence the problem reduces to one in the t-r-plane, where t, r denote the time and radial coordinate respectively. We use the Riemann invariants which lead to a natural formulation of the equations of motion along characteristics. The resulting constraint equations form a two by two system of nonlinear ordinary differential equations. In the first part we show existence and uniqueness of a solution of this system globally away from the center of symmetry r = 0.

Once the constraint equations are solved and therefore characteristic data has been established, a natural follow up question is whether one can find a solution of the equations of motion in a neighborhood of two intersecting characteristic hypersurfaces in the acoustical future of the intersection. This is the content of the second part of the present work.

It is important to note that the solution thus obtained corresponds to a solution in the *t*-*r*-plane only where the jacobian of the transformation from the characteristic coordinates to the *t*-*r*-plane does not vanish (see page 22). Such points of vanishing jacobian represent points in the singular part of the boundary of the maximal development (see for example chapter 2 of [3]). Therefore, such points

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and their range of influence have to be excluded from the solution. Furthermore, in view of obtaining a physically acceptable solution, we note that an even further restriction might apply once a shock solution beyond a point of blowup has been established, the shock lying in the past of the boundary of the maximal development (see [4]).

The present work can be viewed as a first step towards understanding the characterisitic initial value problem for the Euler equations without any symmetry assumptions.

2. Equations of Motion, Characteristic System

We review the basic equations needed for the study of a barotropic fluid in spherical symmetry.

2.1. Equations of Motion in Spherical Symmetry. We denote by w, ρ , p the fluid velocity, the density and the pressure, respectively. We assume a barotropic equation of state, i.e. $p = f(\rho)$, and we assume $f \in C^{\infty}$, $dp/d\rho$, $d^2p/d\rho^2 > 0$. The adiabatic condition decouples and we are left with

(1)
$$\partial_t \rho + \partial_r (\rho w) = -\frac{2\rho w}{r},$$

(2)
$$\partial_t w + w \partial_r w = -\frac{\eta^2}{\rho} \partial_r \rho$$

where we denote by η the sound speed, i.e. $\eta^2 = dp/d\rho$. We assume $\rho > 0$, i.e. we exclude vacuum.

2.2. Riemann Invariants, Characteristic System. Let (see [1])

(3a,b)
$$\alpha \stackrel{\text{def}}{=} \int^{\rho} \frac{\eta(\rho')}{\rho'} d\rho' + w, \qquad \beta \stackrel{\text{def}}{=} \int^{\rho} \frac{\eta(\rho')}{\rho'} d\rho' - w$$

and

(4a,b)
$$c_{\pm} \stackrel{\text{def}}{=} w \pm \eta, \qquad L_{\pm} \stackrel{\text{def}}{=} \partial_t + c_{\pm} \partial_r.$$

We have

(5)
$$L_{+}\alpha = L_{-}\beta = -\frac{\eta(\alpha,\beta)}{r}(\alpha-\beta) \stackrel{\text{def}}{=} F(\alpha,\beta,r).$$

Introducing the coordinates u, v such that u is constant along integral curves of L_+ and v is constant along integral curves of L_- , (5) becomes

(6a,b)
$$\frac{\partial \alpha}{\partial v} = \frac{\partial t}{\partial v} F(\alpha, \beta, r), \qquad \frac{\partial \beta}{\partial u} = \frac{\partial t}{\partial u} F(\alpha, \beta, r).$$

t and r satisfy the Hodograph system

(7a,b)
$$\frac{\partial r}{\partial v} = \frac{\partial t}{\partial v}c_{+}(\alpha,\beta), \qquad \frac{\partial r}{\partial u} = \frac{\partial t}{\partial u}c_{-}(\alpha,\beta)$$

In the following we refer to (6a,b), (7a,b) as the characteristic system of equations (see [2]).

From (3a,b) we have

(8)
$$\frac{\partial(\alpha,\beta)}{\partial(\rho,w)} = \begin{pmatrix} \eta/\rho & 1\\ \eta/\rho & -1 \end{pmatrix}.$$

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Therefore,

(9)
$$\frac{\partial(\rho, w)}{\partial(\alpha, \beta)} = \begin{pmatrix} \rho/2\eta & \rho/2\eta \\ 1/2 & -1/2 \end{pmatrix}.$$

Let now

(10)
$$\chi \stackrel{\text{def}}{=} \alpha - \beta, \qquad \chi^{\dagger} \stackrel{\text{def}}{=} \alpha + \beta.$$

We have

(11)
$$\frac{\partial \alpha}{\partial \chi^{\dagger}} = \frac{\partial \beta}{\partial \chi^{\dagger}} = \frac{1}{2}.$$

Now,

(12)
$$\frac{\partial \eta}{\partial \chi^{\dagger}} = \frac{d\eta}{d\rho} \left\{ \frac{\partial \rho}{\partial \alpha} \frac{\partial \alpha}{\partial \chi^{\dagger}} + \frac{\partial \rho}{\partial \beta} \frac{\partial \beta}{\partial \chi^{\dagger}} \right\}$$
$$= \frac{\rho}{4\eta^2} \frac{d^2 p}{d\rho^2} > 0.$$

Similarly we find $\partial \eta / \partial \chi = 0$. Therefore, $\eta = \eta(\chi^{\dagger})$ and $d\eta / d\chi^{\dagger} > 0$.

3. CHARACTERISTIC INITIAL DATA

We look at a point (t_0, r_0) in the *t*-*r*-plane and denote the outgoing and incoming characteristic originating from this point by C^+ and C^- respectively. We put the origin of the *u*-*v*-coordinates at (t_0, r_0) and we set $t_0 = 0$.

In view of (6a), (7a) the free data on C^+ consists of $\beta^+(v) = \beta(0, v)$, $t^+(v) = t(0, v)$ for an increasing function t^+ . We fix the coordinate v along C^+ by setting $t^+(v) = v$. Then (6a), (7a) constitute the following system of nonlinear ode for the derived data $\alpha^+(v)$, $r^+(v)$ on C^+ :

(13a)
$$\frac{d\alpha}{dv} = -\frac{\eta(\alpha,\beta)}{r}(\alpha-\beta),$$

(13b)
$$\frac{dr}{dv} = \frac{1}{2}(\alpha - \beta) + \eta(\alpha, \beta),$$

where we omitted the superscript + on α , β and r.

In view of (6b), (7b) the free data on C^- consists of $\alpha^-(u) = \alpha(u, 0)$, $t^-(u) = t(u, 0)$ for an increasing function t^- . We fix the coordinate u along C^- by setting $t^-(u) = u$. Then (6b), (7b) constitute the following system of nonlinear ode for the derived data $\beta^-(u)$, $r^-(u)$ on C^- :

(14a)
$$\frac{d\beta}{du} = -\frac{\eta(\alpha,\beta)}{r}(\alpha-\beta),$$

(14b)
$$\frac{dr}{du} = \frac{1}{2}(\alpha - \beta) - \eta(\alpha, \beta),$$

where we omitted the superscript - on α , β and r. (13), (14) are the constraint equations along C^+ , C^- , respectively. The following lemma shows that there exists derived smooth data on C^+ and on C^- as long as C^- stays away from r = 0.

Lemma 1. Let $\alpha^-, \beta^+ \in C^{\infty}(\mathbb{R}^+ \cup \{0\})$ and $r_0 > 0$. Then

i) the system (13) with $\beta = \beta^+$, $\alpha(0) = \alpha^-(0)$ and $r(0) = r_0$ has a solution for $v \in \mathbb{R}^+ \cup \{0\}$,

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ii) for any $\varepsilon > 0$ with $\varepsilon < r_0$ the system (14) with $\alpha = \alpha^-$, $\beta(0) = \beta^+(0)$ and $r(0) = r_0$ has a solution for $u \in [0, \overline{u})$, where $\overline{u} = \sup \left\{ u' \in \mathbb{R}^+ \cup \{0\} : \forall u'' \in [0, u'] : r(u'') > \varepsilon \right\}.$

Proof. Part i). The only possibilities for the system (13) to blow up are

(15)
$$r \to 0, \qquad r \to \infty, \qquad |\alpha| \to \infty.$$

Using $\chi = \alpha - \beta$, the system (13) becomes (we omit the argument of η)

(16a)
$$\frac{d\chi}{dv} = -\frac{\eta}{r}\chi - \frac{d\beta}{dv},$$

(16b)
$$\frac{d\eta}{dv} = \eta + \frac{1}{2}\chi$$

From (16a) we obtain

(17)
$$\chi(v) = \chi(0)e^{-\int_0^v \left(\frac{\eta}{r}\right)(v')dv'} - \int_0^v e^{-\int_{v'}^v \left(\frac{\eta}{r}\right)(v'')dv''} \frac{d\beta}{dv}(v')dv'.$$

Therefore,

(18)
$$|\chi(v)| \le |\chi(0)| + \int_0^v \left| \frac{d\beta}{dv}(v') \right| dv',$$

which implies that the absolute value of χ is bounded. The bound on χ implies that also α is bounded in absolute value, i.e. $|\alpha| \leq C$. Hence also χ^{\dagger} is bounded in absolute value, i.e. $|\chi^{\dagger}| \leq C$. From (9), (10) we have

(19)
$$\frac{d\log\rho}{d\chi^{\dagger}} = \frac{1}{4\eta}$$

In view of $\eta > 0$, we deduce that $\rho \leq C$. This in turn implies that η is bounded. From (16b) we have

(20)
$$r(v) = r_0 + \int_0^v (\eta + \frac{1}{2}\chi)(v')dv'.$$

Therefore, r is bounded from above.

The only possibility for blowup left to study is $r \to 0$. Let us assume

(21)
$$r \to 0 \quad \text{as} \quad v \to v^*.$$

Let $0 < v_1 < v < v^*$. Integrating (16a) on $[v_1, v]$ yields

(22)
$$\chi(v) \ge \chi(v_1) e^{-\int_{v_1}^v \left(\frac{\pi}{r}\right)(v')dv'} - C(v-v_1).$$

Let

(23)
$$\underline{\eta} \stackrel{\text{def}}{=} \inf_{[0,v^*]} \eta.$$

We have

(24)
$$\eta(v) \ge \eta > 0$$

Let $0 < \tilde{C} < \underline{\eta}$. Using the lower bounds for χ and η as given by (22) and (24), respectively, in (16b) and choosing $v_1 \in (0, v^*)$ such that

(25)
$$v_1 \ge v^* - \frac{2}{C}(\underline{\eta} - \tilde{C}),$$

where C is the constant appearing in (22), we obtain

(26)
$$\frac{dr}{dv}(v) \ge \tilde{C} + \frac{1}{2}\chi(v_1)e^{-\int_{v_1}^v \left(\frac{\eta}{r}\right)(v')dv'}$$

Defining

(27)
$$\tilde{\chi}(v) \stackrel{\text{def}}{=} \chi(v_1) e^{-\int_{v_1}^v \left(\frac{\eta}{r}\right)(v')dv'},$$

(26) becomes

(28)
$$\frac{dr}{dv}(v) \ge \tilde{C} + \frac{1}{2}\tilde{\chi}(v)$$

In the case $\chi(v_1) \geq 0$ we have $\tilde{\chi} \geq 0$ and $(dr/dv)(v) \geq \tilde{C} > 0$ for $v \in [v_1, v^*]$, which contradicts (21). We consider the case $\chi(v_1) < 0$. From (27) we see that in $[v_1, v^*]$, $\tilde{\chi}$ is monotonically increasing and $\tilde{\chi} < 0$. We define $\tilde{\phi} \stackrel{\text{def}}{=} -\tilde{\chi}$. Then, on $[v_1, v^*]$, $\tilde{\phi} > 0$, $\tilde{\phi}$ is monotonically decreasing and

(29)
$$\frac{dr}{dv}(v) \ge \tilde{C} - \frac{1}{2}\tilde{\phi}(v).$$

We look at the subcase $2\tilde{C} \geq \tilde{\phi}(v_1)$. In this subcase we have $2\tilde{C} \geq \tilde{\phi}(v)$ for $v \in [v_1, v^*]$, which implies $(dr/dv)(v) \geq 0$ in $[v_1, v^*]$. Together with $r(v_1) > 0$ this contradicts (21).

We look at the subcase $2\tilde{C} < \tilde{\phi}(v_1)$. Since $\tilde{\phi}$ is a monotonically decreasing function, it either drops not below $2\tilde{C}$ or it drops below $2\tilde{C}$ on $[v_1, v^*]$. In the latter situation there is a $v_0 \in [v_1, v^*]$ such that $(dr/dv)(v) \ge 0$ for $v \in [v_0, v^*]$ and we get a contradiction in the same way as in the subcase studied above.

Therefore, we are left to study the situation in which ϕ does not drop below $2\tilde{C}$ in $[v_1, v^*]$. Since $\tilde{\phi}$ is a decreasing function which is bounded from below by $2\tilde{C}$, it tends to a limit as $v \to v^*$. We have

(30)
$$\lim_{v \to v^*} \tilde{\phi}(v) \ge 2\tilde{C}.$$

Now,

(31)
$$-\frac{dr}{dv}(v) \le \frac{1}{2}\tilde{\phi}(v) - \tilde{C} \le \frac{1}{2}\tilde{\phi}(v_1) - \tilde{C}.$$

Integrating this on $[v, v^*]$ for $v \in [v_1, v^*]$ and taking into account the assumption (21), we obtain

(32)
$$r(v) \le \left(\frac{1}{2}\tilde{\phi}(v_1) - \tilde{C}\right)(v^* - v).$$

Using this together with the lower bound on η we deduce

(33)
$$\int_{v_1}^{v} \left(\frac{\eta}{r}\right) (v') dv' \geq \frac{\underline{\eta}}{\frac{1}{2}\tilde{\phi}(v_1) - \tilde{C}} \int_{v_1}^{v} \frac{dv'}{v^* - v'} = \frac{\underline{\eta}}{\frac{1}{2}\tilde{\phi}(v_1) - \tilde{C}} \log\left(\frac{v^* - v_1}{v^* - v}\right).$$

Recalling the definition of $\tilde{\phi}$ we get

(34)

$$\tilde{\phi}(v) = \tilde{\phi}(v_1)e^{-\int_{v_1}^v \left(\frac{\eta}{r}\right)(v')dv'}$$

$$\leq \tilde{\phi}(v_1)\left(\frac{v^* - v}{v^* - v_1}\right)^{\frac{1}{2}\frac{\tilde{\phi}(v_1) - \tilde{C}}}$$

The right hand side tends to 0 as $v \to v^*$ giving us a contradiction to (30).

Part ii). Assuming $r > \varepsilon$ the only possibilities for the system (14) to blow up are

(35)
$$r \to \infty, \qquad |\beta| \to \infty.$$

Using $\chi = \alpha - \beta$, the system (14) becomes (we omit the argument of η)

(36a)
$$\frac{d\chi}{du} = \frac{\eta}{r}\chi + \frac{d\alpha}{du},$$

(36b)
$$\frac{d\eta}{du} = \frac{1}{2}\chi - \eta$$

Let $0 < u_1 < u < u^*$. Integrating (36a) on $[u_1, u]$ yields

(37)
$$\chi(u) = e^{\int_{u_1}^u \left(\frac{u}{r}\right)(u')du'} (\chi(u_1) + F_1(u)),$$

where

(38)
$$F_1(u) \stackrel{\text{def}}{=} \int_{u_1}^u e^{-\int_{u_1}^{u'} \left(\frac{\eta}{r}\right)(u'')du''} \frac{d\alpha}{du}(u')du'.$$

We have

(39)
$$|F_1(u)| \le C(u - u_1)$$

and

(40)
$$|\chi(u)| \le C e^{\int_{u_1}^u \left(\frac{\eta}{r}\right)(u')du'}.$$

Let us assume

(41)
$$|\chi| \to \infty \quad \text{as} \quad u \to u^*$$

In view of the bound (39) we see that there exists $u_1 \in [0, u^*)$ such that for $u \in [u_1, u^*)$ we have either $\chi(u_1) + F_1(u) > 0$ or $\chi(u_1) + F_1(u) < 0$. Using this in (37) we obtain that either $\chi(u) > 0$ for $u \in [u_1, u^*)$ and $\chi \to \infty$ as $u \to u^*$ or $\chi(u) < 0$ for $u \in [u_1, u^*)$ and $\chi \to -\infty$ as $u \to u^*$ respectively.

In the first case, $\chi \to \infty$ as $u \to u^*$ implies $\beta \to -\infty$ as $u \to u^*$ which in turn implies $\chi^{\dagger} \to -\infty$ as $u \to u^*$. In view of (12) we then have $\lim_{u\to u^*} \eta(u) \leq C$. Using this together with $1/r \leq C$ we obtain that the right hand side of (40) with $u = u^*$ is bounded. This contradicts (41).

In the second case, $\chi \to -\infty$ as $u \to u^*$ implies, through (40) and $1/r \leq C$, that

(42)
$$\int_{u_1}^u \eta(u') du' \to \infty \quad \text{as} \quad u \to u^*$$

Integrating (36b) gives

(43)
$$r(u) = r(u_1) + \int_{u_1}^{u} \left(\frac{1}{2}\chi(u') - \eta(u')\right) du'.$$

Using (42) together with $\chi \to -\infty$, we see that $r(u) \to 0$ as $u \to u^*$, contradicting our assumption r > 0.

Therefore χ is bounded and hence so is β , i.e. $|\beta| \leq C$. The bounds on α and β imply that also η is bounded (see (19)). In view of (36b) we then obtain an upper bound for r(u).

The above lemma shows that there is no blowup of α , r along C^+ and of β , r along C^- as long as C^- does not hit the center of symmetry r = 0, thus establishing a continuously differentiable solution of the constraint equations. In the following we show that there is also no blowup for higher order derivatives of α , β , t and r.

We define

(44a,b)
$$\mu \stackrel{\text{def}}{=} \frac{\partial t}{\partial u}, \quad \nu \stackrel{\text{def}}{=} \frac{\partial t}{\partial v}$$

and

(45a,b)
$$\gamma \stackrel{\text{def}}{=} \frac{\partial \alpha}{\partial u}, \qquad \delta \stackrel{\text{def}}{=} \frac{\partial \beta}{\partial v}.$$

In the following we denote by $f_i : i = 1, ..., 10$ given continuously differentiable functions. Taking the derivative of (6a) with respect to u and making use of (6b), (7b) we obtain the following equation along C^+

(46)
$$\frac{d\gamma}{dv} = f_1 \frac{d\mu}{dv} + f_2 \gamma + f_3 \mu$$

Taking the derivative of (6b) with respect to v and making use of (6a), (7a) we obtain the following equation along C^-

(47)
$$\frac{d\delta}{du} = f_4 \frac{d\nu}{du} + f_5 \nu + f_6 \delta.$$

Taking the derivative of (7a) with respect to u and the derivative of (7b) with respect to v, subtracting the resulting equations from each other, we arrive at

(48)
$$\frac{\partial^2 t}{\partial u \partial v} + \frac{1}{c_+ - c_-} \left(\frac{\partial c_+}{\partial u} \nu - \frac{\partial c_-}{\partial v} \mu \right) = 0,$$

where we omitted the arguments of c_{\pm} . (48) becomes, along C^+ , C^- ,

(49)
$$\frac{d\mu}{dv} = f_7 \gamma + f_8 \mu,$$

(50)
$$\frac{d\nu}{du} = f_9\delta + f_{10}\nu,$$

respectively. (46), (49) constitute a system of linear equations for μ , γ which we supplement with the initial conditions $\mu(0) = 1, \gamma(0) = (d\alpha^{-}/du)(0)$. We deduce that μ , γ do not blow up on $\mathbb{R}^{+} \cup \{0\}$. (47), (50) constitute a system of linear equations for ν , δ which we complement with the initial conditions $\nu(0) = 1, \delta(0) = (d\beta^{+}/dv)(0)$. We deduce that ν , δ do not blow up for $u \in [0, \overline{u})$. In view of (7a,b) also $\partial r/\partial u$, $\partial r/\partial v$ do not blow up along C^{+} and C^{-} . Therefore, the derivatives of first order of α , β , t and r do not blow up on C^{+} , C^{-} .

Continuing in a similar manner we see that also all derivatives of higher order satisfy linear equations along C^+ and C^- . We conclude that there is also no blowup in those quantities. We therefore have the following result which establishes characteristic initial data.

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Proposition 1. Let $r_0 > \varepsilon > 0$ and let us be given free data $\beta_0^+, \alpha_0^-, t_0^+, t_0^- \in C^{\infty}(\mathbb{R}^+ \cup \{0\})$ such that $t_+(0) = t^-(0)$. Then there exist unique smooth functions $\alpha_0^+, \alpha_i^{\pm}, \beta_0^-, \beta_i^{\pm}, t_i^{\pm}, r_0^{\pm}, r_i^{\pm}, i \ge 1$, such that $r_0^{\pm}(0) = r_0$ and, for $f \in \{\alpha, \beta, t, r\}$, the functions

(51a)
$$\frac{\partial^{i+j}f}{\partial u^i \partial v^j}(0,v) = \frac{df_i^+}{dv^j}(v): i, j \ge 0,$$

(51b)
$$\frac{\partial^{i+j}f}{\partial u^j \partial v^i}(u,0) = \frac{df_i^-}{du^j}(u): i,j \ge 0,$$

satisfy the characteristic system and derivatives of it along C^+ , C^- and the initial conditions

(52a,b)
$$f_i^+(0) = \frac{d^i f_0^-}{du^i}(0), \qquad f_i^-(0) = \frac{d^i f_0^+}{dv^i}(0) : i \ge 0$$

 C^+ corresponds to $\{(0,v):v\in\mathbb{R}^+\cup\{0\}\}$ and C^- corresponds to $\{(u,0):u\in[0,\overline{u})\},$ where

(53)
$$\overline{u} = \sup\left\{u' \in \mathbb{R}^+ \cup \{0\} : \forall u'' \in [0, u'] : r_0^-(u'') > \varepsilon\right\}.$$

Remark 1. It would suffice to give α_0^- , t_0^- on $[0, \overline{u})$.

4. CHARACTERISTIC INITIAL VALUE PROBLEM

In the following we establish local existence of a solution to the characteristic initial value problem. We assume that, after giving free data, the constraint equations have been solved and in the following we are going to make use of this solution without further reference. We focus on establishing the solution in a region adjacent to C^+ . Establishing the solution in a region adjacent to C^- is analogous. Let us be given data along C^+ up to $v = v^*$ and along C^- up to $u = u^*$.

4.1. Solution in the Corner. We define

(54)
$$I_a \stackrel{\text{def}}{=} \left\{ (0, v) \in \mathbb{R}^2 : 0 \le v \le a \right\},$$

(55)
$$\Pi_{ab} \stackrel{\text{def}}{=} \left\{ (u, v) \in \mathbb{R}^2 : 0 \le u \le a, 0 \le v \le b \right\},$$

(see figure 1).



FIGURE 1. The domain Π_{ab}

Recall that
$$\nu(0, v) = 1$$
. Let
(56) $a_0 \stackrel{\text{def}}{=} \max_{I_{v^*}} |\alpha|, \qquad b_0 \stackrel{\text{def}}{=} \max_{I_{v^*}} |\beta|, \qquad d_0 \stackrel{\text{def}}{=} \max_{I_{v^*}} |\delta|$

and let

(57)
$$r_m \stackrel{\text{def}}{=} \min_{I_{v^*}} r, \qquad r_M \stackrel{\text{def}}{=} \max_{I_{v^*}} r.$$

Let l > 1 and let

(58)
$$A \stackrel{\text{def}}{=} la_0, \qquad B \stackrel{\text{def}}{=} lb_0, \qquad D \stackrel{\text{def}}{=} ld_0$$

Let us choose $0 < h < u^*$ such that for $u \in [0, h]$ we have

(59)
$$|\alpha(u,0)| < A, \qquad |\beta(u,0)| < B, \qquad |\delta(u,0)| < D,$$

(60)
$$\frac{1}{2}r_m < r(u,0) < \frac{3}{2}r_M.$$

Recall that $\mu(u,0) = 1$. In the following we generalize our discussion to the case $\mu(u,0) \neq 1$, the case $\mu(u,0) = 1$ being a trivial subcase. Let

(61)
$$m_0 \stackrel{\text{def}}{=} \sup_{u \in [0,h]} |\mu(u,0)|, \qquad g_0 \stackrel{\text{def}}{=} \sup_{u \in [0,h]} |\gamma(u,0)|.$$

Let

(62)
$$R_l \stackrel{\text{def}}{=} \left\{ (\alpha, \beta) \in \mathbb{R}^2 : |\alpha| \le A, |\beta| \le B \right\},$$

(63)
$$\Omega_l \stackrel{\text{def}}{=} R_l \times \left[\frac{1}{2}r_m, \frac{3}{2}r_M\right].$$

In the following we establishes bounds on $|\gamma|$, $|\mu|$ once bounds for all other quantities have been established.

Making use of the characteristic system we obtain

(64a)
$$\frac{\partial \gamma}{\partial v} = A_1 \nu \gamma + B_1 \mu \delta + C_1 \mu \nu,$$

(64b)
$$\frac{\partial \mu}{\partial v} = A_2 \nu \gamma + B_2 \mu \delta + C_2 \mu \nu,$$

where

(65)
$$A_{1} = \frac{1}{r} \left\{ (\alpha - \beta)(\frac{1}{4} - \frac{1}{2}\eta') - \eta \right\},$$

(66)
$$B_1 = \frac{\alpha - \beta}{2r} (\frac{1}{2} + \eta'),$$

(67)
$$C_1 = \frac{\eta(\alpha - \beta)}{r^2} (\alpha - \beta - 2\eta),$$

(68)
$$A_2 = B_2 = -\frac{1}{2\eta} (\frac{1}{2} + \eta'),$$

(69)
$$C_2 = \frac{\alpha - \beta}{r} (\eta' - \frac{1}{2})$$

and we denote $\eta' \stackrel{\text{def}}{=} d\eta/d\chi^{\dagger}$ (see (10)). Defining

(70)
$$a_1 \stackrel{\text{def}}{=} A_1 \nu, \qquad b_1 \stackrel{\text{def}}{=} B_1 \delta + C_1 \nu,$$

(71)
$$b_2 \stackrel{\text{def}}{=} A_2 \nu, \qquad a_2 \stackrel{\text{def}}{=} B_2 \delta + C_2 \nu,$$

we arrive at

(72)
$$\frac{\partial \gamma}{\partial v} = a_1 \gamma + b_1 \mu,$$

(73)
$$\frac{\partial \mu}{\partial v} = a_2 \mu + b_2 \gamma.$$

Now we define

(74)
$$Q_i \stackrel{\text{def}}{=} \sup_{\substack{(\alpha,\beta,r)\in\Omega_l\\|\nu|\leq l}} |a_i|, \qquad S_i \stackrel{\text{def}}{=} \sup_{\substack{(\alpha,\beta,r)\in\Omega_l\\|\nu|\leq l,|\delta|\leq D}} |b_i|, \qquad i=1,2.$$

Then, the system (72), (73) implies

(75)
$$|\gamma(u,v)| \le e^{vQ_1} \left\{ |\gamma(u,0)| + S_1 \int_0^v |\mu|(u,v')dv' \right\},$$

(76)
$$|\mu(u,v)| \le e^{vQ_2} \left\{ |\mu(u,0)| + S_2 \int_0^v |\gamma|(u,v')dv' \right\}.$$

Substituting one of these equations into the other and making use of Gronwall's inequality we obtain

(77)
$$|\gamma(u,v)| \le f_1(v)|\gamma(u,0)| + f_2(v)|\mu(u,0)|,$$

(78)
$$|\mu(u,v)| \le f_3(v)|\mu(u,0)| + f_4(v)|\gamma(u,0)|,$$

where

(79)
$$f_1(v) = e^{vQ_1} \left\{ 1 + vS_1S_2 e^{v(Q_1 + Q_2)} \int_0^v e^{-S_1S_2 \int_{v'}^v v'' e^{v''(Q_1 + Q_2)} dv''} dv' \right\},$$

,

(80)
$$f_2(v) = vS_1 e^{vQ_2} f_1(v)$$

and analogous expressions hold for f_3 , f_4 . We define

(81)
$$\overline{f}_1(v) \stackrel{\text{def}}{=} e^{vQ_1} \left\{ 1 + v^2 S_1 S_2 e^{v(Q_1 + Q_2)} \right\}$$

(82)
$$\overline{f}_2(v) \stackrel{\text{def}}{=} v S_1 e^{vQ_2} \overline{f}_1(v)$$

and analogously we define $\overline{f}_3, \overline{f}_4$. We have

(83)
$$f_i \le \overline{f}_i : i = 1, \dots, 4.$$

We note that $\overline{f}_i: i = 1, \dots, 4$ are strictly increasing functions satisfying

(84)
$$\lim_{v \to 0} \overline{f}_i = 1 : i = 1, 3, \qquad \lim_{v \to 0} \overline{f}_k = 0 : k = 2, 4.$$

We make the definitions

(85)
$$G \stackrel{\text{def}}{=} \overline{f}_1(v^*)g_0 + \overline{f}_2(v^*)m_0,$$

(86)
$$M \stackrel{\text{def}}{=} \overline{f}_3(v^*)g_0 + \overline{f}_4(v^*)m_0.$$

We have proved the following:

Lemma 2. Let us be given a solution of the characteristic system in Π_{uv} for $u \in (0, h]$, $v \in (0, v^*]$ such that this solution satisfies the bounds

(87) $|\nu| \le l, \quad |\alpha| \le A, \quad |\beta| \le B, \quad |\delta| \le D, \quad \frac{1}{2}r_m \le r \le \frac{3}{2}r_M.$

Then the constants G, M given by (85), (86) respectively, satisfy $G > g_0$, $M > m_0$ and we have

(88) $|\gamma| \le G, \qquad |\mu| \le M,$

with strict inequalities for $v < v^*$. The constants G, M depend on A, B, D, l, r_m , r_M , g_0 , m_0 .

From the characteristic system we have

(89)
$$\alpha(u,v) = \alpha(u,0) + \int_0^v (\nu F)(u,v')dv',$$

(90)
$$\beta(u,v) = \beta(0,v) + \int_0^u (\mu F)(u',v) du'.$$

Introducing the two functions

(91)
$$K \stackrel{\text{def}}{=} \frac{1}{c_+ - c_-} \frac{\partial c_+}{\partial u}, \qquad L \stackrel{\text{def}}{=} \frac{1}{c_+ - c_-} \frac{\partial c_-}{\partial v},$$

with $c_{\pm} = c_{\pm}(\alpha, \beta)$ and

(92)
$$\frac{\partial c_{+}}{\partial u} = \frac{\partial c_{+}}{\partial \alpha} (\alpha, \beta) \frac{\partial \alpha}{\partial u} + \frac{\partial c_{+}}{\partial \beta} (\alpha, \beta) \frac{\partial \beta}{\partial u}$$

and analogous for $\partial c_{-}/\partial v$, we can rewrite (48) as

(93)
$$\frac{\partial^2 t}{\partial u \partial v} + K\nu - L\mu = 0.$$

We now construct a solution of the characteristic system as the limit of a sequence of functions $((\alpha_n, \beta_n, t_n, r_n); n = 0, 1, 2, ...)$ defined on $\Pi_{h\varepsilon}$. The sequence is generated by the following iteration. We first define (α_0, β_0) setting

(94)
$$\alpha_0(u,v) = \alpha(u,0), \qquad \beta_0(u,v) = \beta(0,v),$$

the right hand sides being given by the initial data on C^- , C^+ respectively. We then define $t_0(u, v)$ to be the solution of (see (93))

(95)
$$\frac{\partial^2 t_0}{\partial u \partial v} + K_0 \nu_0 - L_0 \mu_0 = 0,$$

together with the initial data t(u, 0) and t(0, v) = v. We then define (see (7b))

(96)
$$r_0(u,v) = r(0,v) + \int_0^u (\mu_0 c_{-0})(u',v) du'.$$

Then, given the iterate (α_n, β_n) we define the next iterate $(\alpha_{n+1}, \beta_{n+1})$ according to the following. We define t_n to be the solution of (95) with n in the role of 0. Then we define r_n to be the solution of (96) with n in the role of 0. Then we find the next iterate $(\alpha_{n+1}, \beta_{n+1})$ according to (see (89), (90))

(97)
$$\alpha_{n+1}(u,v) = \alpha(u,0) + \int_0^v (\nu_n F_n)(u,v') dv',$$

(98)
$$\beta_{n+1}(u,v) = \beta(0,v) + \int_0^u (\mu_n F_n)(u',v) du',$$

where

(99)
$$F_n = -\frac{\eta(\alpha_n, \beta_n)}{r_n} (\alpha_n - \beta_n).$$

In the following we are going to work with the differences

(100)
$$\alpha'_n(u,v) \stackrel{\text{def}}{=} \alpha_n(u,v) - \alpha(u,0),$$

(101)
$$\beta'_n(u,v) \stackrel{\text{def}}{=} \beta_n(u,v) - \beta(0,v).$$

We note that $\alpha'_n(u,0) = \beta'_n(0,v) = 0$. Let (for the definition of Ω_l see (63))

(102)
$$\overline{F} \stackrel{\text{def}}{=} \sup_{\Omega_l} |F|.$$

Lemma 3. Consider the closed set C in the space $C^1(\Pi_{h\varepsilon}, \mathbb{R}^2)$ of continuously differentiable maps $(u, v) \mapsto (\alpha', \beta')(u, v)$ of $\Pi_{h\varepsilon}$ into \mathbb{R}^2 defined by the conditions

(103)
$$\alpha'(u,0) = \beta'(0,v) = 0$$

and the inequalities

(104)
$$\left|\frac{\partial \alpha'}{\partial u}\right| \le G - g_0, \qquad \left|\frac{\partial \alpha'}{\partial v}\right| \le l\overline{F}, \qquad \left|\frac{\partial \beta'}{\partial u}\right| \le M\overline{F}, \qquad \left|\frac{\partial \beta'}{\partial v}\right| \le D - d_0.$$

If h and ε are sufficiently small, then the sequence $((\alpha'_n, \beta'_n); n = 0, 1, 2, ...)$ is contained in C.

The following proof is given in some detail in order to arrive at the explicit smallness conditions on h which will then be used again in the bootstrap argument where we continue the solution (see below).

Proof. Let $(\alpha'_n, \beta'_n) \in \mathcal{C}$. We have

(105)
$$\begin{aligned} |\alpha_n(u,v)| &\leq |\alpha(u,0)| + \int_0^v \left| \frac{\partial \alpha'_n}{\partial v} \right| (u,v') dv' \\ &\leq \sup_{u \in [0,h]} |\alpha(u,0)| + v l \overline{F}, \end{aligned}$$

(106)
$$\begin{aligned} |\beta_n(u,v)| &\leq |\beta(0,v)| + \int_0^u \left| \frac{\partial \beta'_n}{\partial u} \right| (u',v) du' \\ &\leq b_0 + uM\overline{F}. \end{aligned}$$

On the other hand

(107)
$$\begin{aligned} |\alpha_n(u,v)| &\leq |\alpha(0,v)| + \int_0^u \left| \frac{\partial \alpha'_n}{\partial u}(u',v) + \frac{\partial \alpha}{\partial u}(u',0) \right| du' \\ &\leq a_0 + uG, \end{aligned}$$

(108)
$$\begin{aligned} |\beta_n(u,v)| &\leq |\beta(u,0)| + \int_0^v \left| \frac{\partial \beta'_n}{\partial v}(u,v') + \frac{\partial \beta}{\partial v}(0,v') \right| dv' \\ &\leq \sup_{u \in [0,h]} |\beta(u,0)| + vD. \end{aligned}$$

Therefore, if we choose ε and h sufficiently small such that

(109)
$$\varepsilon \leq \min\left\{\frac{B - \sup_{u \in [0,h]} |\beta(u,0)|}{D}, \frac{A - \sup_{u \in [0,h]} |\alpha(u,0)|}{l\overline{F}}\right\}$$

(110)
$$h \le (l-1) \min\left\{\frac{a_0}{G}, \frac{b_0}{M\overline{F}}\right\},\$$

we have that $(\alpha_n, \beta_n) \in R_l$.

For the following discussion of μ and ν we omit the index *n* since it would be the only index appearing. We consider (95) with *n* in the role of 0. Integrating with

respect to v and u we obtain (recall that $\nu(0, v) = 1$)

(111)
$$\mu(u,v) = \mu(u,0)e^{\int_0^v L(u,v')dv'} - \int_0^v e^{\int_{v'}^v L(u,v'')dv''} (K\nu) (u,v')dv',$$

(112)
$$\nu(u,v) = e^{-\int_0^u K(u',v)du'} + \int_0^u e^{-\int_{u'}^u K(u'',v)du''} (L\mu) (u',v)du'.$$

We define

(113)
$$C_{\pm\alpha} \stackrel{\text{def}}{=} \sup_{R_l} \left| \frac{\partial c_{\pm}}{\partial \alpha} \right|, \quad C_{\pm\beta} \stackrel{\text{def}}{=} \sup_{R_l} \left| \frac{\partial c_{\pm}}{\partial \beta} \right|, \quad C_{+-} \stackrel{\text{def}}{=} \sup_{R_l} \frac{1}{c_{+} - c_{-}}.$$

We have

(114)

$$|L| = \frac{1}{c_{+} - c_{-}} \left| \frac{\partial c_{-}}{\partial v} \right|$$

$$\leq C_{+-} \left(C_{-\alpha} l \overline{F} + C_{-\beta} D \right) \stackrel{\text{def}}{=} \overline{L},$$

$$|K| = \frac{1}{c_{+} - c_{-}} \left| \frac{\partial c_{+}}{\partial u} \right|$$

$$\leq C_{+-} \left(C_{+\alpha} G + C_{+\beta} M \overline{F} \right) \stackrel{\text{def}}{=} \overline{K}.$$

Therefore,

(116)
$$|\mu(u,v)| \le e^{v\overline{L}} \left(m_0 + \overline{K} \int_0^v |\nu|(u,v')dv' \right),$$

(117)
$$|\nu(u,v)| \le e^{uK} \left(1 + \overline{L} \int_0^{\cdot} |\mu|(u',v)du'\right).$$

Substituting (116) into (117) we obtain

(118)
$$T(u,v) \le f_1 + f_2 \int_0^v T(u,v') dv',$$

where

(119)
$$T(u,v) \stackrel{\text{def}}{=} \sup_{u' \in [0,u]} |\nu(u',v)|,$$

(120)
$$f_1(u,v) \stackrel{\text{def}}{=} e^{u\overline{K}} \left(1 + u\overline{L}e^{v\overline{L}}M \right), \qquad f_2(u,v) \stackrel{\text{def}}{=} u\overline{KL}e^{u\overline{K}+v\overline{L}}.$$

Defining

(121)
$$\Sigma_u(v) \stackrel{\text{def}}{=} \int_0^v T(u, v') dv',$$

(118) yields

(122)
$$\frac{d\Sigma_u}{dv}(v) \le f_1(u,v) + f_2(u,v)\Sigma_u(v),$$

which implies

(123)
$$\Sigma_u(v) \le \int_0^v f_1(u, v') e^{\int_{v'}^v f_2(u, v'') dv''} dv'.$$

Using this in (118) yields (putting back the index n)

(124)
$$|\nu_n(u,v)| \le f_1(u,v) + f_2(u,v) \int_0^v f_1(u,v') e^{\int_{v'}^v f_2(u,v'')dv''} dv' \stackrel{\text{def}}{=} F_1(u,v).$$

Substituting this into (116) gives

(125)
$$|\mu_n(u,v)| \le e^{v\overline{L}} \left\{ m_0 + \overline{K} \int_0^v F_1(u,v') dv' \right\} \stackrel{\text{def}}{=} F_2(u,v).$$

We note that

(126)
$$\lim_{u \to 0} F_1 = 1, \qquad \lim_{v \to 0} F_2 = m_0.$$

Now we choose ε , h sufficiently small such that (recall that $m_0 < M, 1 < l$)

(127a,b)
$$F_2 \le M, \quad F_1 \le l$$

respectively, which implies

(128)
$$|\nu_n(u,v)| \le 1, \qquad |\mu_n(u,v)| \le M.$$

Now we look at (96) with n in the role of 0 which is

(129)
$$r_n(u,v) = r(u,0) + \int_0^v (\nu_n c_{+n})(u,v') dv'.$$

This together with (95) with n in the role of 0 gives

(130)
$$r_n(u,v) = r(0,v) + \int_0^u (\mu_n c_{-n})(u',v) du'.$$

Let

(131)
$$c_{\pm}^{\dagger} \stackrel{\text{def}}{=} \sup_{R_l} |c_{\pm}|.$$

If we choose ε and h such that

(132)
$$\varepsilon \leq \frac{1}{c_{+}^{\dagger}l} \min\left\{ \inf_{u \in [0,h]} r(u,0) - \frac{1}{2}r_{m} , \frac{3}{2}r_{M} - \sup_{u \in [0,h]} r(u,0) \right\},$$
(133)
$$h \leq \frac{r_{m}}{2c^{\dagger}M},$$

we obtain

(134)

(135)

$$\frac{1}{2}r_m \le r_n \le \frac{3}{2}r_M,$$

hence $(\alpha_n, \beta_n, r_n) \in \Omega_l$, which implies

$$|F_n| \leq \overline{F}.$$

In view of (97), (98), we have

(136)
$$\left|\frac{\partial \alpha'_{n+1}}{\partial v}\right| \le l\overline{F}, \qquad \left|\frac{\partial \beta'_{n+1}}{\partial u}\right| \le M\overline{F},$$

respectively. From (97), (98) we have

(137)
$$\frac{\partial \alpha'_{n+1}}{\partial u}(u,v) = \int_0^v \left(\nu_n \frac{\partial F_n}{\partial u} + \frac{\partial^2 t_n}{\partial u \partial v} F_n\right)(u,v') dv',$$

(138)
$$\frac{\partial \beta'_{n+1}}{\partial v}(u,v) = \int_0^u \left(\mu_n \frac{\partial F_n}{\partial v} + \frac{\partial^2 t_n}{\partial u \partial v} F_n\right)(u',v) du'.$$

In view of (114), (115), (128) we see that

(139)
$$\left|\frac{\partial^2 t}{\partial u \partial v}\right| \le M\overline{L} + l\overline{K}.$$

Let

(140)
$$F_f \stackrel{\text{def}}{=} \sup_{\Omega_l} \left| \frac{\partial F}{\partial f} \right| : f \in \{\alpha, \beta, r\}.$$

We have

$$(141) \left| \nu_n \frac{\partial F_n}{\partial u} + \frac{\partial^2 t_n}{\partial u \partial v} F_n \right| \leq l \left(F_\alpha G + F_\beta M \overline{F} + F_r c_-^{\dagger} M \right) + \overline{F} (M \overline{L} + l \overline{K}) \stackrel{\text{def}}{=} H_1,$$

$$(142) \left| \mu_n \frac{\partial F_n}{\partial v} + \frac{\partial^2 t_n}{\partial u \partial v} F_n \right| \leq M \left(F_\alpha l \overline{F} + F_\beta D + F_r c_+^{\dagger} l \right) + \overline{F} (M \overline{L} + l \overline{K}) \stackrel{\text{def}}{=} H_2.$$

Choosing ε , h sufficiently small such that (recall that $G > g_0$)

(143a,b)
$$\varepsilon \leq \frac{G-g_0}{H_1}, \qquad h \leq \frac{D-d_0}{H_2},$$

respectively, we obtain

(144)
$$\left|\frac{\partial \alpha'_{n+1}}{\partial u}\right| \le G - g_0, \quad \left|\frac{\partial \beta'_{n+1}}{\partial v}\right| \le D - d_0.$$

In view of (136), (144) the proof is complete.

Lemma 4. Let the hypotheses of lemma 3 be satisfied. If h and ε are sufficiently small, depending on A, B, D, G, M, r_m , r_M , l then the sequence $((\alpha'_n, \beta'_n); n = 0, 1, 2, ...)$ is a contractive sequence in the space $C^1(\Pi_{h\varepsilon}, \mathbb{R}^2)$.

Proof. We use the definition

(145)
$$\Delta_n f \stackrel{\text{def}}{=} f_n - f_{n-1}.$$

Let

(146)
$$\Lambda \stackrel{\text{def}}{=} \max \left\{ \sup_{\Pi_{h\varepsilon}} \left| \frac{\partial \Delta_n \alpha'}{\partial u} \right|, \sup_{\Pi_{h\varepsilon}} \left| \frac{\partial \Delta_n \alpha'}{\partial v} \right|, \sup_{\Pi_{h\varepsilon}} \left| \frac{\partial \Delta_n \beta'}{\partial u} \right| + \sup_{\Pi_{h\varepsilon}} \left| \frac{\partial \Delta_n \beta'}{\partial v} \right| \right\}.$$

In the following we denote by C a constant depending on A, B, D, G, M, r_m , r_M , l. From (97), (98) we have

(147)
$$\left|\frac{\partial \Delta_{n+1} \alpha'}{\partial v}\right| \le C(|\Delta_n \nu| + |\Delta_n \alpha| + |\Delta_n \beta| + |\Delta_n r|),$$

(148)
$$\left|\frac{\partial \Delta_{n+1}\beta'}{\partial u}\right| \le C(|\Delta_n\mu| + |\Delta_n\alpha| + |\Delta_n\beta| + |\Delta_nr|)$$

For the differences $\Delta_n \alpha$, $\Delta_n \beta$ we have

(149)
$$|\Delta_n \alpha| \le v \sup_{\Pi_{h\varepsilon}} \left| \frac{\partial \Delta_n \alpha'}{\partial v} \right|, \qquad |\Delta_n \beta| \le u \sup_{\Pi_{h\varepsilon}} \left| \frac{\partial \Delta_n \beta'}{\partial u} \right|.$$

For the difference $\Delta_n r$ we use (96) with n and n-1 in the role of 0. We obtain

(150)
$$\Delta_n r = \int_0^u (\mu_n \Delta_n c_- + c_{-,n-1} \Delta_n \mu) (u', v) du'.$$

In view of (149), we have

(151)
$$|\Delta_n c_{\pm}| \le C \left(v \sup_{\Pi_{h\varepsilon}} \left| \frac{\partial \Delta_n \alpha'}{\partial v} \right| + u \sup_{\Pi_{h\varepsilon}} \left| \frac{\partial \Delta_n \beta'}{\partial u} \right| \right).$$

Now, from (95) with n and n-1 in the role of 0 we obtain

(152)
$$\frac{\partial^2 \Delta_n t}{\partial u \partial v} + K_n \Delta_n \nu - L_n \Delta_n \mu = \Xi_n,$$

where

(153)
$$\Xi_n \stackrel{\text{def}}{=} \mu_{n-1} \Delta_n L - \nu_{n-1} \Delta_n K.$$

Making use of (151) we get

$$\begin{aligned} |\Delta_{n}K| &\leq \left| \Delta_{n} \left(\frac{1}{c_{+} - c_{-}} \frac{\partial c_{+}}{\partial u} \right) \right| \\ &\leq C \left\{ v \sup_{\Pi_{h\varepsilon}} \left| \frac{\partial \Delta_{n} \alpha'}{\partial v} \right| + u \sup_{\Pi_{h\varepsilon}} \left| \frac{\partial \Delta_{n} \beta'}{\partial u} \right| + \left| \frac{\partial \Delta_{n} \alpha'}{\partial u} \right| + \left| \frac{\partial \Delta_{n} \beta'}{\partial u} \right| \right\} \\ (154) &\leq C\Lambda, \\ |\Delta_{n}L| &\leq \left| \Delta_{n} \left(\frac{1}{c_{+} - c_{-}} \frac{\partial c_{-}}{\partial v} \right) \right| \\ &\leq C \left\{ v \sup_{\Pi_{h\varepsilon}} \left| \frac{\partial \Delta_{n} \alpha'}{\partial v} \right| + u \sup_{\Pi_{h\varepsilon}} \left| \frac{\partial \Delta_{n} \beta'}{\partial u} \right| + \left| \frac{\partial \Delta_{n} \alpha'}{\partial v} \right| + \left| \frac{\partial \Delta_{n} \beta'}{\partial v} \right| \right\} \\ (155) &\leq C\Lambda. \end{aligned}$$

Therefore,

$$(156) |\Xi_n| \le C\Lambda.$$

Integrating (152) yields

(157)
$$\Delta_n \mu(u, v) = -\int_0^v e^{\int_{v'}^v L_n(u, v'') dv''} \left(K_n \Delta_n \nu - \Xi_n \right) (u, v') dv',$$

(158)
$$\Delta_n \nu(u,v) = \int_0^u e^{-\int_{u'}^u K_n(u'',v)du''} \left(L_n \Delta_n \mu + \Xi_n\right) (u',v)du'$$

Substituting (157) into (158) and following a similar procedure as was carried out in the prove of the previous lemma, we arrive at

(159)
$$|\Delta_n \nu| \le C u \Lambda,$$

(160)
$$|\Delta_n \mu| \le C v \Lambda.$$

Using this in (150) we obtain

(161)
$$|\Delta_n r| \le Cu(u+v)\Lambda.$$

Therefore,

(162)
$$\left| \frac{\partial \Delta_{n+1} \alpha'}{\partial v} \right|, \left| \frac{\partial \Delta_{n+1} \beta'}{\partial u} \right| \le C(u+v)\Lambda.$$

To estimate $|\partial \Delta_n \alpha' / \partial u|$ and $|\partial \Delta_n \beta' / \partial v|$ we use (137), (138). To estimate the mixed derivative on the right hand side of the resulting difference, we observe that from (152), (156), (159), (160) we get

(163)
$$\left|\frac{\partial^2 \Delta_n t}{\partial u \partial v}\right| \le C\Lambda.$$

We obtain

(164)
$$\left|\frac{\partial \Delta_{n+1} \alpha'}{\partial u}\right|, \left|\frac{\partial \Delta_{n+1} \beta'}{\partial v}\right| \le C(u+v)\Lambda.$$

In view of (162), (164) and recalling that the constants in these equations depend on $A, B, D, G, M, r_m, r_M, l$, we see that for sufficiently small h and ε , depending on $A, B, D, G, M, r_m, r_M, l$, the sequence contracts in $C^1(\Pi_{h\varepsilon}, \mathbb{R}^2)$.

The two lemmas above show that the sequence (α'_n, β'_n) converges to $(\alpha', \beta') \in \mathcal{C}$ uniformly in $\Pi_{h\varepsilon}$. Therefore we also have uniform convergence of (α_n, β_n) to $(\alpha, \beta) \in C^1(\Pi_{h\varepsilon})$. Now, (159), (160) show the convergence of the derivatives of t_n . Therefore, the pair of integral equations (111), (112) are satisfied in the limit. We denote by t the limit of (t_n) . It then follows that the mixed derivative $\partial^2 t / \partial u \partial v$ satisfies (93). In view of the Hodograph system (7a,b) the partial derivatives of r_n converge uniformly in $\Pi_{h\varepsilon}$ and the limit satisfies the Hodograph system. Let us denote by r the limit of (r_n) . We have thus found a solution of the characteristic initial value problem in $\Pi_{h\varepsilon}$. We note that the solution satisfies the bounds (87), (88).

The two previous lemmas establish the following proposition.

Proposition 2. Let us be given data on C^+ of size a_0 , b_0 , d_0 , r_m , r_M according to (56), (57). Let us be given a constant l > 1. Let us be given data on C^- which agrees with the data on C^+ at (u, v) = (0, 0) and is of size m_0 , g_0 according to (61). Then, for h and ε sufficiently small, depending on the size of the initial data, we have existence of a C^1 solution of the characteristic system in $\Pi_{h\varepsilon}$ which satisfies the bounds (87), (88).

We supplement this proposition with the following uniqueness result.

Proposition 3. Let $(\alpha_1, \beta_1, t_1, r_1)$ and $(\alpha_2, \beta_2, t_2, r_2)$, both in $C^1(\Pi_{h\varepsilon})$, be two solutions of the characteristic system corresponding to the same initial data. Then, for h, ε sufficiently small, depending on the size of the initial data, the two solutions coincide.

Using similar estimates as in the convergence proof above, the proof is straightforward.

4.2. Extension to a Strip. Now we show that the solution is actually given in Π_{hv^*} , i.e. we have existence of a unique continuously differentiable solution in a region adjacent to C^+ which extends over the full domain of the initial data and is of thickness h, where h depends on the size of the initial data in the way given by the above propositions.

Lemma 5. The solution in $\Pi_{h\varepsilon}$ can be continued to Π_{hv^*} .

Proof. As a consequence of the results above, the solution satisfies the following bounds in $\Pi_{h\varepsilon}$:

 $(\mathrm{BA}\sharp) \qquad |\nu| \leq l, \qquad |\alpha| \leq A, \qquad |\beta| \leq B, \qquad |\delta| \leq D, \qquad \tfrac{1}{2}r_m \leq r \leq \tfrac{3}{2}r_M.$ Let

(165)

 $v_2 \stackrel{\text{def}}{=} \sup \left\{ v \in (0, v^*) : (BA\sharp) \text{ holds and the solution is unique for } (u, v) \in \Pi_{hv} \right\}.$

Let us assume $v_2 < v^*$. Since the solution is continuously differentiable it is also unique at $v = v_2$. In the following all statements hold within Π_{hv_2} . From lemma 2 we have

$$(166) \qquad \qquad |\gamma| < G, \qquad |\mu| < M.$$

From now on we will make use of $(BA\sharp)$, (166) without further notice. We look at

(167)
$$\beta(u,v) = \beta(0,v) + \int_0^u (\mu F)(u',v) du'$$

We have

(168)
$$|\beta(u,v)| < b_0 + hM\overline{F} \\ \leq b_0 + (l-1)b_0 = lb_0 = B,$$

where for the second inequality we used (110). We look at

(169)
$$r(u,v) = r(0,v) + \int_0^u (c_-\mu)(u',v)du'$$

We have

(170)
$$r(u,v) < r_M + hc_{\perp}^{\top}M$$
$$\leq r_M + \frac{1}{2}r_m \leq \frac{3}{2}r_M,$$

where for the second inequality we used (133). Similarly we find $r(u,v) > \frac{1}{2}r_m$. Therefore,

L.

$$\frac{1}{2}r_m < r < \frac{3}{2}r_M$$

We have

(172)
$$|L| = \frac{1}{c_+ - c_-} \left| \frac{\partial c_-}{\partial v} \right| \le C_{+-} \left(C_{-\alpha} l \overline{F} + C_{-\beta} D \right) = \overline{L},$$

(173)
$$|K| = \frac{1}{c_+ - c_-} \left| \frac{\partial c_+}{\partial u} \right| < C_{+-} \left(C_{+\alpha} G + C_{+\beta} M \overline{F} \right) = \overline{K}$$

For the definition of \overline{L} , \overline{K} see (114), (115) respectively. We deduce that there exists a constant \overline{K}' such that

$$|K| \le \overline{K}' < \overline{K}.$$

Therefore, the inequalities (116), (117) hold for μ , ν with \overline{K}' in the role of \overline{K} . This then implies that (124) holds with \overline{K}' in the role of \overline{K} , i.e.

(175)
$$|\nu(u,v)| \le F_1'(u,v)$$

where by $F'_1(u, v)$ we denote $F_1(u, v)$ with \overline{K}' in the role of \overline{K} . Since

(176)
$$F_1'(u,v) < F_1(u,v)$$
$$\leq F_1(h,v).$$

together with (127b), we obtain

$$(177) \qquad \qquad |\nu(u,v)| < l.$$

Using (172), (173) together with (166) we obtain

(178)
$$\left| \frac{\partial^2 t}{\partial u \partial v}(u, v) \right| < M\overline{L} + l\overline{K}.$$

We also have

(179)
$$\left| \mu \frac{\partial F}{\partial v} \right| < M \left(F_{\alpha} l \overline{F} + F_{\beta} D + F_{r} c_{+}^{\dagger} l \right).$$

Taking the derivative of (90) with respect to v and using the previous two estimates we obtain (for the definition of H_2 see (142))

(180)
$$\left| \mu \frac{\partial F}{\partial v} + \frac{\partial^2 t}{\partial u \partial v} F \right| < H_2$$

Therefore,

$$\begin{aligned} |\delta(u,v)| &< d_0 + hH_2 \\ (181) &\leq D, \end{aligned}$$

where for the last inequality we used (143b). Finally we consider

(182)
$$\alpha(u,v) = \alpha(0,v) + \int_0^u \gamma(u',v) du'.$$

We have

$$|\alpha(u,v)| < a_0 + hG$$
(183) $\leq A,$

where we used (110). We have therefore established in $\Pi_{h\varepsilon}$:

(BA)
$$|\nu| < l, \quad |\alpha| < A, \quad |\beta| < B, \quad |\delta| < D, \quad \frac{1}{2}r_m < r < \frac{3}{2}r_M,$$

i.e. we have improved (BA \sharp). Therefore, using the result from above, we can solve an initial value problem with corner at $(u, v) = (0, v_2)$, thus establishing a unique solution in $\Pi_{h(v_2+\varepsilon)}$, for some $\varepsilon < v^* - v_2$ which satisfies the bounds (BA \sharp). This implies $v_2 = v^*$.

4.3. Higher Regularity. We now establish uniform bounds in Π_{hv^*} for the partial derivatives of α , β , t and r to arbitrary order.

Lemma 6. The partial derivatives of α , β , t, r to all order are, in absolute value, uniformly bounded in Π_{hv^*} .

Proof. We establish such bounds by induction. Let (P_{n-1}) be the proposition

$$(P_{n-1}) \qquad \qquad \sup_{\Pi_{hv^*}} \left| \frac{\partial^{n-1}}{\partial u^i \partial v^j} (\alpha, \beta, t, r) \right| \le C, \quad i+j=n-1.$$

We have already established proposition (P_1) . Suppose that proposition (P_k) holds for k = 1, ..., n - 1. In the following we denote by F_i a function in Π_{hv^*} which involves only (n - 1)'th order derivatives of α , β , t, r and which is, therefore, uniformly bounded. Functions carrying the same index can change from line to line. We first deal with the mixed derivatives of order n. Let $1 \le i, j \le n - 1$, i + j = n. In view of (93) we have

(184)
$$\frac{\partial^n t}{\partial u^i \partial v^j} = \frac{\partial^{n-2}}{\partial u^{i-1} \partial v^{j-1}} (L\mu - K\nu).$$

Because the right hand side involves only derivatives of order n-1 and lower, by the inductive hypothesis, it is bounded. Therefore

(185)
$$\left|\frac{\partial^n t}{\partial u^i \partial v^j}\right| \le C.$$

In view of (6a) we have

(186)
$$\frac{\partial^{n}\alpha}{\partial u^{i}\partial v^{j}} = \frac{\partial^{n-1}}{\partial u^{i}\partial v^{j-1}} \left(\frac{\partial\alpha}{\partial v}\right) = \frac{\partial^{n-1}}{\partial u^{i}\partial v^{j-1}} \left(\nu F\right)$$
$$= F_{1}\frac{\partial^{n}t}{\partial u^{i}\partial v^{j}} + F_{2}.$$

In view of (6b), the same holds for $\partial^n\beta/\partial u^i\partial v^j$. Therefore, together with (185), we obtain

(187)
$$\left|\frac{\partial^{n}\alpha}{\partial u^{i}\partial v^{j}}\right|, \left|\frac{\partial^{n}\beta}{\partial u^{i}\partial v^{j}}\right| \leq C.$$

We turn to the pure derivatives of order n. By (6a,b) we have

(188)
$$\frac{\partial^n \alpha}{\partial v^n} = \frac{\partial^{n-1}}{\partial v^{n-1}} (\nu F) = F_1 \frac{\partial^n t}{\partial v^n} + F_2,$$

(189)
$$\frac{\partial^n \beta}{\partial u^n} = \frac{\partial^{n-1}}{\partial v^{n-1}} (\mu F) = F_1 \frac{\partial^n t}{\partial u^n} + F_2,$$

while from (89) we have

$$\frac{\partial^{n} \alpha}{\partial u^{n}}(u,v) = \frac{\partial^{n} \alpha}{\partial u^{n}}(u,0) + \int_{0}^{v} \left(\frac{\partial^{n}}{\partial u^{n}}(\nu F)\right)(u,v')dv'$$

$$(190) = F_{1}(u,v) + \int_{0}^{v} \left(F_{2}\frac{\partial^{n+1}t}{\partial u^{n}\partial v} + F_{3}\frac{\partial^{n}F}{\partial u^{n}} + F_{4}\frac{\partial^{n}t}{\partial u^{n-1}\partial v}\right)(u,v')dv'.$$

The mixed derivative of t of order n is taken care of by (185). For the mixed derivative of order n + 1 of t we look at

(191)
$$\frac{\partial^{n+1}t}{\partial u^n \partial v} = \frac{\partial^{n-1}}{\partial u^{n-1}} (L\mu - K\nu)$$
$$= F_1 + F_2 \frac{\partial^n t}{\partial u^n} + F_3 \frac{\partial^n \alpha}{\partial u^n} + F_4 \frac{\partial^n \beta}{\partial u^n},$$

where for the second equality we use that we already have bounds for the mixed derivatives to order n of α , β , t. Using also (189) we arrive at

(192)
$$\frac{\partial}{\partial v} \left(\frac{\partial^n t}{\partial u^n} \right) = F_1 + F_2 \frac{\partial^n t}{\partial u^n} + F_3 \frac{\partial^n \alpha}{\partial u^n}.$$

This implies

(193)
$$\frac{\partial^n t}{\partial u^n}(u,v) = F_1(u,v) + \int_0^v \left(F_2 \frac{\partial^n \alpha}{\partial u^n}\right)(u,v') dv'.$$

Using now

(194)
$$\frac{\partial^n F}{\partial u^n} = F_1 + F_2 \frac{\partial^n \alpha}{\partial u^n} + F_3 \frac{\partial^n \beta}{\partial u^n} + F_4 \frac{\partial^n t}{\partial u^n}$$

together with (191), (193) in (190) we obtain

(195)
$$\frac{\partial^n \alpha}{\partial u^n}(u,v) = F_1(u,v) + \int_0^v \left(F_2 \frac{\partial^n \alpha}{\partial u^n} + F_3 \frac{\partial^n t}{\partial u^n}\right)(u,v')dv'$$

This together with (193) yields the following system of inequalities

(196)
$$\left|\frac{\partial^{n}\alpha}{\partial u^{n}}(u,v)\right| \leq C + C' \int_{0}^{v} \left(\left|\frac{\partial^{n}\alpha}{\partial u^{n}}\right| + \left|\frac{\partial^{n}t}{\partial u^{n}}\right|\right)(u,v')dv',$$

(197)
$$\left|\frac{\partial^{n}t}{\partial u^{n}}(u,v)\right| \leq C + C' \int_{0}^{v} \left|\frac{\partial^{n}\alpha}{\partial u^{n}}\right|(u,v')dv',$$

which implies

(198)
$$\left|\frac{\partial^{n}\alpha}{\partial u^{n}}(u,v)\right|, \left|\frac{\partial^{n}t}{\partial u^{n}}(u,v)\right| \leq C.$$

Similarly we obtain

(199)
$$\left|\frac{\partial^{n}\beta}{\partial v^{n}}(u,v)\right|, \left|\frac{\partial^{n}t}{\partial v^{n}}(u,v)\right| \leq C.$$

In view of (188), (189) these imply

(200)
$$\left| \frac{\partial^n \beta}{\partial u^n}(u,v) \right|, \left| \frac{\partial^n \alpha}{\partial v^n}(u,v) \right| \le C.$$

These together with (185), (187) imply proposition (P_n) .

The above existence, uniqueness, continuation and regularity result can be carried out for a region adjacent to C^- as well. Together with the solution of the constraint equations from the previous section we arrive at the following result.

Theorem 1. Let $r_0 > 0$ and let us be given free data $\beta^+, t^+ \in C^{\infty}[0, v^*], \alpha^-, t^- \in C^{\infty}[0, u^*]$ such that $t_+(0) = t^-(0)$. Then, for h', h'' sufficiently small depending on the size of the data, there exists a unique smooth solution α , β , t, r of the characteristic system of equations for $(u, v) \in \Pi_{u^*h'} \cup \Pi_{h''v^*}$, where

(201)
$$\Pi_{ab} \stackrel{def}{=} \{(u, v) \in \mathbb{R}^2 : 0 \le u \le a, 0 \le v \le b\},\$$

such that

(202)
$$r(0,0) = r_0, \quad t(0,v) = t^+(v), \quad t(u,0) = t^-(u),$$

(203) $\beta(0,v) = \beta^-(v), \quad \alpha(u,0) = \alpha^-(u)$



FIGURE 2. The domain $\Pi_{u^*h'} \cup \Pi_{h''v^*}$

4.4. Solution in the *t*-*r*-plane. We consider the map $(u, v) \mapsto (t, r)$. In view of

(204)
$$\begin{vmatrix} \frac{\partial t}{\partial u} & (\frac{\partial t}{\partial u})c_{-} \\ \frac{\partial t}{\partial v} & (\frac{\partial t}{\partial v})c_{+} \end{vmatrix} = \begin{vmatrix} \mu & \mu c_{-} \\ \nu & \nu c_{+} \end{vmatrix}$$
$$= 2\mu\nu\eta,$$

a solution of the characteristic system corresponds to a solution in the $t\mathchar}r\mathchar$ long as

(205)
$$\mu, \nu > 0.$$

We recall that $\mu(u, 0) = \nu(0, v) = 1$. Let us assume that $\mu(0, v), \nu(u, 0) > 0$. Since we have bounds on the second derivatives of t we deduce that for h', h'' sufficiently small, the solution in $\Pi_{u^*h'} \cup \Pi_{h''v^*}$ corresponds to a solution in the t-r-plane.

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Department of Mathematics, Princeton University $E\text{-}mail\ address:$ lisibach@princeton.edu