Covering Problems and Core Percolations on Hypergraphs

Bruno Coelho Coutinho,[1, *] Hai-Jun Zhou,[2] and Yang-Yu Liu[3, 4, †]

[1]*Instituto de Telecomunicações, Physics of Information and Quantum Technologies Group, Portugal*
[2]*CAS Key Laboratory for Theoretical Physics, Institute of Theoretical Physics,*
Chinese Academy of Sciences, Beijing 100190, China
[3]*Channing Division of Network Medicine, Brigham and Women's*
Hospital and Harvard Medical School, Boston, Massachusetts 02115, USA
[4]*Center for Cancer Systems Biology, Dana-Farber Cancer Institute, Boston, Massachusetts 02115, USA*

We introduce two generalizations of core percolation in graphs to hypergraphs, related to the minimum hyperedge cover problem and the minimum vertex cover problem on hypergraphs, respectively. We offer analytical solutions of these two core percolations for random hypergraphs with arbitrary vertex degree and hyperedge cardinality distributions. We find that for several real-world hypergraphs their two cores tend to be much smaller than those of their randomized counterparts, suggesting that covering problems in those real-world hypergraphs can actually be solved in polynomial time.

As a natural generalization of a graph, a hypergraph consists of vertices and hyperedges [1]. A hyperedge can simultaneously connect any number of vertices, which facilitates a more faithful representation of many real-world networks [2, 3]. For example, given a set of proteins and a set of protein complexes, the corresponding hypergraph naturally captures the information on proteins that interact within a protein complex [4]. For a biochemical reaction system, the hypergraph representation indicates which biomolecules participate in a particular reaction [4, 5]. In computer science, the factorization of complicated global functions of many variables can often be represented by a factor graph, wich can be mapped to a hypergraph [3–8]. In social science, a collaboration network can also be represented by a hypergraph, where vertices represent individuals and hyperedges connect individuals who were involved in a specific collaboration, e.g., a scientific paper, a patent, a consulting task, or an art performance [7, 8].

As in graphs, the degree of a vertex in a hypergraph is the number of hyperedges that connect to it. The number of vertices connected by a hyperedge is called the cardinality of that hyperedge. If all hyperedges have the same cardinality K, the hypergraph is said to be uniform or K-uniform. Note that a graph is just a 2-uniform hypergraph.

The core of a graph — defined as the remainder of the greedy leaf removal (GLR) procedure where leaves (vertices of degree one) and their neighbors are removed iteratively from the graph — has been related to the conductor-insulator transition [9], structural controllability [10], and many combinatorial optimization problems [11]. Indeed, the core size is related to a fundamental combinatorial problem — the minimum vertex cover (MVC) problem, which aims to find the smallest set of vertices in a graph so that every edge is incident to at least one vertex in the set [12]. If the core is absent, then the MVC problem is solvable in polynomial. Otherwise, if the core exists and is extensive in size, then the

MVC problem is generally NP-hard [12, 13]. As the dual of the MVC problem, the minimum edge cover problem aims to find the smallest set of edges so that for every vertex in the graph there is at least one edge incident to it. Both covering problems can be defined similarly on hypergraphs. The minimum edge cover problem on graphs can be computed in polynomial time [14]. Yet, this is not true for hypergraphs, where both the minimum hyperedge cover (MHC) and the MVC problems are generally NP-hard [15]. Note that the MVC and MHC on hypergraphs are related to many real-world problems, e.g., finding the optimal drug combination in pharmacology [16], searching files in a storage systems [17], etc. Typically these problems can be solved using approximate algorithms, e.g., highest-degree-first [16, 18] and simulated annealing [16]. Here we show that those approximate algorithms are not always necessary. To achieve that, we extend the concept of the core in graph to the hypergraph case, and define two cores associated with the MVC and MHC problem, respectively.

Let us consider the MHC problem of the hypergraph $\mathcal{H}_0$ in Fig. 1(a), which has three hyperedges $\{h_1, h_2, h_3\}$ and four vertices $\{v_1, v_2, v_3, v_4\}$. The hyperedge h_3 contains all the vertices in h_1, as well as vertex v_4, thus h_1 is not necessary for the MHC and can be removed, leading to the hypergraph $\mathcal{H}_1$ shown in Fig. 1(b1). In $\mathcal{H}_1$ the vertex v_2 is contained by hyperedge h_2 that contains also vertex v_1. Hence if v_1 is covered, v_2 is also covered. We can therefore remove v_2 from $\mathcal{H}_1$. By iteratively removing vertices and hyperedgse using these rules, we get the hypergraph shown in Fig. 1(b3), for which solving the MHC problem is trivial. The MVC problem is dual to the MHC problem, hence we can obtain a dual set of rules. In the hypergraph $\mathcal{H}_0$, the hyperedge h_3 contains all the vertices in h_1, as well as vertex v_4, thus if h_1 is covered h_3 is also covered. We can therefore remove h_3 and obtain the hypergraph $\mathcal{H}_2$ in Fig. 1(c1), for which we find that vertex v_1 is redundant and can be removed (since v_2 covers the same hyperedge as v_1 and also covers
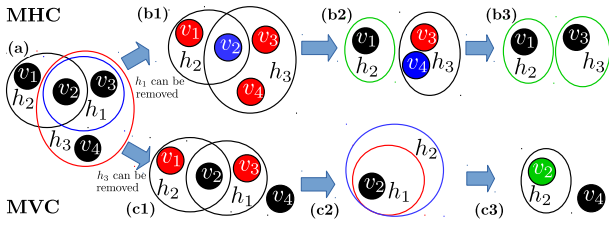
FIG. 1. Generalized greedy leaf removal helps us solve the minimum hyperedge cover (MHC) and the minimum vertex cover (MVC) problems on hypergraphs. Vertices are represented by dots and hyperedges by circles. The green vertices and hyperedges in (b3) and (c3) are the solution to the MHC or MVC problem, respectively.

hyperedge h_1). By iteratively removing vertices and hyperedgse using these rules, we get the hypergraph shown in Fig. 1(c3), for which the MVC problem is trivial.

The example shown in Fig. 1 prompts us to define three sets of hyperedges (or vertices): (1) S is a solution of the MHC (MVC) problem; (2) $\tilde{S}$ contains hyperedges (vertices) that can be determined to be part of S using our approach; and (3) the core h_{core} (v_{core}) contains the hyperedges (vertices) that cannot be determined if they belong to S or not using our approach. If the degree (cardinality) of a vertex (hyperedge) is zero, then it can not be covered. For the remaining vertices (hyperedges) we can find the vertices (hyperedges) that belong to each category based on the following rules. *(Rule-1)*: Consider hyperedges h_1 and h_2 that contain the set of vertices V_1 and V_2. If V_1 and V_2 are not empty sets, and $V_1 \subseteq V_2$, we remove h_1 (or h_2) to solve the MHC (or MVC) problem, respectively. *(Rule-2)*: Consider vertices v_1 and v_2 that are contained by the set of hyperedges H_1 and H_2. If H_1 and H_2 are are not empty sets, and $H_1 \subseteq H_2$, we remove v_2 (or v_1) to solve the MHC (or MVC) problem, respectively. We repeat this process until no more vertices or hyperedges can be removed. In the final hypergraph, hyperedges (vertices) with cardinalty (degree) one belong to $\tilde{S}$ (Fig. 1(b3) and (c3)); hyperedges (vertices) with cardinalty (degree) larger than one belong to h_{core} (v_{core}).

We emphasize that these two rules can be considered as the generalized GLR procedure on hypergraphs, which reduces to the standard GLR procedure on graphs. Note that even if the resulting h_{core} (v_{core}) is very small but non-zero, the generalized GLR procedure is better than approximation algorithms in solving the MHC (MVC) problem, because it explicitly tells us which hyperedges (vertices) belong to the solution S, which do not, and which cannot be determined.

Since the hypergraph core h_{core} is closely related to the MHC problem, we study the corresponding core percolation problem on random hypergraphs. To achieve that, we generalize the mean-field approach proposed for

the graph case [19]. We define two types of removable vertices: a vertex is (i) α-removable if it is or can become a vertex of degree one; (ii) β-removable if its degree is larger than one and belongs to at least one leaf hyperedge. Dually, we define two types of removable hyperedges: a hyperedge is (i) δ-removable if it is or can become a leaf hyperedge; (ii) ϵ-removable if it has cardinality r and is removed because it is connected to $(r-1)$ β-removable vertices. Consider a large uncorrelated random hypergraph $\mathcal{H}$ with arbitrary vertex degree and hyperedge cardinality distributions. We can determine the category of a vertex v in $\mathcal{H}$ by the categories of its neighboring hyperedges in the modified hypergraph $\mathcal{H} \setminus v$ with vertex v and all its hyperedges removed from $\mathcal{H}$, using the following rules: (i) α-removable vertex: all neighboring hyperedges are ϵ-removable; (ii) β-removable vertex: at least one neighboring hyperedge is δ-removable. Similarly, we can determine the category of a hyperedge e in $\mathcal{H}$ by the categories of its neighboring vertices in the modified hypergraph $\mathcal{H} \setminus e$ with hyperedge e and all its vertices removed from $\mathcal{H}$, using the following rules: (iii) δ-removable hyperedge: at least one neighboring vertex is α-removable; (iv) ϵ-removable hyperedge: at least one neighboring vertex is β-removable. Let α (or β) denote the probability that a random neighboring vertex of a random hyperedge e in a hypergraph $\mathcal{H}$ is α-removable (or β-removable) in $\mathcal{H} \setminus e$. Let δ (or ϵ) denote the probability that a random neighbor of a random vertex v in a hypergraph $\mathcal{H}$ is α-removable (or β-removable) in $\mathcal{H} \setminus v$. Then rules (i)-(iv) enable us to derive a set of self-consistent equations:

$$\alpha = \sum_{k=1}^{\infty} Q_n(k)\epsilon^{k-1}, \tag{1}$$

$$1 - \beta = \sum_{k=1}^{\infty} Q_n(k)(1-\delta)^{k-1}, \tag{2}$$

$$1 - \delta = \sum_{r=1}^{\infty} Q_h(r)(1-\alpha)^{r-1}, \tag{3}$$

$$\epsilon = \sum_{r=1}^{\infty} Q_h(r)\beta^{r-1}. \tag{4}$$

Here $Q_n(k)$ ($Q_h(r)$) is the excess degree (cardinality) distribution. The fraction of vertices in h_{core}, denoted as s_h^v, is given by

$$s_h^v = \sum_{k=2}^{\infty} P_n(k) \sum_{l=2}^{k} \binom{k}{l}(1-\delta-\epsilon)^l \epsilon^{k-l} \tag{5}$$

where $P_n(k)$ ($P_h(r)$) is the degree (cardinality) distribution. (See SI Sec.IV.B for the formula of the fraction of hypedeges in h_{core}, denoted as s_h^h.) For hypergraphs with Poisson vertex degree distribution and different hyperedge cardinality distributions, we find that the h_{core}
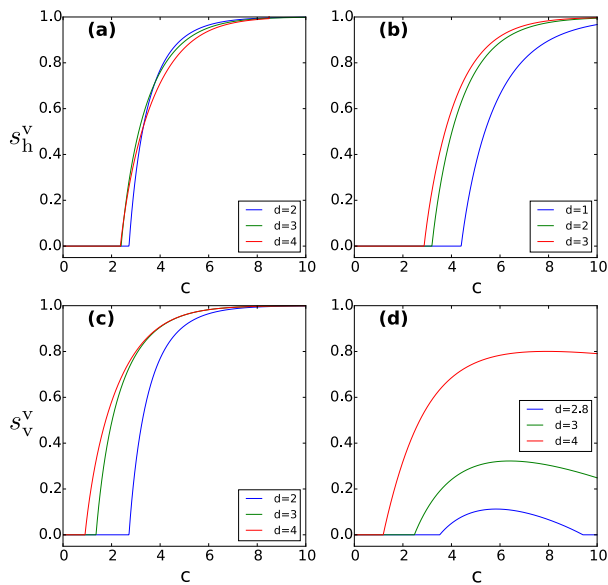
FIG. 2. (Color online) The relative size of the core size hypergraphs with Poisson vertex degree distributions, (a,b) s_h^v. (c,d) s_v^v. In (a) and (c) we consider d-uniform hypergraphs, meaning that all hyperdege have the same cardinality d. In (b) and (d) we consider that the cardinality of the hyperdeges follows a Poisson distribution with average cardinality d.

emerges as a continuous phase transition (see Fig. 2(a) and (b)),

$$s_\mathrm{h}^\mathrm{v} \propto (c - c^*)^{\zeta_1}, \tag{6}$$

with critical exponent $\zeta_1 = 1$ (see SI Sec.IV.E for details). The relation between the critical mean degree c^* (percolation threshold) and the hyperedge mean cardinality d is represented in Fig. 3.

Similarly, the $\mathrm{v_{core}}$ percolation (associated with the MVC problem) can also be analytically studied for random hypergraphs. In this case, the equations on α and β are the same as the $\mathrm{h_{core}}$ percolation case, but for hyperedges we derive the following self-consistent equations:

$$\delta = \sum_{r=1}^\infty Q_\mathrm{h}(r)\alpha^{r-1}, \tag{7}$$

$$1 - \epsilon = \sum_{r=1}^\infty Q_\mathrm{h}(r)(1 - \beta)^{r-1}. \tag{8}$$

The $\mathrm{v_{core}}$ consists of those vertices connected to at least two non-removable hyperedges. Hence, the fraction of vertices in $\mathrm{v_{core}}$ is given by

$$s_\mathrm{v}^\mathrm{v} = \sum_{k=2}^\infty P_\mathrm{n}(k) \sum_{l=2}^k \binom{k}{l} (1 - \delta - \epsilon)^l \epsilon^{k-l}. \tag{9}$$

See SI Sec.IV.B for the formula of the fraction of hypedeges in v_core, s_h^h. For hypergraphs with Poisson
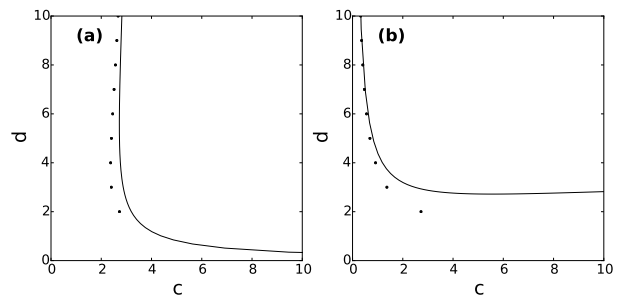

FIG. 3. Phase diagram of the $\mathrm{h_{core}}$ and $\mathrm{v_{core}}$ percolations on hypergraphs with Poisson vertex degree distributions. Black dots and black line represent the phase boundary of d-uniform hypergraphs and hypergraphs with Poisson hyperedge cardinality distribution, respectively. (a) $\mathrm{h_{core}}$. Note that, for d-uniform hypergraph ($d > 1$) with Poisson vertex degree distribution, the critical mean degree (i.e., c^* of the $\mathrm{v_{core}}$ percolation) can be simply related to d as $c^* = \frac{e}{d-1}$, where $e = 2.71828\cdots$ (see Fig. 3(a) black dots and SI Sec.IV.C for details). This result was previously found in [20]. (b) $\mathrm{v_{core}}$.

vertex degree distribution and different hyperedge cardinality distributions, we find that the $\mathrm{v_{core}}$ emerges as a continuous phase transition (see Fig. 2(c,d)):

$$s_\mathrm{v}^\mathrm{v} \sim (c - c^*)^{\zeta_2}, \tag{10}$$

with critical exponent $\zeta_2 = 1$ (see SI Sec.IV.D for details). Fig. 2 (d) shows that for a Poisson-Poisson hypergraph the size of s_v^v starts to decrease at large values of c. By increasing the number of hyperedges connected to a vertex, but keeping the cardinailty distribution constant, the probability of a vertex being connected to a hyperedge with cardinality one increases, and any vertex connected to a hyperedge with cardinality one is automatically removed. This effect is not relevant if the probability that a vertex is connected to a hyperedge with cardinality one is very small, $1 - \exp(-c\, e^{-d}) \ll 1$. For large values of c and d, this effect is only relevant if $c \sim e^d$.

Phase diagrams of the $\mathrm{h_{core}}$ and $\mathrm{v_{core}}$ percolations on hypergraphs with Poisson vertex degree distributions are shown in Fig. 3. Note that the phase diagram of $\mathrm{v_{core}}$ percolation is equal to that of $\mathrm{h_{core}}$ percolation if we interchanged the mean cardinality d with the mean degree c. This is true because the $\mathrm{v_{core}}$ of a hypergraph is the $\mathrm{h_{core}}$ of the dual hypergraph.

We also apply the generalized GLR procedure to compute the $\mathrm{h_{core}}$ and $\mathrm{v_{core}}$ for several real-world hypergraphs: (1) APS consists of articles published in all the Americal Physical Society journals from 1893 to 2010 [21], where individual authors and their joint articles are considered as vertices and hyperedges, respectively. (2) DGT consists of drugs (hyperedges) and their target genes (vertices) as listed in the DrugBank [22]. (3) GMN consists of reactions (hyperedges) and the involved metabolites (vertices) in the genome-scale metabolic net-
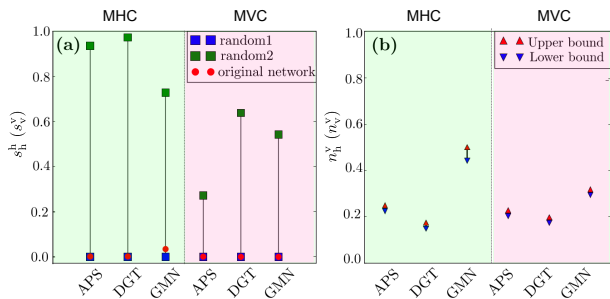
FIG. 4. (a) Fraction of hyperedges s_h^h (or vertices s_v^v) associated with the MHC (MVC) problem for three real-world hypergraphs: APS, DGT and GMN; and their respective randomized counterparts. This set represents those hyperedges (vertices) that cannot be covered optimally using our generalized GLR procedure. (b) Fraction of vertices n_h^v (hyperedges n_v^h) necessary to cover all the hyperedges (vertices) for the three hypergraphs.
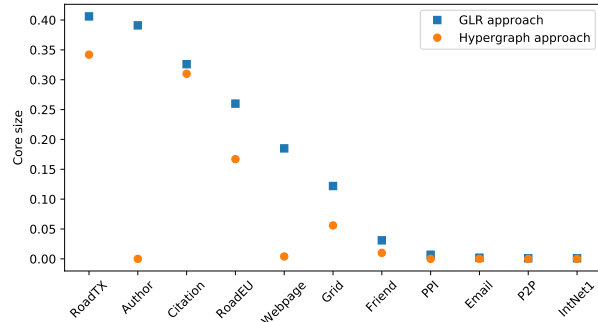

FIG. 5. Relative size of the core associated with the MDS problem for eleven real-world networks, using the GLR procedure introduced in [25] (square) and using our generalized GLR approach (circles).

work of *E. coli* obtained from the BiGG database [23]. We find that for GMN, h_{core} contains 3.4% of hypereges; while v_{core} contains less than 0.2% of vertices. For the other two hypergraphs (APS and DGT), both h_{core} and v_{core} contains less than 0.2% of hyperedges or vertices (Fig. 4a). We also compare the size of each core with that of two randomized counterparts of the real-world hypergraphs. For the first randomization scheme (random1), we preserve both the vertex degree and hyperedge cardinality distributions of the real hypergraph. For the second randomization scheme (random2), we consider a random Poisson-Poisson hypergraph with the same average degree and average cardinality as the real hypergraph. Note that for random1, the size of the core is always zero (blue points in Fig. 4a); while for random2, the size of the core is between 30% and 100%, much bigger than that of the real hypergraph. These results suggest that the degree and cardinality distributions are main factors that explains the small cores of these real-world hypergraphs. Because h_{core} and v_{core} of those real-world hypergraphs are really small, the MHC and MVC problems are effectively solvable. Indeed, as shown in Fig. 4(b), the upper and lower bound of the fraction of vertices (hyperedges) n_h^v (n_v^h) that are necessary to cover all the vertices (hyperedges) are very close to each other.

It turns out that our method can also be used to solve another classical NP-hard combinatorial problem: finding the minimum dominating set (MDS) for graphs. The MDS of a graph is the smallest set of vertices that needs to be occupied so that all unoccupied vertices are adjacent to at least one occupied vertex [24]. Our basic idea is as follows. We consider a hypergraph that has the same set of vertices as the original graph, and one hyperedge i contains all vertices adjacent to a vertex v_i (including v_i itself). Solving the MHC problem on this hypergraph is then equivalent to solve the MDS problem of the original

graph. Our method offers a much more general approach than the greedy algorithm introduced in [25]. Indeed, the two leaf-removal rules introduced in [25] can be considered as special cases of our generalized GLR rules (see SI Sec.IV.A for details). In Fig. 5, we show the size of the cores associated with the dominating set for the eleven real-world networks analyzed in [25]. For most of those networks, our method shows a considerable improvement over the previous method. For some of the networks, our method actually finds no core left.

In physics it is common that a more abstract or general approach actually makes certain complicated problems easier to solve. This is often not true in social systems, biological systems, or complex systems in general. Our results suggest that generalizing graph to hypergraphs is one of the few cases where a small generalization makes a very hard problem easier to solve. Indeed, we show that our generalized GLR procedure and the corresponding hypergraph cores can help us solve various NP-hard covering problems in a systematic and universal way. If we aim to find a simple solution for complex problems, this is really an exciting result, indicating that hypergraphs might be the right way to represent complex networked systems. Our results open a new set of tools to analyze complex networked systems. It also raises a very important question: why do hypergraph cores of real systems tend to be small or absent? We anticipate our work will trigger more research activities in addressing this intriguing question.

———

* bruno.c.coutinho@gmail.com
† yyl@channing.harvard.edu

[1] A. Bretto, *Hypergraph Theory: An Introduction* (Springer Publishing Company, Incorporated, 2013).

[2] O. T. Courtney and G. Bianconi, Phys. Rev. E **93**, 062311 (2016).

[3] G. Bianconi and C. Rahmede, Phys. Rev. E **93**, 032315 (2016).

[4] S. Klamt, U.-U. Haus, and F. Theis, PLoS computational biology **5**, e1000385 (2009).

[5] E. Ramadan, A. Tarafdar, and A. Pothen, in *Parallel and Distributed Processing Symposium, 2004. Proceedings. 18th International* (April) pp. 189–196.

[6] F. R. Kschischang, B. J. Frey, and H.-A. Loeliger, IEEE Trans. Inf. Theory **47**, 498 (2001).

[7] C. Taramasco, J.-P. Cointet, and C. Roth, Scientometrics **85**, 721 (2010).

[8] L. Krumov, C. Fretter, M. Mller-Hannemann, K. Weihe, and M.-T. Houtt, The European Physical Journal B **84**, 535 (2011).

[9] M. Bauer and O. Golinelli, Phys. Rev. Lett. **86**, 2621 (2001).

[10] Y.-Y. Liu, J.-J. Slotine, and A.-L. Barabasi, Nature **473**, 167 (2011).

[11] R. M. Karp and M. Sipser, in *Foundations of Computer Science, 1981. SFCS '81. 22nd Annual Symposium on* (1981) pp. 364–375.

[12] M. Weigt and A. K. Hartmann, Phys. Rev. Lett. **84**, 6118 (2000).

[13] A. K. Hartmann and M. Weigt, Journal of Physics A: Mathematical and General **36**, 11069 (2003).

[14] M. R. Garey and D. S. Johnson, *Computers and Intractability: A Guide to the Theory of NP-Completeness (Series of Books in the Mathematical Sciences)*, first edition ed. (W. H. Freeman, 1979).

[15] V. Repiský, *Covering edges of a hypergraph: complexity and applications*, Master's thesis, Comenius University, Faculty of Mathematics, Physics and Informatics (2007).

[16] A. Vazquez, BMC Systems Biology **3**, 1 (2009).

[17] W. Kautz and R. Singleton, IEEE Transactions on Information Theory **10**, 363 (1964).

[18] D. S. Johnson, Journal of Computer and System Sciences **9**, 256 (1974).

[19] Y.-Y. Liu, E. Csóka, H. Zhou, and M. Pósfai, Phys. Rev. Lett. **109**, 205703 (2012).

[20] S. Takabe and K. Hukushima, Phys. Rev. E **89**, 062139 (2014).

[21] R. Sinatra, D. Wang, P. Deville, C. Song, and A.-L. Barabasi, in *APS Meeting Abstracts*, Vol. 1 (2014) p. 17004.

[22] D. S. Wishart, C. Knox, A. C. Guo, S. Shrivastava, M. Hassanali, P. Stothard, Z. Chang, and J. Woolsey, Nucleic Acids Research **34**, D668 (2006).

[23] J. Schellenberger, J. O. Park, T. M. Conrad, and B. Ø. Palsson, BMC Bioinformatics **11**, 1 (2010).

[24] T. W. Haynes, P. J. Slater, and S. T. Hedetniemi, *Fundamentals of domination in graphs* (New York : Marcel Dekker, 1998).

[25] J.-H. Zhao, Y. Habibulla, and H.-J. Zhou, Journal of Statistical Physics **159**, 1154 (2015).

Supplementary Information

Bruno Coelho Coutinho,[1] Hai-Jun Zhou,[2] and Yang-Yu Liu[3,4]

[1]*Instituto de Telecomunicações, Physics of Information*
and Quantum Technologies Group, Portugal

[2]*CAS Key Laboratory for Theoretical Physics, Institute of Theoretical Physics,*
Chinese Academy of Sciences, Beijing 100190, China

[3]*Channing Division of Network Medicine,*
Brigham and Women's Hospital and Harvard Medical School, Boston, Massachusetts 02115, USA

[4]*Center for Cancer Systems Biology, Dana-Farber*
Cancer Institute, Boston, Massachusetts 02115, USA

CONTENTS

ORGANIZATION

In Sec.I we introduce the concept of a hypergraph and different types of cores and perco-lation processes in graphs. In Sec.II we explore the giant connected component and in Sec.III a generelized version of K-core percolation for hypergraphs. Although not fundamental to the main paper, it provides standard tools to study percolation transitions in hypergraphs. In Sec.IV he show detailed calculation underlying the results in the main text. In Sec.V we compare the results of our analytical calculation with the results of numerical simulations.

I. INTRODUCTION

Despite the ubiquity of hypergraphs in different fields, fundamental structural properties of hypergraphs have not been fully understood. Most of the previous works focus on uniform hypergraphs [1–3], ignoring the fact that hyperedges could have a wide range of cardinalities. In this work, we systematically study the percolation transitions on hypergraphs with arbitrary vertex degree and hyperedge cardinality distributions. We are particularly interested in the emergence of a giant component, the K-core, and the core in hypergraphs (see Fig. S1). Those special subgraphs have been extensively studied in the graph case and play very important roles in many network properties [4, 5]. A giant component of a graph is a connected component that contains a finite fraction of the entire graph's vertices, which is relevant to structural robustness and resilience of networks [6, 7]. The K-core of a graph is obtained by recursively removing vertices with degree less than K, as well as edges incident to them. The K-core has been used to identify influential spreaders in complex networks [8]. The core of a graph is the remainder of the greedy leaf removal (GLR) procedure: leaves (vertices of degree one) and their neighbors are removed iteratively from the graph. The emergence of the core in a graph has been related to the conductor-insulator transition [9], structural controllability [10], and many combinatorial optimization problems [11].

We can naturally extend the definition of giant component to the hypergraph case. Yet, to obtain the K-core in a hypergraph, we have to specify how to remove hyperedges containing vertices of degree less than K. To achieve that, we introduce the (K, S)-core defined as the largest fraction of the hypergraph where each hyperedge contains at least S vertices and each vertex belongs to at least K hyperedges in the subset. The (K, S)-core is obtained by recursively removing vertices with degree less than K and hyperedges with cardinality less than S.

(a) Original hypergraph
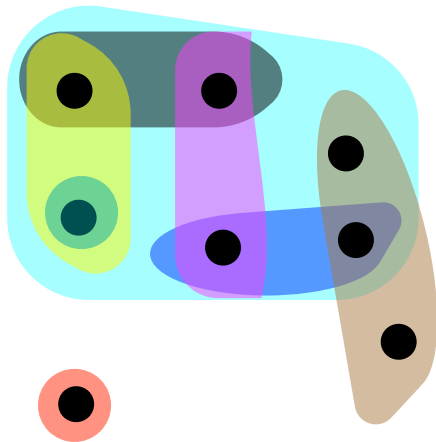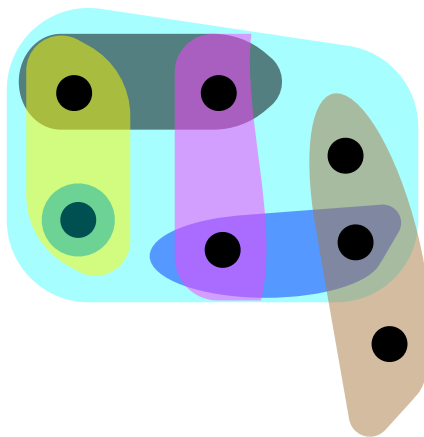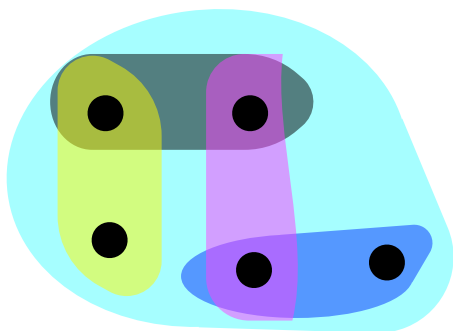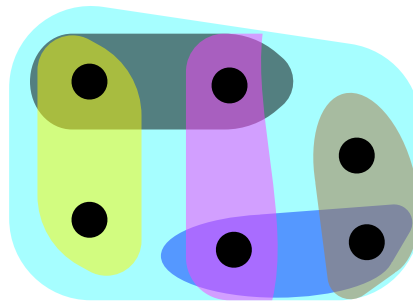
(b) Giant connected component

(c) $(2, S_{\max})$-Core

(d) $(2, 2)$-Core

FIG. S1. (Color online) Example of the different percolations studied in the work. (a) shows the original hypergraph. (b), (c) and (d) shows the $(2, S_{\max})$-core, $(2, 2)$-core, where $S_{\max} \equiv \max(r, 2)$.

II. GIANT COMPONENT

A giant connected component of a hypergraph is a connected component that contains a constant fraction of the entire vertices. In the mean-field picture, we can derive a set of self-consistent equations to calculate the relative size of the GCC, using the generating function formalism [12]. Let μ represent the probability that a randomly selected vertex from a randomly chosen hyperedge is not connected via other hyperedges with the GCC. Dually, let ψ represent the probability that a randomly chosen hyperedge connecting to a randomly chosen vertex is not connected via other vertices with the GCC. Then we have

$$\mu = \sum_{k=1}^{\infty} Q_{\mathrm{n}}(k)\psi^{k-1} \tag{S1}$$

$$\psi = \sum_{r=1}^{\infty} Q_{\mathrm{h}}(r)\mu^{r-1}. \tag{S2}$$

Here $Q_{\mathrm{n}}(k) \equiv kP_{\mathrm{n}}(k)/c$ is the excess degree distribution of vertices, i.e., the degree distribution for the vertices in a randomly chosen hyperedge. $P_{\mathrm{n}}(k)$ is the vertex degree distribution, and $c = c_1$ is the mean degree of the vertices. In general we define $c_m \equiv \sum_{k=0}^{\infty} k^m P_{\mathrm{n}}(k)$. $Q_{\mathrm{h}}(r) \equiv rP_{\mathrm{h}}(r)/d$ is the excess cardinality distribution of hyperedges, i.e., the cardinality distribution for the hyperedges connected to a randomly chosen vertex. $P_{\mathrm{h}}(r)$ is the hyperedge cardinality distribution, and $d = d_1$ is the mean cardinality of the hyperdges. In general we define $d_m \equiv \sum_{r=0}^{\infty} r^m P_{\mathrm{h}}(r)$.

The relative size of the GCC is then given by

$$s_{\mathrm{g}} = 1 - \sum_{k=0}^{\infty} P_{\mathrm{n}}(k)\psi^k. \tag{S3}$$

Fig. S2 shows the analytical result of s_{g} as a function of the mean degree c for hypergraphs with Poisson vertex degree distribution and different hyperedge cardinality distributions. Clearly the giant component in hypergraphs emerges as a continuous phase transition with scaling behavior

$$s_{\mathrm{g}} \sim (c - c^*)^{\eta} \tag{S4}$$

for $c - c^* \to 0^+$, where c^* is the critical value of mean degree (i.e., the percolation threshold) and η is the critical exponent associated with the critical singularity.

The condition for the percolation transition can be determined by rewriting Eqs. S1 as

$$F_{\mathrm{g}} = 0, \tag{S5}$$

where we define

$$F_{\mathrm{g}} \equiv \sum_{k=1}^{\infty} Q_{\mathrm{n}}(k)\psi^{k-1} - \mu. \tag{S6}$$

At the critical point when $\mu^* = 1$ and

$$\left.\frac{\partial F_{\mathrm{g}}}{\partial \mu}\right|_{\mu=\mu^*=1} = 0, \tag{S7}$$

we obtain

$$\frac{d_2 - d}{d}\frac{c_2 - c}{c} > 1. \tag{S8}$$

where c^* and c_2^* are the hyperdegree first and second moment, respectively, at the critical point.

Note that a similar relation has been found for uniform hypergraphs [13]. In the graph case ($d = 2$ for all edges) we recover the classical result $\frac{c_2-c}{c} > 1$ [6, 14].

The critical exponent η can be calculated by consider a point around the critical point, such as, $\mu = 1 - \zeta$ with $\zeta = 0^+$ and $c = c^* + \chi$ with $\chi = 0^+$. We can define the function from Eq. S6 as a function of μ and c (i.e. $F_{\mathrm{g}}(\mu, c)$). By expanding $F_{\mathrm{g}}(\mu, c)$ around the point $(\mu, c) = (1, c^*)$ and combining it with the result from Eq. S7 at the critical point, we can rewrite Eq. S6 as

$$\left.\frac{\partial F_{\mathrm{g}}(\mu, c)}{\partial c}\right|_{(1,c^*)} \chi + \left.\frac{\partial^2 F_{\mathrm{g}}(\mu, c)}{\partial^2 \mu}\right|_{(1,c^*)} \zeta^2 + \left.\frac{\partial^2 F_{\mathrm{g}}(\mu, c)}{\partial^2 c}\right|_{(1,c^*)} \chi^2$$

$$-2\left.\frac{\partial^2 F_{\mathrm{g}}(\mu, c)}{\partial \mu \partial c}\right|_{(1,c^*)} \chi\zeta + (\cdots) = 0. \tag{S9}$$

For $\mu = 1$,

$$F_{\mathrm{g}}(1, c) = 1 - c/c = 0, \tag{S10}$$

it follows that

$$\left.\frac{\partial^n F_{\mathrm{g}}(\mu, c)}{\partial^n c}\right|_{(1,c^*)} = 0 \tag{S11}$$

for any positive integer n. Let us assume that there are no diverging moments for the hyperedge cardinality distribution. We can truncate our expansion of F_{g} at order 2 for ζ and χ. This implies that

$$\zeta \sim \chi. \tag{S12}$$

Let us expand s_g as a function of ζ,

$$s_g = s_g(\mu)|_{\mu=\mu^*} - \left.\frac{\partial s_g(\mu)}{\partial \mu}\right|_{\mu=\mu^*} \zeta + \left.\frac{\partial^2 s_g(\mu)}{\partial^2 \mu}\right|_{\mu=\mu^*} \zeta^2 + (\cdots). \tag{S13}$$

For $\mu^* = 1$, $s_g(\mu^*) = 0$. We obtain

$$s_\mathrm{g} \sim \chi, \tag{S14}$$

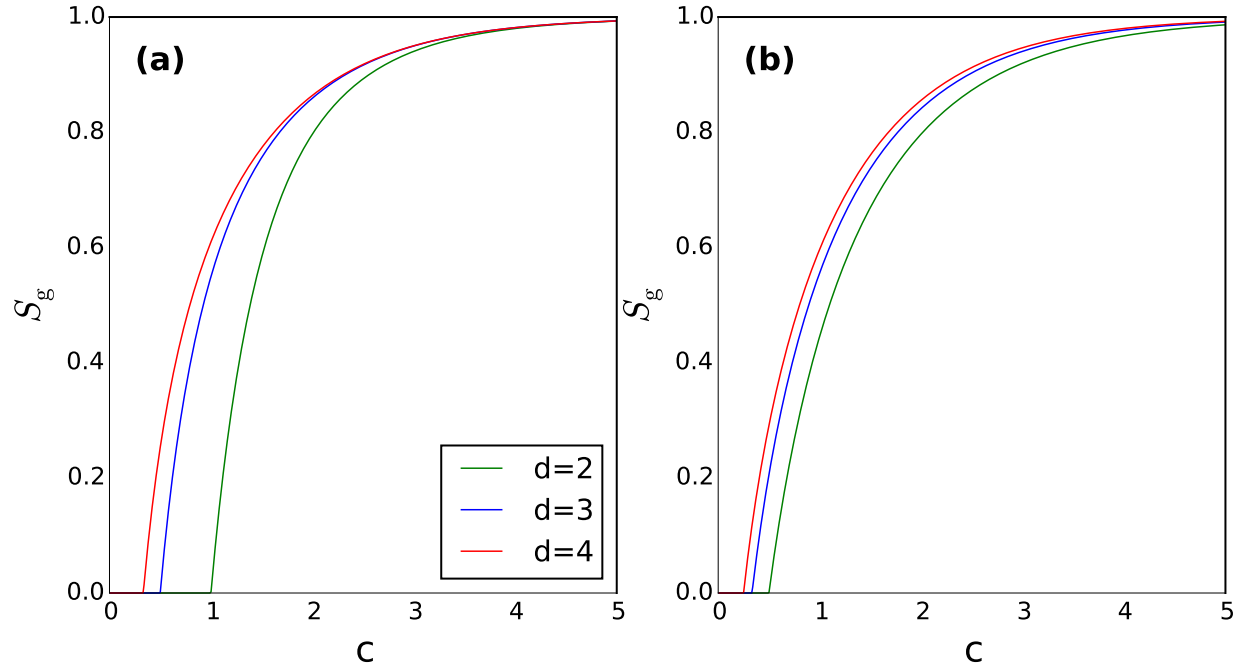This correspond to he same exponent $\eta = 1$ as in the graph case [6, 15].

FIG. S2.   (Color online) The relative size of the GCC s_g as a function of the mean degree c for hypergraphs with Poisson vertex degree distribution. (a) d-uniform hypergraphs with $d = 2, 3, 4$; (b) hypergraphs with Poisson hyperedge cardinality distribution and mean cardinality $d = 2, 3, 4$.

III. (K, S)-CORE ON HYPERGRAPHS

The (K, S)-core of a hypergraph is obtained by recursively removing vertices with degree less than K and hyperedge with cardinality less than S. A hyperedge with cardinality r is removable if at least $r - S + 1$ vertices connected to it are also removable and a vertex with degree k is removable if at least $k - K + 1$ hyperedges connected to it are also removable. One can remove a vertex or a hyperedge from the hypergraph, and see what is the probability of a neighboring hyperedge or vertex, respectively, being removable. This allows us to derive a set of self-consistent equations:

$$\alpha = \sum_{k=1}^{\infty} Q_{\mathrm{n}}(k) \sum_{l=k+1-S}^{k-1} \binom{k-1}{l} \delta^l (1-\delta)^{k-1-l}, \tag{S15}$$

$$\delta = \sum_{r=1}^{\infty} Q_{\mathrm{h}}(r) \sum_{l=r+1-K}^{r-1} \binom{r-1}{l} \alpha^l (1-\alpha)^{r-1-l}. \tag{S16}$$

Here α and δ are, respectively, the probability that a vertex or a hyperedge is removable. From now on we will focus on the case of $K = 2$. Then Eq. (S15) reduces to

$$\alpha = \sum_{k=1}^{\infty} Q_{\mathrm{n}}(k) \delta^{k-1}. \tag{S17}$$

A. $K = 2$ and $S = S_{\max}$

The (K, S)-core defined with $K = 2$ and $S = S_{\max} \equiv \max(r, 2)$, (where $S_{\max} = 2$ if $r < 2$ and and $S_{\max} = r$ if $r \geq 2$ and r is the intial cardinality of an hyperedge before any removal process) is obtained by recursively removing all vertices with degree one as well as the hyperedges containing them, and all hyperedges with cardinality smaller than two. Hyperedges with cardinality one or zero do not connect any nodes, thus have no meaning in what cores are concerned. In this case the threshold S depends on the cardinally of the hyperedges. Furthermore, if one of the vertices of any hyperedge is removed the hyperedge is also removed. (Note that the $(2, S_{\max})$-core has been defined in literature [2] simply as 2-core, and discontinuous 2-core percolation is found in d-uniform hypergraphs with $d > 2$.) In this case, Eq. (S16) reduces to

$$1 - \delta = \sum_{r=2}^{\infty} Q_{\mathrm{h}}(r)(1-\alpha)^{r-1}. \tag{S18}$$

The relative size of the $(2, S_{\mathrm{max}})$-core is given by the probability that a randomly chosen vertex is connected to at least two non-removable hyperedges:

$$s_{2\mathrm{c}} = \sum_{k=2}^{\infty} P_{\mathrm{h}}(k) \sum_{l=2}^{k} \binom{k}{l} (1 - \delta)^l \delta^{k-l}. \tag{S19}$$

Fig. S3 shows the analytical result of $s_{2\mathrm{c}}$ as a function of the mean degree c for hypergraphs with Poisson vertex degree distribution and different hyperedge cardinality distributions. We find that, depending on the mean hyperedge cardinality d, the $(2, S_{\mathrm{max}})$-core emerges as either a continuous or a hybrid phase transition, with scaling behavior

$$s_{2\mathrm{c}} - s_{2\mathrm{c}}^* \sim (c - c^*)^{\zeta} \tag{S20}$$

for $c - c^* \to 0^+$, where c^* is the percolation threshold and ζ is the critical exponent. $s_{2\mathrm{c}}^*$ is the $(2, S_{\mathrm{max}})$-core relative size right at the critical point: $s_{2\mathrm{c}}^* = 0$ for continuous phase transition and non-zero for hybrid phase transitions. The percolation threshold c^* can be calculated by defining

$$F_{2\mathrm{c}} \equiv \sum_{k=1}^{\infty} Q_{\mathrm{n}}(k) \delta^{k-1} - \alpha = 0. \tag{S21}$$

If we consider $S = \max(r, 2)$, δ, defined in Eq. S18, and we combine Eqs. S21, S25 and S18, we obtain

$$1 = \sum_{k=1}^{\infty} \sum_{r=2}^{\infty} Q_{\mathrm{n}}^*(k) Q_{\mathrm{h}}(r)(k-1)(r-1) \delta^{*k-2}(1 - \alpha^*)^{r-2}, \tag{S22}$$

where $Q_{\mathrm{n}}^*(k)$ is the connectivity distribution with the critical mean hyperdegree c^*. δ^* and α^* are the values of δ and α at the critical point, respectively. The phase transition is continuous if for $\delta^* = \alpha^* = 1$,

$$\left. \frac{\partial^2 F_{2\mathrm{c}}}{\partial^2 \alpha} \right|_{\alpha=1} > 0, \tag{S23}$$

that reduces to,

$$2 Q_{\mathrm{n}}(3)(c_2^* - c^*) - Q_{\mathrm{n}}(2)^2(c_3^* - 3 c_2^* + 2 c^*) < 0. \tag{S24}$$

where c_3^* and c_2^* is third and second moment of the degree distribution at the critical point, respectively. The phase transition is only continuous if at the critical point

$$\left. \frac{\partial F_{2\mathrm{c}}}{\partial \alpha} \right|_{(\alpha, c_1)=(\alpha^*, c^*)} = 0. \tag{S25}$$

From this condition we obtain that the critical point for a continuous phase transition is given by,

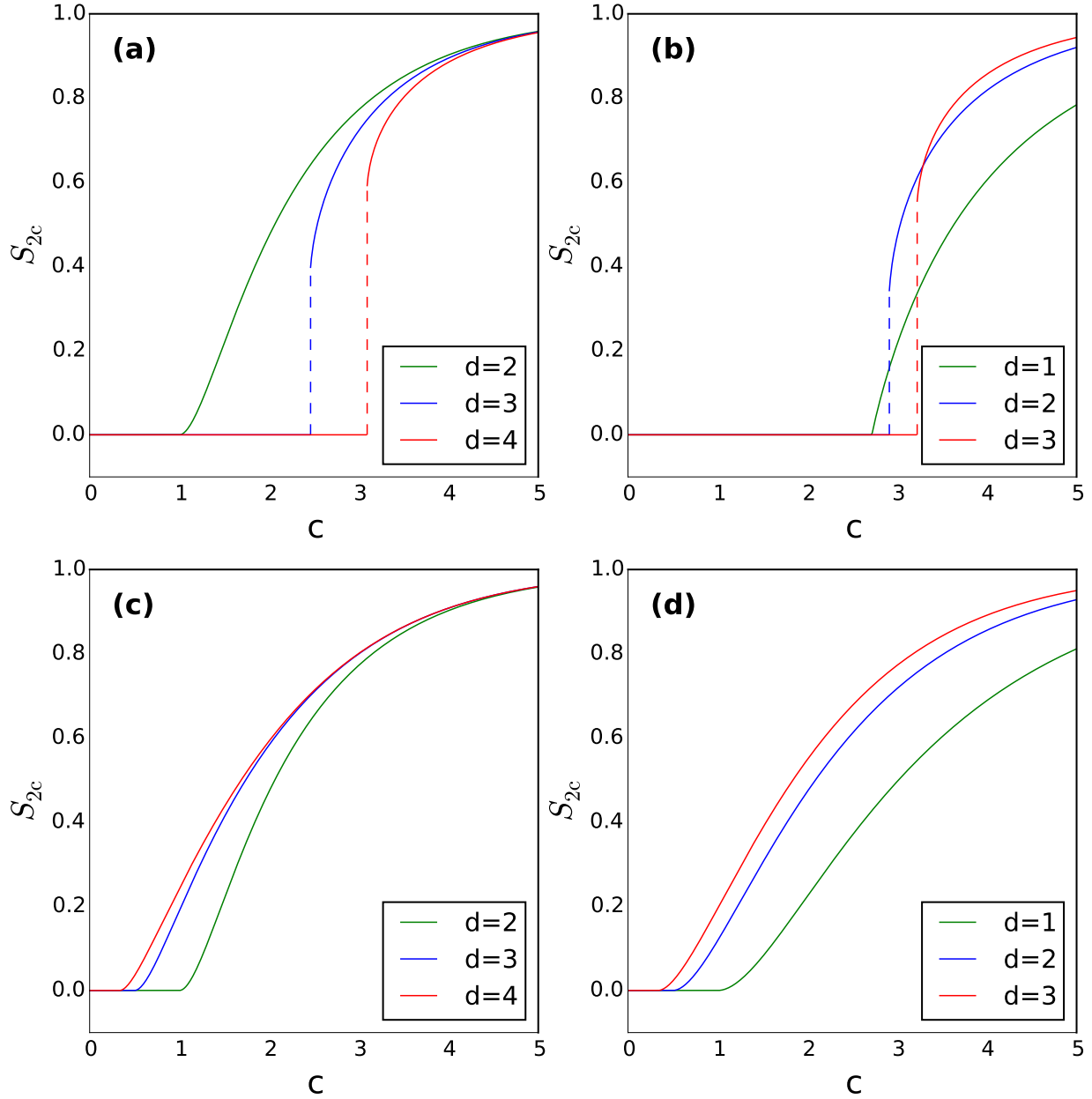$$Q_{\mathrm{h}}(2) \frac{c_2^* - c^*}{c^*} = 1. \tag{S26}$$

FIG. S3. (Color online) The relative size of $(2, S)$-core s_{2c} as a function of the mean degree c for hypergraphs with Poisson degree distribution, for $S = \max(r - 1, 2)$ (a) and (b), and $S = 2$, (b) and (d). (a) and (c) d-uniform hypergraphs . (b) and (d) hypergraphs with Poisson hyperedge cardinality distribution.

Let us assume a Poisson distribution of hyperdegrees, in this case $c_2^* - c^* = c^{*2}$ and $c_3^* - 3c_2^* + 2c^* = c^{*2}$. Combine these relations with Eqs. S24 and S26, Eq. S26 reduces to

$$2Q_{\mathrm{h}}(3) < Q_{\mathrm{h}}(2). \tag{S27}$$

If the hyperedge cardinality follows a Poisson distribution, there is a continuous phase transition when $d_1 < 1$, witch corresponds to

$$d_1 < \bar{d}, \tag{S28}$$

where $\bar{d} \equiv 1$. Let us consider points around the critical point, such as $\alpha = \alpha^* - \zeta$ with $\zeta = 0^+$ and $c = c^* + \chi$ with $\chi = 0^+$. We can define the function from Eq. S21 as a function of α and χ (i.e. $F_{2c}(\alpha, c)$). By expanding $F_g(\alpha, \chi)$ around the point $(\alpha, c) = (\alpha^*, c^*)$ and combining it with result from Eq S25 at the critical point, we can rewrite Eq. S6 as

$$\left. \frac{\partial F_{2c}(\alpha, c)}{\partial c} \right|_{(\alpha^*, c^*)} \chi + \left. \frac{\partial^2 F_{2c}(\alpha, c)}{\partial^2 \alpha} \right|_{(\alpha^*, c^*)} \zeta^2 + \left. \frac{\partial^2 F_{2c}(\alpha, c_1)}{\partial^2 c} \right|_{(\alpha^*, c^*)} \chi^2 - 2 \left. \frac{\partial^2 F_{2c}(\alpha, c)}{\partial \alpha \partial c} \right|_{(\alpha^*, c^*)} \chi\zeta + (\cdots) = 0. \tag{S29}$$

For the continuous phase transition, $\alpha^* = 1$, this implies

$$\left. \frac{\partial^n F_{2c}(\mu, c_1)}{\partial^n c} \right|_{(1, c^*)} = 0 \tag{S30}$$

for any positive integer n. Thus,

$$\zeta \sim \chi. \tag{S31}$$

Let us expand s_{2c} as a function of ζ,

$$s_{2c} = s_{2c}(\mu)|_{\alpha=\alpha^*} - \left. \frac{\partial s_{2c}(\alpha)}{\partial \alpha} \right|_{\alpha=\alpha^*} \zeta + \left. \frac{\partial^2 s_{2c}(\alpha)}{\partial^2 \alpha} \right|_{\alpha=\alpha^*} \zeta^2 + (\cdots), \tag{S32}$$

For $\alpha^* = 1$, $s_{2c}(\alpha^*) = 0$ and

$$\left. \frac{\partial s_{2c}(\alpha)}{\partial \alpha} \right|_{\alpha=\alpha^*=1} = 0. \tag{S33}$$

We obtain

$$s_{2c} \sim \zeta^2 \Rightarrow s_{2c} \sim \chi^2, \tag{S34}$$

for $s_{2c}^* = 0$ and $\eta = 2$. For the discontinuous phase transition, $\alpha^* \neq 1$, we have

$$\left. \frac{\partial F_{2c}(\alpha, c)}{\partial c} \right|_{(1, c^*)} \neq 0, \tag{S35}$$

$s_{2c}(\alpha) \neq 0$ and

$$\left. \frac{\partial s_{2c}(\alpha)}{\partial \alpha} \right|_{\alpha=\alpha^*} \neq 0. \tag{S36}$$

It implies

$$\zeta \sim \chi^{1/2}, \tag{S37}$$

and

$$s_{2c} - s_{2c}^* \sim \chi^{1/2}, \tag{S38}$$

that is equivalent to Eq. S20 for $\eta = 1/2$. A discontinuous phase transition with a critical exponent smaller than one is considered as hybrid phase transition.

At the critical mean cardinality, $d = \bar{d}$, $\alpha^* = 1$, but

$$\left. \frac{\partial^2 F_{2c}}{\partial^2 \alpha} \right|_{\alpha=1} = 0. \tag{S39}$$

In this case we have to take into account terms of order 3 in the expansion of Eq. S29. We obtain,

$$\zeta \sim \chi^{1/2}, \tag{S40}$$

and,

$$s_{2c} \sim \chi, \tag{S41}$$

We can summarize that for d-uniform hypergraphs the $(2, S_{\max})$-core percolation is (i) continuous with critical exponent $\zeta = 2$ if $d = 2$; and (ii) hybrid with critical exponent $\zeta = 1/2$ if $d > 2$ (which is consistent with a previous work [2]). For hypergraphs where both the vertex degeree and hyperedge cardnality distributions are Poissonian, the $(2, \max(r, 2))$-core percolation is (i) continuous with critical exponent $\zeta = 2$ if $d < \bar{d} = 1$; (ii) continous with critical exponent $\zeta = 1$ if $d = \bar{d}$; and (iii) hybrid with critical exponent $\zeta = 1/2$ if $d > \bar{d}$. The same set of critical exponents was found for the heterogeneous-K-core [16].

B. $K = 2$ and $S = 2$

In this section we study the $(2, 2)$-core. A similar definition of removable hyperedges was used in [3], where the core obtained from the GLR procedure is used to study the vertex cover problem in uniform hypergraphs.

In this case, Eq. (S16) reduces to

$$\delta = \sum_{r=1}^{\infty} Q_{\mathrm{h}}(r) \alpha^{r-1}. \tag{S42}$$

The relative size of the $(2, 2)$-core can be calculated by considering the probability that a randomly chosen vertex is connected to at least two non-removable hyperedges and the
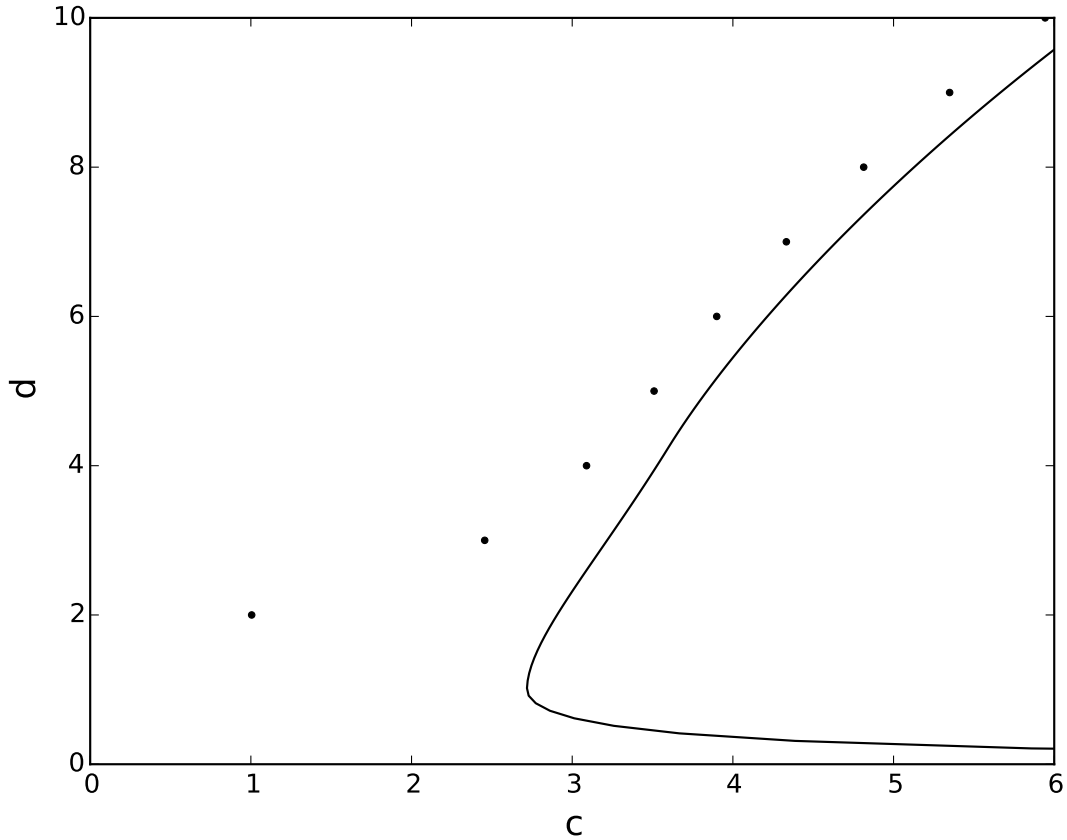
FIG. S4. Phase diagram of $(2, \max(r, 2))$-core percolation on hypergraphs with Poisson vertex degree distributions. Black circles and black line represents the phase boundary of d-uniform hypergraphs and hypergraphs with Poisson hyperedge cardinality distribution, respectively.

probability that a degree-one vertex is connected to a hyperedge with less than $(r-2)$ other degree-one vertices. This results in

$$s_{2c} = \sum_{k=2}^{\infty} P_n(k) \sum_{l=2}^{k} \binom{k}{l} (1 - \delta)^l \delta^{k-l}. \tag{S43}$$

Eqs. (S19) and (S42) have the same critical point as Eqs. (S17) and (S16). Therefore, for $(2, 2)$-core we recover the result found in the graph case that both the $(2, 2)$-core and the GCC emerge at the same critical point [6]. We can compute the critical point by combining Eq. S21, S42 and S30 at $\alpha = \delta = 1$, obtaining

$$\frac{d_2 - d_1}{d_1} \frac{c_2^* - c^*}{c^*} = 1, \tag{S44}$$

In this case the phase transition is always continuous (see solid lines in Fig. S3 c and d) and for the studied hyperedge cardinality and vertex degree distributions we have $\eta = 2$.

As before, we assume that the moments of the vertex degree and hyperedge cardinality distributions do not diverge. Let us consider points around the critical point, such as, $\alpha = 1 - \zeta$ with $\zeta = 0^+$ and $c = c^* + \chi$ with $\chi = 0^+$. We can define the function from Eqs. S21 as a function of α and χ(i.e. $F_{2c}(\alpha, c)$). By expanding $F_g(\alpha, \chi)$ around the point $(\alpha, c) = (1, c^*)$, and combining it with result from Eq. S25 at the critical point, we can rewrite Eq. S6 as

$$\left.\frac{\partial^2 F_{2c}(\alpha, c)}{\partial^2 \alpha}\right|_{(1,c^*)} \zeta^2 - 2 \left.\frac{\partial^2 F_{2c}(\alpha, c)}{\partial\alpha\partial c}\right|_{(1,c^*)} \chi\zeta + (\cdots) = 0. \tag{S45}$$

Note that for this case the Eq. S30 is still valid. We obtain

$$\zeta \sim \chi. \tag{S46}$$

By combining this equations with the fact that $s_{2c}(\alpha^*) = 0$ and the result from Eq. S33, we obtain

$$s_{2c} \sim \chi^2. \tag{S47}$$

IV.   CORE PERCOLATION ON HYPERGRAPHS

In Sec. IV A we show the relation between the hyperedge covering problems and the dominating set problem Sec. IV A We also show how our algorithm is a generalization of the greedy leaf removals procedure [17] used to solve the dominant set problem in polynomial time, and that the core obtain trough our methodology is always smaller than produced by the greedy leaf removal. In Sec. IV B to Sec. IV E we show the detailed derivations for the equations in the main text.

A.   Relation between hyperedge and vertex covering problems and the minimum dominating set problem

The minimum dominating set (MDS) is the smallest set of vertices, D of a graph G, so that every vertex in G is adjacent to at least one vertex in D. The MDS can be mapped to a minimum hyperedge cover in a hypergraph composed by the same vertices as in G and for each vertex v_1 there is an hyperedge, h_1, that contains v_1 and all neighbors of v_1 (see Figure S5). Therefore if we find the set of hyperedges $[v_i, ..., v_j]$, that covers all vertices in this hypergraph, the dominanting set of graph G is given by $[v_i, ..., v_j]$. Our method is a generalization of the method proposed in [17]. Note that our method doesn't focus onleaves, i.e., vertices with degree one. As shown in Figure S5, even given the fact that there is no one-degree vertices, our method can still solve the MDS exactly in polynomial time.

We emphasize that the greedy leaf removals rules described in [17] are special cases of our method. Let us consider the first rule: if vertex v_i is an unobserved leaf vertex (which has only a single neighbor, say v_j), then occupying v_j but leaving v_i empty must be an optimal strategy. And let us now consider the hypergraph with the same nodes as the original graph, and hyperedges h_i that represent the vertices observed by node v_i. For a leaf v_i the set of vertices that are covered by hyperedge h_i is a subset or equal set to the ones covered by hyperedge h_j, i.e. $V_i \subseteq V_j$, then h_i can be removed. After this, the hyperedges that cover v_i are a subset of or equal to the set of hyperedges that cover h_j ($H_i \subseteq H_j$), and a subset or equal set of hyperedges that cover the other neighbors of v_j, in the original network, v_k ($H_i \subseteq H_k$). Then vertices v_j and v_k are removed. Note that all observed nodes are removed from the network, in our method, since they do not need to be observed anymore, but we
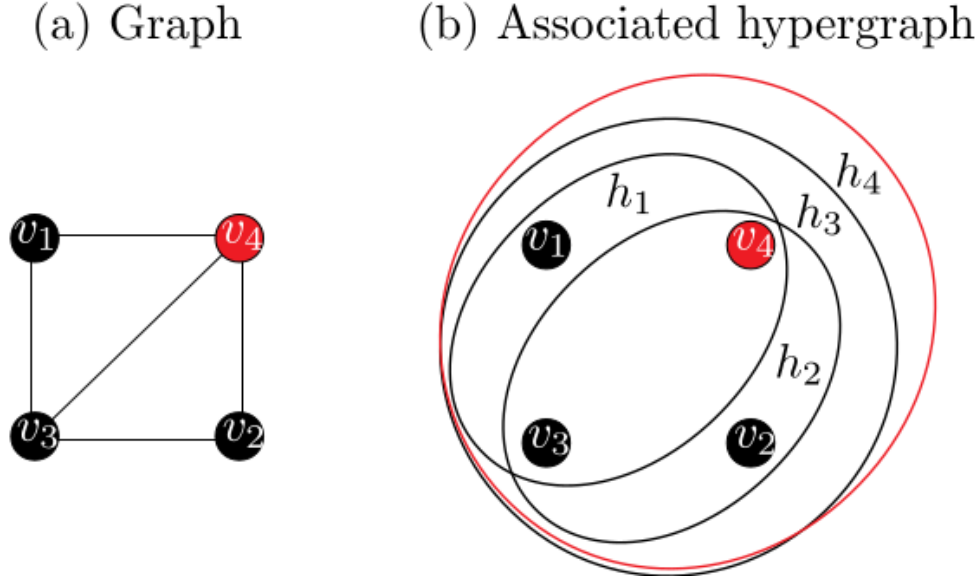
(a) Graph (b) Associated hypergraph

FIG. S5. A graph and the associated hypergraph. Hyperedge h_i represents vertices observed by vertice v_i. Solving the hypergraph's minimum hyperedge cover set is equivalent to solve the graph's dominating set. The red vertex represents the solution of the minimum dominating set problem in graph (a), the red edge the solution of the minimum hyperedge cover set problem.

still keep the hyperedges associated with them because they can still be used to cover other vertices. After this process the cardinality of hyperedge h_j is one, meaning that h_j is part of a minimum hyperedge cover set. In terms of dominating set it means that occupying node j is an optimal strategy. Let us now consider the second rule : if vertex v_i is an unobserved leaf vertex (which has only a single neighbor, say v_j), then occupying v_j but leaving v_i empty must be an optimal strategy, and if v_i is an empty but observed vertex and at most one of its adjacent vertices is unobserved, then it must be an optimal strategy not to occupy v_i. We emphasize that in our hyperedge all observed vertices v_k are automatically removed from the network. If all except at most one neighbor of vertex i is unobserved, v_j, then h_i can also be removed because all vertices covered by hyperedge v_i are also covered by vertex, v_j ($H_i \subseteq H_j$). In terms of dominating set it means that not occupying vertice v_i is an optimal strategy.

Table I shows the size of the cores associated with the dominating set, for the eleven networks analyzed in [17]. For most networks our method shows a considerable improvement, . For some of them, our method actually find no core left.

Networks	N	M	s_{glr}	s_{d}
RoadEU	1177	1417	0.260	0.167
PPI	2361	6646	0.007	0.000
Grid	4941	6594	0.122	0.056
IntNet1	6474	12572	0.001	0.000
Author	23133	93439	0.391	0.000
Citation	34546	420877	0.326	0.310
P2P	62586	147892	0.001	0.000
Friend	196591	950327	0.031	0.001
Email	265214	364481	0.002	0.000
Webpage	875713	4322051	0.185	0.004
RoadTX	1379917	1921660	0.406	0.342

TABLE I. Normalized size of the cores associated with the minimum dominanting set problem for eleven real-world networka, using the greedy leaf removal procedure from[17], s_{glr}, and using our approach s_{d}. N and M are, respectively, the total number of nodes and links in the network.

B. Number of hyperedges contained in the core

In the main text we characterize the size of the core as the number of vertices contained in the core, we can also compute the number of hyperedges in each core, s'_v and s'_h. We can use the fact that the v_{core} of a hypergraph is h_{core} of the dual hypergraph. Thus, we can get Eqs. (1) (2) (7) and (8) in the main text by the following transformation

$$\alpha \rightarrow \delta \tag{S48}$$

$$\delta \rightarrow \beta \tag{S49}$$

$$\beta \rightarrow \epsilon \tag{S50}$$

$$\epsilon \rightarrow \alpha \tag{S51}$$

on Eqs. (1) to (4) from the main text. Because the vertices of an hypergraph are hyperedges in the dual one, we obtain that s'_{h} is given by

$$s_{\text{h}}^{\text{h}} = \sum_{k=2}^{\infty} P_{\text{h}}(r) \sum_{l=2}^{k} \binom{k}{l} (1 - \beta - \alpha)^l \alpha^{k-l}, \tag{S52}$$

where α, β, δ and γ are obtained by solving Eqs. (1),(2) (7) and (8) in the main text. Using the same argument we can conclude that s_h' is given by

$$s_v^h = \sum_{k=2}^{\infty} P_h(r) \sum_{l=2}^{k} \binom{k}{l} (1 - \alpha - \beta)^l \beta^{k-l}, \tag{S53}$$

where α, β, δ and γ are obtained by solving Eqs. (1),(2) (7) and (8) in the main text.

C. v_{core} for uniform hypergraphs - critical point

There is a simple relation between v_{core} critical point and the mean cardinality d_1 for uniform hypergraphs with $d_1 > 1$. Let us start by rewriting Eq. 1 in the main text as

$$F_{c1}(\alpha, c) = 0 \tag{S54}$$

where we define

$$F_{c1}(\alpha, c) \equiv \sum_{k=1}^{\infty} Q_n(k) \epsilon^{k-1} - \alpha. \tag{S55}$$

For uniform-Poisson hypergraph Eqs (1), (2) and (7) and (8) in the main text can be reduced to

$$x = e^{-c(d_1-1)y} \tag{S56}$$

$$y = e^{-c(d_1-1)x}, \tag{S57}$$

where we define $x \equiv \alpha^{r-1}$ and $y \equiv (1 - \beta)^{r-1}$. The function defined in Eq. S55 can be written as a function of x and c,

$$F_{c1}(x, c) \equiv \exp\left(-c\,(d_1 - 1)\,e^{-c\,(d_1-1)\,x}\right) - x. \tag{S58}$$

The critical point is given by $\alpha^* = (c^*\,(d_1 - 1))^{1/(1-d_1)}$ and $c^* = e/(d_1 - 1)$.

D. v_{core} - critical exponent

Eqs. 1 and 2, 7 and 8 in the main text can be written as

$$\alpha = A(1 - z) \tag{S59}$$

$$z = B(h) \tag{S60}$$

$$h = A(1 - \delta) \tag{S61}$$

$$\delta = B(\alpha) \tag{S62}$$

where we define $z \equiv 1-\epsilon$, $h \equiv 1-\beta$, $A(1-z) \equiv \sum_{k=1}^{\infty} Q_{\mathrm{n}}(k)(1-z)^{k-1}$ and $B(h) \equiv \sum_{r=2}^{+\infty} Q_r(r)h^{r-1}$.

$F_{\mathrm{v}}(\alpha, c)$ can be written as

$$F_{\mathrm{v}}(\alpha, c) = H(H(\alpha, c), c) - \alpha, \tag{S63}$$

where we define

$$H(\alpha, c) \equiv A(1 - B(\alpha), c1). \tag{S64}$$

This function has the same form for the graph case, (see Eq. S60 from [18]). The only difference is the way we define the function $H(\alpha, c)$. Nevertheless it obeys the same relation, and therefore has the same critical exponent $\zeta_1 = 1$, i.e:

$$\left. \frac{\partial H(\alpha, c)}{\partial \alpha} \right|_{(\alpha, c) = (\alpha^*, c^*)} = -1, \tag{S65}$$

$$\left. \frac{\partial F_{\mathrm{h}}(\alpha, c)}{\partial c} \right|_{(\alpha, c) = (\alpha^*, c^*)} = 0, \tag{S66}$$

$$\left. \frac{\partial^2 F_{\mathrm{h}}(\alpha, c)}{\partial^2 \alpha} \right|_{(\alpha, c) = (\alpha^*, c^*)} = 0. \tag{S67}$$

From an expansion of F_{h} as a function of $(c - c^*)$, for $c - c^* = 0^+$, it follows that

$$\alpha - \alpha^* \sim (c - c^*)^{1/2}, \tag{S68}$$

$$\beta - \beta^* \sim (c - c^*)^{1/2}, \tag{S69}$$

$$1 - \beta - \alpha \sim (c - c^*)^{1/2}, \tag{S70}$$

and

$$s_v^{\mathrm{v}} \sim (c - c^*), \tag{S71}$$

witch is equivalent to Eq. (6) in the main text, for $\zeta_1 = 1$.

E. h_{core} - critical exponent

Eqs. (1) to (4) in the main text can be written as

$$\alpha = A(\epsilon) \tag{S72}$$

$$\epsilon = B(1 - u) \tag{S73}$$

$$u = A(v) \tag{S74}$$

$$v = B(1 - \alpha) \tag{S75}$$

where we define $u \equiv 1 - \beta$ and $v \equiv 1 - \delta$. The function $F_{c2}(\alpha, c) \equiv A(\epsilon, c) - \alpha$ can be written as

$$F_{\mathrm{h}}(\alpha, c) = G(G(\alpha, c), c) - \alpha, \tag{S76}$$

where we define

$$G(\alpha, c) \equiv A(B(1 - u), c1). \tag{S77}$$

This function has the same form as Eqs. S64. The only difference is how we define the function $G(\alpha, c)$. Nevertheless it obeys the same relations as Eqs. S65 to S70, thus,

$$s_{\mathrm{h}}^{\mathrm{v}} \sim (c - c^*)^{\zeta_2}, \tag{S78}$$

with $\zeta_2 = \zeta_1 = 1$.

V.   SIMULATIONS

We compare our analytical calculations with random hypergraphs generated in the following ways.

1.   Poisson-Uniform hypergraphs

We first generate random hypergraphs where all hyperedges have the same cardinality d and the vertex hyperdegrees follow a Possion distribution with average value c_1. We can generate this hypergraphs by creating N nodes and L hyperedges of cardinality d. We then fill the hyperedges with random selected vertices from our list of nodes. In the end

$$c = dL/N. \tag{S79}$$

2.   Poisson-Poisson hypergraphs

In the Poisson-Poisson hypergraphs we generate random hypergraphs where and both the vertex degrees and hyperedge cardinalities follow a Possion distribution with average c (or d), respectively. We can generate this hypergraphs by creating N vertices and L hyperedges. We then select random vertices and add it to a random selected hyperedge until the average hyperdegree of the network is

$$c = d \times L/N. \tag{S80}$$

A.   Results

Figures S6 to S10 show that numerical simulations agree well with the analytical calculations. In the simulations we used hypergraphs of size 10^4 vertices.
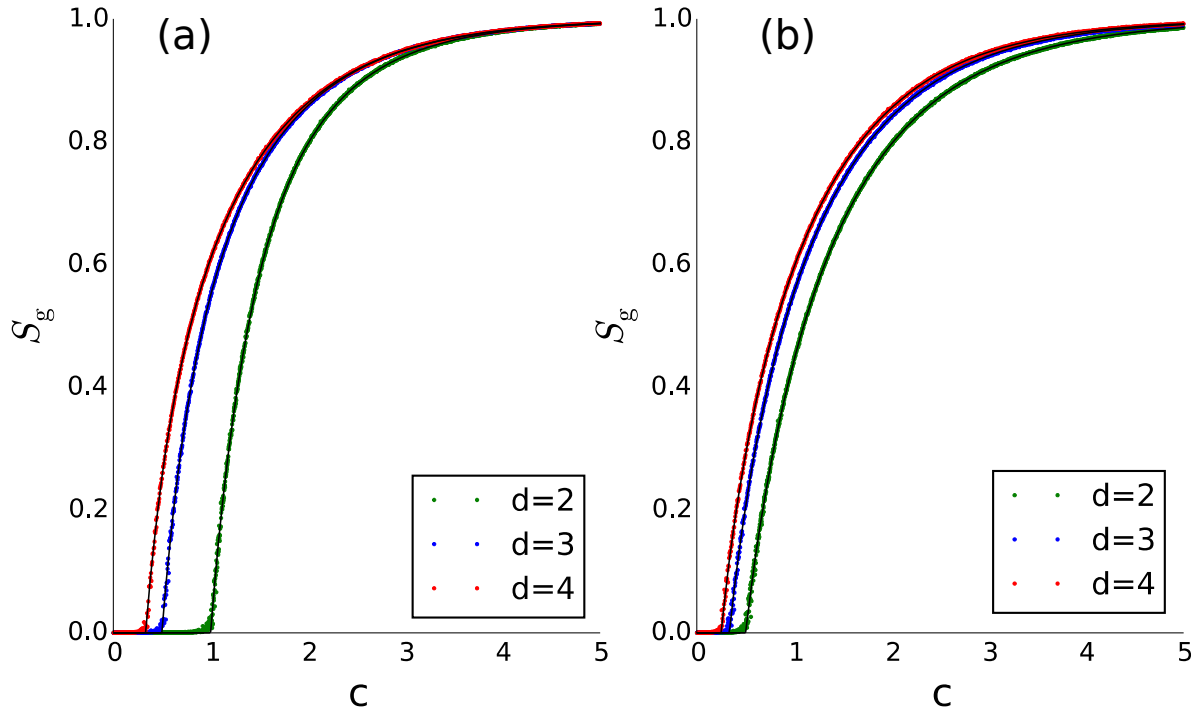
FIG. S6. The relative size of the giant connected component, s_{g}, for hypergraphs with Poisson vertex degree distributions and mean degree c. Black solid lines: theoretical results. Dots: numerical results from hypergraphs with 10^4 vertices. (a) d-uniform hypergraphs where all hyperdege have the same cardinality d. (b) the hyperedge cardinality follows a Poisson distribution with average cardinality d.
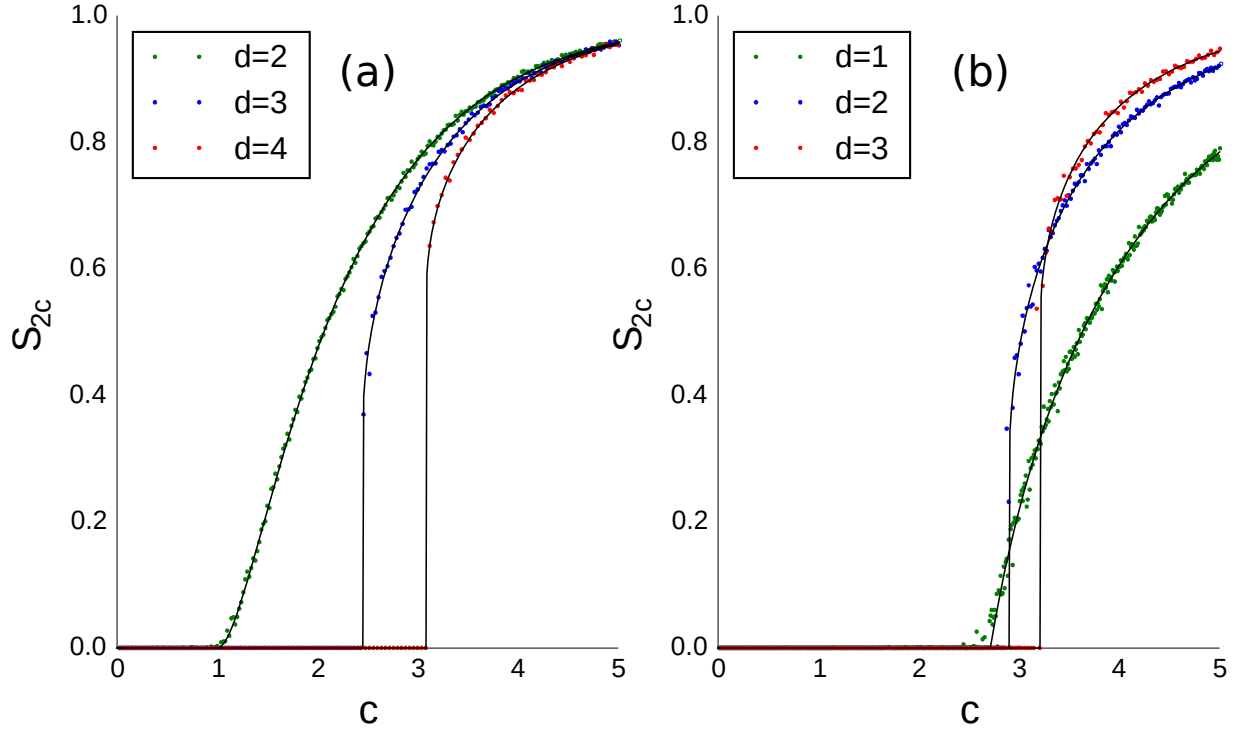
FIG. S7. (a) The relative size of the $(2, S)$-core with $S = \max(r - 1, 2)$ for hypergraphs with Poisson vertex degree distributions and mean degree c. Black solid lines: theoretical results. Dots: numerical results from hypergraphs with 10^4 vertices. (a) d-uniform hypergraphs where all hyperdege have the same cardinality d. (b) the hyperedge cardinality follows a Poisson distribution with average cardinality d.
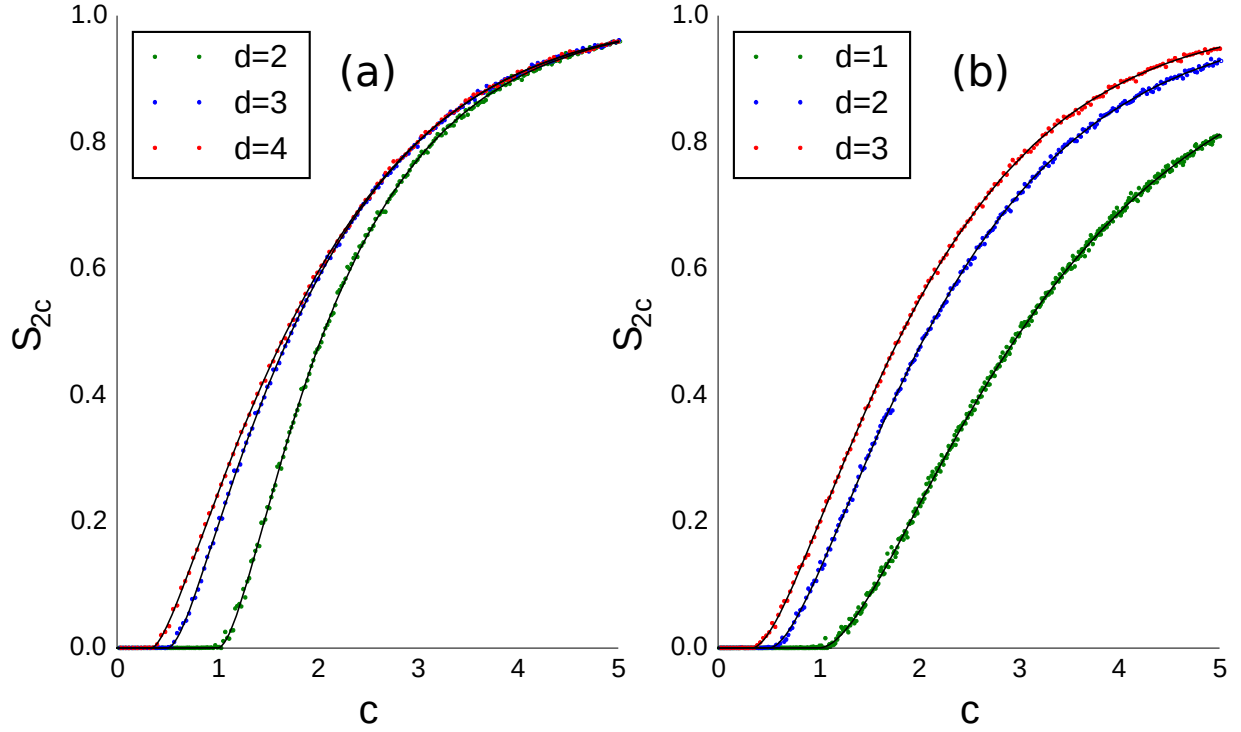
FIG. S8. The relative size of the $(2, S)$-core with $S = 2$ for hypergraphs with Poisson vertex degree distributions and mean degree c. Black solid lines: theoretical results. Dots: numerical results from hypergraphs with 10^4 vertices. (a) d-uniform hypergraphs where all hyperdege have the same cardinality d. (b) the hyperedge cardinality follows a Poisson distribution with average cardinality d.
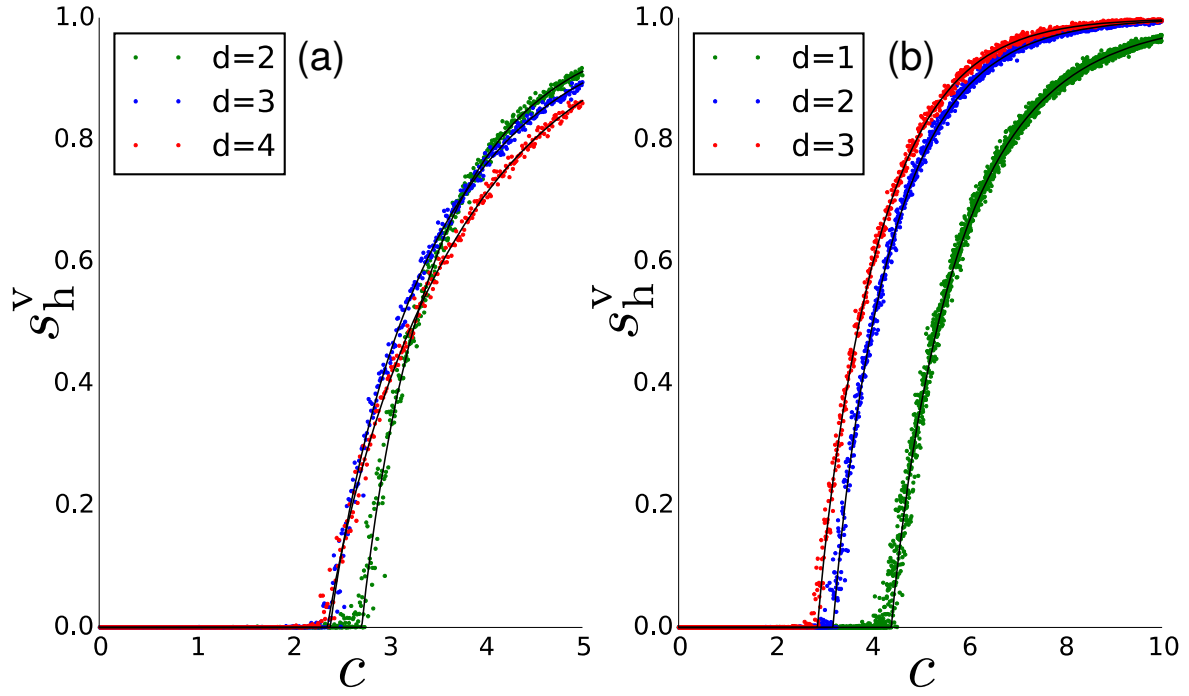
FIG. S9. h_{core} **obtained from simulations.** The relative size of the h-core for hypergraphs with Poisson vertex degree distributions and mean degree c. Black solid lines: theoretical results. Dots: numerical results from hypergraphs with 10^5 vertices. (a) d-uniform hypergraphs where all hyperdege have the same cardinality d. (b) the hyperedge cardinality follows a Poisson distribution with average cardinality d.
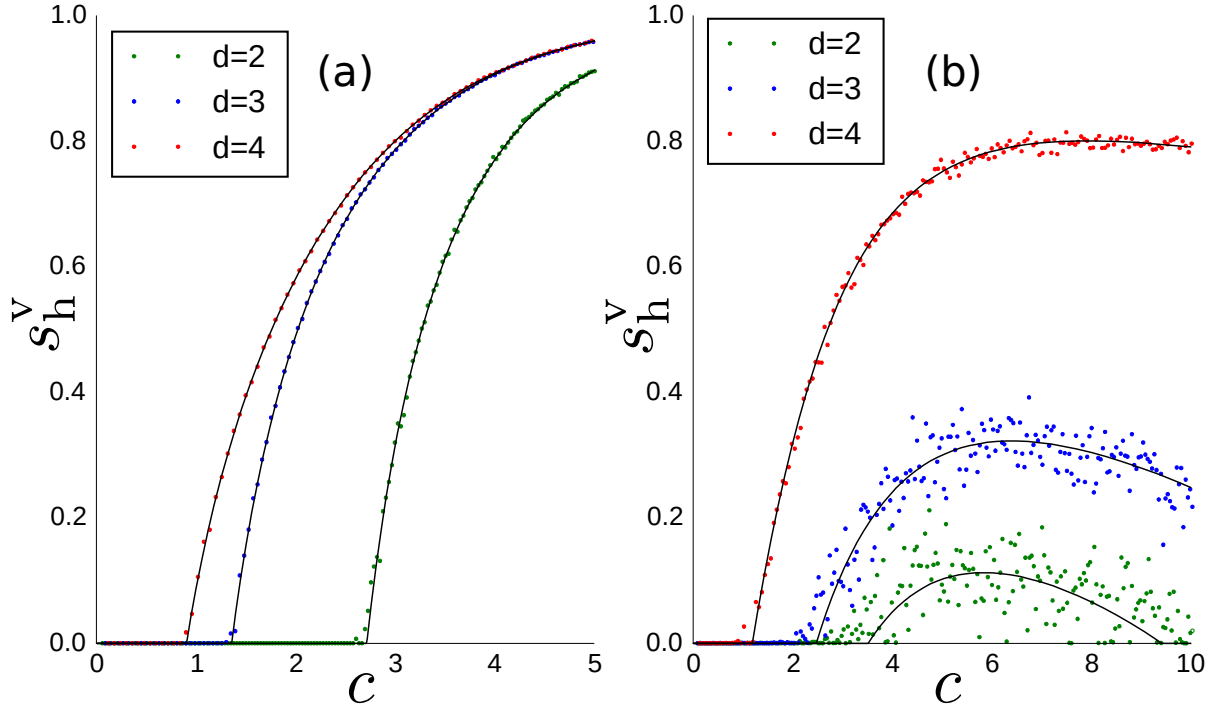
FIG. S10. v_{core} **obtained from simulations.** The relative size of the v-core for hypergraphs with Poisson vertex degree distributions and mean degree c. Black solid lines: theoretical results. Dots: numerical results from hypergraphs with 10^5 vertices. (a) d-uniform hypergraphs where all hyperdege have the same cardinality d. (b) the hyperedge cardinality follows a Poisson distribution with average cardinality d. The large fluctuation for $d = 2$ and 3 are due to the relatively large number of hypereges with cardinalty 1, and the chance that if a edge is connected to a hypereges with cardinalty 1 is automatically removed.

VI.   CONTRIBUTIONS AND AUTHOR INFORMATION

Contributions. Y.-Y.L. and H.-J.Z conceived the project. All authors designed the research. B.C.C. performed all the analytical calculations and numerical simulations. All authors analyzed the results. B.C.C. and Y.-Y.L. wrote the manuscript. H.-J.Z. edited the manuscript.

Author Information. The authors declare no competing financial interests. Correspondence and requests for materials should be addressed to Y.-Y.L. (yyl@channing.harvard.edu), or B.C.C. (coutinho.b@husky.neu.ed).

[1] Christina Goldschmidt. Critical random hypergraphs: the emergence of a giant set of identifiable vertices. Ann/ Prob, pages 1573–1600, 2005.

[2] Colin Cooper. The cores of random hypergraphs with a given degree sequence. Random Structures and Algorithms, 25(4):353–375, 2004.

[3] Satoshi Takabe and Koji Hukushima. Minimum vertex cover problems on random hypergraphs: Replica symmetric solution and a leaf removal algorithm. Phys. Rev. E, 89:062139, Jun 2014.

[4] S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes. k-core organization of complex networks. Phys. Rev. Lett., 96:040601, Feb 2006.

[5] J. Ignacio Alvarez-hamelin, Alain Barrat, and Alessandro Vespignani. Large scale networks fingerprinting and visualization using the k-core decomposition. In Advances in Neural Information Processing Systems 18, pages 41–50. MIT Press, 2006.

[6] S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes. Critical phenomena in complex networks. Rev. Mod. Phys., 80:1275–1335, Oct 2008.

[7] P. Erdös and A. Rényi. On random graphs, I. Publicationes Mathematicae (Debrecen), 6:290–297, 1959.

[8] Maksim Kitsak, Lazaros K. Gallos, Shlomo Havlin, Fredrik Liljeros, Lev Muchnik, H. Eugene Stanley, and Hernan A. Makse. Identification of influential spreaders in complex networks. Nat Phys, 6(11):888–893, 11 2010.

[9] M. Bauer and O. Golinelli. Exactly solvable model with two conductor-insulator transitions driven by impurities. Phys. Rev. Lett., 86:2621–2624, Mar 2001.

[10] Yang-Yu Liu, Jean-Jacques Slotine, and Albert-Laszlo Barabasi. Controllability of complex networks. Nature, 473(7346):167–173, May 2011.

[11] Richard M. Karp and M. Sipser. Maximum matching in sparse random graphs. In Foundations of Computer Science, 1981. SFCS '81. 22nd Annual Symposium on, pages 364–375, Oct 1981.

[12] M. E. J. Newman, S. H. Strogatz, and D. J. Watts. Random graphs with arbitrary degree distributions and their applications. Phys. Rev. E, 64:026118, Jul 2001.

[13] Marc Mezard and Andrea Montanari. Information, Physics, and Computation. Oxford University Press, Inc., New York, NY, USA, 2009.

[14] Michael Molloy and Bruce Reed. A critical point for random graphs with a given degree sequence. Random Structures and Algorithms, 6:161–179, 1995.

[15] Béla Bollobás and Oliver Riordan. Erdős Centennial, chapter The Phase Transition in the Erdős-Rényi Random Graph Process, pages 59–110. Springer Berlin Heidelberg, Berlin, Heidelberg, 2013.

[16] Davide Cellai, Aonghus Lawlor, Kenneth A. Dawson, and James P. Gleeson. Tricritical point in heterogeneous k-core percolation. Phys. Rev. Lett., 107:175703, Oct 2011.

[17] Jin-Hua Zhao, Yusupjan Habibulla, and Hai-Jun Zhou. Statistical mechanics of the minimum dominating set problem. Journal of Statistical Physics, 159(5):1154–1174, 2015.

[18] Yang-Yu Liu, Endre Csóka, Haijun Zhou, and Márton Pósfai. Core percolation on complex networks. Phys. Rev. Lett., 109:205703, Nov 2012.