### Atmospheric Quantum Channels with Weak and Strong Turbulence

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The free-space transfer of high-fidelity optical signals between remote locations has many applications, including both classical and quantum communication, precision navigation, clock synchronization, etc. The physical processes that contribute to signal fading and loss need to be carefully analyzed in the theory of light propagation through the atmospheric turbulence. Here we derive the probability distribution for the atmospheric transmittance including beam wandering, beam shape deformation, and beam-broadening effects. Our model, referred to as the elliptic-beam approximation, applies to weak, weak-to-moderate, and strong turbulence and hence to the most important regimes in atmospheric communication scenarios.

Introduction.– The transmission of quantum light to remote receivers recently attracted great interest in connection with the implementation of quantum communication protocols over large distances. Experimental advances in this field allowed one to demonstrate the successful quantum-light transmission over horizontal communication links [\[1](#page-4-0)[–7\]](#page-4-1) and paved the way for the realization of ground-to-satellite quantum communication [\[8–](#page-4-2) [10\]](#page-4-3). The main obstacle for the transmission of quantum light in free space is the atmospheric turbulence, which leads to spatial and temporal variations of the refractive index of the channel. The transmitted signal is usually measured by detectors with a finite aperture. Typically, the recorded data are contaminated by fluctuating losses due to beam wandering, beam broadening, scintillation, and degradation of coherence.

The theory of classical light propagation through the atmosphere is well developed [\[11](#page-4-4)[–16\]](#page-4-5). Some progress was also achieved in the theory of free-space propagation of quantum light [\[17–](#page-4-6)[24\]](#page-4-7). The atmosphere is considered as a quantum channel characterized by fluctuating transmission properties. In terms of the Glauber-Sudarshan P function  $[25-27]$  $[25-27]$ , which is a quasiprobability as it may attain negativities, the relation between input  $P_{in}(\alpha)$  and output  $P_{\text{out}}(\alpha)$  states can be written as [\[22,](#page-4-10) [24\]](#page-4-7)

$$
P_{\text{out}}(\alpha) = \int_{0}^{1} d\eta \, \mathcal{P}(\eta) \frac{1}{\eta} P_{\text{in}}\left(\frac{\alpha}{\sqrt{\eta}}\right). \tag{1}
$$

Here  $\mathcal{P}(\eta)$  is the probability distribution of the transmittance (PDT),  $\eta$  being the intensity transmittance. Hence, the description of quantum-light propagation through the turbulent atmosphere merely reduces to identifying this probability distribution. In Ref. [\[24\]](#page-4-7) we have derived the PDT for the case when the leading effect of fluctuating losses in the atmosphere is beam wandering, as it is the case for weak turbulence.

In this Letter, we present a substantially extended model of PDT, based on the elliptic-beam approximation that incorporates effects of beam wandering, broadening, and deformation. Our theory properly describes atmospheric quantum channels in the limits of relatively weak turbulence, as in experiments in Erlangen with an atmospheric link of 1.6 km length [\[7,](#page-4-1) [28\]](#page-4-11). For the case of strong turbulence, our theory also yields a reasonable agreement with the log-normal model [\[13–](#page-4-12)[20\]](#page-4-13), which has been verified in experiments on the Canary Islands [\[4\]](#page-4-14). Most importantly, our elliptic-beam model overcomes the deficiency of physical inconsistencies inherent in the lognormal distribution.

The aperture transmittance.– Temporal and spatial variations of temperature and pressure in atmospheric turbulent flows cause random fluctuations of the refraction index of the air. Consequently, the atmosphere acts as a source of losses for transmitted photons which are measured at the receiver by a detection module with a finite aperture. The transmitted signal is degraded by effects like beam wandering, broadening, deformation, and others. Let us consider a Gaussian beam that propagates along the z axis onto the aperture plane at distance  $z = L$ . In general, the fluctuating intensity transmittance of such a signal is given by [\[24\]](#page-4-7)

<span id="page-0-1"></span>
$$
\eta = \int_{\mathcal{A}} d^2 \mathbf{r} \, I(\mathbf{r}; L), \tag{2}
$$

where  $A$  is the aperture area and  $I(\mathbf{r};L)$  is the normalized intensity with respect to the full  $\mathbf{r} = \{x, y\}$  plane.

<span id="page-0-2"></span>The Gaussian beam underlies turbulent disturbances along the propagation path. Within our model we assume that these disturbances lead to beam wandering and deformation of the beam profile into an elliptical form. This is justified for weak turbulence, when speckles play no essential role. For strong turbulence the beam shape is the result of many small spatially averaged distortions. The intensity of the elliptic beam at the aperture plane is given by

<span id="page-0-0"></span>
$$
I(\mathbf{r};L) = \frac{2}{\pi\sqrt{\det \mathbf{S}}} \exp\left[-2(\mathbf{r}-\mathbf{r}_0)^{\mathrm{T}} \mathbf{S}^{-1}(\mathbf{r}-\mathbf{r}_0)\right],\quad(3)
$$

with  $\mathbf{r} = (x \ y)^T$ . It is characterized by the beam-centroid position  $\mathbf{r}_0 = (x_0 \ y_0)^{\mathrm{T}}$  and the real, symmetric, positivedefinite spot-shape matrix S. The eigenvalues of this

matrix,  $W_i^2$ ,  $i=1, 2$ , are squared semiaxes of the elliptic spot. The semiaxis  $W_1$  has an angle  $\phi \in [0, \pi/2)$  relative to the x axis, and the set  $\{W_1^2, W_2^2, \phi\}$  uniquely describes the orientation and the size of the ellipse.

For an elliptic-beam profile, the transmittance  $\eta$  is obtained by substituting Eq. [\(3\)](#page-0-0) into Eq. [\(2\)](#page-0-1). The resulting integral cannot be evaluated analytically. Here we adapt the technique proposed in Ref. [\[24\]](#page-4-7) to derive an analytical approximation. For this purpose we consider the displacement of the beam centroid to the point  $\mathbf{r}_0 = (r_0 \cos \varphi_0 \ r_0 \sin \varphi_0)^{\mathrm{T}}$ . Regarding the transmittance  $\eta$  as a function of  $r_0$ , for given  $\chi = \phi - \varphi_0$ , we observe that it behaves similar to the transmittance of the circular Gaussian beam with the effective squared spot radius

$$
W_{\text{eff}}^{2}(\chi) = 4a^{2} \left[ \mathcal{W} \left( \frac{4a^{2}}{W_{1}W_{2}} e^{\frac{a^{2}}{W_{1}^{2}} \left\{ 1 + 2\cos^{2}\chi \right\}} \times e^{\frac{a^{2}}{W_{2}^{2}} \left\{ 1 + 2\sin^{2}\chi \right\}} \right) \right]^{-1}, \tag{4}
$$

where  $W(\xi)$  is the Lambert W function [\[29\]](#page-4-15) and a is the aperture radius. In this case the transmittance is approximated by

$$
\eta = \eta_0 \exp\left\{-\left[\frac{r_0/a}{R\left(\frac{2}{W_{\rm eff}(\phi-\varphi_0)}\right)}\right]^{\lambda\left(\frac{2}{W_{\rm eff}(\phi-\varphi_0)}\right)}\right\}.
$$
 (5)

Here  $\eta_0$  is the transmittance for the centered beam, i.e. for  $r_0=0$ ,

$$
\eta_0 = 1 - I_0 \left( a^2 \left[ \frac{1}{W_1^2} - \frac{1}{W_2^2} \right] \right) e^{-a^2 \left[ \frac{1}{W_1^2} + \frac{1}{W_2^2} \right]}
$$
  
\n
$$
-2 \left[ 1 - e^{-\frac{a^2}{2} \left( \frac{1}{W_1} - \frac{1}{W_2} \right)^2} \right]
$$
  
\n
$$
\times \exp \left\{ - \left[ \frac{\frac{(W_1 + W_2)^2}{|W_1^2 - W_2^2|}}{R \left( \frac{1}{W_1} - \frac{1}{W_2} \right)} \right]^{\lambda \left( \frac{1}{W_1} - \frac{1}{W_2} \right)} \right\},
$$
 (6)

 $R(\xi)$  and  $\lambda(\xi)$  are scale and shape functions, respectively,

$$
R(\xi) = \left[ \ln \left( 2 \frac{1 - \exp[-\frac{1}{2}a^2 \xi^2]}{1 - \exp[-a^2 \xi^2] I_0(a^2 \xi^2)} \right) \right]^{-\frac{1}{\lambda(\xi)}},\tag{7}
$$

$$
\lambda(\xi) = 2a^2 \xi^2 \frac{e^{-a^2 \xi^2} I_1(a^2 \xi^2)}{1 - \exp[-a^2 \xi^2] I_0(a^2 \xi^2)} \times \left[ \ln \left( 2 \frac{1 - \exp[-\frac{1}{2} a^2 \xi^2]}{1 - \exp[-a^2 \xi^2] I_0(a^2 \xi^2)} \right) \right]^{-1}, \quad (8)
$$

and  $I_i(\xi)$  is the modified Bessel function of *i*th order. Since  $\phi$  is defined by modulo  $\pi/2$ , the transmittance  $\eta$  is a  $\pi/2$ -periodical function of  $\phi$ . For the limit  $W_1^2 = W_2^2$ , Eq. [\(5\)](#page-1-0) reduces to the transmission coefficient of a Gaussian beam with a circular profile [\[24\]](#page-4-7). For details of the approximation see Supplemental Material [\[30\]](#page-4-16) and Ref. [\[31\]](#page-4-17).

The probability distribution of the transmittance.– The aperture transmittance  $\eta$ , cf. Eq. [\(5\)](#page-1-0), is a function of five real parameters,  $\{x_0, y_0, \Theta_1, \Theta_2, \phi\}$ , randomly changed by the atmosphere, where  $W_i^2 = W_0^2 \exp \Theta_i$ , and  $W_0$  is the initial beam-spot radius. For these parameters we assume a Gaussian approximation, with  $\phi$  being a  $\pi/2$ -periodical wrapped Gaussian variable [\[32\]](#page-4-18). We restrict our attention to isotropic turbulence. In this case the wrapped Gaussian distribution for  $\phi$  reduces to a uniform one and its correlations with other parameters vanishes. In the reference frame with  $\langle \mathbf{r}_0 \rangle = 0$ , there are also no correlations of  $x_0$ ,  $y_0$ , and  $\Theta_i$ .

The variances  $\langle \Delta x_0^2 \rangle = \langle \Delta y_0^2 \rangle = \langle x_0^2 \rangle$ , which describe beam wandering, are expressed in terms of the classical field correlation function of the fourth order,  $\Gamma_4(\mathbf{r}_1, \mathbf{r}_2)$  $\langle I(\mathbf{r}_1;L)I(\mathbf{r}_2;L)\rangle$ , in the aperture plane (see, e.g., [\[13–](#page-4-12) [16,](#page-4-5) [33–](#page-4-19)[37\]](#page-4-20)):

<span id="page-1-1"></span>
$$
\langle x_0^2 \rangle = \int_{\mathbb{R}^4} \mathrm{d}^4 \mathbf{r} \, x_1 x_2 \, \Gamma_4(\mathbf{r}_1, \mathbf{r}_2), \tag{9}
$$

<span id="page-1-0"></span>where  $d^4\mathbf{r} = d^2\mathbf{r}_1 d^2\mathbf{r}_2$ . The means and the (co)variances of  $\Theta_i$  are functions of the means and the (co)variances (first and second moments) of  $W_i^2$ :

$$
\langle \Theta_i \rangle = \ln \left[ \frac{\langle W_i^2 \rangle}{W_0^2} \left( 1 + \frac{\langle (\Delta W_i^2)^2 \rangle}{\langle W_i^2 \rangle^2} \right)^{-1/2} \right], \qquad (10)
$$

<span id="page-1-2"></span>
$$
\langle \Delta \Theta_i \Delta \Theta_j \rangle = \ln \left[ 1 + \frac{\langle \Delta W_i^2 \Delta W_j^2 \rangle}{\langle W_i^2 \rangle \langle W_j^2 \rangle} \right].
$$
 (11)

In general, the evaluation of  $\langle W_i^2 \rangle$  and  $\langle \Delta W_i^2 \Delta W_j^2 \rangle$  in Eqs. [\(10\)](#page-1-1) and [\(11\)](#page-1-2) is almost intractable. However the assumptions of Gaussianity and isotropy enable to express these quantities in a tractable form as (for details cf. the Supplemental Material [\[30\]](#page-4-16))

$$
\langle W_i^2 \rangle = 4 \left[ \int_{\mathbb{R}^2} d^2 \mathbf{r} \, x^2 \Gamma_2(\mathbf{r}) - \langle x_0^2 \rangle \right],\tag{12}
$$

$$
\langle W_i^2 W_j^2 \rangle = 8 \Big[ -8 \,\delta_{ij} \langle x_0^2 \rangle^2 - \langle x_0^2 \rangle \langle W_i^2 \rangle \tag{13}
$$
\n
$$
+ \int d^4 \mathbf{r} \left[ x_1^2 x_2^2 \left( 4 \delta_{ij} - 1 \right) - x_1^2 y_2^2 \left( 4 \delta_{ij} - 3 \right) \right] \Gamma_4(\mathbf{r}_1, \mathbf{r}_2) \Big],
$$

where  $\Gamma_2(\mathbf{r}) = \langle I(\mathbf{r};L) \rangle$  is the classical field correlation function of the second order.

 $\mathbb{R}^4$ 

Therefore, the means and the covariance matrix of the random vector  $\mathbf{v} = (x_0, y_0 \Theta_1 \Theta_2)^T$ , i.e.  $\mu_i = \langle v_i \rangle$  and  $\Sigma_{ij} = \langle \Delta v_i \Delta v_j \rangle$ , respectively, are expressed in terms of classical field correlation functions  $\Gamma_2$  and  $\Gamma_4$ . These functions are important characteristics of atmospheric channels, which are widely discussed in the literature; see, e.g., [\[13–](#page-4-12)[16,](#page-4-5) [33–](#page-4-19)[35\]](#page-4-21). In the Supplemental Mate-rial [\[30\]](#page-4-16), we derive  $\mu_i$  and  $\Sigma_{ij}$  for horizontal links by using the phase approximation of the Huygens-Kirchhoff

method and the Kolmogorov turbulence spectrum [\[33–](#page-4-19) [35\]](#page-4-21).

With the given assumptions, the PDT in Eq. [\(1\)](#page-0-2) reads as

$$
\mathcal{P}(\eta) = \frac{2}{\pi} \int_{\mathbb{R}^4} d^4 \mathbf{v} \int_0^{\pi/2} d\phi \, \rho_G(\mathbf{v}; \boldsymbol{\mu}, \boldsymbol{\Sigma}) \delta \left[ \eta - \eta \left( \mathbf{v}, \phi \right) \right], \tag{14}
$$

where  $\eta(\mathbf{v}, \phi)$  is the transmittance defined by Eq. [\(5\)](#page-1-0) as a function of random parameters and  $\rho_G(\mathbf{v}; \boldsymbol{\mu}, \Sigma)$  is the Gaussian probability density of the vector v with the mean  $\mu$  and the covariance matrix  $\Sigma$ . In general, the PDT can be evaluated with the Monte Carlo method. For this purpose, one has to simulate the Gaussian random vector **v** and the uniformly distributed angle  $\phi$ . For practical purposes, we apply the Rayleigh distribution for  $r_0$ , a uniform distribution for  $\chi$ , and a Gaussian one for  $\Theta_i$ . The obtained values should be substituted in the transmittance; cf. Eq. [\(5\)](#page-1-0). Within the standard procedure of estimation, one obtains the mean value of any function of the transmittance,  $\langle f(\eta) \rangle$ . The PDT can be obtained within the smooth-kernel method [\[38\]](#page-5-0). The cumulative probability distribution,  $\mathcal{F}(\eta) = \int_0^{\eta} d\eta' \mathcal{P}(\eta')$ , and the exceedance  $\overline{\mathcal{F}}(\eta) = 1 - \mathcal{F}(\eta)$  are estimated by the technique of empirical distribution functions [\[39\]](#page-5-1).

From weak to strong turbulence. Let us distinguish the regimes of weak, moderate, and strong turbulence, through the values of Rytov parameter  $\sigma_R^2$  < 1,  $\sigma_R^2$   $\approx$ 1...10, and  $\sigma_{\rm R}^2 \gg 1$ , respectively. The Rytov parameter is defined as  $\sigma_R^2 = 1.23 C_n^2 k^{\frac{7}{6}} L^{\frac{11}{6}}$ , where  $C_n^2$  is the atmospheric index-of-refraction structure constant and  $k$  is the optical wave number; for more details and the corresponding motivation, see Ref. [\[13\]](#page-4-12). For weak turbulence the atmosphere mainly causes beam wandering. In this case Eq. [\(14\)](#page-2-0) reduces to the log-negative Weibull distribution [\[24\]](#page-4-7). For the weak-to-moderate transition and for strong turbulence, broadening and deformation of the beam occur, resulting in a smooth PDT. More problematic is the evaluation of **v** and  $\Sigma$  for the moderate-tostrong turbulence transition. Hence we will restrict our considerations to the ranges of weak-to-moderate and strong turbulence.

Figure [1](#page-2-1) shows the probability  $P(\eta)$  derived by the elliptic-beam approximation for the conditions of weakto-moderate turbulence. This distribution is compared with the corresponding ones obtained from the beamwandering model [\[24\]](#page-4-7) and from the log-normal model, see Supplemental Material [\[30\]](#page-4-16). The inset shows the experimental data given in Ref. [\[28\]](#page-4-11). It is obvious that the elliptic model yields the best agreement with the measured data.

The log-normal distribution is quite popular for modeling atmospheric turbulence effects [\[13](#page-4-12)[–20\]](#page-4-13). Usually this model is applied for the description of intensity fluctuations in one spatial point. In Fig. [1,](#page-2-1) the log-normal model is applied to the signal detection with a finite aperture; see Supplemental Material [\[30\]](#page-4-16). The dashed line

<span id="page-2-0"></span>

<span id="page-2-1"></span>Figure 1. The PDTs (a) and the corresponding exceedances (b): elliptic-beam approximation, log-normal, and beam wandering [\[24\]](#page-4-7). The shaded area in (a) shows the experimental PDT from Ref. [\[28\]](#page-4-11). The inset in (b) shows the tail of the exceedance. For the log-normal exceedance, it extends to the unphysical region,  $\eta > 1$  (shaded gray). Further parameters: wavelength 809 nm, initial spot radius  $W_0$ =20 mm, propagation distance 1.6 km, Rytov parameter  $\sigma_R^2$ =1.5, aperture radius  $a = 40$  mm, deterministic attenuation of 1.25 dB.

in Fig. [1](#page-2-1) (a) shows that the log-normal distribution differs significantly from the measured PDT. Moreover, the log-normal PDT is not limited to the physically allowed interval  $\eta \in [0, 1]$  $\eta \in [0, 1]$  $\eta \in [0, 1]$ . This feature is clearly seen in Fig. 1 (b) where the exceedance functions  $\overline{\mathcal{F}}(\eta)$ , i.e., the probability that the transmittance exceeds the value of  $\eta$ , are shown for the elliptic model, the beam-wandering model, and the log-normal model. As was shown in Ref.[\[23,](#page-4-22) [24\]](#page-4-7), the tails of  $\overline{\mathcal{F}}$  with large values of  $\eta$  are important for preserving nonclassical properties of transmitted light, which are overestimated by the log-normal model.

It has been shown in experiments with coherent light propagating through a 144 km atmospheric channel on the Canary Islands [\[4\]](#page-4-14) that the log-normal distribution in its physical domain demonstrates a good agreement with the experimental data under the conditions of strong turbulence. In Fig. [2,](#page-3-0) we compare the PDTs derived from the elliptic-beam approximation with the ones obtained in the beam-wandering and the log-normal models. Although in this case we consider a short propagation distance, the turbulence is quite strong. Similar conditions may occur, e.g., for the case of near-to-ground propagation on a hot summer day.

From Fig. [2](#page-3-0) one can clearly conclude that the beamwandering model strongly differs from the log-normal distribution and, consequently, it cannot well describe the strong turbulence scenario. However, the elliptic-beam model gives a reasonable agreement with the log-normal distribution in the physical domain of the latter. From this fact, one may conclude that our model consistently describes also the case of strong turbulence. A clear advantage of the elliptic-beam model is that the corresponding PDT does not attain nonzero values in the unphysical domain,  $\eta$  > 1, which is the case for the log-normal distribution. Hence, the usage of the elliptic-beam model gives physically consistent results, whereas the log-normal distribution may yield unphysical artifacts, e.g., the creation of photons by the atmosphere [\[19\]](#page-4-23). Such artifacts may cause an overestimation of the security of quantum communication protocols. Finally, we note that in some cases beam wandering is suppressed by tracking procedures [\[1,](#page-4-0) [8\]](#page-4-2). Under such conditions, the beam-wandering model is no longer useful but the elliptic-beam model does apply.



<span id="page-3-0"></span>Figure 2. The PDTs and the corresponding exceedances, similar to those shown in Fig. [1,](#page-2-1) but for the case of strong turbulence. Further parameters: wavelength 780 nm, initial spot radius  $W_0 = 50$ mm, propagation distance  $2 \text{ km}$ , Rytov parameter  $\sigma_R^2 = 31.5$ , aperture radius  $a = 150$  mm and no deterministic attenuation.

Application: quadrature squeezing.– The PDT [\(14\)](#page-2-0) in the elliptic-beam approximation allows one to analyze the quantum properties of light transmitted through the turbulent atmosphere by means of the input-output relation  $(1)$ . As an example, we analyze the squeezing properties after a weak-to-moderate turbulent atmospheric channel. We consider the 1.6 km link in the city of Erlangen [\[7\]](#page-4-1). The transmitter generates squeezed light  $(-2.4 \text{ dB})$  at  $\lambda = 780 \text{ nm}$  and sends it through the link with the Rytov parameter  $\sigma_R^2 = 2.6$ . The receiver detects



<span id="page-3-1"></span>Figure 3. Transmitted value of squeezing as a function of the postselection threshold  $\eta_{\min}$ . Initially squeezed light (to  $-2.4$  dB,  $\lambda = 780$  nm, spot radius  $W_0 = 25$  mm) is sent through a 1.6 km atmospheric link ( $\sigma_R^2 = 2.6$ ) and detected with an aperture radius of  $a = 75$  mm. The deterministic attenuation is 1.9 dB. The output signal is squeezed by −0.95 dB. With the postselection protocol, the squeezing value can be improved depending on the postselection threshold  $\eta_{\min}$ . The theory is shown for the elliptic-beam, log-normal, and beam-wandering models, compared with the experimental results (shaded area lies within the error bars) from Ref.[\[7\]](#page-4-1).

The postselection procedure of transmission events with  $\eta \geq \eta_{\text{min}}$  yields larger detected values for the transmitted squeezing. In Fig. [3,](#page-3-1) we compare the values of detected squeezing as functions of postselection thresholds  $\eta_{\min}$ , for the experimental values given in Ref. [\[7\]](#page-4-1). The beam-wandering model yields smaller values of postselected squeezing as detected in the experiment, as it does not properly describe the distribution tails for high values of  $\eta$ ; cf. Fig. [1.](#page-2-1) The postselected values of squeezing calculated within the elliptic-beam approximation agree very well within the error bars with the experimentally measured values. The shown log-normal model gives the correct values for the first two moments of  $\eta$ , but it differs in higher moments from the experimentally measured distribution. This feature allows one to obtain the correct value of squeezing for the transmitted signal from the log-normal model. However, this model completely fails to describe the postselection procedure, where the higher moments play a dominant role.

Summary and Conclusions.– We have introduced a model for the atmospheric turbulence effects on quantum light, which is based on an elliptic-beam approximation. Surprisingly, it yields a reasonable agreement with experiments for the conditions of weak-to-moderate turbulence. In this case, we get an excellent description of the transfer of squeezed light through a 1.6 km channel, analyzed with data postselection.

For the case of strong turbulence, we have shown that our theory gives a reasonable agreement with the lognormal distribution. In experiments using a 144 km channel under strong turbulence conditions, the log-normal model also yields a proper description of the transmission of coherent light. Hence, our theory describes in a unified manner the quantum-light transfer through atmospheric channels under dissimilar turbulence conditions. The case of the transition regime of moderate-to-strong turbulence requires further research.

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### SUPPLEMENTAL MATERIAL ATMOSPHERIC QUANTUM CHANNELS WITH WEAK AND STRONG TURBULENCE

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The supplement is structured as follows:

In Sec. [A](#page-6-0) we discuss the properties of Gaussian elliptical beams. In Sec. [B](#page-7-0) we derive the analytic expression for the transmittance of the elliptical beam through the circular aperture. In Sec. [C](#page-8-0) the statistical properties of the elliptical beam transmitted through turbulence are discussed in Gaussian approximation. In Sec. [D](#page-9-0) we discuss the simplifications which arise from the assumption that the atmospheric turbulence is isotropic. Here we derive the formulas that connect the statistical characteristics of the elliptical beam in the isotropic atmosphere with the field correlation functions. In Sec. [E](#page-12-0) the phase approximation of the Huygens-Kirchhoff method is presented and the general expressions for field correlation functions are derived. In Sec. [F](#page-13-0) and in Sec. [G](#page-15-0) we derive the means and (co)variances connected with beam wandering and beam shape deformation, respectively. These results are evaluated for limits of weak and strong turbulence and are summarized in the table in Sec. [H.](#page-17-0) Finally, in Sec. [I](#page-17-1) the log-normal distribution for the beam transmittance is considered.

### <span id="page-6-0"></span>A. ELLIPTIC BEAMS

In this Section we discuss the properties of elliptical beams, which are crucial for the consideration of light transferring through the turbulent atmosphere. In the paraxial approximation the beam amplitude  $u(\mathbf{r}, z)$  satisfies the equation, cf. Ref. [\[15\]](#page-4-24),

$$
2ik\frac{\partial u(\mathbf{r},z)}{\partial z} + \Delta_{\mathbf{r}}u(\mathbf{r},z) + 2k^2\delta n(\mathbf{r},z)u(\mathbf{r},z) = 0, \text{ (A1)}
$$

where k is the wave number,  $\delta n(\mathbf{r}, z)$  is a small fluctuating part of the index of air refraction,  $\mathbf{r} = (x \ y)^T$  is the vector of transverse coordinates. The boundary condition in the transmitter plane  $z = 0$  for the initially Gaussian beam is given by

$$
u(\mathbf{r}, z=0) = u_0(\mathbf{r}) = \sqrt{\frac{2}{\pi W_0^2}} \exp\left[-\frac{1}{W_0^2}|\mathbf{r}|^2 - \frac{ik}{2F}|\mathbf{r}|^2\right].
$$
\n(A2)

Here  $W_0$  is the beam spot radius, F is the wavefront radius in the center of the transmitting aperture at  $z=0$ . The intensity of light is defined as

<span id="page-6-4"></span>
$$
I(\mathbf{r}, z) = |u(\mathbf{r}, z)|^2.
$$
 (A3)

This function can be chosen in the normalized form

$$
\int_{\mathbb{R}^2} d^2 \mathbf{r} \, I(\mathbf{r}, z) = 1,\tag{A4}
$$

and Eq.  $(A1)$  implies that this norm preserves for any z. For our purposes it is also important that  $I(\mathbf{r}, z) \geq 0$ .

Consider the transverse Fourier transform of intensity,

<span id="page-6-3"></span>
$$
C(\mathbf{k}, z) = \int_{\mathbb{R}^2} d^2 \mathbf{r} I(\mathbf{r}, z) e^{i\mathbf{k} \cdot \mathbf{r}},
$$
 (A5)

where k·r denotes the scalar product of two vectors. Similar to the cumulative expansion in the probability theory one writes

<span id="page-6-2"></span>
$$
\ln C(\mathbf{k}, z) = i\mathbf{k} \cdot \mathbf{r}_0 - \frac{1}{8} \mathbf{k}^{\mathrm{T}} \mathbf{S} \mathbf{k} + \dots,
$$
 (A6)

where

<span id="page-6-9"></span>
$$
\mathbf{r}_0 = \int_{\mathbb{R}^2} d^2 \mathbf{r} \, \mathbf{r} \, I(\mathbf{r}, z) \tag{A7}
$$

<span id="page-6-1"></span>is the beam-centroid position,

<span id="page-6-6"></span>
$$
\mathbf{S} = \begin{pmatrix} S_{xx} & S_{xy} \\ S_{xy} & S_{yy} \end{pmatrix} \tag{A8}
$$
\n
$$
= 4 \int_{\mathbb{R}^2} d^2 \mathbf{r} \left[ (\mathbf{r} - \mathbf{r}_0)(\mathbf{r} - \mathbf{r}_0)^{\mathrm{T}} \right] I(\mathbf{r}, z)
$$

is the spot-shape matrix. Within the elliptic-beam approximation we suppose that the expansion [\(A6\)](#page-6-2) in the aperture plane can be restricted by the second (Gaussian) term. Substituting it into the inversion of Eq. [\(A5\)](#page-6-3),

<span id="page-6-5"></span>
$$
I(\mathbf{r}, z) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} d^2 \mathbf{r} C(\mathbf{k}, z) e^{-i\mathbf{k} \cdot \mathbf{r}}.
$$
 (A9)

<span id="page-6-11"></span>one gets for the intensity of the elliptic beam

$$
I(\mathbf{r}, z) = \frac{2}{\pi \sqrt{\det \mathbf{S}}} \exp \left[ -2(\mathbf{r} - \mathbf{r}_0)^{\mathrm{T}} \mathbf{S}^{-1} (\mathbf{r} - \mathbf{r}_0) \right]. \tag{A10}
$$

In the particular case, when the spot-shape matrix is proportional to the identity matrix, this expression is reduced to the intensity of a circular Gaussian beam.

Two eigenvalues,  $W_1^2$  and  $W_2^2$ , of the spot-shape matrix S correspond to two semi-axes of the beam ellipse. They are related to the elements of the matrix S as

<span id="page-6-10"></span><span id="page-6-7"></span>
$$
S_{xx} = W_1^2 \cos^2 \phi + W_2^2 \sin^2 \phi,
$$
 (A11)

$$
S_{yy} = W_1^2 \sin^2 \phi + W_2^2 \cos^2 \phi, \tag{A12}
$$

<span id="page-6-8"></span>
$$
S_{xy} = \frac{1}{2} \left( W_1^2 - W_2^2 \right) \sin 2\phi, \tag{A13}
$$



<span id="page-7-1"></span>Figure A1. (Color online) The aperture of radius a and the elliptical beam profile with the half-axis  $W_1$  rotated on the angle  $\phi$  relative to the x-axis and on the angle  $\chi$  relative to the  $r_0$ -associated axis are shown. The beam centroid is situated in the point  $\mathbf{r}_0$  with the polar coordinates  $(r_0, \varphi_0)$ . The  $x'-y'$  coordinate system is associated with the elliptical beam centroid.

where  $\phi \in [0, \pi/2)$  is the angle between the x-axis and the ellipse semi-axis related to  $W_1^2$ . The set of three parameters  $(W_1^2, W_2^2, \phi)$  uniquely defines all possible orientations of the ellipse.

The introduced representation of the ellipse assumes that we do not distinguish between major and minor semi-axes of the ellipse. The semi-axis related to  $W_1^2$  is defined as being situated in first and third quarter-planes of the  $x'$ -y' coordinate system, cf. Fig. [A1,](#page-7-1) while  $W_2^2$  is in the second and fourth ones. Within this definition the values of  $W_1^2$  and  $W_2^2$  are not ordered.

### <span id="page-7-0"></span>B. APERTURE TRANSMITTANCE FOR ELLIPTIC BEAMS

In this Section we derive in details an analytical approximation for the transmittance of elliptic beams through a circular aperture. For the aperture situated in the point  $z=L$  the transmittance is determined via the expression, cf. Ref. [\[24\]](#page-4-7),

<span id="page-7-2"></span>
$$
\eta = \int_{\mathcal{A}} d^2 \mathbf{r} \, I(\mathbf{r}, L), \tag{B1}
$$

where  $I(\mathbf{r}, L)$  is the normalized intensity defined by Eq. [\(A3\)](#page-6-4) and integration is performed in the aperture opening area. Substituting Eq. [\(A10\)](#page-6-5) into Eq. [\(B1\)](#page-7-2) and considering the structure of spot-shape matrix S, cf. Eqs.  $(A8)$  and  $(A11)-(A13)$  $(A11)-(A13)$  $(A11)-(A13)$ , one gets for the transmittance,

$$
\eta = \frac{2}{\pi W_1 W_2} \int_0^a dr \, r \int_0^{2\pi} d\varphi \, e^{-2A_1 \left(r \cos \varphi - r_0\right)^2}
$$
\n
$$
\times e^{-2A_2 r^2 \sin^2 \varphi} e^{-2A_3 \left(r \cos \varphi - r_0\right) r \sin \varphi}.\tag{B2}
$$

Here a is the aperture radius,  $r, \varphi$  are polar coordinates for the vector r,

<span id="page-7-3"></span>
$$
x = r \cos \varphi, \tag{B3}
$$

$$
y = r\sin\varphi,\tag{B4}
$$

 $r_0$ ,  $\varphi_0$  are polar coordinates for the vector  $\mathbf{r}_0$ ,

$$
x_0 = r_0 \cos \varphi_0,\tag{B5}
$$

$$
y_0 = r_0 \sin \varphi_0,\tag{B6}
$$

$$
A_1 = \left(\frac{\cos^2(\phi - \varphi_0)}{W_1^2} + \frac{\sin^2(\phi - \varphi_0)}{W_2^2}\right), \quad \text{(B7)}
$$

$$
A_2 = \left(\frac{\sin^2(\phi - \varphi_0)}{W_1^2} + \frac{\cos^2(\phi - \varphi_0)}{W_2^2}\right), \quad (B8)
$$

$$
A_3 = \left(\frac{1}{W_1^2} - \frac{1}{W_2^2}\right) \sin 2(\phi - \varphi_0),\tag{B9}
$$

and  $\phi$  is defined with the modulo  $\pi/2$  such that  $\eta$  in Eq. [\(B2\)](#page-7-3) is a  $\pi/2$ -periodical function of  $\phi$ .

For the given angle  $\chi = \phi - \varphi_0$  the transmittance  $\eta$  as a function of  $r_0$  has a behavior similar to the transmittance of the circular Gaussian beam with a certain effective spot-radius  $W_{\text{eff}}(\chi)$ . Applying the method developed in Ref. [\[24\]](#page-4-7) one can write the corresponding approximation,

<span id="page-7-6"></span>
$$
\eta = \eta_0 \exp\left\{-\left[\frac{r_0/a}{R\left(\frac{2}{W_{\rm eff}(\phi-\varphi_0)}\right)}\right]^{\lambda\left(\frac{2}{W_{\rm eff}(\phi-\varphi_0)}\right)}\right\}.
$$
\n(B10)

Here  $\eta_0$  is the beam transmittance at  $r_0=0$ , and

$$
R(\xi) = \left[ \ln \left( 2 \frac{1 - \exp[-\frac{1}{2}a^2 \xi^2]}{1 - \exp[-a^2 \xi^2] I_0(a^2 \xi^2)} \right) \right]^{-\frac{1}{\lambda(\xi)}}, \quad (B11)
$$

<span id="page-7-5"></span><span id="page-7-4"></span>
$$
\lambda(\xi) = 2a^2 \xi^2 \frac{e^{-a^2 \xi^2} I_1(a^2 \xi^2)}{1 - \exp[-a^2 \xi^2] I_0(a^2 \xi^2)} \times \left[ \ln \left( 2 \frac{1 - \exp[-\frac{1}{2} a^2 \xi^2]}{1 - \exp[-a^2 \xi^2] I_0(a^2 \xi^2)} \right) \right]^{-1}
$$
(B12)

are scale and shape functions, respectively.

The transmittance  $\eta_0$  is obtained from Eq. [\(B2\)](#page-7-3) by setting the beam-centroid position  $r_0 = 0$ ,

$$
\eta_0 = \frac{2}{\pi W_1 W_2} \int_0^a dr \, r \int_0^{2\pi} d\varphi \, e^{-\left\{\frac{1}{W_1^2} + \frac{1}{W_2^2}\right\} r^2}
$$
\n
$$
\times e^{-\left|\frac{1}{W_1^2} - \frac{1}{W_2^2}\right| r^2 \cos 2(\varphi - \tilde{\varphi})}
$$
\n
$$
= \frac{2}{|W_1 W_2|} \int_0^{a^2} dt \, e^{-\left\{\frac{1}{W_1^2} + \frac{1}{W_2^2}\right\} t} \, I_0\left(\left|\frac{1}{W_1^2} - \frac{1}{W_2^2}\right| t\right),
$$
\n(B13)

where  $\widetilde{\varphi} = \frac{1}{2} \arctan[A_3/(A_1-A_2)]$ . It is expressed in terms of the incomplete Lipshitz-Hankel integral of Ref [31] of the incomplete Lipshitz-Hankel integral, cf. Ref. [\[31\]](#page-4-17), as

$$
I_{e_0}(a, z) = \int_0^z dt e^{-at} I_0(t),
$$
 (B14)

that results in

$$
\eta_0 = \frac{2W_1W_2}{|W_1^2 - W_2^2|} \mathbf{I}_{e_0} \left( \frac{W_1^2 + W_2^2}{|W_1^2 - W_2^2|}, a^2 \frac{|W_1^2 - W_2^2|}{W_1^2 W_2^2} \right). \tag{B15}
$$

The incomplete Lipshitz-Hankel integral can be evaluated numerically. However, using the relation between the incomplete Lipshitz-Hankel  $(I_{e_0})$  and Weber  $(Q_0)$  integrals [\[31\]](#page-4-17), we can rewrite Eq. [\(B15\)](#page-8-1) as

$$
\eta_0 = 1 - e^{-a^2 \frac{W_1^2 + W_2^2}{W_1^2 W_2^2}} \Big[ I_0 \Big( a^2 \frac{|W_1^2 - W_2^2|}{W_1^2 W_2^2} \Big) + 2 \widetilde{Q}_0 \Big( a^2 \frac{(W_1 + W_2)^2}{2W_1^2 W_2^2}, a^2 \frac{|W_1^2 - W_2^2|}{W_1^2 W_2^2} \Big) \Big].
$$
 (B16)

In Ref. [\[24\]](#page-4-7), an analytical approximation for  $\tilde{Q}_0$  is derived. Applying here the same procedure for the approximation of the incomplete Weber integral in Eq. [\(B16\)](#page-8-2) one obtains

$$
\eta_0 = 1 - I_0 \left( a^2 \frac{W_1^2 - W_2^2}{W_1^2 W_2^2} \right) e^{-a^2 \frac{W_1^2 + W_2^2}{W_1^2 W_2^2}} \n-2 \left[ 1 - e^{-\frac{a^2}{2} \left( \frac{1}{W_1} - \frac{1}{W_2} \right)} \right] \n\times \exp \left[ -\left\{ \frac{\frac{(W_1 + W_2)^2}{|W_1^2 - W_2^2|}}{R \left( \frac{1}{W_1} - \frac{1}{W_2} \right)} \right\}^{\lambda \left( \frac{1}{W_1} - \frac{1}{W_2} \right)} \right], \quad (B17)
$$

where  $R(\xi)$  and  $\lambda(\xi)$  are defined by Eqs. [\(B11\)](#page-7-4) and [\(B12\)](#page-7-5), respectively. For the case when  $W_1^2 = W_2^2 = W^2$ , Eqs. [\(B16\)](#page-8-2) and [\(B17\)](#page-8-3) are reduced to  $\eta_0=1-e^{-2a^2/W^2}$ that is the maximal transmittance of the circular beam, cf. Ref. [\[24\]](#page-4-7).

In order to get an approximate value for the effective spot-radius  $W_{\text{eff}}(\chi)$  we assume that the intensity of the

corresponding circular beam is equal to the intensity of the elliptic beam at the aperture plane, i.e.

<span id="page-8-4"></span>
$$
\frac{1}{W_{\text{eff}}^2(\chi)} e^{-\frac{2}{W_{\text{eff}}^2(\chi)} (r^2 + r_0^2 + 2 r r_0 \cos \varphi)} = \frac{1}{W_1 W_2}
$$
\n
$$
\times e^{-2A_1(\chi) r_0^2} e^{2r_0 r} \{2A_1(\chi) \cos \varphi + A_3(\chi) \sin \varphi\}
$$
\n
$$
\times e^{-2r^2} \{A_2(\chi) + [A_1(\chi) - A_2(\chi)] \cos^2 \varphi + \frac{A_3(\chi)}{2} \sin 2\varphi\}.
$$
\n(B18)

In the most general case this equality cannot be satisfied exactly. However, we can find such a value of  $W_{\text{eff}}(\chi)$ that Eq. [\(B18\)](#page-8-4) will be fulfilled approximately. For this purpose we expand both sides of this equation in series with respect to  $e^{i\varphi}$ . Then we equate the zeroth-order terms of these expansions at the point  $r=r_0=a$ . This results in the expression

$$
4\frac{a^2}{W_{\text{eff}}^2(\chi)} + \ln\left[\frac{W_{\text{eff}}^2(\chi)}{a^2}\right] - 2a^2 \left[\frac{1}{W_1^2} + \frac{1}{W_2^2}\right] -a^2 \left[\frac{1}{W_1^2} + \frac{1}{W_2^2}\right] \cos 2\chi - \ln\left(\frac{W_1 W_2}{a^2}\right) = 0.
$$
 (B19)

<span id="page-8-1"></span>Solving this equation with respect to  $W_{\text{eff}}(\chi)$  one gets

<span id="page-8-5"></span>
$$
W_{\text{eff}}^{2}(\chi) = 4a^{2} \left[ \mathcal{W} \left( \frac{4a^{2}}{W_{1}W_{2}} e^{\frac{a^{2}}{W_{1}^{2}} \left\{ 1 + 2\cos^{2}\chi \right\}} \times e^{\frac{a^{2}}{W_{2}^{2}} \left\{ 1 + 2\sin^{2}\chi \right\}} \right) \right]^{-1}, \qquad (B20)
$$

where  $W(x)$  is the Lambert function [\[29\]](#page-4-15).

<span id="page-8-2"></span>In Fig. [B1](#page-9-1) we compare the transmittance  $\eta$  obtained by numerical integration of Eq. [\(B2\)](#page-7-3) and its analytical approximation. The approximation, cf. Eq. [\(B10\)](#page-7-6), gives a reasonable accuracy especially in the case of small beam ellipticity. It is also important to note that  $W_{\text{eff}}^2(\phi - \varphi_0)$ and  $\eta$ , cf. Eqs. [\(B20\)](#page-8-5) and [\(B17\)](#page-8-3), respectively, are  $\pi/2$ periodical functions of the  $\phi$ , since this angle is defined with the modulo  $\pi/2$ .

### <span id="page-8-0"></span>C. GAUSSIAN APPROXIMATION

In this Section we discuss in detail the statistical properties of elliptic beams and discuss the applicability of the Gaussian approximation. Any spot in the ellipticbeam approximation at the aperture plane is uniquely described by the set of five parameters  $(x_0, y_0, W_1^2, W_2^2, \phi)$ . While the beam passes through the turbulent atmosphere, these parameters are randomly changed. Each part of the path slightly contributes in these values. Also it is important to note that these parameters can be correlated.

<span id="page-8-3"></span>Random fluctuations of the beam-centroid position  $r_0$ , i.e. the parameters  $x_0$  and  $y_0$ , lead to the effect of beam wandering. These parameters can be considered as affected by an additive noise during the propagation. A large number of small additive contributions is a good argument for using the Gaussian approximation for the beam-centroid position, cf. Ref. [\[24\]](#page-4-7).



<span id="page-9-1"></span>Figure B1. (Color online) The transmittance of the elliptical beam (half-axes  $|W_1|$ ,  $|W_2|$ ) through the circular aperture (radius *a*) as a function of the beam-centroid displacement  $r_0$ : (a)  $|W_1| = 0.2a, |W_2| = 0.1a, \chi = \pi/3$ ; (b)  $|W_1| = a, |W_2| = 0.9a$ ,  $\chi = \pi/4$ ; (c)  $|W_1|=1.8a, |W_2|=1.7a, \chi = \pi/5$ . The solid line for  $\eta$  is obtained by numerical calculation, the dashed line represents the analytical approximation, cf. Eq. [\(B10\)](#page-7-6).

### C.1. Wrapped Gaussian model for  $\phi$

Similar argumentations work for the angle  $\phi$ . This parameter can also be considered as affected by a large number of the small additive contributions. An important difference is that the angle  $\phi$  is a  $\pi/2$ -periodical variable. For this reason one should use in this case the wrapped Gaussian distribution, cf. Ref. [\[32\]](#page-4-18),

<span id="page-9-2"></span>
$$
\rho(\phi) = \frac{1}{\sqrt{2\pi} \sigma_{\phi}} \sum_{k=-\infty}^{+\infty} \exp\left[-\frac{\left(\phi - \mu_{\phi} + \frac{\pi}{2}k\right)^2}{2\sigma_{\phi}^2}\right], \quad \text{(C1)}
$$

where  $\mu_{\phi}$  is the mean direction and  $\sigma_{\phi}$  is the unwrapped standard deviation. For  $\sigma_{\phi} \rightarrow +\infty$  Eq. [\(C1\)](#page-9-2) becomes the probability density of the uniform distribution.

# C.2. Multiplicative-noise model for  $W_i^2$

In this model one assumes that each small kth part of the atmospheric channel multiplicatively changes values of  $W_i^2$ ,  $i=1,2$ , with the factor  $\varepsilon_i^k \in \mathbb{R}^+$ . As a result at the aperture plane the value of  $W_i^2$  is

$$
W_i^2 = W_0^2 \prod_{k=1}^N \varepsilon_i^k, \qquad i=1,2. \tag{C2}
$$

The large number  $N$  of small random contributions gives a good argument for assuming  $W_i^2$  log-normally distributed.

Let us introduce the random parameters

$$
\Theta_i = \ln \frac{W_i^2}{W_0^2}.\tag{C3}
$$

In framework of the considered model these parameters yield a two-fold normal distribution. For the complete characterization of this distribution we need the means and the (co)variances of  $\Theta_i$ . They can be expressed in terms of the means and the (co)variances of  $W_i^2$  as

<span id="page-9-7"></span>
$$
\langle \Theta_i \rangle = \ln \left[ \frac{\langle W_i^2 \rangle}{W_0^2 \left( 1 + \frac{\langle (\Delta W_i^2)^2 \rangle}{\langle W_i^2 \rangle^2} \right)^{1/2}} \right], \quad (C4)
$$

$$
\langle \Delta \Theta_i \Delta \Theta_j \rangle = \ln \left( 1 + \frac{\langle \Delta W_i^2 \Delta W_j^2 \rangle}{\langle W_i^2 \rangle \langle W_j^2 \rangle} \right), \quad i, j = 1, 2
$$
\n(C5)

which can be used for the corresponding calculations.

### <span id="page-9-0"></span>D. ISOTROPY OF TURBULENCE

In this Section we discuss simplifications, which follow from the assumption that the atmospheric turbulence is isotropic. We also assume that

<span id="page-9-8"></span><span id="page-9-4"></span><span id="page-9-3"></span>
$$
\langle \mathbf{r}_0 \rangle = 0, \tag{D1}
$$

i.e. beam wandering fluctuations are placed around the reference-frame origin. We consider the field intensity at the aperture plane,  $I(\mathbf{r}, L)$ , as a stochastic field characterized by the probability density functional  $\rho[I(\mathbf{r}, L)].$ The above assumptions mean that

$$
\rho\left[I(O\,\mathbf{r},L)\right] = \rho\left[I(\mathbf{r},L)\right],\tag{D2}
$$

where O is a representation of the  $O(2)$  group. In the following we consider important consequences from Eqs. [\(D1\)](#page-9-3) and [\(D2\)](#page-9-4).

#### D.1. Uniform distribution for the angle  $\phi$

A clear consequence from the isotropy assumption is the fact that the angle parameter  $\phi$  appears to be uniformly distributed. This fact is a consequence from Eq. [\(D2\)](#page-9-4). Indeed, according to this requirement the probability density  $\rho(\phi)$  does not depend on the choice of the reference frame, i.e. for any angle  $\zeta$ 

$$
\rho(\phi + \zeta) = \rho(\phi). \tag{D3}
$$

This equation holds true only for the uniform distribution. For details of circular distributions see Ref. [\[32\]](#page-4-18).

### D.2. Correlations between linear and angle parameters

<span id="page-9-6"></span>Let  $\bf{v}$  be a random vector, which consists of variables  $v_i$  with the support  $\mathbb{R}$ ,

<span id="page-9-5"></span>
$$
\mathbf{v} = \begin{pmatrix} x_0 & y_0 & \Theta_1 & \Theta_2 \end{pmatrix}^\mathrm{T} . \tag{D4}
$$

The parameters  $v_i$ ,  $i=1, ..., 4$  of Eq. [\(D4\)](#page-9-5) and the angle parameter  $\phi$  are distributed according to the two-fold normal distribution, which is wrapped for  $\phi$ , cf. Section [C,](#page-8-0)

$$
\rho(v_i, \phi) = \frac{1}{2\pi\sqrt{\det \Sigma_{v_i, \phi}}} \sum_{k=-\infty}^{+\infty} \exp\left(-\frac{1}{2} \nu_k^{\mathrm{T}} \Sigma_{v_i, \phi}^{-1} \nu_k\right),\tag{D5}
$$

where

$$
\nu_k = \left( v_i - \langle v_i \rangle \phi - \mu_\phi + \frac{\pi}{2} k \right)^{\mathrm{T}}, \quad (D6)
$$

 $\mu_{\phi}$  is the mean direction of  $\phi$ 

$$
\Sigma_{v_i, \phi} = \begin{pmatrix} \sigma_{v_i}^2 & s\sigma_{v_i}\sigma_{\phi} \\ s\sigma_{v_i}\sigma_{\phi} & \sigma_{\phi}^2 \end{pmatrix}
$$
 (D7)

is the covariance matrix,  $\sigma_{v_i}^2$  is the standard deviation of  $v_i$ ,  $\sigma_{\phi}^2$  is the unwrapped variance of  $\phi$ , and s is the correlation coefficient.

The considered probability distribution can also be rewritten in the form, cf. Ref. [\[32\]](#page-4-18),

$$
\rho(v_i, \phi) = \frac{1}{\sqrt{2\pi}\sigma_{v_i}} e^{-\frac{1}{2}\frac{(v_i - \langle v_i \rangle)^2}{\sigma_{v_i}^2}} \frac{2}{\pi} \Big\{ 1 \qquad (D8)
$$

$$
+ 2 \sum_{n=1}^{\infty} e^{-8(1-s^2)\sigma_{\phi}^2 n^2} \cos \Big[ 4n \big(\phi - \mu_{\phi} - s \frac{\sigma_{\phi}}{\sigma_{v_i}} [v_i - \langle v_i \rangle] \big) \Big] \Big\}.
$$

As it has been already shown, in the case of isotropic turbulence the marginal distribution for  $\phi$  is uniform. This corresponds to the case of  $\sigma_{\phi}^2 \rightarrow +\infty$ . If the correlation is imperfect, i.e.  $s^2 \neq 1$ , Eq. [\(D8\)](#page-10-0) is factorized in the normal distribution for  $v_i$  and the uniform distribution for  $\phi$ ,

$$
\rho(v_i, \phi) = \frac{1}{\sqrt{2\pi}\sigma_{v_i}} e^{-\frac{1}{2}\frac{(v_i - \langle v_i \rangle)^2}{\sigma_{v_i}^2}} \frac{2}{\pi}, \quad i = 1, ..., 4.
$$
 (D9)

Hence, for the isotropic turbulence correlations between the angle  $\phi$  and the linear parameters vanish.

#### <span id="page-10-7"></span>D.3. Correlations between beam-centroid position and spot-shape parameters

Consider the random variables  $\Theta_i$ , *i*=1, 2, which describe the spot shape, cf. Eq. [\(C3\)](#page-9-6). We will be interested in the correlations  $\langle \Delta \Theta_i \Delta \mathbf{r}_0 \rangle$ . With the considered assumption

$$
\langle \Delta \Theta_i \, \Delta \mathbf{r}_0 \rangle = \langle \Theta_i \, \mathbf{r}_0 \rangle, \qquad i = 1, 2, \qquad \text{(D10)}
$$

because the beam centroid is fluctuating around the reference-frame origin, cf. Eq. [\(D1\)](#page-9-3).

By using the definition of  $r_0$ , cf. Eq. [\(A7\)](#page-6-9), the correlation coefficient is written as

$$
\langle \Delta \Theta_i \, \Delta \mathbf{r}_0 \rangle = \int_{\mathbb{R}^2} \mathrm{d}^2 \mathbf{r} \, \mathbf{r} \, \langle \Theta_i \, I(\mathbf{r}, L) \rangle. \tag{D11}
$$

The assumption of isotropy, cf. Eq. [\(D2\)](#page-9-4), results in the statement that  $\langle \Theta_i I(\mathbf{r}, L) \rangle$  is invariant with respect to the rotations in the  $(x, y)$  plane. Hence, this function has the central symmetry. This leads to the conclusion that

<span id="page-10-2"></span>
$$
\langle \Delta \Theta_i \, \Delta \mathbf{r}_0 \rangle = 0,\tag{D12}
$$

because the integral in Eq. [\(D11\)](#page-10-1) vanishes.

We assume that  $\Theta_i$  and  $\mathbf{r}_0$  are Gaussian variables, cf. Section [C.](#page-8-0) Together with Eq. [\(D12\)](#page-10-2) this yields

$$
\langle F(\Theta_i)G(\mathbf{r}_0)\rangle = \langle F(\Theta_i)\rangle \langle G(\mathbf{r}_0)\rangle. \tag{D13}
$$

Here  $F$  and  $G$  are arbitrary functions.

# D.4. Moments and (co)variances of  $W_i^2$

<span id="page-10-0"></span>In Section [C](#page-8-0) it has been shown that for the characterization of probability distributions for elliptic beams we need among other first and second moments for  $W_i^2$ ,  $i=1, 2$ , cf. Eqs. [\(C4\)](#page-9-7) and [\(C5\)](#page-9-8). In general, the calculation of these moments is a complicated task, which requires non-Gaussian functional integration. Here we will show that the assumption of turbulence isotropy essentially simplifies this problem such that the moments are expressed in terms of field correlation functions of the second and fourth orders.

# <span id="page-10-6"></span><span id="page-10-5"></span>D.4.1. First moments of  $W_i^2$

We start the consideration with averaging the elements of the matrix  $S$ , cf. Eq. [\(A8\)](#page-6-6), by the atmosphere states,

<span id="page-10-4"></span>
$$
\langle S_{xx} \rangle =
$$
\n
$$
4 \left[ \int_{\mathbb{R}^2} d^2 \mathbf{r} \, x^2 \Gamma_2(\mathbf{r}; L) - \int_{\mathbb{R}^4} d^2 \mathbf{r}_1 d^2 \mathbf{r}_2 \, x_1 x_2 \Gamma_4(\mathbf{r}_1, \mathbf{r}_2; L) \right],
$$
\n(D14)

$$
\langle S_{yy}\rangle =
$$
\n
$$
4\left[\int_{\mathbb{R}^2} d^2 \mathbf{r} \, y^2 \Gamma_2(\mathbf{r}; L) - \int_{\mathbb{R}^4} d^2 \mathbf{r}_1 d^2 \mathbf{r}_2 \, y_1 y_2 \Gamma_4(\mathbf{r}_1, \mathbf{r}_2; L)\right],
$$
\n(D15)

$$
\langle S_{xy}\rangle =
$$
\n
$$
4\left[\int_{\mathbb{R}^2} d^2 \mathbf{r} \, xy \Gamma_2(\mathbf{r}; L) - \int_{\mathbb{R}^4} d^2 \mathbf{r}_1 d^2 \mathbf{r}_2 \, x_1 y_2 \Gamma_4(\mathbf{r}_1, \mathbf{r}_2; L)\right].
$$
\n(D16)

Here

<span id="page-10-9"></span><span id="page-10-8"></span><span id="page-10-3"></span>
$$
\Gamma_2(\mathbf{r}; z) = \langle I(\mathbf{r}, z) \rangle = \langle u^*(\mathbf{r}, z) u(\mathbf{r}, z) \rangle, \quad \text{(D17)}
$$

<span id="page-10-1"></span>
$$
\Gamma_4(\mathbf{r}_1, \mathbf{r}_2; z) = \langle I(\mathbf{r}_1, z)I(\mathbf{r}_2, z) \rangle
$$
\n
$$
= \langle u^*(\mathbf{r}_1, z)u(\mathbf{r}_1, z)u^*(\mathbf{r}_2, z)u(\mathbf{r}_2, z) \rangle
$$
\n(D18)

are the field correlation functions of the second and fourth orders, respectively. The isotropy assumption, cf. Eq. [\(D2\)](#page-9-4), results in the equalities,

$$
\langle S_{xx} \rangle = \langle S_{yy} \rangle, \tag{D19}
$$

$$
\langle S_{xy}\rangle = 0,\tag{D20}
$$

which means that the averaged beam has a circular shape. Equation [\(D20\)](#page-11-0) is a consequence of the fact that due to the turbulence isotropy  $\Gamma_2(\mathbf{r};L)$  and  $\int_{\mathbb{R}^2} dx_2 dy_1 x_1 y_2 \Gamma_4(\mathbf{r}_1, \mathbf{r}_2; L)$  have a symmetry in planes  $(x, y)$  and  $(x_1, y_2)$ , respectively. This symmetry implies that the integrals in Eq. [\(D16\)](#page-10-3) appear to have zero values.

Combining Eqs. [\(A13\)](#page-6-8) and [\(D20\)](#page-11-0) one gets

$$
\left\langle W_1^2 \right\rangle = \left\langle W_2^2 \right\rangle, \tag{D21}
$$

where we have used the fact that the angle  $\phi$  does not correlate with  $W_i^2$ , cf. Eq. [\(D9\)](#page-10-4). Similarly, averaging Eqs. [\(A11\)](#page-6-7) and [\(A12\)](#page-6-10) one gets

$$
\left\langle W_{1/2}^2 \right\rangle = \left\langle S_{xx/yy} \right\rangle. \tag{D22}
$$

This equation together with Eqs. [\(D14\)](#page-10-5) and [\(D15\)](#page-10-6) express the first moments of  $W_i^2$  in terms of the field correlation functions  $\Gamma_2$  and  $\Gamma_4$ .

# D.4.2. Second moments of  $W_i^2$

Similar argumentations enable us to express the second moments of  $W_i^2$ ,  $i=1, 2$ , in terms of field correlation functions. For this purpose we multiply Eq. [\(A13\)](#page-6-8) by  $(W_1^2 + W_2^2)$  and average it,

$$
\langle S_{xy}W_1^2 \rangle + \langle S_{xy}W_2^2 \rangle = \frac{1}{2} \Big( \langle W_1^4 \rangle - \langle W_2^4 \rangle \Big) \langle \sin 2\phi \rangle .
$$
\n(D23)

Here

$$
\langle S_{xy} W_i^2 \rangle = 4 \left[ \int_{\mathbb{R}^2} d^2 \mathbf{r} \, xy \, \langle W_i^2 I(\mathbf{r}, L) \rangle \right] \tag{D24}
$$

$$
- \int_{\mathbb{R}^4} d^2 \mathbf{r}_1 d^2 \mathbf{r}_2 \, x_1 y_2 \, \langle W_i^2 I(\mathbf{r}_1, L) I(\mathbf{r}_2, L) \rangle \right].
$$

The isotropy condition [\(D2\)](#page-9-4) implies that the functions  $\langle W_i^2 I(\mathbf{r}, L) \rangle$  and  $\int_{\mathbb{R}^2} dx_2 dy_1 \langle W_i^2 I(\mathbf{r}_1, L) I(\mathbf{r}_2, L) \rangle$  have such a symmetry in  $(x, y)$  and  $(x_1, y_2)$  planes, respectively, that the integrals in Eq. [\(D24\)](#page-11-1) are zeros. This means that the left-hand side of Eq. [\(D23\)](#page-11-2) is also zero, which results in

$$
\langle W_1^4 \rangle = \langle W_2^4 \rangle. \tag{D25}
$$

The assumption of isotropy also implies that

$$
\langle S_{xx}^2 \rangle = \langle S_{yy}^2 \rangle, \tag{D26}
$$

i.e. the second moments of  $S_{xx/yy}$  are equal.

<span id="page-11-0"></span>Equations [\(A11\)](#page-6-7) and [\(A12\)](#page-6-10) enable to express the moments  $\langle S_{xx/yy}^2 \rangle$  and  $\langle S_{xx}S_{yy} \rangle$  in terms of the moments  $\langle W^4_{1/2} \rangle$  and  $\langle W^2_1 W^2_2 \rangle$ ,

<span id="page-11-3"></span>
$$
\langle S_{xx/yy}^2 \rangle = \frac{3}{4} \langle W_{1/2}^4 \rangle + \frac{1}{4} \langle W_1^2 W_2^2 \rangle, \tag{D27}
$$

<span id="page-11-4"></span>
$$
\langle S_{xx} S_{yy} \rangle = \frac{1}{4} \langle W_{1/2}^4 \rangle + \frac{3}{4} \langle W_1^2 W_2^2 \rangle, \tag{D28}
$$

where we have utilized the absence of correlations between  $W_{1/2}^2$  and the angle  $\phi$ , cf. Eq. [\(D9\)](#page-10-4). Inverting Eqs. [\(D27\)](#page-11-3) and [\(D28\)](#page-11-4) one gets

<span id="page-11-8"></span>
$$
\langle W_{1/2}^4 \rangle = \frac{3}{2} \langle S_{xx/yy}^2 \rangle - \frac{1}{2} \langle S_{xx} S_{yy} \rangle, \tag{D29}
$$

<span id="page-11-9"></span>
$$
\langle W_1^2 W_2^2 \rangle = -\frac{1}{2} \langle S_{xx/yy}^2 \rangle + \frac{3}{2} \langle S_{xx} S_{yy} \rangle. \tag{D30}
$$

Since the moments  $\langle S_{xx/yy}^2 \rangle$  and  $\langle S_{xx}S_{yy} \rangle$  can be expressed in terms of field correlation functions, we get a tool for obtaining the moments  $\langle W_{1/2}^4 \rangle$  and  $\langle W_1^2 W_2^2 \rangle$ .

<span id="page-11-7"></span>The straightforward expressions for  $\langle S_{xx/yy}^2 \rangle$  and  $\langle S_{xx}S_{yy}\rangle$  contain the even-order field correlation functions up to  $\Gamma_8$ . Analytical methods are quite involved for evaluation of sixth- and eight-order functions. By using the assumptions of Gaussianity for the beam parameters and isotropic properties of the turbulence we can rewrite these expressions in terms of field correlation functions  $\Gamma_2$  and  $\Gamma_4$  only.

The moment  $\langle S_{xx}^2 \rangle$  is obtained from Eq. [\(A8\)](#page-6-6) by squaring and averaging  $S_{xx}^2$ ,

$$
\langle S_{xx}^2 \rangle = 16 \left( \int_{\mathbb{R}^4} d^2 \mathbf{r}_1 d^2 \mathbf{r}_2 x_1^2 x_2^2 \Gamma_4(\mathbf{r}_1, \mathbf{r}_2; L) + \langle x_0^4 \rangle \right)
$$
\n
$$
-2 \left\langle x_0^2 \int_{\mathbb{R}^2} d^2 \mathbf{r} x^2 I(\mathbf{r}; L) \right\rangle \right),
$$
\n(D31)

<span id="page-11-2"></span><span id="page-11-1"></span>and similarly for the moment  $\langle S_{yy}^2 \rangle$ . The second term on the right-hand side of this expression contains the field correlation function  $\Gamma_8$ . However, assuming that the beam-centroid coordinate,  $x_0$ , is a Gaussian variable and utilizing Eq. [\(D1\)](#page-9-3), this term can be written as

<span id="page-11-10"></span><span id="page-11-6"></span><span id="page-11-5"></span>
$$
\langle x_0^4 \rangle = 3 \langle x_0^2 \rangle^2. \tag{D32}
$$

Here

$$
\langle x_0^2 \rangle = \int_{\mathbb{R}^4} d^2 \mathbf{r}_1 d^2 \mathbf{r}_2 x_1 x_2 \Gamma_4(\mathbf{r}_1, \mathbf{r}_2; L), \tag{D33}
$$

which is expressed in terms of the field correlation function  $\Gamma_4$ .

Consider the third term in right-hand side of Eq. [\(D31\)](#page-11-5). By using Eqs. [\(A8\)](#page-6-6), [\(A11\)](#page-6-7), [\(D9\)](#page-10-4), and [\(D32\)](#page-11-6) one gets

$$
\left\langle x_0^2 \int_{\mathbb{R}^2} d^2 \mathbf{r} \, x^2 I(\mathbf{r}; L) \right\rangle = \frac{1}{4} \langle x_0^2 W_{1/2}^2 \rangle + 3 \langle x_0^2 \rangle^2.
$$
 (D34)

Because the assumption of isotropy results in the fact that the beam-centroid coordinate  $x_0$  does not correlate with the spot-shape parameters, cf. Section [D.3,](#page-10-7) we can write

$$
\langle x_0^2 W_{1/2}^2 \rangle = \langle x_0^2 \rangle \langle S_{xx/yy} \rangle, \tag{D35}
$$

where we have also used Eq. [\(D22\)](#page-11-7). Next, the expression for the moment  $\langle S_{xx}^2 \rangle$ , cf. Eq. [\(D31\)](#page-11-5), in terms of the second- and fourth-order field correlation functions reads as

$$
\langle S_{xx}^2 \rangle = 16 \left( \int_{\mathbb{R}^4} d^2 \mathbf{r}_1 d^2 \mathbf{r}_2 x_1^2 x_2^2 \Gamma_4(\mathbf{r}_1, \mathbf{r}_2; L) \right) \tag{D36}
$$

$$
-3 \langle x_0^2 \rangle^2 - \frac{1}{2} \langle x_0^2 \rangle \langle S_{xx} \rangle \right).
$$

Similar considerations should be applied for the calculation of the moment  $\langle S_{xx}S_{yy}\rangle$ , taking into account that  $\langle x_0^2 y_0^2 \rangle = \langle x_0^2 \rangle^2.$ 

Finally, we substitute the obtained expressions for the moments  $\langle S_{xx}^2 \rangle$  and  $\langle S_{xx}S_{yy} \rangle$  in Eqs. [\(D29\)](#page-11-8) and [\(D30\)](#page-11-9). This results in relations for the moments  $\langle W_{1/2}^4 \rangle$  and  $\langle W_1^2 W_2^2 \rangle$  in terms of field correlation functions  $\Gamma_2$  and  $\Gamma_4$ ,

$$
\langle W_{1/2}^4 \rangle = 8 \left( 3 \int_{\mathbb{R}^4} d^2 \mathbf{r}_1 d^2 \mathbf{r}_2 x_1^2 x_2^2 \Gamma_4(\mathbf{r}_1, \mathbf{r}_2; L) \right) \tag{D37}
$$

$$
- \int_{\mathbb{R}^4} d^2 \mathbf{r}_1 d^2 \mathbf{r}_2 x_1^2 y_2^2 \Gamma_4(\mathbf{r}_1, \mathbf{r}_2; L) - 8 \langle x_0^2 \rangle^2 - \langle x_0^2 \rangle \langle S_{xx} \rangle \right),
$$

$$
\langle W_1^2 W_2^2 \rangle = 8 \left( 3 \int_{\mathbb{R}^4} d^2 \mathbf{r}_1 d^2 \mathbf{r}_2 x_1^2 y_2^2 \Gamma_4(\mathbf{r}_1, \mathbf{r}_2; L) \right) \quad (D38)
$$

$$
- \int_{\mathbb{R}^4} d^2 \mathbf{r}_1 d^2 \mathbf{r}_2 x_1^2 x_2^2 \Gamma_4(\mathbf{r}_1, \mathbf{r}_2; L) - \langle x_0^2 \rangle \langle S_{xx} \rangle \right).
$$

Here  $\langle S_{xx} \rangle$  and  $\langle x_0^2 \rangle$  are given by Eqs. [\(D14\)](#page-10-5) and [\(D33\)](#page-11-10), respectively.

### <span id="page-12-0"></span>E. PHASE APPROXIMATION OF THE HUYGENS-KIRCHHOFF METHOD

The parameters, which characterize statistical properties of elliptic beams, are expressed in terms of the field correlation functions  $\Gamma_2$  and  $\Gamma_4$ , see Section [D.](#page-9-0) Here we briefly discuss the method of obtaining these functions as proposed in Ref. [\[34\]](#page-4-25). We start from the paraxial equation, cf. Eq. [\(A1\)](#page-6-1), which describes the beam amplitude,  $u(\mathbf{r}, z)$  and the corresponding boundary condition,  $u_0(\mathbf{r}')$ , cf. Eq. [\(A2\)](#page-6-11). For our purposes this equation is represented in such an integral form,

$$
u(\mathbf{r}, z) = \int_{\mathbb{R}^2} d^2 \mathbf{r}' u_0(\mathbf{r}') G_0(\mathbf{r}, \mathbf{r}'; z, 0) G_1(\mathbf{r}, \mathbf{r}'; z, 0)
$$

$$
+ \frac{i}{2k} \int_0^z dz' \int_{\mathbb{R}^2} d^2 \mathbf{r}' u(\mathbf{r}', z') G_0(\mathbf{r}, \mathbf{r}'; z, z') \Delta' G_1(\mathbf{r}, \mathbf{r}'; z, z').
$$
(E1)

Here

$$
G_0(\mathbf{r}, \mathbf{r}'; z, z') = \frac{k}{2\pi i(z - z')} \exp\left[\frac{ik|\mathbf{r} - \mathbf{r}'|^2}{2(z - z')}\right], \quad \text{(E2)}
$$

<span id="page-12-7"></span><span id="page-12-3"></span>
$$
G_1(\mathbf{r}, \mathbf{r}'; z, z') = \exp\Big[iS(\mathbf{r}, \mathbf{r}'; z, z')\Big],\tag{E3}
$$

$$
S(\mathbf{r}, \mathbf{r}'; z, z') = k \int_{z'}^{z} d\xi \, \delta n \left( \mathbf{r} \frac{\xi - z'}{z - z'} + \mathbf{r}' \frac{z - \xi}{z - z'}, \xi \right), \quad \text{(E4)}
$$

<span id="page-12-8"></span>and  $\Delta'$  is the transverse Laplace operator acting on functions of r'.

The phase approximation assumes that we consider the zero-order approximation for the solution of Eq. [\(E1\)](#page-12-1) in the aperture plane  $z=L$ , i.e.

<span id="page-12-2"></span>
$$
u(\mathbf{r},L) = \int_{\mathbb{R}^2} d^2 \mathbf{r}' u_0(\mathbf{r}') G_0 G_1(\mathbf{r},\mathbf{r}';L,0).
$$
 (E5)

Substituting this expression in the definition of the field correlation functions, cf. Eqs. [\(D17\)](#page-10-8) and [\(D18\)](#page-10-9), one gets

<span id="page-12-10"></span>
$$
\Gamma_{2n}(\mathbf{r}_1,\ldots,\mathbf{r}_n;L) =
$$
\n
$$
\int_{\mathbb{R}^{4n}} d^2 \mathbf{r}'_1 \ldots d^2 \mathbf{r}'_{2n} u_0(\mathbf{r}'_1) u_0^*(\mathbf{r}'_2) \ldots u_0(\mathbf{r}'_{2n-1}) u_0^*(\mathbf{r}'_{2n})
$$
\n
$$
\times \mathcal{G}_{2n,0}(\mathbf{r}_1,\ldots,\mathbf{r}_n,\mathbf{r}'_1,\ldots,\mathbf{r}'_{2n};L,0)
$$
\n
$$
\times \langle \mathcal{G}_{2n,1}(\mathbf{r}_1,\ldots,\mathbf{r}_n,\mathbf{r}'_1,\ldots,\mathbf{r}'_{2n};L,0) \rangle,
$$

<span id="page-12-9"></span>where  $n=1, 2, \ldots$ ,

$$
\mathcal{G}_{2n,i}(\mathbf{r}_1,\ldots,\mathbf{r}_n,\mathbf{r}'_1,\ldots,\mathbf{r}'_{2n};L,0) =
$$
\n
$$
\prod_{k=1}^n G_i(\mathbf{r}_k,\mathbf{r}'_{2k-1};L,0) G_i^*(\mathbf{r}_k,\mathbf{r}'_{2k};L,0),
$$
\n(E7)

and  $i=0, 1$ . The assumption that  $\delta n(\mathbf{r}; z)$  is a Gaussian stochastic field enables to average  $\mathcal{G}_{2n,1}$  in Eq. [\(E6\)](#page-12-2), such that

<span id="page-12-6"></span>
$$
\langle \mathcal{G}_{2n,1}(\mathbf{r}_1,\ldots,\mathbf{r}_n,\mathbf{r}'_1,\ldots,\mathbf{r}'_{2n};L,0)\rangle =
$$
(E8)  

$$
\exp\Big[\frac{1}{2}\sum_{k=2}^{2n}\sum_{l=1}^{k-1}(-1)^{k+l}\mathcal{D}_S(\mathbf{r}_l,\mathbf{r}_k;\mathbf{r}'_l,\mathbf{r}'_k;L,0)\Big].
$$

Here

<span id="page-12-5"></span>
$$
\mathcal{D}_S(\mathbf{r}_l, \mathbf{r}_k; \mathbf{r}'_l, \mathbf{r}'_k; L, 0)
$$
\n
$$
= \left\langle \left[ S(\mathbf{r}_l, \mathbf{r}'_l; L, 0) - S(\mathbf{r}_k, \mathbf{r}'_k; L, 0) \right]^2 \right\rangle
$$
\n(E9)

is the structure function of phase fluctuations of a spherical wave propagating in turbulence.

<span id="page-12-1"></span>The correlation function for the index of refraction in the Markovian approximation, cf. e.g. Ref. [\[15\]](#page-4-24), reads as

<span id="page-12-4"></span>
$$
\langle \delta n(\mathbf{r}; z) \delta n(\mathbf{r}'; z') \rangle
$$
\n
$$
= 2\pi \delta(z - z') \int_{\mathbb{R}^2} d^2 \kappa \, \Phi_n(\kappa, z) e^{i\kappa \cdot (\mathbf{r} - \mathbf{r}')}.
$$
\n(E10)

Here  $\Phi_n(\kappa, z)$  is the spectrum of turbulence, which we use in the Kolmogorov form, see Ref. [\[11\]](#page-4-4),

$$
\Phi_n(\kappa, z) = 0.033 C_n^2(z) \kappa^{-\frac{11}{3}},\tag{E11}
$$

and  $C_n^2(z)$  is the refractive index structure constant. Inserting Eqs.  $(E4)$ ,  $(E10)$  and  $(E11)$  in Eq.  $(E9)$ , we arrive at the following expression for the phase structure function

$$
\mathcal{D}_S(\mathbf{r}, \mathbf{r}') = 2\rho_0^{-\frac{5}{3}} \int_0^1 d\xi \left| \mathbf{r} \xi + \mathbf{r}'(1 - \xi) \right|^{\frac{5}{3}}, \quad \text{(E12)}
$$

where we assume that  $C_n^2$  is constant for the horizontal link,

$$
\mathcal{D}_S(\mathbf{r}_k - \mathbf{r}_l, \mathbf{r}_k' - \mathbf{r}_l') = \mathcal{D}_S(\mathbf{r}_l, \mathbf{r}_k; \mathbf{r}_l', \mathbf{r}_k'; L, 0), \quad \text{(E13)}
$$

is a simplified notion for the structure function of phase fluctuations,

$$
\rho_0 = (1.5 C_n^2 k^2 L)^{-3/5}
$$
 (E14)

is the radius of spatial coherence of a plane wave in the atmosphere.

Finally we substitute Eqs. [\(E12\)](#page-13-2), [\(E13\)](#page-13-3) into Eq. [\(E8\)](#page-12-6). Then substituting Eqs. [\(E2\)](#page-12-7) and [\(E8\)](#page-12-6) into Eq. [\(E6\)](#page-12-2) and performing some trivial integrations, we evaluate the field correlation functions for  $n = 1, 2$ ,

$$
\Gamma_2(\mathbf{r}) = \frac{k^2}{4\pi^2 L^2} \int_{\mathbb{R}^2} d^2 \mathbf{r}' e^{-\frac{g^2 |\mathbf{r}'|^2}{2W_0^2} - 2i \frac{\Omega}{W_0^2} \mathbf{r} \cdot \mathbf{r}' - \frac{1}{2} \mathcal{D}_S(0, \mathbf{r}')} (E15)
$$

and

$$
\Gamma_4(\mathbf{r}_1, \mathbf{r}_2) = \frac{2k^4}{\pi^2 (2\pi)^3 L^4 W_0^2} \int_{\mathbb{R}^6} d^2 \mathbf{r}_1' d^2 \mathbf{r}_2' d^2 \mathbf{r}_3'
$$
  
\n
$$
\times e^{-\frac{1}{W_0^2} (|\mathbf{r}_1'|^2 + |\mathbf{r}_2'|^2 + g^2 |\mathbf{r}_3'|^2) + 2i \frac{\Omega}{W_0^2} [1 - \frac{L}{F}] \mathbf{r}_1' \cdot \mathbf{r}_2'}
$$
  
\n
$$
\times e^{-2i \frac{\Omega}{W_0^2} (\mathbf{r}_1 - \mathbf{r}_2) \cdot \mathbf{r}_2' - 2i \frac{\Omega}{W_0^2} (\mathbf{r}_1 + \mathbf{r}_2) \cdot \mathbf{r}_3'}
$$
  
\n
$$
\times \exp\left[\frac{1}{2} \sum_{j=1,2} \left\{ \mathcal{D}_S(\mathbf{r}_1 - \mathbf{r}_2, \mathbf{r}_1' + (-1)^j \mathbf{r}_2') \right\} - \mathcal{D}_S(\mathbf{r}_1 - \mathbf{r}_2, \mathbf{r}_1' + (-1)^j \mathbf{r}_3') \right\} \right].
$$
  
\n(E16)

Here

$$
\Omega = \frac{kW_0^2}{2L} \tag{E17}
$$

is the Fresnel number of the transmitter aperture and  $g^2=1+\Omega^2[1-\frac{L}{F}]^2$  is the generalized diffraction beam parameter.

# <span id="page-13-5"></span><span id="page-13-0"></span>F. BEAM WANDERING

<span id="page-13-1"></span>In this Section we derive the beam-wandering variance for weak and strong turbulence regimes. The beamwandering variance  $\langle x_0^2 \rangle$  is evaluated by substituting Eq. [\(E16\)](#page-13-4) into Eq. [\(D33\)](#page-11-10)

$$
\langle x_0^2 \rangle = \frac{2k^4}{\pi^2 (2\pi)^3 L^4 W_0^2} \int_{\mathbb{R}^{10}} d^2 \mathbf{R} d^2 \mathbf{r} d^2 \mathbf{r}'_1 d^2 \mathbf{r}'_2 d^2 \mathbf{r}'_3
$$
  
 
$$
\times \left( R_x^2 - \frac{r_x^2}{4} \right) e^{-\frac{1}{W_0^2} (|\mathbf{r}'_1|^2 + |\mathbf{r}'_2|^2 + g^2 |\mathbf{r}'_3|^2)} e^{-4i \frac{\Omega}{W_0^2} \mathbf{R} \cdot \mathbf{r}'_3}
$$
  
 
$$
\times e^{2i \frac{\Omega}{W_0^2} [1 - \frac{L}{F}] \mathbf{r}'_1 \cdot \mathbf{r}'_2 - 2i \frac{\Omega}{W_0^2} \mathbf{r} \cdot \mathbf{r}'_2} \mathcal{J}(\mathbf{r}, \mathbf{r}'_1, \mathbf{r}'_2, \mathbf{r}'_3), \qquad (F1)
$$

<span id="page-13-2"></span>with

<span id="page-13-8"></span><span id="page-13-3"></span>
$$
\mathcal{J}(\mathbf{r}, \mathbf{r}'_1, \mathbf{r}'_2, \mathbf{r}'_3)
$$
\n
$$
= \exp \Big[ \rho_0^{-\frac{5}{3}} \int_0^1 d\xi \sum_{j=1,2} \Big( \big| \mathbf{r}\xi + [\mathbf{r}'_1 + (-1)^j \mathbf{r}'_2] (1-\xi) \big|^{\frac{5}{3}}
$$
\n
$$
- \big| \mathbf{r}\xi + [\mathbf{r}'_1 + (-1)^j \mathbf{r}'_3] (1-\xi) \big|^{\frac{5}{3}} - (1-\xi)^{\frac{5}{3}} \big| \mathbf{r}'_2 + (-1)^j \mathbf{r}'_3 \big|^{\frac{5}{3}} \Big) \Big],
$$
\n(F2)

<span id="page-13-6"></span>where we have used the variables  $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$  and  $\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2)/2$ . We integrate over the variables **R** and  $\mathbf{r}'_3$ using the properties of Dirac delta function, which occurs in the integral representation of Eq. [\(F1\)](#page-13-5). For example one can show that

<span id="page-13-10"></span>
$$
\int_{\mathbb{R}^4} d^2 \mathbf{R} d^2 \mathbf{r}'_3 \mathbf{R}^2 e^{-4i \frac{\Omega}{W_0^2} \mathbf{R} \cdot \mathbf{r}'_3} f(\mathbf{r}'_3)
$$
\n
$$
= -\frac{(2\pi)^2 W_0^8}{(4\Omega)^4} \Delta_{\mathbf{r}'_3}^2 f(\mathbf{r}'_3) \Big|_{\mathbf{r}'_3=0}, \quad \text{(F3)}
$$

<span id="page-13-9"></span>where  $\Delta_{\mathbf{r}'_3}^2$  is the transverse Laplace operator and  $f(x)$  is an arbitrary function. We arrive at

$$
\langle x_0^2 \rangle = \frac{2\Omega^2}{(2\pi)^3 W_0^6} \int_{\mathbb{R}^6} d^2 \mathbf{r} d^2 \mathbf{r}_1' d^2 \mathbf{r}_2' \left( \frac{g^2 W_0^2}{2\Omega^2} - r_x^2 \right)
$$
  
\n
$$
\times e^{-\frac{1}{W_0^2} (|\mathbf{r}_1'|^2 + |\mathbf{r}_2'|^2)} e^{2i \frac{\Omega}{W_0^2} [1 - \frac{L}{F}] \mathbf{r}_1' \cdot \mathbf{r}_2' - 2i \frac{\Omega}{W_0^2} \mathbf{r} \cdot \mathbf{r}_2'}
$$
  
\n
$$
\times \exp \left[ \rho_0^{-\frac{5}{3}} \int_0^1 d\xi \left( \sum_{j=1,2} |\mathbf{r}\xi + [\mathbf{r}_1' + (-1)^j \mathbf{r}_2'] (1 - \xi) |^{\frac{5}{3}}
$$
  
\n
$$
-2 |\mathbf{r}\xi + \mathbf{r}_1' (1 - \xi) |^{\frac{5}{3}} - 2(1 - \xi)^{\frac{5}{3}} |\mathbf{r}_2' |^{\frac{5}{3}} \right).
$$
 (F4)

<span id="page-13-4"></span>Let us consider the cases of weak and strong turbulence separately.

### <span id="page-13-7"></span>F.1. Weak turbulence

<span id="page-13-11"></span>The weak turbulence is characterized by large values of the parameter  $\rho_0$ , cf. Eq. [\(E14\)](#page-13-6) together with the dependence on the Rytov parameter in [\(F7\)](#page-14-0). Hence, we can expand the last exponent of [\(F4\)](#page-13-7) into series with respect to  $\rho_0^{-\frac{5}{3}}$  up to the first order. The first term of the

$$
\langle x_0^2 \rangle = \frac{2\Omega^2 \rho_0^{-\frac{5}{3}}}{(2\pi)^3 W_0^6} \int_{\mathbb{R}^6} d^2 \mathbf{r} d^2 \mathbf{r}'_1 d^2 \mathbf{r}'_2 \left( \frac{g^2 W_0^2}{2\Omega^2} - r_x^2 \right) \times e^{-\frac{1}{W_0^2} (|\mathbf{r}'_1|^2 + |\mathbf{r}'_2|^2)} e^{2i \frac{\Omega}{W_0^2} [1 - \frac{L}{F}] \mathbf{r}'_1 \cdot \mathbf{r}'_2 - 2i \frac{\Omega}{W_0^2} \mathbf{r} \cdot \mathbf{r}'_2} \times \int_0^1 d\xi \left( \sum_{j=1,2} |\mathbf{r}\xi + [\mathbf{r}'_1 + (-1)^j \mathbf{r}'_2] (1 - \xi) \right)^{\frac{5}{3}} -2 |\mathbf{r}\xi + \mathbf{r}'_1 (1 - \xi)|^{\frac{5}{3}} - 2(1 - \xi)^{\frac{5}{3}} |\mathbf{r}'_2|^{\frac{5}{3}}.
$$
 (F5)

Performing the multiple integrations in this equation, one derives for the beam-wandering variance for a focused beam,  $L=F$  (defined in Eq. [\(A2\)](#page-6-11)), for weak turbulence the result

$$
\langle x_0^2 \rangle = 0.94 C_n^2 L^3 W_0^{-\frac{1}{3}} = 0.33 W_0^2 \sigma_R^2 \Omega^{-\frac{7}{6}}, \qquad \text{(F6)}
$$

where

$$
\sigma_R^2 = 1.23 C_n^2 k^{\frac{7}{6}} L^{\frac{11}{6}} = 0.82 \rho_0^{-\frac{5}{3}} k^{-\frac{5}{6}} L^{\frac{5}{6}}
$$
 (F7)

is the Rytov parameter [\[11\]](#page-4-4).

### <span id="page-14-3"></span><span id="page-14-2"></span><span id="page-14-1"></span>F.2. Strong turbulence

For the case of strong turbulence the parameter  $\rho_0$  is small. The exponential in Eq. [\(F2\)](#page-13-8),  $\mathcal{J}(\mathbf{r}, \mathbf{r}'_1, \mathbf{r}'_2, \mathbf{r}'_3)$ , significantly differs from zero in the following regions:

$$
|\mathbf{r}'_2|(1-\xi) \gg \rho_0, \quad |\mathbf{r}'_3|(1-\xi), \ |\mathbf{r}\xi + \mathbf{r}'_1(1-\xi)| \lesssim \rho_0; \quad \text{(F8)}
$$

$$
|\mathbf{r}\xi+\mathbf{r}'_1(1-\xi)| \gg \rho_0, \quad |\mathbf{r}'_2|(1-\xi), |\mathbf{r}'_3|(1-\xi) \lesssim \rho_0;
$$
 (F9)

$$
|\mathbf{r}'_2|(1-\xi), \ |\mathbf{r}'_3|(1-\xi), \ |\mathbf{r}\xi + \mathbf{r}'_1(1-\xi)| \lesssim \rho_0. \tag{F10}
$$

This function is negligibly small provided that any of the conditions

<span id="page-14-5"></span>
$$
|\mathbf{r}_3'|(1-\xi) \gg \rho_0, |\mathbf{r}_2'|(1-\xi), |\mathbf{r}\xi + \mathbf{r}_1'(1-\xi)| \lesssim \rho_0; |\mathbf{r}\xi + \mathbf{r}_1'(1-\xi)|, |\mathbf{r}_2'|(1-\xi) \gg \rho_0, |\mathbf{r}_3'|(1-\xi) \lesssim \rho_0; |\mathbf{r}\xi + \mathbf{r}_1'(1-\xi)|, |\mathbf{r}_3'|(1-\xi) \gg \rho_0, |\mathbf{r}_2'|(1-\xi) \lesssim \rho_0; \quad \text{(F11)} |\mathbf{r}_2'|(1-\xi), |\mathbf{r}_3'|(1-\xi) \gg \rho_0, |\mathbf{r}\xi + \mathbf{r}_1'(1-\xi)| \lesssim \rho_0; |\mathbf{r}\xi + \mathbf{r}_1'(1-\xi)|, |\mathbf{r}_2'|(1-\xi), |\mathbf{r}_3'|(1-\xi) \gg \rho_0
$$

<span id="page-14-4"></span><span id="page-14-0"></span>holds true. The function [\(F2\)](#page-13-8) can be approximated then as

$$
\mathcal{J}(\mathbf{r}, \mathbf{r}'_{1}, \mathbf{r}'_{2}, \mathbf{r}'_{3}) = \exp\left[-\rho_{0}^{-\frac{5}{3}}\int_{0}^{\frac{1}{2}} d\xi \sum_{j=1,2} |\mathbf{r}\xi + [\mathbf{r}'_{1} + (-1)^{j} \mathbf{r}'_{3}](1-\xi)|^{\frac{5}{3}}\right] \sum_{n=0}^{\infty} \frac{\rho_{0}^{-\frac{5}{3}n}}{n!} \Big\{ \sum_{j=1,2} \Big( \int_{0}^{\frac{1}{2}} d\xi \left| \mathbf{r}\xi + [\mathbf{r}'_{1} + (-1)^{j} \mathbf{r}'_{2}](1-\xi)|^{\frac{5}{3}} \Big) \Big\}^{n} + \exp\left[-\frac{3}{8}\rho_{0}^{-\frac{5}{3}}\left|\mathbf{r}'_{2} + (-1)^{j} \mathbf{r}'_{3}\right|^{\frac{5}{3}}\right] \sum_{n=0}^{\infty} \frac{\rho_{0}^{-\frac{5}{3}n}}{n!} \Big\{ \sum_{j=1,2} \int_{0}^{\frac{1}{2}} d\xi \Big( |\mathbf{r}\xi + [\mathbf{r}'_{1} + (-1)^{j} \mathbf{r}'_{2}](1-\xi)|^{\frac{5}{3}} \Big) \Big\}^{n} + \exp\left[-\frac{3}{8}\rho_{0}^{-\frac{5}{3}}\left|\mathbf{r}'_{2} + (-1)^{j} \mathbf{r}'_{3}\right|^{\frac{5}{3}}\right] \sum_{n=0}^{\infty} \frac{\rho_{0}^{-\frac{5}{3}n}}{n!} \Big\{ \sum_{j=1,2} \int_{0}^{\frac{1}{2}} d\xi \left| \mathbf{r}\xi + [\mathbf{r}'_{1} + (-1)^{j} \mathbf{r}'_{3}](1-\xi)|^{\frac{5}{3}} \Big\} \Big\}
$$
\n
$$
+ \sum_{n=0}^{\infty} \frac{\left(\frac{3}{8}\right)^{n}}{n!} \rho_{0}^{-\frac{5}{3}n} \Big\{ \sum_{j=1,2} \int_{0}^{\frac{1}{2}} d\xi \left| \mathbf{r}\xi + [\mathbf{r}'_{1} + (-1)^{j} \mathbf{r}'_{3}](1-\xi)|^{\frac{5}{3}} \
$$

 $\overline{1}$ 

Here the first term on the right hand side accounts for the contributions from the regions [\(F8\)](#page-14-1) and [\(F9\)](#page-14-2). If one substitutes the latter into Eqs. [\(F1\)](#page-13-5) and [\(F2\)](#page-13-8) and performs integrations, then the region [\(F10\)](#page-14-3) would be counted twice. Therefore, the last term on the right hand side of [\(F12\)](#page-14-4) is introduced to eliminate the aforementioned double-counting. It is worth to mention that already the first  $(n = 0, 1)$  terms of expansion [\(F12\)](#page-14-4) give a good approximation of the function  $J$ , cf. Ref. [\[34\]](#page-4-25).

Substituting the right-hand side of Eq. [\(F12\)](#page-14-4) into Eq. [\(F1\)](#page-13-5) and integrating over the variables  $\mathbf{R}$ ,  $\mathbf{r}'_3$  as it is described above, we obtain

$$
\langle x_{0}^{2} \rangle = \frac{2\Omega^{2}}{(2\pi)^{3}W_{0}^{6}} \int_{\mathbb{R}^{6}} d^{2} \mathbf{r} d^{2} \mathbf{r}'_{1} d^{2} \mathbf{r}'_{2} \left( \frac{g^{2}W_{0}^{2}}{2\Omega^{2}} - r_{x}^{2} \right) e^{-\frac{1}{W_{0}^{2}} (|\mathbf{r}'_{1}|^{2} + |\mathbf{r}'_{2}|^{2})} e^{2i \frac{\Omega}{W_{0}^{2}} [1 - \frac{L}{F}|\mathbf{r}'_{1} \cdot \mathbf{r}'_{2} - 2i \frac{\Omega}{W_{0}^{2}} \mathbf{r} \cdot \mathbf{r}'_{2}} \times \left\{ \exp \left[ -\rho_{0}^{-\frac{5}{3}} \int_{0}^{1} d\xi \sum_{j=1,2} |\mathbf{r}\xi + [\mathbf{r}'_{1} + (-1)^{j} \mathbf{r}'_{3}] (1 - \xi)|^{\frac{5}{3}} \right] \left( 1 - \frac{3}{4} \rho_{0}^{-\frac{5}{3}} |\mathbf{r}'_{2}|^{\frac{5}{3}} + \rho_{0}^{-\frac{5}{3}} \sum_{j=1,2} \int_{0}^{1} d\xi |\mathbf{r}\xi + [\mathbf{r}'_{1} + (-1)^{j} \mathbf{r}'_{2}] (1 - \xi)|^{\frac{5}{3}} \right)
$$
  
+ 
$$
\exp \left[ -\frac{3}{4} \rho_{0}^{-\frac{5}{3}} |\mathbf{r}'_{2}|^{\frac{5}{3}} \right] \left( 1 - 2 \rho_{0}^{-\frac{5}{3}} \int_{0}^{1} d\xi |\mathbf{r}\xi + \mathbf{r}'_{1} (1 - \xi)|^{\frac{5}{3}} + \rho_{0}^{-\frac{5}{3}} \int_{0}^{1} d\xi \sum_{j=1,2} |\mathbf{r}\xi + [\mathbf{r}'_{1} + (-1)^{j} \mathbf{r}'_{2}] (1 - \xi)|^{\frac{5}{3}} \right)
$$
  
- 
$$
\exp \left[ -\rho_{0}^{-\frac{5}{3}} \left( \frac{3}{4} |\mathbf{r}'_{2}|^{\frac{5}{3}} + 2 \int_{0}^{1} d\xi |\mathbf{r}\xi + \mathbf{r}'_{1} (1 - \xi)|^{\frac{5
$$

The evaluation of Eq. [\(F13\)](#page-15-1) is simplified further with the use of the approximation [\[14\]](#page-4-26)

$$
\exp\left[-\left(\frac{|\mathbf{r}|}{\rho_0 \Omega}\right)^{\frac{5}{3}}\right] = \exp\left[-\left(\frac{|\mathbf{r}|}{\rho_0 \Omega}\right)^2\right],\qquad\text{(F14)}
$$

which gives good accuracy for small values of  $\rho_0$ , cf.Ref. [\[35\]](#page-4-21).

Consecutive integration of [\(F13\)](#page-15-1) yields for the colli-mated beam [\(A2\)](#page-6-11) with  $F \gg L$  the following result

$$
\langle x_0^2 \rangle = 1.78 C_n^{\frac{8}{5}} L^{\frac{37}{15}} k^{-\frac{1}{15}} = 0.75 W_0^2 \sigma_R^{\frac{8}{5}} \Omega^{-1}.
$$
 (F15)

A similar expression has been obtained by using the Markovian-random-process approximation, cf. Ref. [\[35\]](#page-4-21) and the references therein.

### <span id="page-15-0"></span>G. BEAM-SHAPE DISTORTION

In this Section we derive the expressions for the moments  $\langle W_{1/2}^2 \rangle$ ,  $\langle W_{1/2}^4 \rangle$  and  $\langle W_1^2 W_2^2 \rangle$  for weak and strong turbulence regimes. From Eqs. [\(D22\)](#page-11-7), [\(D36\)](#page-12-8)-[\(D38\)](#page-12-9) one can see that these moments are expressed through the integrals containing the field correlation functions  $\Gamma_2$  and  $\Gamma_4$ . The first moments of  $W_{1/2}^2$  defined by Eqs. [\(D14\)](#page-10-5), [\(D15\)](#page-10-6), and [\(D22\)](#page-11-7) contain the following integral

$$
\int_{\mathbb{R}^2} d^2 \mathbf{r} x^2 \Gamma_2(\mathbf{r}) = \frac{W_0^2}{\pi^2 \Omega^4} \int_{\mathbb{R}^4} d^2 \mathbf{r} d^2 \mathbf{r}' x^2 e^{-\frac{g^2}{2 \Omega^2} |\mathbf{r}'|^2}
$$
\n
$$
\times \exp\left[-\frac{2i}{\Omega} \mathbf{r} \cdot \mathbf{r}' - \rho_0^{-\frac{5}{3}} W_0^{\frac{5}{3}} \int_0^1 d\xi (1-\xi)^{\frac{5}{3}} \left(\frac{|\mathbf{r}'|}{\Omega}\right)^{\frac{5}{3}}\right].
$$
 (G1)

Here we have used Eqs. [\(E15\)](#page-13-9) and [\(E13\)](#page-13-3). The second moments of  $W_{1/2}^2$  defined in Eqs. [\(D37\)](#page-12-10), [\(D38\)](#page-12-9) contain

<span id="page-15-1"></span>the integrals

<span id="page-15-6"></span><span id="page-15-3"></span>
$$
\int_{\mathbb{R}^{4}} d^{2} \mathbf{r}_{1} d^{2} \mathbf{r}_{2} x_{1}^{2} x_{2}^{2} \Gamma_{4}(\mathbf{r}_{1}, \mathbf{r}_{2}) = \frac{\Omega^{2}}{2(2\pi)^{3} W_{0}^{6}} \int_{\mathbb{R}^{6}} d^{2} \mathbf{r} d^{2} \mathbf{r}'_{1} d^{2} \mathbf{r}'_{2}
$$
\n
$$
\times \left( \frac{3g^{4} W_{0}^{4}}{4\Omega^{4}} - \frac{g^{2} W_{0}^{2}}{\Omega^{2}} r_{x}^{2} + r_{x}^{4} \right) e^{-\frac{1}{W_{0}^{2}} \left( |\mathbf{r}'_{1}|^{2} + |\mathbf{r}'_{2}|^{2} \right)}
$$
\n
$$
\times \exp \left[ 2i \frac{\Omega}{W_{0}^{2}} \left( 1 - \frac{L}{F} \right) \mathbf{r}'_{1} \cdot \mathbf{r}'_{2} - 2i \frac{\Omega}{W_{0}^{2}} \mathbf{r} \cdot \mathbf{r}'_{2} \right] \qquad (G2)
$$
\n
$$
\times \exp \left[ \rho_{0}^{-\frac{5}{3}} \int_{0}^{1} d\xi \left( \sum_{j=1,2} |\mathbf{r}\xi + [\mathbf{r}'_{1} + (-1)^{j} \mathbf{r}'_{2}] (1 - \xi)|^{\frac{5}{3}}
$$
\n
$$
-2 |\mathbf{r}\xi + \mathbf{r}'_{1} (1 - \xi)|^{\frac{5}{3}} - 2(1 - \xi)^{\frac{5}{3}} |\mathbf{r}'_{2}|^{\frac{5}{3}} \right) \right]
$$

<span id="page-15-5"></span>and

$$
\int_{\mathbb{R}^{4}} d^{2} \mathbf{r}_{1} d^{2} \mathbf{r}_{2} x_{1}^{2} y_{2}^{2} \Gamma_{4}(\mathbf{r}_{1}, \mathbf{r}_{2}) = \frac{\Omega^{2}}{2(2\pi)^{3} W_{0}^{6}} \int_{\mathbb{R}^{6}} d^{2} \mathbf{r} d^{2} \mathbf{r}'_{1} d^{2} \mathbf{r}'_{2}
$$
\n
$$
\times \left( \frac{g^{4} W_{0}^{4}}{4\Omega^{4}} + \frac{g^{2} W_{0}^{2}}{\Omega^{2}} r_{x}^{2} + r_{x}^{2} r_{y}^{2} \right) e^{-\frac{1}{W_{0}^{2}} \left( |\mathbf{r}'_{1}|^{2} + |\mathbf{r}'_{2}|^{2} \right)}
$$
\n
$$
\times \exp \left[ 2i \frac{\Omega}{W_{0}^{2}} \left( 1 - \frac{L}{F} \right) \mathbf{r}'_{1} \cdot \mathbf{r}'_{2} - 2i \frac{\Omega}{W_{0}^{2}} \mathbf{r} \cdot \mathbf{r}'_{2} \right] \qquad (G3)
$$
\n
$$
\times \exp \left[ \rho_{0}^{-\frac{5}{3}} \int_{0}^{1} d\xi \left( \sum_{j=1,2} |\mathbf{r}\xi + [\mathbf{r}'_{1} + (-1)^{j} \mathbf{r}'_{2}] (1 - \xi) \right)^{\frac{5}{3}}
$$
\n
$$
-2 |\mathbf{r}\xi + \mathbf{r}'_{1} (1 - \xi)|^{\frac{5}{3}} - 2(1 - \xi)^{\frac{5}{3}} |\mathbf{r}'_{2}|^{\frac{5}{3}} \right].
$$

<span id="page-15-2"></span>Here we have used the definition of  $\Gamma_4$  given in Eq. [\(E16\)](#page-13-4) and performed the four-fold integration in a similar way as in Eq. [\(F1\)](#page-13-5), with the aid of formulas similar to [\(F3\)](#page-13-10).

### <span id="page-15-4"></span>G.1. Weak turbulence

In the limit of weak turbulence we derive, by substituting Eqs.  $(G1)$ ,  $(F6)$  in Eqs.  $(D14)$  and  $(D22)$ , the following result for the first moment of  $W_{1/2}^2$ :

$$
\langle W_{1/2}^2 \rangle = \frac{W_0^2}{\Omega^2} + 2.96 W_0^2 \sigma_R^2 \Omega^{-\frac{7}{6}}.
$$
 (G4)

In Eq. [\(G1\)](#page-15-2) we have used the approximations  $(|{\bf r}'|/\Omega)^{\frac{5}{3}} \approx$  $\left(\frac{|\mathbf{r}'|}{\Omega}\right)^2$ , cf. [\[14\]](#page-4-26) and  $\int_0^1 \mathrm{d}\xi f(\xi) \approx f(0)$ , cf. [\[36\]](#page-4-27). The first term in Eq. [\(G4\)](#page-16-0) describes the diffraction broadening in free space and the second term gives the amount of diffraction broadening in turbulence.

The second order moments of  $W_{1/2}^2$  are evaluated by substituting Eqs. [\(F6\)](#page-14-5), [\(G2\)](#page-15-3), [\(G4\)](#page-16-0) in Eq. [\(D37\)](#page-12-10) and correspondingly Eqs. [\(F6\)](#page-14-5), [\(G3\)](#page-15-4), [\(G4\)](#page-16-0) in Eq. [\(D38\)](#page-12-9). We evaluate the integrals in Eqs. [\(G2\)](#page-15-3) and [\(G3\)](#page-15-4) by expanding the last exponents into series with respect to  $\rho_0^{-\frac{5}{3}}$  up to the second order and consecutive integration. For a focused beam  $(L=F)$  we obtain

$$
\int_{\mathbb{R}^4} d^2 \mathbf{r}_1 d^2 \mathbf{r}_2 x_1^2 x_2^2 \Gamma_4(\mathbf{r}_1, \mathbf{r}_2)
$$
\n
$$
= \frac{W_0^4}{16\Omega^4} + 0.58 W_0^4 \sigma_R^2 \Omega^{-\frac{19}{6}} + 1.37 W_0^4 \sigma_R^4 \Omega^{-\frac{7}{3}}, \quad (G5)
$$

$$
\int_{\mathbb{R}^4} d^2 \mathbf{r}_1 d^2 \mathbf{r}_2 x_1^2 y_2^2 \Gamma_4(\mathbf{r}_1, \mathbf{r}_2)
$$
\n
$$
= \frac{W_0^4}{16\Omega^4} + 0.51 W_0^4 \sigma_R^2 \Omega^{-\frac{19}{6}} + 1.145 W_0^4 \sigma_R^4 \Omega^{-\frac{7}{3}}. (G6)
$$

<span id="page-16-0"></span>The corresponding (co)variances are evaluated as

$$
\langle (\Delta W_{1/2}^2)^2 \rangle = 1.2 W_0^4 \sigma_R^2 \Omega^{-\frac{19}{6}} + 0.17 W_0^4 \sigma_R^4 \Omega^{-\frac{7}{3}},
$$
 (G7)

$$
\langle \Delta W_1^2 \Delta W_2^2 \rangle = -0.8 W_0^4 \sigma_R^2 \Omega^{-\frac{19}{6}} - 0.05 W_0^4 \sigma_R^4 \Omega^{-\frac{7}{3}}, \quad (G8)
$$

correspondingly. The correlation function for weak turbulence is negative, i.e. the shape of the ellipse is deformed in such a way that the increase of the beam width along one half-axis of the ellipse causes the decrease of the width in the complementary direction.

#### <span id="page-16-2"></span>G.2. Strong turbulence

For strong turbulence the first moment of  $W_{1/2}^2$  is evaluated by substituting Eqs. [\(G1\)](#page-15-2) and [\(F15\)](#page-15-5) in Eqs. [\(D14\)](#page-10-5), [\(D22\)](#page-11-7). We also use the approximation [\(F14\)](#page-15-6) for evaluating [\(G1\)](#page-15-2) to obtain

$$
\langle W_{1/2}^2 \rangle = \gamma W_0^2 + 1.71 W_0^2 \sigma_R^{\frac{12}{5}} \Omega^{-1} - 2.99 W_0^2 \sigma_R^{\frac{8}{5}} \Omega^{-1}, \quad (G9)
$$

where  $\gamma = (1+\Omega^2)/\Omega^2$ . It is also assumed that  $\Omega > 1$ , cf. Eq. [\(E17\)](#page-13-11).

<span id="page-16-1"></span>For calculating the (co)variances of  $W_{1/2}^2$  we firstly evaluate the integrals in [\(G2\)](#page-15-3) and [\(G3\)](#page-15-4) by using the approximation [\(F12\)](#page-14-4) in the way described in Section [F.](#page-13-0) Within this approximation one gets, for example

$$
\int_{\mathbb{R}^4} d^2 \mathbf{r}_1 d^2 \mathbf{r}_2 x_1^2 x_2^2 \Gamma_4(\mathbf{r}_1, \mathbf{r}_2) = \frac{\Omega^2}{2(2\pi)^3 W_0^6} \int_{\mathbb{R}^6} d^2 \mathbf{r} d^2 \mathbf{r}_1' d^2 \mathbf{r}_2' \left(\frac{3}{4} \gamma^2 W_0^4 - \gamma W_0^2 r_x^2 + r_x^4\right) \times \exp\left[-\frac{1}{W_0^2} (|\mathbf{r}_1'|^2 + |\mathbf{r}_2'|^2) + 2i \frac{\Omega}{W_0^2} \mathbf{r}_1' \cdot \mathbf{r}_2' - 2i \frac{\Omega}{W_0^2} \mathbf{r} \cdot \mathbf{r}_2'\right]
$$
\n(G10)

$$
\times \Biggl\{ \exp \Bigl[ -\rho_0^{-\frac{5}{3}} \int_0^1 \mathrm{d}\xi \sum_{j=1,2} \bigl| \mathbf{r}\xi + [\mathbf{r}_1' + (-1)^j \mathbf{r}_3'] (1-\xi) \bigr|^{\frac{5}{3}} \Bigr] \Bigl( 1 - \frac{3}{4} \rho_0^{-\frac{5}{3}} \bigl| \mathbf{r}_2' \bigr|^{\frac{5}{3}} + \rho_0^{-\frac{5}{3}} \sum_{j=1,2} \int_0^1 \mathrm{d}\xi \bigl| \mathbf{r}\xi + [\mathbf{r}_1' + (-1)^j \mathbf{r}_2'] (1-\xi) \bigr|^{\frac{5}{3}} \Bigr) \Biggr\}
$$
  
+ 
$$
\exp \Bigl[ -\frac{3}{4} \rho_0^{-\frac{5}{3}} \bigl| \mathbf{r}_2' \bigr|^{\frac{5}{3}} \Bigr] \Bigl( 1 - 2 \rho_0^{-\frac{5}{3}} \int_0^1 \mathrm{d}\xi \bigl| \mathbf{r}\xi + \mathbf{r}_1' (1-\xi) \bigr|^{\frac{5}{3}} + \rho_0^{-\frac{5}{3}} \int_0^1 \mathrm{d}\xi \sum_{j=1,2} |\mathbf{r}\xi + [\mathbf{r}_1' + (-1)^j \mathbf{r}_2'] (1-\xi) \bigr|^{\frac{5}{3}} \Bigr)
$$
  
- 
$$
\exp \Bigl[ -\rho_0^{-\frac{5}{3}} \Bigl( \frac{3}{4} \bigl| \mathbf{r}_2' \bigr|^{\frac{5}{3}} + 2 \int_0^1 \mathrm{d}\xi \bigl| \mathbf{r}\xi + \mathbf{r}_1' (1-\xi) \bigr|^{\frac{5}{3}} \Bigr) \Bigr] \Bigl( 1 + \rho_0^{-\frac{5}{3}} \int_0^1 \mathrm{d}\xi \sum_{j=1,2} |\mathbf{r}\xi + [\mathbf{r}_1' + (-1)^j \mathbf{r}_2'] (1-\xi) \bigr|^{\frac{5}{3}} \Bigr) \Biggr\}.
$$

Performing the multiple integration in [\(G10\)](#page-16-1), one derives

×

$$
\int_{\mathbb{R}^4} d^2 \mathbf{r}_1 d^2 \mathbf{r}_2 x_1^2 x_2^2 \Gamma_4(\mathbf{r}_1, \mathbf{r}_2)
$$
  
=  $\gamma^2 \frac{W_0^4}{16} + 4.34 \gamma W_0^4 \sigma_R^{\frac{12}{5}} \Omega^{-1}$  (G11)

and similarly

<span id="page-16-4"></span>
$$
\int_{\mathbb{R}^4} d^2 \mathbf{r}_1 d^2 \mathbf{r}_2 x_1^2 y_2^2 \Gamma_4(\mathbf{r}_1, \mathbf{r}_2)
$$
  
=  $\gamma^2 \frac{W_0^4}{16} + 3.16 \gamma W_0^4 \sigma_R^{\frac{12}{5}} \Omega^{-1}.$  (G12)

<span id="page-16-3"></span>Finally, substituting Eqs. [\(F15\)](#page-15-5), [\(G9\)](#page-16-2), [\(G11\)](#page-16-3) and [\(G12\)](#page-16-4)

into Eqs. [\(D37\)](#page-12-10) and [\(D38\)](#page-12-9) we obtain

$$
\left\langle (\Delta W_{1/2}^2)^2 \right\rangle = 13.14 \gamma W_0^4 \sigma_R^{\frac{12}{5}} \Omega^{-1}
$$
 (G13)

and

$$
\left\langle \Delta W_1^2 \Delta W_2^2 \right\rangle = 0.65 \gamma W_0^4 \sigma_R^{\frac{12}{5}} \Omega^{-1}.
$$
 (G14)

It is worth to note that in contrast to weak turbulence case the covariance [\(G14\)](#page-17-2) is positive, i.e. the shape of the beam profile of the ellipse is deformed in such a way that the increase of beam width along one half-axis of the ellipse causes the increase in the complimentary direction.

### <span id="page-17-0"></span>H. MEAN VALUES AND COVARIANCE MATRIX ELEMENTS

<span id="page-17-3"></span>Table I. Mean values and elements of the covariance matrix of the vector v, are given for horizontal links, in terms of the transmitter beam spot radius,  $W_0$ , the Fresnel parameter of the beam,  $\Omega = \frac{kW_0^2}{2L}$ , and the Rytov parameter,  $\sigma_R^2 = 1.23 C_n^2 k^{\frac{7}{6}} L^{\frac{11}{6}}$ . Here k is the beam wave-number, L is the propagation distance,  $C_n^2$   $[m^{-\frac{2}{3}}]$  is the structure con-<br>stant of the refrective index of the six and  $\alpha = (1+\Omega^2)/\Omega^2$ stant of the refractive index of the air, and  $\gamma = (1+\Omega^2)/\Omega^2$ .



Strong turbulence

$$
\langle \Theta_{1/2} \rangle \quad \ln \left[ \frac{\left( \gamma + 1.71 \sigma_R^{\frac{12}{5}} \Omega^{-1} - 2.99 \sigma_R^{\frac{8}{5}} \Omega^{-1} \right)^2}{\sqrt{\left( \gamma + 1.71 \sigma_R^{\frac{12}{5}} \Omega^{-1} - 2.99 \sigma_R^{\frac{8}{5}} \Omega^{-1} \right)^2 + 3.24 \gamma \sigma_R^{\frac{12}{5}} \Omega^{-1}}} \right]
$$
\n
$$
\langle \Delta x_0^2 \rangle, \langle \Delta y_0^2 \rangle \qquad \qquad 0.75 \, W_0^2 \sigma_R^{\frac{8}{5}} \Omega^{-1}
$$
\n
$$
\langle \Delta \Theta_{1/2}^2 \rangle \qquad \qquad \ln \left[ 1 + \frac{13.14 \gamma \sigma_R^{12/5} \Omega^{-1}}{\left( \gamma + 1.71 \sigma_R^{\frac{12}{5}} \Omega^{-1} - 2.99 \sigma_R^{\frac{8}{5}} \Omega^{-1} \right)^2} \right]
$$
\n
$$
\langle \Delta \Theta_1 \Delta \Theta_2 \rangle \qquad \qquad \ln \left[ 1 + \frac{0.65 \gamma \sigma_R^{12/5} \Omega^{-1}}{\left( \gamma + 1.71 \sigma_R^{\frac{12}{5}} \Omega^{-1} - 2.99 \sigma_R^{\frac{8}{5}} \Omega^{-1} \right)^2} \right]
$$

The Table [I](#page-17-3) lists the non-zero means and covariance matrix elements of the four-dimensional Gaussian distribution for the random vector v defined in Eq. [\(D4\)](#page-9-5). <span id="page-17-2"></span>We list the results for weak and strong turbulence. The weak turbulence results can be applied, e.g., for short propagation distances with  $\sigma_R^2 \lesssim 1$ . In near-to-ground propagation the latter condition is fulfilled for optical frequencies for night-time communication. The strong turbulence results are applied for short distance communication,  $\sigma_R^2 \gg 1$ . For a near-to-ground communication scenario this corresponds to the day-time operation at clear sunny days.

#### <span id="page-17-1"></span>I. LOG-NORMAL MODEL

The log-normal probability distribution for transmittance is

$$
\mathcal{P}(\eta) = \frac{1}{\eta \sigma \sqrt{2\pi}} \exp\left[-\frac{\left(-\ln \eta - \mu\right)^2}{2\sigma^2}\right] \tag{I1}
$$

where

$$
\mu = -\ln\left(\frac{\langle \eta \rangle^2}{\sqrt{\langle \eta^2 \rangle}}\right) \tag{I2}
$$

and

$$
\sigma^2 = \ln\left(\frac{\langle \eta^2 \rangle}{\langle \eta \rangle^2}\right),\tag{I3}
$$

are parameters of the log-normal distribution. The parameters  $\mu$  and  $\sigma$  are the functions of the first and second moments of transmittance

<span id="page-17-5"></span><span id="page-17-4"></span>
$$
\langle \eta \rangle = \int_{\mathcal{A}} d^2 \mathbf{r} \Gamma_2(\mathbf{r}), \tag{I4}
$$

$$
\langle \eta^2 \rangle = \int_{\mathcal{A}} d^2 \mathbf{r}_1 d^2 \mathbf{r}_2 \Gamma_4(\mathbf{r}_1, \mathbf{r}_2), \tag{I5}
$$

where the field coherence functions  $\Gamma_2$  and  $\Gamma_4$  are given by Eqs. [\(E15\)](#page-13-9) and [\(E16\)](#page-13-4), respectively. Here the integration is performed over the circular aperture opening area A.

The first moment of transmittance [\(I4\)](#page-17-4) is evaluated explicitly as

$$
\langle \eta \rangle = 1 - \exp\left[ -\frac{2a^2}{\langle W^2 \rangle} \right],\tag{I6}
$$

where a is the aperture radius and

$$
\langle W^2 \rangle = \langle S_{xx} \rangle + 4 \langle x_0^2 \rangle \tag{I7}
$$

is the so called "long-term" beam mean-square radius [\[15\]](#page-4-24). Here  $\langle S_{xx} \rangle$  and  $\langle x_0^2 \rangle$  are defined by Eqs. [\(D14\)](#page-10-5) and [\(D33\)](#page-11-10) respectively. For weak turbulence from Eqs. [\(G4\)](#page-16-0) and [\(F6\)](#page-14-5) we evaluate  $\langle W^2 \rangle = W_0^2 \Omega^{-2} + 4.33 W_0^2 \sigma_R^2 \Omega^{-\frac{7}{6}}$ . However, the integration of Eq. [\(I5\)](#page-17-5) is more involved. In this Letter we evaluated Eq. [\(I5\)](#page-17-5) numerically.