

Chapman-Enskog expansion for the Vicsek model of self-propelled particles

Thomas Ihle

Institut für Physik, Ernst-Moritz-Arndt-Universität Greifswald, 17489 Greifswald, Germany

Department of Physics, North Dakota State University, Fargo, ND 58108-6050, USA

E-mail: thomas.ihle@uni-greifswald.de

Abstract. Using the standard Vicsek model, I show how the macroscopic transport equations can be systematically derived from microscopic collision rules. The approach starts with the exact evolution equation for the N -particle probability distribution, and after making the mean-field assumption of Molecular Chaos leads to a multi-particle Enskog-type equation. This equation is treated by a non-standard Chapman-Enskog expansion to extract the macroscopic behavior. The expansion includes terms up to third order in a formal expansion parameter ϵ , and involves a fast time scale. A self-consistent closure of the moment equations is presented that leads to a continuity equation for the particle density and a Navier-Stokes-like equation for the momentum density. Expressions for all transport coefficients in these macroscopic equations are given explicitly in terms of microscopic parameters of the model. The transport coefficients depend on specific angular integrals which are evaluated asymptotically in the limit of infinitely many collision partners, using an analogy to a random walk. The consistency of the Chapman-Enskog approach is checked by an independent calculation of the shear viscosity using a Green-Kubo relation.

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1. Introduction

Recently, there has been a large interest in active many-particle systems such as bird flocks [1], swarming bacteria [2], chemically powered nanorods [3], microtubule mixtures [4] and actin networks [5] driven by molecular motors. These systems display very interesting behaviors such as pattern formation, collective motion and non-equilibrium phase transitions [6, 7]. Due to the many degrees of freedom, theoretical and numerical investigations are usually based on coarse-grained macroscopic transport equations for the slow variables. In many cases, the general form of these equations are designed by symmetry arguments or Onsager relations which means that the exact coefficients of the terms in them remain undetermined, see for example the Toner-Tu theory [8, 9]. This can lead to models with many free parameters which are time-consuming to explore numerically. In addition, in complicated systems it is possible to overlook nontrivial symmetries which would cause some of these terms to vanish. It is also not clear how these many terms depend on a few microscopic parameters, and brute force scanning of the parameters could waste computer time in regions of the parameter space which are not compatible with the underlying microscopic system.

Therefore, it is very desirable to have a direct, rigorous derivation of these macroscopic equations from the microscopic rules. Systems of interest are often described by simplified models, such as the Vicsek-model (VM) for self-propelled agents, [10, 11, 12], which are suited for computer simulations but keep the essential physics. The time evolution in such models is often discrete, that is, there is a non-zero time step, τ , and there is some type of generalized interaction between the simulated objects. For instance, in the VM, at every time step an agent interacts with all agents in a circle of radius r_0 around it to adopt its flying direction towards the mean direction of the other agents plus some noise. Interpreting agents as “particles”, this alignment corresponds to a genuine multi-particle interaction which is not pairwise additive.

In this paper, I will present how to systematically derive macroscopic evolution equations from the microscopic rules of the standard Vicsek-model using techniques from kinetic theory. One of the techniques applied in this paper is a non-standard Chapman-Enskog expansion that keeps a fast time scale. This approach can be easily generalized to other particle-based models of active soft matter [7] with a discrete time step and multi-particle interactions. In previous publications [13, 14, 15, 16], the results of this derivation have already been discussed and used without showing the technical details. For example, in table 2 of Ref. [16] the density dependence of the transport coefficients in the large density limit was given. In this paper, I will focus on the details of the derivation. This includes presenting a self-consistent closure of the moment equations as well as determining the coefficients of nonlinear terms and gradient terms in the hydrodynamic equations up to a predefined order.

Note, that similar derivations of macroscopic equations for Vicsek-like models and similar closures have been performed by other groups, see for example Refs. [17, 18, 19, 20, 21, 22, 23, 24]. However, these authors did not explicitly treat the

standard VM with multi-body interactions and discrete time step as done in this paper. Furthermore, they used methods to extract the macroscopic behavior, which differ from the ones used here. Recently, the results presented in this paper and the ones obtained by the Boltzmann-approach of Refs. [17, 18] were compared in detail and critically debated in a series of publications, Refs. [15, 24, 25, 26, 27]. However, a comprehensive comparative study of the different approaches combined with computer-assisted quantitative verification of the derived transport coefficients is still lacking and will be left for the future.

2. The Vicsek model

The two-dimensional Vicsek-model (VM) [10, 11, 12] consists of N point particles at global number density ρ_0 , which move at constant speed v_0 . The positions and velocities of the particles are given by $\mathbf{x}_i(t)$ and $\mathbf{v}_i(t)$, respectively. The particles undergo discrete-time dynamics with time step τ . The evolution consists of two steps: streaming and collision. Note, that the term ‘‘collision’’ is not to be taken literally. Instead, it just denotes any interaction that changes the momentum of a particle. In the streaming step all positions are updated according to

$$\mathbf{x}_i(t + \tau) = \mathbf{x}_i(t) + \tau \mathbf{v}_i(t). \quad (1)$$

Because the particle speeds stay the same at all times, the velocities are parameterized by the ‘‘flying’’ angles, θ_i , $\mathbf{v}_i = v_0(\cos \theta_i, \sin \theta_i)$. In the collision step, the directions θ_i are modified. Particles align with their neighbors within a fixed distance r_0 plus some external noise: a circle of radius r_0 is drawn around the focal particle i , and the average direction Φ_i of motion of the particles (including particle i) within the circle is determined according to

$$\Phi_i = \text{Arg} \left(\sum_{\{j\}} e^{i\theta_j} \right) \quad (2)$$

Eq. (2) means that the vector sum of all particle velocities in every circle is computed and the direction of the resulting vector is taken as average angle Φ_i . Once all average directions Φ_i are known, the new directions follow as

$$\theta_i(t + \tau) = \Phi_i + \xi_i \quad (3)$$

where ξ_i is the so-called angular noise. The random numbers ξ_i are uniformly distributed in the interval $[-\eta/2, \eta/2]$. The model uses parallel updating, and, in this paper, I will also assume the so-called standard Vicsek-model which uses a forward-updating rule. This amounts to using the already updated positions $\mathbf{x}_i(t + \tau)$ in determining the average directions Φ_i . The noise strength η is an important parameter of the VM. Another relevant parameter is the average particle number M that can be found inside a circle of radius r_0 . Thus, $M = \rho_0 \pi r_0^2$ where ρ_0 is the global number density. On the mean-field level, the average number of particles a particular focal particle is encountering

inside its collision circle, is also given by M . The number density, ρ_0 , itself is not a relevant parameter because the particles have zero volume and the spatial “extension” of a particle is described by the range of the alignment interaction, r_0 , instead. One of the few meaningful dimensionless quantities to be formed by the number density is given by M , which describes the ratio of the interaction range to the average particle distance $1/\sqrt{\rho_0}$.

3. Kinetic theory

In this paper the details of how to systematically derive kinetic and hydrodynamic equations for particle-based models with a discrete time dynamics and very general collision rules are presented. No linearizations or BGK-relaxation-type approximations (Bhatnagar-Gross-Krook), [28], are required. Consequently, nonlinearities and gradient terms of the hydrodynamic fields can in principle be recovered to arbitrary order with no free parameters. The main approximations involved are the assumption of Molecular chaos and that the system is not too inhomogeneous. The former assumption amounts to a mean-field approach; the latter means that gradients in the hydrodynamic variables are small enough to justify a gradient expansion.

The kinetic approach presented here relies on a properly designed ensemble of macroscopically identical copies of the system. This allows the definition of the N -particle probability density, $P(\theta^{(N)}, \mathbf{X}^{(N)}, t)$ where $\mathbf{X}^{(N)} \equiv (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N)$, and $\theta^{(N)} \equiv (\theta_1, \theta_2, \dots, \theta_N)$. Here, the velocities $\mathbf{V}^{(N)} \equiv (\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_N)$, are given in terms of the angles, $\mathbf{v}_i = v_0(\cos \theta_i, \sin \theta_i)$. Then, $P_N \prod_i d\mathbf{x}_i d\theta_i$ gives the fraction of those members of the ensemble within which particle 1 is found in the phase space element $d\mathbf{x}_1 d\theta_1$ around position \mathbf{x}_1 and angle θ_1 while simultaneously particle 2 is in the element $d\mathbf{x}_2 d\theta_2$ around position \mathbf{x}_2 and angle θ_2 , and so on. Macroscopic quantities, such as the density field $\rho(\mathbf{r}, t)$ or the two-time, two-point correlation function $g_2(\mathbf{r}, \phi, \mathbf{r}', \phi', t')$ can then be defined as ensemble averages over their microscopic counterparts. To describe quantities that merely depend on one position and one time, only the one-point microscopic phase space density

$$\Psi_1 = \sum_i \delta(\theta - \theta_i) \delta(\mathbf{x} - \mathbf{x}_i) \quad (4)$$

is needed. See Refs. [29, 30] for more information on phase space densities. When viewed at fixed \mathbf{x} and θ for one particular member of the ensemble, Ψ_1 is strongly fluctuating in time t due to the underlying particle dynamics. The ensemble average over Ψ_1 defines the one-particle density f ,

$$f(\mathbf{x}, \theta, t) = \langle \Psi_1 \rangle = \int \Psi_1 P(\theta^{(N)}, \mathbf{X}^{(N)}, t) \prod_i^N d\mathbf{x}_i d\theta_i, \quad (5)$$

Additional integrations over the angle θ yield the macroscopic number and momentum density, $\rho(\mathbf{x}, t)$ and $\mathbf{w}(\mathbf{x}, t)$, respectively:

$$\rho(\mathbf{x}, t) = \int_0^{2\pi} f(\mathbf{x}, \theta, t) d\theta$$

$$\mathbf{w}(\mathbf{x}, t) = \int_0^{2\pi} \mathbf{v} f(\mathbf{x}, \theta, t) d\theta \quad (6)$$

with $\mathbf{v} = v_0(\cos\theta, \sin\theta)$. In that sense, $\rho(\mathbf{x}, t)$ denotes the average density at position \mathbf{x} , where the average is performed over all members of the ensemble. The goal of a mean-field kinetic theory is to first derive a closed equation for $f(\mathbf{x}, \theta, t)$, which is then either analyzed directly or used to derive hydrodynamic equations for macroscopic fields such as ρ and \mathbf{w} .

The starting point of the kinetic approach presented here, called phase space approach (PSA) in Ref. [15], is not an approximate equation for f such as the Boltzmann-, Enskog- or Vlasov-equations. Instead, one starts at a more fundamental level with an exact evolution equation for the N-particle probability density P_N ,

$$P(\theta^{(N)}, \mathbf{X}^{(N)} + \tau \mathbf{V}^{(N)}, t + \tau) = C \diamond P(\theta^{(N)}, \mathbf{X}^{(N)}, t) \quad (7)$$

where the collision operator C encodes the microscopic collision rules,

$$\begin{aligned} C \diamond P(\theta^{(N)}, \mathbf{X}^{(N)}, t) &= \frac{1}{\eta^N} \int_{-\eta/2}^{\eta/2} d\xi^{(N)} \int_0^{2\pi} d\tilde{\theta}^{(N)} P(\tilde{\theta}^{(N)}, \mathbf{X}^{(N)}, t) \\ &\times \prod_{i=1}^N \hat{\delta}(\theta_i - \xi_i - \Phi_i(\tilde{\theta}^{(N)}, \mathbf{X}^{(N)}; r_0)). \end{aligned} \quad (8)$$

Here, $\hat{\delta}(x) = \sum_{m=-\infty}^{\infty} \delta(x + 2\pi m)$ is the periodically continued delta function which is needed to map angles outside the range $[0, 2\pi]$ back to this interval. The mean angle Φ_i depends on the precollisional angles $\tilde{\theta}_i$ of the particles which are located in a circle of radius r_0 around particle i . Thus, Φ_i depends also implicitly on the interaction range r_0 and on the positions of the particles. This mean angle Φ_i is defined in Eq. (2). The first integral in Eq. (8) averages over the individual angular noises, ξ_i .

Eq. (7) describes a Markov-chain in phase space. It is exact and contains the microscopic details of the multi-body collision rules introduced in Sec. 2. An alternative but equivalent approach is to construct an exact evolution equation for the microscopic quantity Ψ_1 itself. This approach was pioneered by Klimontovich [29] and is often used in plasma physics [31]. Very recently, it has also been applied to Vicsek-like models [21]. However, I believe that complicated multi-particle collisions which cannot be expressed as a sum of pairwise interactions are technically easier to treat in the current approach, even at mean-field level. This belief is supported by the observation that the authors of Ref. [21] chose to modify the interaction rules of the VM to make them pairwise additive, which is not required for PSA.

Because the phase space approach is based on an ensemble average over infinitely many members, the resulting kinetic and hydrodynamic equations do not and must not contain explicit noise-terms. This is because whether noise terms are required or not depends on the way how the main kinetic quantity, which is very often denoted by the same symbol $f(\mathbf{x}, \theta)$, is defined. If one were to derive an approximate equation for the strongly fluctuating microscopic density Ψ_1 itself, for example by defining $\tilde{f} = \langle \Psi_1 \rangle + \mu$ where μ is supposed to model the fluctuating difference $\Psi_1 - \langle \Psi_1 \rangle$ which must be noisy

by definition, the resulting kinetic equation for \tilde{f} would need an explicit noise term. In this work, the final kinetic equation is for the one-particle density f defined in Eq. (5), whose evolution equation is noise-free. However, if going beyond the mean-field approximation [32], this equation will also depend on two-point, three-point and higher correlation functions. A thorough discussion on when noise terms are required can be found in Ref. [33].

Similar to the BBGKY-hierarchy in classical mechanics, it is possible to derive a hierarchy of evolution equations for n -particle correlation functions with $n = 1, 2 \dots N$. from Eqs. (7,8). In Ref. [32] the first two members of this hierarchy were considered.

In this paper, I use the simplest and most common way to close this hierarchy and assume that there are no correlations among particles *prior* to the collisions. This means that the probability distribution P just before the collision step is approximated by a product of identical one-particle probability distributions:

$$P(\theta^{(N)}, \mathbf{X}^{(N)}, t) = \prod_{i=1}^N P_1(\theta_i, \mathbf{x}_i, t) \quad (9)$$

This is the assumption of Molecular Chaos which is reasonable at moderate to large noise strength η and/or at a large ratio of mean free path, $\Lambda = v\tau$, to the radius of interaction, r_0 . More discussions on the validity of this Ansatz in active matter can be found in Refs. [32, 34, 35]. On the technical level, for a time-discrete model such as the VM, the factorization is only used on the r.h.s of Eq. (7). The Molecular Chaos approximation has a long history in the kinetic theory of gases. Its proper use, [36], in a formal derivation of the Boltzmann equation for a regular gas with time-continuous dynamics from the BBGKY-hierarchy is quite intricate and involves further assumptions, see for example Refs. [37, 38, 39].

Using the factorization, Eq. (9), an Enskog-like equation for the one-particle probability distribution can be obtained. This is achieved by first multiplying Eq. (7) by the phase space density Ψ_1 , Eq. (4), and subsequently integrating over all particle positions x_i and angles θ_i . This amounts to an ensemble average of Ψ_1 and leads to the distribution function $f(\theta, \mathbf{x})$. The set of variables (θ, \mathbf{x}) will be called field variables which have to be distinguished from the phases of individual particles, (θ_i, \mathbf{x}_i) . Each term in the sum, Eq. (4), gives only a non-zero contribution if a particular particle j resides at location \mathbf{x} , that is $\mathbf{x}_j = \mathbf{x}$. This particle is called the *focal* particle. Since all N particles behave the same and have the same physical properties, it suffices to merely consider the first term in Ψ_1 . This amounts to taking particle 1 as the focal particle, that is, $j = 1$ and $\mathbf{x}_1 = \mathbf{x}$. The contributions from the other terms in Eq. (4) are included by an overall factor of N . This factor of N is incorporated into the definition of the one-particle distribution function, $f(\theta, \mathbf{x}, t) = NP_1(\theta, \mathbf{x}, t)$, and, in the thermodynamic limit $N \rightarrow \infty$ will drop entirely from the kinetic equations.

Although the position and the post-collisional angle θ_1 of particle 1 is now fixed in our calculation, all possible configurations of the remaining $N - 1$ particles have to be considered. The post-collisional angle of the focal particle depends on these

configurations. For example, if only particle 3 is inside the collision circle of particle 1, the alignment rule describes a binary collision and has a different effect as if, for example, particles 2, 6, and 9 were within interaction range and participate in a four-particle interaction. In order to properly sum over all possible configurations, the integration over the locations of particles 2, 3, \dots , N is split into integrations over particles *inside* the collision circle around the focal particles, and integrals over positions that are *outside*. Mathematically, this is described by the identity

$$\int_{\text{all space}} d\mathbf{x}_2 d\mathbf{x}_3 d\mathbf{x}_4 \dots d\mathbf{x}_N \rightarrow \sum_{n=1}^N C_n \int_{\text{out}} d\mathbf{x}_{n+1} d\mathbf{x}_{n+2} \dots d\mathbf{x}_N \int_{\text{in}} d\mathbf{x}_2 d\mathbf{x}_3 \dots d\mathbf{x}_n \quad (10)$$

where C_n is a combinatorial factor. This factor counts the number of possibilities $n - 1$ particles can be picked from the available particles $N - 1$ particles and assigned to the collision circle, yielding

$$C_n = \frac{(N - 1)!}{(n - 1)!(N - n)!} . \quad (11)$$

The subscripts “in” and “out” describe integrations over the inside and outside of the collision circle, respectively.

Consider a location \mathbf{x} which will also be the location of the focal particle $j = 1$. Because the particles have zero volume, the collision circle around \mathbf{x} can contain any particle number n between one and N . There are C_n possibilities to assign n particles to that circle. For every such microstate, one integrates over the positions and angles of all other $N - n$ particles outside this circle. This is easily done, because the collisions that affect particle 1 only couple the n particles inside its interaction circle. Assuming an inhomogeneous density distribution $\rho(x)$, Eq. (6), and integrating one particle over all pre-collisional angles and all possible positions *outside* the collision circle in the area $\tilde{A} = A - \pi r_0^2$ gives the contribution

$$\int_0^{2\pi} d\theta \int_{\tilde{A}} d\mathbf{x} P_1(\mathbf{x}, \theta) = \frac{1}{N} \int_0^{2\pi} d\theta \int_{\tilde{A}} d\mathbf{x} f(\mathbf{x}, \theta) = \frac{1}{N} \int_{\tilde{A}} d\mathbf{x} \rho(x, t) = \frac{1}{N} \left(N - \int_{\odot} dx \rho(x, t) \right) \quad (12)$$

where the subscript \odot denotes an integration over the collision circle centered around $\mathbf{x} = \mathbf{x}_1$. These integrations over the $N - n$ particles outside the circle lead to a total factor $(1 - M_R(\mathbf{x})/N)^{N-n}$, where $M_R(\mathbf{x})$ is the position-dependent average number of particles in the collision circle,

$$M_R(\mathbf{x}, t) = \int_{\odot} \rho(\mathbf{x}_2, t) d\mathbf{x}_2 \quad (13)$$

Note, that M_R is actually a *functional* of the density, ρ , and hence also a functional of the distribution function f itself. Thus, M_R can differ from the constant global value $\langle M_R \rangle = M = N/A$. Neglecting this seemingly small difference would lead to spurious gradient terms in the macroscopic equations.

In the large N limit, one obtains the Poisson distribution for the number of particles in a circle,

$$\frac{N!}{n!(N-n)!} \left(1 - \frac{M_R}{N}\right)^{N-n} \approx \frac{N^n}{n!} e^{-M_R}, \quad (14)$$

and finally arrives at an Enskog-like equation,

$$f(\mathbf{x} + \tau \mathbf{v}, \theta, t + \tau) = C_E \diamond f(\mathbf{x}, \theta, t), \quad (15)$$

where C_E is an Enskog collision operator for multi-particle collisions defined by

$$C_E \diamond f(\mathbf{x}, \theta, t) = \frac{1}{\eta} \int_{-\eta/2}^{\eta/2} d\xi \left\langle \left\langle \sum_{n=1}^N \frac{e^{-M_R}}{n!} n \right. \right. \\ \left. \left. \times f(\tilde{\theta}_1, \mathbf{x}, t) \hat{\delta}(\theta - \xi - \Phi_i) \prod_{i=2}^n f(\tilde{\theta}_i, \mathbf{x}_i, t) \right\rangle_{\tilde{\theta}} \right\rangle_x. \quad (16)$$

Here, $\langle \dots \rangle_x = \int_{\odot} \dots d\mathbf{x}_2 d\mathbf{x}_3 \dots d\mathbf{x}_n$ denotes the integration over all positions of the particles $2, 3, \dots, n$ inside the collision circle, and $\langle \dots \rangle_{\tilde{\theta}} = \int_0^{2\pi} \dots d\tilde{\theta}_1 d\tilde{\theta}_2 \dots d\tilde{\theta}_n$ refers to the integration over the pre-collisional angles of all n particles inside the circle.

The form of the collision operator can be understood as follows: (i) it is a sum over all possible particle numbers n that could be found inside the interaction circle of particle 1, weighted by the Poisson-probability. The Poisson-property is a result of the Molecular chaos approximation. (ii) the additional factor n accounts for the fact that it can be any of the n particles that is pinned at position x and whose direction is updated to θ in the collision, i.e. there is n possibilities to pick one of the particles, (iii) one has to integrate over all the other $n - 1$ particles's positions inside the circle and all possible pre-collisional velocities which would result in the desired outcome-direction θ of the focal particle, (iv) finally there is an average over the distribution of the noise that is applied to the focal particle.

The consistency of the kinetic equation can be tested by setting all distribution functions in $C_E \diamond f$ equal to the homogeneous, disordered solution $f_0 = \rho_0/(2\pi)$, with a homogeneous density $\rho_0 = \int_0^{2\pi} d\theta f_0 = \rho_0$. Then, all integrations and summations in the collision integral can be performed exactly by using the series expression of the exponential function, $e^M = \sum_n M^n/n!$ and the integral representation of the δ -function,

$$\delta(\mathbf{v} - \mathbf{v}_0) = \int_{-\infty}^{\infty} e^{i\mathbf{k}(\mathbf{v}-\mathbf{v}_0)} \frac{d\mathbf{k}}{(2\pi)^d} \quad (17)$$

One finds that f_0 is a fixed point of the integral equation, $f_0 = C_E \diamond f_0$, thus f_0 is a homogeneous steady state distribution of the VM at all values of the noise η . Not passing this test would mean that the collision operator violates mass conservation.

4. Deriving hydrodynamic equations

4.1. Background

Extracting macroscopic behavior from kinetic equations has a long tradition in the kinetic theory of gases and plasmas. One of the methods to achieve this is the

Chapman-Enskog expansion (CE) which involves expansions in small temporal and spatial gradients of hydrodynamic fields [40, 41, 42, 43]. The key assumption is that after a few collisions which can involve rapid changes of the distribution function f , the system reaches a “hydrodynamic state” where local equilibrium is achieved. In this state, f is assumed to be a functional of the slow hydrodynamic fields and should depend only indirectly on space and time through those fields. The hydrodynamic variables are the lowest angular moments of f , such as density and momentum density. The Chapman-Enskog assumption is equivalent to the claim that the first few moments suffice to describe the system on large length and time scales.

Since f is uniquely defined by all its moments, this assumption would be justified if either all higher moments are negligibly small or that they are “enslaved” to the lower moments, meaning that they could be expressed as functionals of the lower moments. Enslavement of the higher modes can be assured if there is a clear separation of time scales. This is usually achieved by explicitly keeping those modes in the Chapman-Enskog expansion that either fulfill a conservation law or become soft close to a transition. Hence, the choice of the first few moments is not arbitrary in CE.

CE takes the local stationary state as a reference state and expands around it in powers of the hydrodynamic gradients. To systematically account for these gradients a dimensionless ordering parameter ϵ is introduced, which is set to unity at the end of the calculation. The physical meaning of this parameter is that it is supposedly proportional to the Knudsen number, e.g. the ratio of the mean free path to the length scale over which hydrodynamic fields change considerably.

4.2. Chapman-Enskog expansion

The Chapman-Enskog procedure starts with a Taylor expansion of the l.h.s of Eq. (15) around (\mathbf{x}, θ, t) ,

$$f(\mathbf{x} + \tau \mathbf{v}, \theta, t + \tau) = \sum_{k=0}^{\infty} \frac{\tau^k (\partial_t + e_\alpha \partial_\alpha)^k}{k!} f(\mathbf{x}, \theta, t), \quad (18)$$

with $\mathbf{v} = (e_x, e_y)$. Then, spatial gradients are scaled as $\partial_\alpha \rightarrow \epsilon \partial_\alpha$, and multiple time scales t_i are introduced in the temporal gradients. For the VM, the following scaling was chosen,

$$\partial_t = \partial_{t_0} + \epsilon \partial_{t_1} + \epsilon^2 \partial_{t_2} + \epsilon^3 \partial_{t_3} \dots \quad (19)$$

This choice differs from the usual set of equations for fluid flow [44, 45, 46, 47] because of the fast time scale t_0 which is not multiplied by a power of ϵ and contributes time derivatives of all orders. The reason to introduce this time scale is that momentum is not locally conserved in the VM. Therefore, the macroscopic momentum transport equation must have a source term; it cannot be written as a continuity equation, and a new non-hydrodynamic time scale should come into play. Momentum can still change even if spatial gradients of density and momentum are zero. Therefore, in a gradient expansion with expansion parameter ϵ , this new scale must be of order $O(\epsilon^0)$ to be

consistent with the assumption that spatial gradients are proportional to some non-zero power of ϵ . Expansions that contain all powers of ∂_{t_0} can be conveniently summed up by the time evolution operator

$$T \equiv \exp(\tau \partial_{t_0}) \quad (20)$$

which shifts the time-argument of a function by the discrete time step τ , $T \circ f(t) = f(t + \tau) + O(\epsilon)$.

The CE proceeds with expanding the distribution function f and the collision integral C_E , e.g. the right hand side of Eq. (15), in powers of ϵ ,

$$\begin{aligned} f &= f_0 + \epsilon f_1 + \epsilon^2 f_2 + \epsilon^3 f_3 + \dots \\ C_E &= C_0 + \epsilon C_1 + \epsilon^2 C_2 + \epsilon^3 C_3 + \dots \end{aligned} \quad (21)$$

Inserting this into Eqs. (16, 18), and collecting terms of the same order in ϵ up to third order yields a hierarchy of evolution equations for the f_i .

$$O(\epsilon^0): T f_0 = C_0 \quad (22)$$

$$O(\epsilon^1): T \circ [f_1 + L f_0] = C_1 \quad (23)$$

$$O(\epsilon^2): T \circ \left[f_2 + L f_1 + \frac{1}{2!} L^2 f_0 + \tau \partial_{t_2} f_0 \right] = C_2 \quad (24)$$

$$O(\epsilon^3): T \circ \left[f_3 + L f_2 + \frac{1}{2!} L^2 f_1 + \frac{1}{3!} L^3 f_0 + \tau \partial_{t_3} f_0 + \tau \partial_{t_2} \circ L f_0 + \tau \partial_{t_2} f_1 \right] = C_3 \quad (25)$$

$$\text{with } D \equiv \tau (\partial_{t_1} + e_\alpha \partial_\alpha)$$

All spatial derivatives are contained in the ‘‘convective’’ time derivative D which is of order ϵ . Due to the absence of momentum conservation and Galilean invariance this set of equations is very different from the usual one. It is not *a priori* evident whether the scaling ansatz for the time derivatives is correct. However, it turns out that this choice is compatible with the microscopic collision rules and avoids inconsistencies if additionally the expansion of the distribution function f is identified as an angular Fourier series,

$$f = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \epsilon^3 f_3 \dots \quad \text{with} \quad (26)$$

$$f_0(\mathbf{x}, t) = \frac{\rho(\mathbf{x}, t)}{2\pi} \quad (27)$$

$$f_n(\mathbf{x}, \theta, t) = \frac{1}{\pi v_0^n} [a_n(\mathbf{x}, t) \cos(n\theta) + b_n(\mathbf{x}, t) \sin(n\theta)] \quad \text{for } n > 0, \quad (28)$$

if the analysis is restricted to the vicinity of the order-disorder transition, and if a redefinition of the collision integrals C_1 and C_3 is performed, as explained later in Sec. 4.4. Hence, the reference state f_0 of the CE, i.e. the leading order contribution to f , coincides with the zero mode of the Fourier series. Thus, the Chapman-Enskog expansion is performed around the disordered state where particles have no preferred direction.

We are seeking a hydrodynamic description of the first two moments of f , namely the particle density ρ and the momentum density vector \mathbf{w} ,

$$\rho = \int_0^{2\pi} f d\theta = \int_0^{2\pi} f_0 d\theta \quad (29)$$

$$w_x = \rho u_x = \int_0^{2\pi} e_x f d\theta = \int_0^{2\pi} v_0 \cos \theta f_1 d\theta = a_1 \quad (30)$$

$$w_y = \rho u_y = \int_0^{2\pi} e_y f d\theta = \int_0^{2\pi} v_0 \sin \theta f_1 d\theta = b_1 \quad (31)$$

where the microscopic velocity vector is given by

$$\mathbf{v} \equiv (e_x, e_y) = v_0(\cos \theta, \sin \theta). \quad (32)$$

The particular choice to expand in a Fourier-series means that the density is given by the lowest order distribution function f_0 alone, and the momentum density is described by the next higher order term, f_1 . This is different to usual hydrodynamic models like the Lattice-Boltzmann method for an simple fluid [45, 48], where the momentum density is given by a moment of f_0 and not f_1 . Thus, in first order one finds,

$$f = \frac{\rho}{2\pi} + \frac{1}{\pi v_0} [w_x \cos \theta + w_y \sin \theta] + O(\epsilon^2) \quad (33)$$

Multiplying the hierarchy of evolution equations (22–25) by powers of the microscopic velocity vector \mathbf{v} and integrating over θ gives a set of equations for the time development of the density and the moments a_i and b_i . A number of moments of f and the collision operator C_E occur in these equations and will be evaluated in the next sections. After this, the nontrivial closure of the hierarchy of moment equations will be discussed and equations for the hydrodynamic fields will be derived.

4.3. Moments of the distribution functions and the collision operator

The following moments are needed in the derivation of the macroscopic equations:

$$\langle e_\alpha e_\beta f_0 \rangle = \delta_{\alpha\beta} \frac{v_0^2}{2} \rho \quad (34)$$

$$\langle e_\alpha e_\beta f_2 \rangle = (\delta_{\alpha\beta x} - \delta_{\alpha\beta y}) \frac{a_2}{2} + (1 - \delta_{\alpha\beta}) \frac{b_2}{2} \quad (35)$$

$$\langle e_\alpha e_\beta e_\gamma f_1 \rangle = (\delta_{\alpha\beta} w_\gamma + \delta_{\alpha\gamma} w_\beta + \delta_{\gamma\beta} w_\alpha) \frac{v_0^2}{4} \quad (36)$$

$$\langle e_\alpha e_\beta e_\gamma e_\delta f_0 \rangle = (\delta_{\alpha\beta} \delta_{\gamma\delta} + \delta_{\alpha\gamma} \delta_{\beta\delta} + \delta_{\alpha\delta} \delta_{\beta\gamma}) \frac{v_0^4 \rho}{8} \quad (37)$$

The brackets denote angular integration, $\langle \dots \rangle = \int_0^{2\pi} \dots d\theta$. The f'_n s are given in Eq. (28). They depend on the angular Fourier coefficients of f but also depend on the cosine and sine of the angle $n\theta$. Therefore, odd moments of f_0 such as $\langle e_\alpha f_0 \rangle$ and even moments of f_1 such as $\langle f_1 \rangle$ and $\langle e_\alpha e_\beta f_1 \rangle$ are equal to zero.

In addition to the moments of f_n , the first few velocity moments of the collision operator are needed and are calculated in the next section. In this paper I will restrict myself to the case of large mean free path; that is I assume that the mean travel distance $\Lambda = \tau v_0$ is large compared to the interaction radius r_0 , [49]. Hence, the ‘‘collisional contributions’’ to the transport coefficients will be neglected here. These contributions are known from the Enskog theory of dense fluids, see Ref. [51]. They take into account the transfer of energy and momentum via the intermolecular potential and arise because

the size of colliding molecules is not neglected anymore. These collisional contributions have been studied in a variety of particle-based models, such as Multi-Particle Collision Dynamics [52, 58] and Direct Simulation Monte Carlo [53] where they become important if the interaction range is larger than the mean free path.

The restriction to large Λ is not a principal limitation of the current approach but will simplify the evaluations of the collision operator in the following section.

4.3.1. First order moment: $\langle e_\alpha C_1 \rangle$ To evaluate moments of the collision operator, the Fourier series for f , Eq. (28), is inserted into the collision operator, Eq. (16). Multiplication with powers of the velocities $\mathbf{v} = (e_x, e_y)$ and integration over the angle θ leads to the disappearance of the Dirac delta function in the collision operator. In the limit of $\Lambda \gg r_0$, variations of f across the interaction circle are neglected. As a result, the integrations over the interaction circle become trivial and just lead to powers of $A = \pi r_0^2$. The remaining angular integrations lead to integrals such as K_C^1 , which I call K -integrals. These integrals are evaluated in Sec. 5 and given in Table 1.

Keeping only terms linear in ϵ leads to

$$\langle e_\alpha C_1 \rangle = \lambda w_\alpha \quad (38)$$

$$\lambda = \frac{4}{\eta} \sin\left(\frac{\eta}{2}\right) e^{-M_R} \sum_{n=1}^{\infty} \frac{n^2 M_R^{n-1}}{n!} K_C^1(n) \quad (39)$$

M_R is the mean local particle number, $M_R = \int_{\odot} \rho(\mathbf{x}) d\mathbf{x}$ where the integration goes over the interaction circle. The coefficient λ can be simplified in the limit $M_R \rightarrow \infty$. Recall the first moments of the Poisson distribution:

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{e^{-M_R}}{n!} M_R^n &= 1 \\ \sum_{n=0}^{\infty} \frac{e^{-M_R}}{n!} n M_R^n &= M_R \end{aligned} \quad (40)$$

and express the sum in Eq. (39) as an average over the Poisson distribution

$$e^{-M_R} \sum_{n=1}^{\infty} \frac{n^2 M_R^{n-1}}{n!} K_C^1(n) = e^{-M_R} \sum_{n=0}^{\infty} \frac{M_R^n}{n!} h(n) \quad (41)$$

by means of the function $h(n) = (n+1) K_C^1(n+1)$. Expanding h around $n = M_R$ gives

$$\begin{aligned} e^{-M_R} \sum_{n=0}^{\infty} \frac{M_R^n}{n!} h(n) &= e^{-M_R} \sum_{n=0}^{\infty} \frac{M_R^n}{n!} (h(M_R) \\ &\quad + (n - M_R) h'(M_R) + O(h''(M_R))) \approx h(M_R). \end{aligned} \quad (42)$$

because the term proportional to h' vanishes due to Eq. (40). Applying this idea to Eqs. (39) and using the expression for $K_C^1(n) \sim \sqrt{\pi}/(4\sqrt{n})$ in the limit $M_R \rightarrow \infty$, see Table 1, gives the asymptotic expression,

$$\lambda \sim \frac{1}{\eta} \sin\left(\frac{\eta}{2}\right) \sqrt{M_R \pi} \quad \text{for } M_R \gg 1 \quad (43)$$

More details on the expansion of all transport coefficients in the large M_R limit are given in Ref. [16]. In the opposite limit of low density, $M_R \ll 1$, one finds

$$\lambda \sim \frac{2}{\eta} \sin\left(\frac{\eta}{2}\right) \left(1 + M_R \left[\frac{4}{\pi} - 1\right] + O(M_R^2)\right) \quad (44)$$

4.3.2. Second order moment: $\langle e_\alpha e_\beta C_2 \rangle$ First, let us evaluate the case $\alpha = \beta = x$. The result is

$$\begin{aligned} \langle e_x^2 C_2 \rangle &= \frac{1}{\eta} \sin(\eta) \sum_{n=0}^N \frac{e^{-M_R}}{n!} n \left\{ 2n M_R^{n-1} K_{2c}^{11} a_2 + 4A M_R^{n-2} \binom{n}{2} \right. \\ &\quad \left. \times [K_{cc}^{11} w_x^2 + K_{ss}^{11} w_y^2] \right\} \end{aligned} \quad (45)$$

The angular integrals such as K_{cc}^{11} and K_{2c}^{11} are calculated in Sec. 5. Using $K_{cc}^{11} = K_{cs}^{12}$ and $K_{ss}^{11} = -K_{cs}^{12}$ one finds,

$$\langle e_x^2 C_2 \rangle = \frac{p}{2} a_2 + \frac{q}{2} (w_x^2 - w_y^2) \quad (46)$$

$$p = \frac{4}{\eta} \sin(\eta) \sum_{n=1}^N \frac{e^{-M_R}}{n!} n^2 M_R^{n-1} K_{2c}^{11}(n) \quad (47)$$

$$q = \frac{4A}{\eta} \sin(\eta) \sum_{n=2}^N \frac{e^{-M_R}}{n!} n^2 (n-1) M_R^{n-2} K_{cs}^{12}(n) \quad (48)$$

with the auxiliary quantities p and q . According to Eq. (201) in Sec. 6, the quantity p can be physically interpreted as the decay rate of the kinetic stress correlations. By means of Eq. (42) the sums in Eqs. (47) and (48) can be approximated for large M_R as,

$$p \approx \frac{4}{\eta} \sin(\eta) M_R K_{2c}^{11}(M_R) \quad (49)$$

$$q \approx \frac{4A}{\eta} \sin(\eta) (M_R - 1) K_{cs}^{12}(M_R) \quad (50)$$

Finally, since the asymptotic behavior for K_{2c}^{11} and K_{cs}^{12} is known, one finds in the limit $M_R \rightarrow \infty$:

$$p \sim \frac{1}{2\eta} \sin(\eta) \quad (51)$$

$$q \sim \frac{A}{2\eta} \sin(\eta) \quad (52)$$

Because the values of the K -integrals are also known for small $n = 1, 2, 3, \dots$, see Sec. 5, the behavior at small density $M_R \ll 1$ can be extracted easily as,

$$p \sim \frac{1}{\eta} \sin(\eta) (1 - M_R + O(M_R^2)) \quad (53)$$

$$q \sim \frac{A}{\eta} \sin(\eta) (1 + M_R(-1 + 12K_{cs}^{12}(3)) + O(M_R^2)) \quad (54)$$

$$= \frac{A}{\eta} \sin(\eta) (1 - 0.41344 M_R + O(M_R^2)) \quad (55)$$

Note that in the low density limit, $M_R \ll 1$, the main contribution to p comes from the “self-interaction” of a single particle which just randomly changes its flying direction without interference from another particle. In contrast, the first contribution to q comes from a binary collision, i.e. from just having two particles in an interaction circle. The “self-interaction”, leading to the dominant term in Eq. (53) and also in Eq. (44), might not be present in other versions of the VM. For example, one could decide to not perform a “collision” if no other particle is found in a circle around a given particle. For larger densities, this little algorithmic detail would not make a difference, but for low densities, the coefficients p and λ would differ from Eq.(53) and (44), respectively. In fact, in this case, both p and λ would go to unity for $M_R \rightarrow 0$. The result $\lambda \rightarrow 1$ makes perfect physical sense: On one hand, for $M_R \rightarrow 0$ we can assume that a given particle will never meet another one, thus it would never change direction according to this modified collision rule. Hence, its momentum is exactly conserved. On the other hand, the signature of exact momentum conservation is $\lambda = 1$ which is what we found here.

Because of $\langle C_2 \rangle = 0$ and $e_x^2 + e_y^2 = v_0^2$ one sees that

$$\langle e_y^2 C_2 \rangle = -\langle e_x^2 C_2 \rangle \quad (56)$$

A calculation similar to the one for $\langle e_x^2 C_2 \rangle$ but now with $\alpha = x$ and $\beta = y$ yields,

$$\begin{aligned} \langle e_x e_y C_2 \rangle = \\ \frac{1}{\eta} \sin(\eta) \sum_{n=0}^N \frac{e^{-M_R}}{n!} n \left\{ 2n M_R^{n-1} K_{2s}^{12} b_2 + 8A M_R^{n-2} \binom{n}{2} K_{cs}^{12} w_x w_y \right\} \end{aligned} \quad (57)$$

Since $K_{2s}^{12} = K_{2c}^{11}$, see Sec. 5, one finds,

$$\langle e_x e_y C_2 \rangle = \frac{p}{2} b_2 + q w_x w_y \quad (58)$$

where the coefficients q and p are identical to the ones calculated above.

4.3.3. Third order moment: $\langle e_\alpha C_3 \rangle$ The expansion of the distribution function f in Fourier modes, see Eq. (28), is inserted in the collision operator, Eq. (16), and only terms of order ϵ^3 are kept. As before, in the limit of $\lambda \gg r_0$, variations of f across the interaction circle are neglected. Hence, all arguments \mathbf{x}_i in f can be formally replaced by \mathbf{x} , and the integrations over the interaction circle become trivial. The result is for $\alpha = x$:

$$\begin{aligned} \langle e_x C_3 \rangle = \frac{2}{\eta v_0^2} \sin\left(\frac{\eta}{2}\right) \sum_{n=0}^N \frac{e^{-M_R}}{n!} n \left\{ 8A^2 M_R^{n-3} \binom{n}{3} \right. \\ \times [w_x^3 K_{ccc}^1 + 3w_x w_y^2 K_{css}^1] \\ \left. + 4A M_R^{n-2} n(n-1) K_{c2c}^1 [w_x a_2 + w_y b_2] \right\} \end{aligned} \quad (59)$$

With $K_{css}^1 = K_{ccc}^1/3$ one finds

$$\langle e_x C_3 \rangle = \Gamma w_x w^2 + S(w_x a_2 + w_y b_2) \quad (60)$$

$$\Gamma = \frac{8A^2}{3\eta v_0^2} \sin \frac{\eta}{2} \sum_{n=3}^N \frac{e^{-M_R}}{n!} n^2(n-1)(n-2) M_R^{n-3} K_{ccc}^1(n) \quad (61)$$

$$S = \frac{8A}{\eta v_0^2} \sin \frac{\eta}{2} \sum_{n=2}^N \frac{e^{-M_R}}{n!} n^2(n-1) M_R^{n-2} K_{c2c}^1(n) \quad (62)$$

with the auxiliary quantities Γ and S . A is the area of the interaction circle, $A = \pi r_0^2$.

The coefficients Γ and S can be simplified in the limit $M_R \rightarrow \infty$ by means of Eq. (42).

$$\Gamma \approx \frac{8A^2}{3\eta v_0^2} \sin \frac{\eta}{2} \frac{(M_R - 1)(M_R - 2)}{M_R} K_{ccc}^1(M_R) \quad (63)$$

$$S \approx \frac{8A}{\eta v_0^2} \sin \frac{\eta}{2} (M_R - 1) K_{c2c}^1(M_R) \quad (64)$$

Finally, since the asymptotic behavior for K_{ccc}^1 and K_{c2c}^1 is known, see Table 1, one has in the limit of large M_R :

$$\Gamma \approx -\frac{A^2}{4\eta v_0^2} \sqrt{\frac{\pi}{M_R}} \sin \frac{\eta}{2} \quad (65)$$

$$S \approx -\frac{A}{4\eta v_0^2} \sqrt{\frac{\pi}{M_R}} \sin \frac{\eta}{2} \quad (66)$$

As before, a low density expansion of the coefficients can also be done for $M_R \ll 1$,

$$\Gamma = \frac{8A^2}{\eta v_0^2} \sin \frac{\eta}{2} K_{ccc}^1(3) (1 + M_R[-1 + 4K_{ccc}^1(4)/3K_{ccc}^1(3)] + O(M_R^2)) \quad (67)$$

$$\approx -0.55736 \frac{A^2}{\eta v_0^2} \sin \frac{\eta}{2} (1 - 0.3943 M_R + O(M_R^2)) \quad (68)$$

$$S = \frac{16A}{\eta v_0^2} \sin \frac{\eta}{2} K_{c2c}^1(2) (1 + M_R[-1 + 3K_{c2c}^1(3)/2K_{c2c}^1(2)] + O(M_R^2)) \quad (69)$$

$$\approx -\frac{8A}{3\pi\eta v_0^2} \sin \frac{\eta}{2} (1 - 0.63893 M_R + O(M_R^2)) \quad (70)$$

4.4. Reevaluation and closure of the moment equations

Before actually deriving hydrodynamic equations, several technical issues need to be resolved. Most Chapman-Enskog expansions are performed to second order and not to third order as done here. This leads to a complication which is hardly discussed in the literature, probably because it only occurs at orders higher than two. It is the fact that the time evolution operators in the Chapman-Enskog expansion do not typically commute (see Eq. (95) as an example). Writing the time derivative as a sum of different time derivatives in Eq. (19) is equivalent to an splitting of the evolution equation into separate parts which have to be put together after the derivation has been completed. This artificial splitting seems to be the reason for the unusual noncommutative property of the time derivatives. In order to include noncommutativity but at the same time keeping the notation compact, I use the “ \circ ” symbol whenever two time derivatives that need special consideration “hit each other”. The \circ symbol is a symmetrization

operator and means that all time derivatives in the term under consideration must be symmetrized. For example, with this notation one would have,

$$\partial_{t_0}^2 \circ \partial_{t_1} = \partial_{t_1} \circ \partial_{t_0}^2 = \partial_{t_0} \circ \partial_{t_1} \circ \partial_{t_0} = \frac{1}{3} (\partial_{t_0}^2 \partial_{t_1} + \partial_{t_0} \partial_{t_1} \partial_{t_0} + \partial_{t_1} \partial_{t_0}^2) \quad (71)$$

Three terms occur because there is three distinct permutations. As another example, the expression $\partial_{t_0}^2 \circ \partial_{t_1} \circ \partial_{t_2}$ stands for $4!/2! = 12$ terms obtained by permuting the four operators $(\partial_{t_0}, \partial_{t_0}, \partial_{t_1}, \partial_{t_2})$. Note, that the symmetrization is not forced but occurs naturally when carefully performing the Chapman-Enskog expansion in terms of noncommutative time evolution operators.

The other technical subtlety is, that for reasons explained further below, C_1 and C_3 , in Eqs. (23, 25) are replaced by \tilde{C}_1 and \tilde{C}_3 ,

$$\begin{aligned} \tilde{C}_1 &= C_1 - C_1 \left(1 - \frac{1}{\lambda}\right) \\ \tilde{C}_3 &= C_3 + C_1 \left(1 - \frac{1}{\lambda}\right) \end{aligned} \quad (72)$$

Eq. (38) shows that the first moment of the “undressed” first order collision contribution C_1 , is proportional to the momentum density, $\langle e_\alpha C_1 \rangle = \lambda w_\alpha$. Analyzing the final macroscopic equations reveal the physical interpretation of the prefactor λ : It is an (ensemble-averaged) amplification factor of the momentum density, where λ equal to unity means that the collisions conserve momentum. This is typically not the case in the VM, unless directly at the order-disorder transition point.

The condition $\lambda = 1$ is identical to the condition for the mean-field bifurcation of a homogeneous disordered solution into an ordered solution [13, 15, 54]. For a given fixed noise η , one can find a critical mean particle number $M_{R,crit}$ by the condition $\lambda(\eta, M_{R,crit}) = 1$. This means that the local order parameter w (which happens to be equal to the macroscopic momentum) will grow from an infinitesimal initial value if the local density is quenched above the critical density. This growth is initially exponentially but will be saturated by nonlinear effects. In the opposite scenario, w will decay to zero if a homogeneous ordered system is quenched to a lower density $M_R < M_{R,crit}$. In a first attempt to derive hydrodynamic equations without the redefinition of Eq. (72) it turned out that closing the hierarchical set of moment equations is especially easy near the order-disorder transition where $\lambda = 1$. In the vicinity of this bifurcation, higher order moments of f are small; they are enslaved to lower order moments, and a useful macroscopic description is already obtained by the equation for the density (continuity equation) and the momentum density (Navier-Stokes-like equation). Here, enslavement means that a given higher order moment such as a_2 is approximately given by a function of lower order moments and their spatial derivatives, and that the difference between the time derivative of this function and the true time derivative of the higher moment is of order ϵ or smaller.

Near the order-disorder phase transition, one has $1 - \lambda \ll 1$, which can be expressed as $1 - \lambda = a_0 + a_1 \epsilon + a_2 \epsilon^2 + \dots$. One has to keep in mind that ϵ is a formal *ordering parameter* which is set to unity at the end of the calculation. This leaves ambiguities in

choosing the a_i , for example one could decide to use $a_0 = a_1 = 0$ or $a_0 = a_2 = 0$. Here, I choose $a_0 = a_1 = 0$, which means $1 - \lambda = O(\epsilon^2)$. This choice can be justified *a posteriori* by analyzing the final equation for the momentum density, Eq. (132), in a homogeneous stationary state, which gives $|\vec{w}| = \sqrt{(\lambda - 1)/q_3} = O(\epsilon)$. Hence, in the current scaling, the momentum density is predicted to be of order ϵ , which is consistent with Eq. (33) and the identification of f_n with the angular Fourier coefficients, Eqs. (26,28). Other choices for $1 - \lambda$ should in principle lead to the same final set of equations, however certain terms would show up at different orders of the derivation and the simple Fourier Ansatz for f_n probably would have to be modified.

Using the bare value C_1 in the first order expression Eq. (23) together with the assumption $1 - \lambda = O(\epsilon^2)$ would introduce an inconsistency since C_1 contains higher order contributions. Eq. (72) shifts this higher order part into C_3 . Note, that this is just a redistribution between different approximation levels since the sum stays the same: $\tilde{C}_1 + \tilde{C}_3 = C_1 + C_3$. The following derivations will assume proximity to the transition point. Thus, the macroscopic equations derived in this work are unlikely to be valid at very small noise – far away from the transition. According to Eqs. (72, 38) one has now

$$\langle e_\alpha \tilde{C}_1 \rangle = w_\alpha \quad (73)$$

That is, at first order in ϵ , the collisions conserve momentum in the ensemble-averaged sense, which is to be expected near the transition.

4.5. Density and momentum evolution for time scales t_0 and t_1

In this paragraph, the first two moments of the first few equations of the hierarchy, (22-24), are evaluated. The resulting equations for the time evolution of density ρ and momentum density \mathbf{w} will be used later to simplify the higher order evolution equations.

Because of particle number conservation the collision operator has the following properties: $\langle C_0 \rangle = \rho$ and $\langle C_n \rangle = 0$ for $n > 0$. Here, the brackets denote an average over the angle θ , $\langle \dots \rangle = \int_0^{2\pi} \dots d\theta$. Hence, if the zeroth moment, i.e. the angular average, of the set of equations (22) is taken, one finds

$$O(\epsilon^0): \quad T\rho = (1 + \tau\partial_{t_0} + \frac{\tau^2}{2}\partial_{t_0}^2 + \dots)\rho = \rho \quad (74)$$

$$O(\epsilon^1): \quad \langle T \circ Lf_0 \rangle = \tau \left\{ \partial_{t_1}\rho + \frac{1}{2} [\partial_{t_1}(\partial_{t_0}\rho) + \partial_{t_0}(\partial_{t_1}\rho)] \right. \quad (75)$$

$$\left. + \frac{1}{6} [\partial_{t_0}^2(\partial_{t_1}\rho) + \partial_{t_1}\partial_{t_0}(\partial_{t_0}\rho) + \partial_{t_0}\partial_{t_1}(\partial_{t_0}\rho)] + \dots \right\} = 0 \quad (76)$$

The first equation is solved by

$$\partial_{t_0}\rho = 0 \quad (77)$$

Using this result, the solution of the second equation is

$$\partial_{t_1}\rho = 0 \quad (78)$$

Next, the evolution equation for the momentum density \mathbf{w} is derived by multiplying Eqs. (22–25) by the velocity component e_α and integrating over θ . In order ϵ^0 , the trivial, but consistent result $0 = 0$ occurs, because the first moment of C_0 vanishes, $\langle e_\alpha C_0 \rangle = 0$, which can be proven easily. In order ϵ^1 one finds

$$T[w_\alpha + \tau \partial_\beta \langle e_\alpha e_\beta f_0 \rangle] = \langle e_\alpha \tilde{C}_1 \rangle = w_\alpha \quad (79)$$

Here, Eq. (73) was used, which states that momentum conservation is approximately realized near the order-disorder transition; deviations from this conservation law will show up in higher order equations. According to Eq. (34), $\langle e_\alpha e_\beta f_0 \rangle = \delta_{\alpha\beta} v_0^2 \rho_0 / 2$, and Eqs. (20,77) it follows that

$$\exp(\tau \partial_{t_0}) w_\alpha + \frac{\tau v_0^2}{2} \partial_\alpha \rho = w_\alpha \quad (80)$$

Because $\partial_{t_0} \rho = 0$, this equation is solved by:

$$\partial_{t_0} w_\alpha = -\partial_\alpha \left(\frac{v_0^2}{2} \rho \right) \quad (81)$$

which is the Euler-equation with an ideal gas pressure $p_{id} = k_B T \rho$ and temperature $k_B T = v_0^2 / 2$ (the particle mass is assumed to be one). This is the expected result for two-dimensional particles with constant speed v_0 and kinetic energy $v_0^2 / 2$. Note, that without the assumption of $1 - \lambda \ll 1$, a momentum source term $\sim w_\alpha$ would appear,

$$\partial_{t_0} w_\alpha = \mu w_\alpha + \nu \partial_\alpha \rho \quad (82)$$

and an interesting logarithmic dependence on λ would follow,

$$\mu = \frac{1}{\tau} \ln \lambda \quad (83)$$

$$\nu = \frac{v_0^2}{2} \frac{\ln \lambda}{1 - \lambda}, \quad (84)$$

which, however, is not used in the remainder of this paper. Finally, an equation for $\partial_{t_1} w_\alpha$, is obtained by taking the first moment of the $O(\epsilon^2)$ equation (24). Remembering that both $\partial_{t_0} \rho$ and $\partial_{t_1} \rho$ vanish, one finds

$$\partial_{t_1} w_\alpha = 0. \quad (85)$$

4.6. Density evolution for time scales t_2 and t_3

Taking the zeroth moment of the $O(\epsilon^2)$ equation of the hierarchy gives

$$O(\epsilon^2): \quad T \circ \left[\tau \partial_\alpha w_\alpha + \frac{1}{2} \tau^2 \partial_\alpha \partial_\beta \langle e_\alpha e_\beta f_0 \rangle + \tau \partial_{t_2} \rho \right] = 0 \quad (86)$$

Inserting the second moment of f_0 given in Eq. (34) and using $\partial_{t_0} \rho = 0$ results in

$$T \partial_\alpha w_\alpha + R \partial_{t_2} \rho + \frac{\tau v_0^2}{4} \partial_\alpha^2 \rho = 0 \quad (87)$$

where a new time evolution operator R was introduced,

$$R = \left(1 + \frac{\tau}{2!} \partial_{t_0} + \frac{\tau^2}{3!} \partial_{t_0}^2 + \dots \right) \quad (88)$$

The first term in Eq. (87) can be replaced by applying the T -operator to Eq. (81), which gives

$$T\partial_\alpha w_\alpha = \partial_\alpha w_\alpha - \frac{\tau v_0^2}{2} \partial_\alpha^2 \rho, \quad (89)$$

and insertion into Eq. (87) yields,

$$\partial_\alpha w_\alpha + R\partial_{t_2}\rho - \frac{\tau v_0^2}{4} \partial_\alpha^2 \rho = 0 \quad (90)$$

Finally, using $\partial_{t_0}\rho = 0$ and $\partial_{t_0}w_\alpha = -v_0^2\partial_\alpha\rho/2$ this equation is solved by the expected continuity equation,

$$\partial_{t_2}\rho + \partial_\alpha w_\alpha = 0 \quad (91)$$

Note, that all unphysical diffusive terms, $\sim \partial_\alpha^2\rho$, present in the starting expression, Eq. (87) have exactly cancelled which is non-trivial in systems with forces and/or absence of momentum conservation. A useful relation which follows from eq. (91) or (90) is

$$T \circ \partial_{t_2}\rho = R\partial_{t_2}\rho = -\partial_\beta w_\beta + \frac{\tau v_0^2}{4} \partial_\beta^2 \rho \quad (92)$$

In order to get the final density evolution equation, Eq. (85) is needed to evaluate the zeroth moment of the $O(\epsilon^3)$ equations. After a short calculation one finds,

$$\partial_{t_3}\rho = 0 \quad (93)$$

Collecting all four different time derivatives, Eqs. (77,78,91,93), the final result for the density evolution is

$$\partial_t\rho = (\partial_{t_0} + \partial_{t_1} + \partial_{t_2} + \partial_{t_3})\rho + O(\epsilon^4) = -\partial_\alpha w_\alpha + O(\epsilon^4). \quad (94)$$

We are now in position to discuss the noncommutativity of the time derivatives again. A simple example is the commutator $[\partial_{t_0}, \partial_{t_2}]$: when applied to the density ρ , a non-zero result is obtained,

$$[\partial_{t_0}, \partial_{t_2}]\rho = \partial_{t_0}\partial_{t_2}\rho - \partial_{t_2}\partial_{t_0}\rho = \frac{v_0^2}{2} \nabla^2 \rho. \quad (95)$$

4.7. Momentum density evolution for time scale t_2

To find $\partial_{t_2}w_\alpha$, Eq. (25) is multiplied by e_α and the angular average is taken. This leads to

$$\begin{aligned} & T \left[\partial_\beta \langle e_\alpha e_\beta f_2 \rangle + \frac{\tau}{2} \partial_\beta \partial_\delta \langle e_\alpha e_\beta e_\delta f_1 \rangle + \frac{\tau^2}{6} \partial_\beta \partial_\delta \partial_\gamma \langle e_\alpha e_\beta e_\gamma e_\delta f_0 \rangle \right] \\ & + T \circ \left[\frac{\tau}{2} \partial_{t_1}^2 w_\alpha + \frac{\tau^2}{2} \partial_{t_1}^2 \partial_\alpha \left(\frac{v_0^2}{2} \rho \right) + \tau \partial_{t_2} \partial_\alpha \left(\frac{v_0^2}{2} \rho \right) + \partial_{t_2} w_\alpha \right] \\ & = \frac{\langle e_\alpha \tilde{C}_3 \rangle}{\tau} \end{aligned} \quad (96)$$

Using Eqs. (77, 78, 81, 85) one can show that the fourth and fifth term on the l.h.s. vanish. The sixth term is evaluated by means of Eqs. (92) as

$$\frac{\tau v_0^2}{2} \partial_\alpha [T \circ \partial_{t_2}\rho] = \frac{\tau v_0^2}{2} \partial_\alpha \left[-\partial_\beta w_\beta + \frac{\tau v_0^2}{4} \partial_\beta^2 \rho \right] \quad (97)$$

The evaluation of the seventh terms gives

$$T \circ \partial_{t_2} w_\alpha = R \partial_{t_2} w_\alpha + \frac{\tau v_0^2}{4} \partial_\alpha \partial_\beta w_\beta - \frac{\tau^2 v_0^4}{24} \partial_\alpha \partial_\beta^2 \rho \quad (98)$$

where R is defined in Eq. (88). The moments in the first three terms are given in Eqs. (23 – 25), and after inserting everything, Eq. (96) for $\alpha = x$ becomes:

$$\begin{aligned} & T \left[\frac{1}{2} (\partial_x a_2 + \partial_y b_2) + \frac{\tau v_0^2}{8} (\nabla^2 w_x + 2 \partial_x \nabla \mathbf{w}) + \frac{\tau^2 v_0^4}{16} \partial_x \nabla^2 \rho \right] \\ & + R \partial_{t_2} w_x + \frac{\tau v_0^2}{4} \partial_x \nabla \mathbf{w} - \frac{\tau^2 v_0^4}{24} \partial_x \nabla^2 \rho + \frac{\tau v_0^2}{2} \partial_x \left\{ -\nabla \mathbf{w} + \frac{\tau v_0^2}{4} \nabla^2 \rho \right\} \\ & = \frac{\langle e_x \tilde{C}_3 \rangle}{\tau} \end{aligned} \quad (99)$$

Using $\partial_{t_0} \rho = 0$ and $\partial_{t_0} w_x = -v_0^2 \partial_x \rho / 2$, one has

$$\begin{aligned} T \partial_x \nabla^2 \rho &= \partial_x \nabla^2 \rho \\ T \nabla^2 w_x &= \nabla^2 w_x - \frac{\tau v_0^2}{2} \partial_x \nabla^2 \rho \\ T \partial_x \nabla \mathbf{w} &= \partial_x \nabla \mathbf{w} - \frac{\tau v_0^2}{2} \partial_x \nabla^2 \rho \end{aligned} \quad (100)$$

which helps simplifying the terms in brackets in Eq. (99) with the result

$$T \left[\frac{1}{2} (\partial_x a_2 + \partial_y b_2) \right] + \frac{\tau v_0^2}{8} \nabla^2 w_x - \frac{\tau^2 v_0^4}{24} \partial_x \nabla^2 \rho + R \partial_{t_2} w_x = \frac{\langle e_x \tilde{C}_3 \rangle}{\tau} \quad (101)$$

In order to simplify the following analysis, a rescaling of time and space is used from now on. Dimensionless spatial coordinates and dimensionless time are defined by

$$\tilde{x}_\alpha = \frac{x_\alpha}{\Lambda} \quad (102)$$

$$\tilde{t} = \frac{t}{\tau} \quad (103)$$

where $\Lambda = v_0 \tau$ is the mean free path. The density and momentum density are nondimensionalized accordingly by

$$\begin{aligned} \tilde{\rho}_\alpha &= \rho \Lambda^2 \\ \tilde{w}_\alpha &= w_\alpha \frac{\Lambda^2}{v_0} \end{aligned} \quad (104)$$

The operators T and R , and the amplitudes a_2 and b_2 are rescaled consistently,

$$\begin{aligned} \tilde{T} &= 1 + \partial_{\tilde{t}_0} + \frac{1}{2!} \partial_{\tilde{t}_0}^2 + \frac{1}{3!} \partial_{\tilde{t}_0}^3 + \dots \\ \tilde{R} &= 1 + \frac{1}{2!} \partial_{\tilde{t}_0} + \frac{1}{3!} \partial_{\tilde{t}_0}^2 + \frac{1}{4!} \partial_{\tilde{t}_0}^3 + \dots \\ \tilde{a}_2 &= a_2 \tau^2 \\ \tilde{b}_2 &= b_2 \tau^2 \end{aligned} \quad (105)$$

The coefficients occurring in the evaluation of moments are rescaled as

$$\tilde{\Gamma} = \Gamma \frac{v_0^2}{\Lambda^4}$$

$$\begin{aligned}\tilde{S} &= S \frac{v_0^2}{\Lambda^2} \\ \tilde{q} &= \frac{q}{\Lambda^2}\end{aligned}\tag{106}$$

The coefficients λ and p are already dimensionless. To simplify notation, the tilde in the rescaled quantities will be omitted in the remainder of the paper.

Using Eq. (60) and rescaling, Eq. (101) now reads

$$\begin{aligned}\frac{T}{2}(\partial_x a_2 + \partial_y b_2) + \frac{1}{8}\nabla^2 w_x - \frac{1}{24}\partial_x \nabla^2 \rho + R \partial_{t_2} w_x \\ = \Gamma w_x w^2 + S(w_x a_2 + w_y b_2) + (\lambda - 1)w_x\end{aligned}\tag{107}$$

The last term comes from the relation $\langle e_\alpha \tilde{C}_3 \rangle = \langle e_\alpha C_3 \rangle + \langle e_\alpha C_1 \rangle (1 - 1/\lambda)$, see Eq. (72). It is interesting to note that all terms containing the divergence of \mathbf{w} , $\sim \partial_x \nabla \mathbf{w}$ have cancelled like in the continuous, low density approach by Bertin [17, 18]. Furthermore, note that a_2 and b_2 , the second order moments of the distribution function f appear, which, in general, could make it difficult to obtain a closed set of macroscopic equations. Fortunately, it turns out that near the order-disorder bifurcation closure is possible; for this purpose additional equations for a_2 and b_2 are derived by multiplying the $O(\epsilon^2)$ member of the hierarchy Eq. (24) by $e_\alpha e_\beta$, and taking the angular average. For the special case $\alpha = \beta = x$ one finds a relation for (the rescaled) a_2 ,

$$\frac{T a_2}{2} + \frac{T}{2} \circ \partial_{t_2} \rho = \langle e_x^2 C_2 \rangle - \frac{T}{4} [3\partial_x w_x + \partial_y w_y] - \frac{1}{16} [3\partial_x^2 \rho + \partial_y^2 \rho]\tag{108}$$

A similar calculation for $\alpha = x$ and $\beta = y$ gives an expression for (the rescaled) b_2 ,

$$\frac{T}{2} b_2 = \langle e_x e_y C_2 \rangle - \frac{T}{4} [\partial_x w_y + \partial_y w_x] - \frac{1}{8} \partial_x \partial_y \rho\tag{109}$$

Combining both relations and working out the effect of the T -operators one gets

$$\frac{T}{2}(\partial_x a_2 + \partial_y b_2) = \partial_x \langle e_x^2 C_2 \rangle + \partial_y \langle e_x e_y C_2 \rangle - \frac{1}{4} \nabla^2 w_x + \frac{1}{16} \partial_x \nabla^2 \rho\tag{110}$$

which is inserted into Eq. (107), yielding

$$\begin{aligned}R \partial_{t_2} w_x = \Gamma w_x w^2 + S(w_x a_2 + w_y b_2) + (\lambda - 1)w_x \\ - \partial_x \langle e_x^2 C_2 \rangle - \partial_y \langle e_x e_y C_2 \rangle + \frac{1}{8} \nabla^2 w_x - \frac{1}{48} \partial_x \nabla^2 \rho\end{aligned}\tag{111}$$

Substituting the remaining moments of the collision operator C_2 , one realizes that the higher order moments a_2 and b_2 are still part of this evolution equation for the lower order moment w_x . That means, if closure is possible at this level, there must be a way to express both a_2 and b_2 as functionals of ρ and \mathbf{w} . To obtain such relations, the moments of the collision operator, Eqs. (46, 58) are inserted into Eqs. (108, 109). This gives decoupled differential equations for a_2 and b_2 , which are of infinite order in the time scale t_0 ,

$$(T - p)a_2 - q(w_x^2 - w_y^2) + \frac{1}{2}(\partial_x w_x - \partial_y w_y) - \frac{1}{8}(\partial_x^2 - \partial_y^2)\rho = 0\tag{112}$$

$$(T - p)b_2 - 2q w_x w_y + \frac{1}{2}(\partial_x w_y + \partial_y w_x) - \frac{1}{4} \partial_x \partial_y \rho = 0\tag{113}$$

Inspection of equations similar to Eqs. (96,108,109) but now for arbitrary α, β , shows that they can be expressed in a very compact and convenient way in terms of vectors and tensors of rank two. This is not very surprising since the system has rotational symmetry and there should be a rotational-invariant formulation of the macroscopic equations. The evolution equation, Eq. (111), becomes

$$R \partial_{t_2} \vec{w} = (\lambda - 1) \vec{w} + \Gamma w^2 \vec{w} + S \boldsymbol{\sigma}_1 \cdot \vec{w} - \nabla \cdot \boldsymbol{\sigma} + \frac{1}{8} \nabla^2 \vec{w} - \frac{1}{48} \nabla \nabla^2 \rho \quad (114)$$

$$\text{with } \boldsymbol{\sigma} = \frac{1}{2} (p \boldsymbol{\sigma}_1 + q \boldsymbol{\Omega}_3) \quad (115)$$

where $\boldsymbol{\sigma}_1$ is a traceless, symmetric tensor with $\sigma_{1,xx} = -\sigma_{1,yy} = a_2$ and $\sigma_{1,xy} = \sigma_{1,yx} = b_2$. In this derivation, the velocity moments of C_2 were expressed in terms of the flux tensor $\boldsymbol{\sigma}$, $\langle e_\alpha e_\beta C_2 \rangle = \sigma_{\alpha\beta}$. Eqs. (112, 113) turn out to be the evolution equations for the components of the tensor $\boldsymbol{\sigma}_1$,

$$(T - p) \boldsymbol{\sigma}_1 = q \boldsymbol{\Omega}_3 - \frac{1}{2} \boldsymbol{\Omega}_1 + \frac{1}{8} \boldsymbol{\Omega}_2 \quad (116)$$

The $\boldsymbol{\Omega}_i$ are defined by

$$\Omega_{1,\alpha\beta} = \partial_\alpha w_\beta + \partial_\beta w_\alpha - \frac{2}{d} \delta_{\alpha\beta} \partial_\gamma w_\gamma \quad (117)$$

$$\Omega_{2,\alpha\beta} = 2 \partial_\alpha \partial_\beta \rho - \frac{2}{d} \delta_{\alpha\beta} \partial_\gamma^2 \rho \quad (118)$$

$$\Omega_{3,\alpha\beta} = 2 w_\alpha w_\beta - \frac{2}{d} \delta_{\alpha\beta} w^2 \quad (119)$$

where $d = 2$ is the dimension. $\boldsymbol{\Omega}_1$ is actually the regular stress tensor of a two-dimensional fluid. Due to the property $\partial_{t_0} \rho = \partial_{t_0}^2 w_\alpha = 0$, and $\partial_{t_0} w_\alpha = -\partial_\alpha \rho / 2$ (Eq. (81) in rescaled form) Eq. (116) can be solved by the following Ansatz,

$$\boldsymbol{\sigma}_1 = \sum_{i=1}^5 C_i \boldsymbol{\Omega}_i \quad (120)$$

which includes two new tensors

$$\begin{aligned} \Omega_{4,\alpha\beta} &= w_\alpha \partial_\beta \rho + w_\beta \partial_\alpha \rho - \frac{2}{d} \delta_{\alpha\beta} w_\gamma \partial_\gamma \rho \\ \Omega_{5,\alpha\beta} &= 2 (\partial_\alpha \rho) (\partial_\beta \rho) - \frac{2}{d} \delta_{\alpha\beta} (\partial_\gamma \rho)^2 \end{aligned} \quad (121)$$

It is easy to see that all $\boldsymbol{\Omega}_i$ are actually tensors since they can be written as direct products of vectors. By means of Eqs. (115,120) this also proves the previous assumption that $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}$ are tensors. Note, that all involved tensors are traceless and symmetric. The Ansatz for the solution, Eq. (120), with just a finite number of terms is successful due to the following relations,

$$\begin{aligned} \partial_{t_0} \boldsymbol{\Omega}_1 &= -\frac{1}{2} \boldsymbol{\Omega}_2 \\ \partial_{t_0} \boldsymbol{\Omega}_2 &= 0 \\ \partial_{t_0} \boldsymbol{\Omega}_3 &= -\boldsymbol{\Omega}_4 \end{aligned}$$

$$\begin{aligned}
\partial_{t_0}^2 \Omega_3 &= \frac{1}{2} \Omega_5 \\
\partial_{t_0} \Omega_4 &= -\frac{1}{2} \Omega_5 \\
\partial_{t_0} \Omega_5 &= 0.
\end{aligned} \tag{122}$$

Hence, all time derivatives of order three and higher with respect to t_0 vanish. The coefficients C_i in Eq. (120) are found as

$$\begin{aligned}
C_1 &= \frac{1}{2(p-1)} \\
C_2 &= -\frac{p+1}{8(1-p)^2} \\
C_3 &= \frac{q}{1-p} \\
C_4 &= \frac{q}{(1-p)^2} \\
C_5 &= \frac{q(1+p)}{4(1-p)^3}
\end{aligned} \tag{123}$$

This concludes the solution of the evolution equation, Eq. (116), for the tensor σ_1 and its components, Eqs. (112, 113). This means, the higher moments a_2 and b_2 are indeed enslaved to the hydrodynamic fields ρ and \mathbf{w} because the solution, Eq. (120 – 123), explicitly shows how they are uniquely determined (at this order in ϵ) by these fields and their spatial gradients.

Further analysis of Eq. (114) is simplified by the observation that *all* terms can be expressed by means of the Ω_i . One finds,

$$R \partial_{t_2} \vec{w} = ([\lambda - 1] \mathbf{I} + \Gamma \Omega_3 + S \sigma_1) \cdot \vec{w} - \nabla \cdot \left(\sigma - \frac{1}{8} \Omega_1 + \frac{1}{48} \Omega_2 \right) \tag{124}$$

where \mathbf{I} is the unity tensor. The momentum evolution at time scale t_2 can then be found by the Ansatz

$$\partial_{t_2} \vec{w} + \nabla \cdot \mathbf{H} = (u_1 \mathbf{I} + \mathbf{Q}_1) \cdot \vec{w} + (u_2 \mathbf{I} + \mathbf{Q}_2) \cdot \nabla \rho \tag{125}$$

One finds $u_1 = \lambda - 1$ and $u_2 = (\lambda - 1)/4$. The momentum flux tensor \mathbf{H} and the tensors $\mathbf{Q}_1, \mathbf{Q}_2$ are expanded as

$$\mathbf{H} = \sum_{i=1}^5 h_i \Omega_i \quad \mathbf{Q}_1 = \sum_{i=1}^5 q_i \Omega_i \quad \mathbf{Q}_2 = \sum_{i=1}^5 k_i \Omega_i \tag{126}$$

The coefficients h_i, q_i and k_i are determined by inserting Eq. (125) into Eq. (124) and using the algebra of t_0 derivatives, Eq. (122) with the result,

$$\begin{aligned}
h_1 &= \frac{p}{2} C_1 - \frac{1}{8} = \frac{1+p}{8(p-1)} \\
h_2 &= \frac{p}{2} C_2 + \frac{p}{8} C_1 - \frac{1}{96} = -\frac{p^2 + 10p + 1}{96(p-1)^2} \\
h_3 &= \frac{p}{2} C_3 + \frac{q}{2} = -\frac{q}{2(p-1)}
\end{aligned}$$

$$\begin{aligned}
h_4 &= \frac{p}{2}C_4 + \frac{p}{4}C_3 + \frac{q}{4} = \frac{q(1+p)}{4(p-1)^2} \\
h_5 &= \frac{p}{2}C_5 + \frac{p}{8}C_4 + \frac{p}{48}C_3 + \frac{q}{48} = -\frac{q(p^2 + 10p + 1)}{48(p-1)^3}
\end{aligned} \tag{127}$$

and

$$\begin{aligned}
q_1 &= SC_1 = \frac{S}{2(p-1)} \\
q_2 &= S \left(C_2 + \frac{C_1}{4} \right) = -\frac{S}{4(p-1)^2} \\
q_3 &= SC_3 + \Gamma = \Gamma - \frac{Sq}{p-1} \\
q_4 &= S \left(C_4 + \frac{C_3}{2} \right) + \frac{\Gamma}{2} = \frac{\Gamma}{2} - \frac{Sq(p-3)}{2(p-1)^2} \\
q_5 &= S \left(C_5 + \frac{C_4}{4} + \frac{C_3}{24} \right) + \frac{\Gamma}{24} = \frac{\Gamma}{24} - \frac{Sq(p^2 - 2p + 13)}{24(p-1)^3}
\end{aligned} \tag{128}$$

$$\begin{aligned}
k_1 &= \frac{SC_1}{4} = \frac{S}{8(p-1)} \\
k_2 &= S \left(\frac{C_1}{24} + \frac{C_2}{4} \right) = -\frac{S(p+5)}{96(p-1)^2} \\
k_3 &= \frac{SC_3}{4} + \frac{\Gamma}{4} = \frac{\Gamma}{4} - \frac{Sq}{4(p-1)} \\
k_4 &= S \left(\frac{C_4}{4} + \frac{C_3}{12} \right) + \frac{\Gamma}{12} = \frac{\Gamma}{12} - \frac{Sq(p-4)}{12(p-1)^2} \\
k_5 &= \frac{q_5}{4} + \frac{k_4}{4} - \frac{q_4}{12} - \frac{k_3}{12} + \frac{q_3}{32} = -\frac{Sq(p+5)}{48(p-1)^3}
\end{aligned} \tag{129}$$

Finally, the total momentum evolution up to cubic order in ϵ is obtained by combining the different time derivatives and setting ϵ equal to unity, $\partial_t \vec{w} = \partial_{t_0} \vec{w} + \partial_{t_1} \vec{w} + \partial_{t_2} \vec{w} + O(\epsilon^4)$,

$$\begin{aligned}
&\partial_t \vec{w} + \nabla \cdot \mathbf{H} \\
&= -\frac{1}{2} \left(1 - \frac{\lambda - 1}{2} \right) \nabla \rho + (\lambda - 1) \vec{w} + \mathbf{Q}_1 \cdot \vec{w} + \mathbf{Q}_2 \cdot \nabla \rho
\end{aligned} \tag{130}$$

Note, that this equation was derived at close proximity to the critical point, $\lambda \rightarrow 1$. In the special case of constant density, $\nabla \rho = 0$, only two of the five $\mathbf{\Omega}_i$ tensors, $\mathbf{\Omega}_1$ and $\mathbf{\Omega}_3$, are non-zero, and all the density dependent coefficients are constant. Then the Navier-Stokes-like equation for the momentum-density, Eq. (130), simplifies to

$$\partial_t \vec{w} + h_3 \nabla \cdot \mathbf{\Omega}_3 = (\lambda - 1) \vec{w} - h_1 \nabla \cdot \mathbf{\Omega}_1 + (q_1 \mathbf{\Omega}_1 + q_3 \mathbf{\Omega}_3) \cdot \vec{w} \tag{131}$$

This can also be written more explicitly as

$$\begin{aligned}
&\partial_t \vec{w} + (2h_3 - q_1)(\vec{w} \cdot \vec{\nabla}) \vec{w} + (2h_3 + q_1)(\vec{\nabla} \cdot \vec{w}) \vec{w} - \left(h_3 + \frac{q_1}{2} \right) \vec{\nabla} (|\vec{w}|^2) \\
&= (\lambda - 1) \vec{w} - h_1 \nabla^2 \vec{w} + q_3 |\vec{w}|^2 \vec{w}
\end{aligned} \tag{132}$$

The prefactor $\lambda - 1$ determines the linear growth rate of the momentum, which is consistent with the interpretation of λ as an amplification factor in Sec. 4.4. The growth is limited by a cubic nonlinearity with coefficient q_3 which is always negative. The coefficients h_3 and q_1 control the strength of the convective terms which are quadratic in \vec{w} and contain one gradient operator. These terms are more complicated than the one occurring in the conventional Navier-Stokes equation because of the absence of Galilean invariance.

The coefficient h_1 is proportional to the shear viscosity. More precisely, after returning to dimensional quantities, see Eqs. (104-106), the Chapman-Enskog expansion predicts the following kinematic shear viscosity,

$$\nu^{kin} = -v_0^2 \tau h_1 = \frac{v_0^2 \tau}{8} \frac{1+p}{1-p}, \quad (133)$$

where p is defined in Eq. (47). The viscosity is positive as expected because $|p| < 1$ for all noise values $\eta > 0$. Note, that this is only the so-called kinetic contribution to the shear viscosity because the evaluation of the collision integrals in Sec. 4.3 was done in the limit of large mean free path $r_0 \ll v_0 \tau$. It can be shown, [55], that similar to dense fluids [51] and other particle-based models [52, 53], there is an additional collisional contribution, ν^{coll} , due to collisional momentum transfer,

$$\nu = \nu^{kin} + \nu^{coll}, \quad (134)$$

This contribution is proportional to r_0^2/τ , and therefore dominates the total viscosity ν at small mean free path. The kinetic contribution was also derived by means of a Green-Kubo relation and is considered in detail in Sec. 6.

4.8. Discussion

The original equations for the evolution of the momentum density, Eqs. (111,114), still depend on higher order moments, and require closure. The non-trivial closure of the moment equations presented by Eqs. (120 – 123) does not simply assume that all higher moments are zero or constant; they still evolve in time but their dynamics is completely prescribed by the dynamics of the lower moments – the hydrodynamic fields ρ and \vec{w} . Note, that this differs from the moment closures proposed by other authors such as Ref. [18, 23, 56]. These authors assume $\dot{f}_2 = 0$.

The closure proposed in this paper was significantly simplified by restricting the analysis to the vicinity of the order-disorder threshold where λ , Eq. (39), is close to one. More specifically, the scaling assumption $1 - \lambda = O(\epsilon^2)$ allows one to express the time evolution of the moments of the higher order distribution functions f_2 and f_3 in terms of gradients of the hydrodynamic fields. This means these functions depend on time only implicitly through their functional dependence on f_0 and f_1 . Thus, at order $O(\epsilon^3)$ and near the flocking threshold, I found that the moments f_2 and f_3 are enslaved to f_0 and f_1 , whereas even higher functions such as f_4 can be neglected at this order. This results in a consistent closure of the hierarchy equations and leads to two hydrodynamic equations, as postulated in the Toner-Tu theory [9].

The transport coefficients of the final hydrodynamic equation, Eq. (130), have no explicit dependence on the radius of the collision circle r_0 . This is a consequence of the assumption of large mean free path, $v_0\tau \gg r_0$, which I used in the evaluation of the collisional integrals in Sec. 5. However, this radius will enter implicitly through M_R , the average number of particles per interaction disc, $M_R = \rho\pi r_0^2$, where ρ is the local particle number density. The collisional contributions to the transport coefficients, which explicitly depend on r_0 and are relevant at small mean free paths, will be left for future studies.

The transport coefficients were evaluated analytically in the large density limit $M \gg 1$ in Ref. [16] and simple expressions were given. The time-dependent nonlinear hydrodynamic equations were also integrated numerically in Ref. [14], and their stability was discussed in Ref. [15].

In Sec. 6 it is shown that the usual Green-Kubo relation for the kinetic part of the viscosity gives a result identical to Eq. (133), if one identifies the temperature $k_B T$ of the VM-“fluid” by $v_0^2/2$ (assuming a particle mass $m = 1$). This temperature definition is also consistent (up to terms of order ϵ^2) with the pressure gradient, derived in the Chapman-Enskog expansion because the first term on the r.h.s. of Eq. (130) can be interpreted as the gradient of an ideal gas pressure in an isothermal system. After undoing the nondimensionalization given by Eqs. (104-106), one reads off the following pressure

$$p_{id} = \rho k_B T = \rho \frac{v_0^2}{2} \left(1 - \frac{\lambda - 1}{2} \right) = \rho \frac{v_0^2}{2} + O(\epsilon^2) \quad (135)$$

This also means that the speed of sound in the VM, near the order-disorder transition where $\lambda \approx 1$, is given by the isothermal speed of sound

$$c_T = \sqrt{k_B T} = v_0 \sqrt{2} + O(\epsilon^2) \quad (136)$$

In this sense, the soliton-like density waves analyzed in Ref. [14] appear to be always supersonic.

5. Evaluation of angular integrals

In order to obtain explicit expressions for the transport coefficients of the hydrodynamic equations in Sec. 4.3, certain moments of the collision operator, for example $\langle e_x e_y C_2 \rangle$, are needed. The core part of these moments are n -dimensional integrals whose calculation for small n , and also asymptotically for very large n will be presented in this section. The integrals are of the following type,

$$I = \frac{1}{(2\pi)^n} \int_0^{2\pi} d\alpha^{(n)} g_1(\cos \Phi_i, \sin \Phi_i) g_2(\cos \alpha_i, \sin \alpha_j) \quad (137)$$

where $\alpha^{(n)} = (\alpha_1, \alpha_2, \dots, \alpha_n)$, and $n = 1, 2, 3, \dots$ is the number of actual particles in a collision cell. g_1 is a product of cosine and sine of the average angle Φ_i , g_2 is a product of cosine and sine of the angles α_i . I will use the notation $K_{\alpha\beta\gamma}^{\mu\nu}$ for these integrals. The upper index $\mu, \nu = 1, 2$ refers to the appearance of $\cos \Phi_i$ and $\sin \Phi_i$ in

the function g_1 ; the lower index describes the product of sine and cosine of the angles α_i . For example, $(\mu, \nu) = (1, 2)$ means $g_1 = \cos \Phi_i \sin \Phi_i$, and $(\mu, \nu) = (1, 1)$ means $g_1 = \cos \Phi_i \cos \Phi_i = \cos^2 \Phi_i$. For the lower index I will use the symbols c , s , $2c$ and $2s$ to describe the function g_2 . For example, $(\alpha, \beta) = (c, 2s)$ means $g_2 = \cos \alpha_1 \sin 2\alpha_2$. With this definition, one has for example:

$$K_{c,2c}^{1,1}(n) = \frac{1}{(2\pi)^n} \int_0^{2\pi} d\alpha^{(n)} \cos^2 \Phi_1 \cos \alpha_1 \cos 2\alpha_2 \equiv K_{c2c}^{11} \quad (138)$$

$$K_{c,c,c}^2(n) = \frac{1}{(2\pi)^n} \int_0^{2\pi} d\alpha^{(n)} \sin \Phi_1 \cos \alpha_1 \cos \alpha_2 \cos \alpha_3 \equiv K_{ccc}^2 \quad (139)$$

$$(140)$$

In order to determine the average angle Φ_1 , the local order parameter vector $\mathbf{L}_n = (L_{n,x}, L_{n,y})$ is defined,

$$\mathbf{L}_n = \sum_{i=1}^n \mathbf{e}_i \quad (141)$$

where $\mathbf{e}_i = v_0(\cos \alpha_i, \sin \alpha_i)$ is the velocity vector for particle i , and v_0 is the constant speed of a particle. The sine and cosine of the average angle follow as usual, $\cos \Phi_1 = L_x/|L|$ and $\sin \Phi_1 = L_y/|L|$. Alternatively, the average angle can be defined directly as $\Phi_1 = \text{atan}(L_y/L_x)$.

5.1. Calculations for finite n

For $n = 2$, the integrals are two-dimensional, and it is possible to use trigonometric sum rules to simplify the integral. Using,

$$\frac{L_x(2)}{v} = \cos \alpha_1 + \cos \alpha_2 = 2 \cos \frac{\alpha_1 + \alpha_2}{2} \cos \frac{\alpha_1 - \alpha_2}{2} \quad (142)$$

$$\frac{L_y(2)}{v} = \sin \alpha_1 + \sin \alpha_2 = 2 \sin \frac{\alpha_1 + \alpha_2}{2} \cos \frac{\alpha_1 - \alpha_2}{2} \quad (143)$$

one finds that

$$\Phi_1 = \frac{\alpha_1 + \alpha_2}{2} \quad \text{for } |\alpha_1 - \alpha_2| < \pi \quad (144)$$

$$\Phi_1 = \frac{\alpha_1 + \alpha_2}{2} + \pi \quad \text{for } |\alpha_1 - \alpha_2| > \pi \quad (145)$$

for $0 \leq \alpha_i \leq 2\pi$. The integral over α_1 and α_2 can then be split into four parts,

$$\begin{aligned} \int_0^{2\pi} d\alpha_1 \int_0^{2\pi} d\alpha_2 \dots &= \int_0^\pi d\alpha_1 \left(\int_0^{\alpha_1+\pi} d\alpha_2 \dots + \int_{\alpha_1+\pi}^{2\pi} d\alpha_2 \right) \\ &+ \int_\pi^{2\pi} d\alpha_1 \left(\int_{\alpha_1-\pi}^{2\pi} d\alpha_2 \dots + \int_0^{\alpha_1-\pi} d\alpha_2 \right) \end{aligned} \quad (146)$$

where in the first and third part $|\alpha_1 - \alpha_2| < \pi$, and in the second and fourth term one has $|\alpha_1 - \alpha_2| > \pi$. All functions under the integral are now products of sine and cosine of a linear combination $a\alpha_1 + b\alpha_2$. Therefore, all collision moments for $n = 2$ can be evaluated analytically. These exact results are given in the ($n = 2$) column of Tab. 1.

	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 10$	$n \rightarrow \infty$
K_c^1	1/2	1/ π	0.2624	0.2249	0.2008	0.141	$n^{-1/2}\sqrt{\pi}/4$
K_{2c}^{11}	1/4	0	0.0544989	0.02700	0.02724	0.0127	$n^{-1}/8$
K_{cs}^{12}	—	1/8	0.04888	0.03716	0.02784	0.0131	$n^{-1}/8$
K_{c2c}^1	—	-1/(6 π)	-0.01277	-0.009402	-0.005810	-0.00143	$-n^{-3/2}\sqrt{\pi}/32$
K_{ccc}^1	—	—	-0.06967	-0.03159	-0.02074	-0.00439	$-3n^{-3/2}\sqrt{\pi}/32$

Table 1. Results for important K -Integrals as a function of particle number n . For details see Sec. 5.

For $n \geq 3$, to my knowledge, no simple addition theorem can be used to simplify the calculations, and the calculations become very tedious. I was able to find an analytical solution for special cases at $n = 3$, see Sec. 5.3, but due to the difficulty level I will not explore this further. It is faster to solve these integrals numerically, as given in Table 1. Furthermore, exact asymptotic solutions can be found in the limit of infinite dimensions, $n \rightarrow \infty$. By comparison to the numerical solutions one realizes that the asymptotic formulas are still quite accurate for $n \approx 5$ and larger. The details of these asymptotic calculations will be presented further below.

Table 1 gives analytical and numerical results for the collision moments. The results for $n = 3, 4, 5$ were obtained by a straightforward numerical integration with constant intervals, the numerical results for $n = 10$ were obtained by simple Monte-Carlo integration. It is difficult to achieve good numerical accuracy for $n > 10$, since the integrand is strongly oscillating and no regions of the angular space ($\alpha_1, \dots, \alpha_n$) can be neglected, hence importance sampling methods, like the Metropolis algorithm, can not be used. The results for $n = 1, n = 2$ and $n \rightarrow \infty$ are exact.

All other similar moments are either zero or related by the ones in the table. One finds,

$$\begin{aligned}
K_{css}^1 &= K_{ccc}^1/3 \\
K_s^2 &= K_c^1 \\
K_{cc}^{11} &= -K_{ss}^{11} = -K_{cc}^{22} = K_{ss}^{22} = K_{cs}^{12} \\
K_{2c}^{11} &= -K_{2c}^{22} = K_{2s}^{12} \\
K_{c2c}^1 &= K_{s2s}^1 = K_{c2s}^2 = -K_{s2c}^2 \\
K_{ccs}^2 &= K_{css}^1 \\
K_{sss}^2 &= K_{ccc}^1.
\end{aligned} \tag{147}$$

For symmetry reasons the following integrals vanish for all n :

$$\begin{aligned}
K_{css}^1 &= K_{sss}^1 = K_{3c}^1 = K_{3s}^1 = K_{c2s}^1 = K_{s2c}^1 = 0 \\
K_{ccc}^2 &= K_{css}^2 = K_{3c}^2 = K_{3s}^2 = K_{c2c}^2 = K_{s2s}^2 = 0 \\
K_{2c}^1 &= K_{cc}^1 = K_{ss}^1 = K_{cs}^1 = K_{c2c}^2 = K_{s2s}^2 = 0
\end{aligned}$$

$$\begin{aligned}
K_{2c}^{12} &= K_{cs}^{11} = K_{2s}^{11} = K_{2s}^{22} = K_{cc}^{12} = K_{ss}^{12} = 0 \\
K_{3c}^{11} &= K_{3s}^{12} = K_{3c}^{12} = K_{3c}^{22} = K_{3s}^{22} = 0 \\
K_{ccc}^{11} &= K_{c2c}^{11} = K_{css}^{11} = K_{sss}^{11} = K_{ccs}^{11} = 0 \\
K_{cc}^{12} &= K_{scc}^{12} = K_{ssc}^{12} = K_{sss}^{12} = 0
\end{aligned} \tag{148}$$

Higher moments, such as K_{c3c}^{11} can be non-zero (-0.002811 for $n = 3$) but occur at higher order than ϵ^3 in the Chapman-Enskog expansion. Note, that K_{cs}^{11} has a non-monotonic behavior, and more interestingly, exactly vanishes for binary collisions, $n = 2$.

5.2. Calculations for $n \rightarrow \infty$

The main idea of how to obtain the asymptotic behavior is based on an analogy to a random walk or a Gaussian chain: The individual velocity vectors \mathbf{e}_i can be seen as segments of fixed length v_0 forming a random chain. The local order parameter vector \mathbf{L}_n , see Eq. (141), translates then into the end-to-end vector for this chain of n segments. It is well known that in the limit of infinite chain length, the probability density to realize a certain end-to-end vector is Gaussian,

$$p(\mathbf{L}) = \frac{1}{\pi v_0^2 n} e^{-L^2/(v_0^2 n)}, \tag{149}$$

due to the central limit theorem. In particular, the average chain length $\langle |\mathbf{L}| \rangle$ is of order $v_0 n^{1/2}$, i.e. is large for large n . This means $1/L$ is small for most chain realizations and will be used as an expansion parameter in the following. Eq. (149) contains the obvious fact that all directions of the vector \mathbf{L} are equally probable, i.e. the probability density for the angle β of the order parameter vector is the same for all angles between zero and 2π .

5.2.1. One-angle calculations I will start by considering integrals where the integrand contains only *one* angular argument, α_1 , in addition to functions of the average angle Φ_1 . The order parameter vector can be split up in the following way:

$$\mathbf{L}_n = \mathbf{L}_{n-1} + \mathbf{e}_1 \tag{150}$$

with the vector $\mathbf{e}_1 = v_0(\cos \alpha_1, \sin \alpha_1)$. The components of \mathbf{L}_{n-1} can be expressed in terms of the angle β and length L_{n-1} of this vector, $L_{n-1,x} = c L_{n-1}$, $L_{n-1,y} = s L_{n-1}$, where $c = \cos \beta$ and $s = \sin \beta$. As a result, the cosine of the average angle can be written as,

$$\cos \Phi_1 = \frac{L_{n-1}c + v_0c_1}{\sqrt{(L_{n-1}c + v_0c_1)^2 + (L_{n-1}s + v_0s_1)^2}} \tag{151}$$

with $c_1 = \cos \alpha_1$, $s_1 = \sin \alpha_1$. A similar expression follows for $\sin \Phi_1$. Now the $n - 1$ integrations over the angles $\alpha_2, \alpha_3 \dots \alpha_n$ can be replaced by an integration over all possible lengths L_{n-1} , which are between zero and $(n - 1)v_0$ and over the orientation angle β of the end-to-end vector. In the limit of infinite dimensions, $n \rightarrow \infty$, the upper limit of

the integral over all possible lengths goes to infinity and we can use the simple Gaussian expressions, Eq. (149) for the probability density. Finally, for example K_{2c}^{11} becomes,

$$K_{2c}^{11} = \frac{1}{2\pi} \int_0^{2\pi} d\beta \int_0^\infty p(L_{n-1}) L_{n-1} dL_{n-1} \\ \times \int_0^{2\pi} d\alpha_1 \cos 2\alpha_1 \frac{(L_{n-1}c + v_0c_1)^2}{(L_{n-1}c + v_0c_1)^2 + (L_{n-1}s + v_0s_1)^2} \quad n \rightarrow \infty \quad (152)$$

where $p(L_{n-1})$ is given in Eq. (149). After introducing the dimensionless variable $L = L_{n-1}/v_0$ this can be written by means of the function $h(L)$ as

$$K_{2c}^{11} = \frac{1}{\pi(n-1)} \int_0^\infty h(L) L e^{-L^2/(n-1)} dL, \quad \text{with} \quad (153)$$

$$h(L) = \frac{1}{2\pi} \int_0^{2\pi} d\beta \int_0^{2\pi} d\alpha_1 \cos 2\alpha_1 \frac{(Lc + c_1)^2}{(Lc + c_1)^2 + (Ls + s_1)^2} \quad (154)$$

I found that $h(L)$ has a very simple structure with a cusp at $L = 1$: $h(L) = \frac{\pi}{2}(1 - L^2)$ for $0 \leq L \leq 1$ and $h(L) = 0$ for $L \geq 1$, which can be easily checked for $L = 0$ and $L = 1$ but is nontrivial to prove for arbitrary L . This simple quadratic expression agreed perfectly with a numerical evaluation of $h(L)$. Since $h = 0$ for $L \geq 1$, the upper limit in Eq. (153) can be reduced to one, and in the limit of infinite n , the exponential factor $\exp(-L^2/n)$ goes to one for all L in the integration range, and $n - 1$ is replaced by n . The asymptotic result is therefore:

$$K_{2c}^{11} = \frac{1}{\pi n} \int_0^1 \frac{\pi}{2} L(1 - L^2) dL = \frac{1}{8n} \quad (155)$$

Similar calculations were performed for other integrals, such as K_{2s}^{12} , leading to the same asymptotic behavior $\sim n^{-1}$. Simulations showed that the integral K_c^1 however has a weaker decay than $\sim n^{-1}$ for $n \rightarrow \infty$ and will therefore be analyzed in the following. Applying the same transformation from individual angular coordinates, (α_2, α_3) to (β, L) with $L = L_{n-1}/v_0$, one finds,

$$K_c^1 = \frac{1}{2\pi} \int_0^{2\pi} d\beta \int_0^\infty p(L) L dL \int_0^{2\pi} d\alpha_1 \cos \alpha_1 \cos \Phi_1 \quad n \rightarrow \infty \quad (156)$$

where $\cos \Phi_1$ is given in Eq. (151). For $L > 1$ the square root is expanded in terms of order $(1/L)^m$. This gives

$$\frac{1}{\sqrt{(L_{n-1}c + v_0c_1)^2 + (L_{n-1}s + v_0s_1)^2}} \\ = \frac{1}{\sqrt{L^2 + 1}} \left(1 - \frac{L}{L^2 + 1} (cc_1 + ss_1) + O(L^{-2}) \right) \quad (157)$$

The integral over L is split into two parts with L_c fixed but much larger than one,

$$K_c^1 = \int_0^{L_c} dL \dots + \int_{L_c}^\infty dL \dots \quad (158)$$

because in the second term it is justified to use the expansion of the square root. The integration over the angular coordinates, β and α_1 can now be performed easily for this

term. One obtains,

$$\begin{aligned}
 K_c^1 &= \frac{1}{2\pi} \int_0^{2\pi} d\beta \int_0^{L_c} p(L)L dL \int_0^{2\pi} d\alpha_1 \cos \alpha_1 \cos \Phi_1 \\
 &+ \frac{\pi}{2} \int_{L_c}^{\infty} \frac{L dL p(L)}{\sqrt{L^2 + 1}} \left[2 - \frac{L^2}{L^2 + 1} \right]
 \end{aligned} \tag{159}$$

In the limit $n \rightarrow \infty$, $p(L)$ is proportional n^{-1} ; $\cos \Phi_1$ is always of order one, hence one sees that the first integral decays at least as quickly as n^{-1} . On the other hand, as it turns out, the second integral decays slower than this, thus the leading asymptotic behavior of K_c^1 is given by the second integral. In particular, after inserting $p(L)$, replacing $n - 1$ by n and the substitution $L = xn^{1/2}$, one finds

$$K_c^1 \sim \frac{1}{2n^{1/2}} \int_{L_c n^{-1/2}}^{\infty} \frac{x dx}{\sqrt{x^2 + n^{-1}}} \left[2 - \frac{x^2}{x^2 + n^{-1}} \right] e^{-x^2} \tag{160}$$

In the limit $n \rightarrow \infty$, this turns into a simple Gaussian integral,

$$K_c^1 \sim \frac{1}{2n^{1/2}} \int_0^{\infty} e^{-x^2} dx = \frac{1}{4} \sqrt{\frac{\pi}{n}} \tag{161}$$

5.2.2. Two-angle calculations Here I will consider integrals where the integrand contains *two* angular arguments, α_1 and α_2 . Examples are, K_{cs}^{12} , K_{cc}^{11} and K_{ss}^{11} . Similar to Eq. (150), the end-to-end vector \mathbf{L} is split up into three contributions

$$\mathbf{L}_n = \mathbf{L}_{n-2} + \mathbf{e}_1 + \mathbf{e}_2, \tag{162}$$

and the integration over the $n - 2$ angles $\alpha_3, \alpha_4, \dots, \alpha_n$ is replaced by an integration over the length and the angle β of the vector \mathbf{L}_{n-2} . Analogous to the previous section, in the limit of $n \rightarrow \infty$ one finds for example

$$\begin{aligned}
 K_{cs}^{12} &= \frac{1}{\pi(n-2)} \int_0^{\infty} g(L) L e^{-L^2/(n-2)} dL, \quad \text{with} \\
 g(L) &= \frac{1}{(2\pi)^2} \int_0^{2\pi} d\beta \int_0^{2\pi} d\alpha_1 \\
 &\quad \times \int_0^{2\pi} d\alpha_2 \sin \alpha_1 \cos \alpha_2 \frac{(Ls + s_1 + s_2)(Lc + c_1 + c_2)}{(Lc + c_1 + c_2)^2 + (Ls + s_1 + s_2)^2}
 \end{aligned} \tag{163}$$

It turns out that $g(L) = 0$ for $L \geq 2$ and that

$$\frac{1}{\pi} \int_0^2 L h(L) dL = \frac{1}{8} \tag{164}$$

Because of that, the asymptotic behavior is identical to the one of K_{2c}^{11} , and one has

$$K_{cs}^{12} = \frac{1}{8n}, \quad \text{for } n \rightarrow \infty. \tag{165}$$

A similar analysis leads to

$$\begin{aligned}
 K_{cc}^{11} &= \frac{1}{8n} \\
 K_{ss}^{11} &= -\frac{1}{8n}, \quad \text{for } n \rightarrow \infty.
 \end{aligned} \tag{166}$$

5.2.3. *Three-angle calculations* The end-to-end vector \mathbf{L} is split up into four contributions

$$\mathbf{L}_n = \mathbf{L}_{n-3} + \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3, \quad (167)$$

and the integration over the $n - 3$ angles $\alpha_4, \alpha_5, \dots, \alpha_n$ is replaced by an integration over the length and direction of \mathbf{L}_{n-3} . Asymptotically one finds

$$\begin{aligned} K_{ccc}^1 &= \frac{1}{\pi(n-3)} \int_0^\infty q(L) L e^{-L^2/(n-3)} dL, \quad \text{with} \\ q(L) &= \frac{1}{(2\pi)^3} \int_0^{2\pi} d\beta \int_0^{2\pi} d\alpha_1 \int_0^{2\pi} d\alpha_2 \int_0^{2\pi} d\alpha_3 \\ &\quad \times c_1 c_2 c_3 \frac{(Lc + c_1 + c_2 + c_3)}{\sqrt{(Lc + c_1 + c_2 + c_3)^2 + (Ls + s_1 + s_2 + s_3)^2}} \end{aligned} \quad (168)$$

This time $q(L)$ does not vanish exactly above a certain finite value of L . In order to proceed, for $L > L_c \approx 3$, the square root can be expanded in a convergent series

$$\begin{aligned} &\frac{1}{\sqrt{(Lc + c_1 + c_2 + c_3)^2 + (Ls + s_1 + s_2 + s_3)^2}} = \frac{1}{\sqrt{L^2 + 3}} \frac{1}{\sqrt{1 + g}} \\ &\approx \frac{1}{\sqrt{L^2 + 3}} \left(1 - \frac{1}{2}g + \frac{3}{8}g^2 - \frac{15}{48}g^3 + \dots\right) \\ &g = \frac{2L}{L^2 + 3} [cc_1 + cc_2 + cc_3 + ss_1 + ss_2 + ss_3] \\ &\quad + \frac{2}{L^2 + 3} [s_1 s_2 + s_1 s_3 + s_2 s_3 + c_1 c_2 + c_1 c_3 + c_2 c_3] \end{aligned} \quad (169)$$

This expansion must be done to third order in g to capture the leading asymptotic behavior of the integral,

$$\begin{aligned} q(L) &= \frac{\phi_0}{L^3} + O(L^{-4}) \quad \text{with} \\ \phi_0 &= \frac{3\pi}{32}, \quad \text{for } L > L_c \end{aligned} \quad (170)$$

For $0 \leq L \leq L_c$ the expansion breaks down, the function $q(L)$ varies strongly, shows one minimum and one maximum, and the contribution to the final integral over L cannot be neglected. However, it turns out that the total integral vanishes, which was confirmed by numerical integration:

$$\int_0^\infty L q(L) dL = 0 \quad (171)$$

This condition is sufficient to obtain an exact formula for the asymptotic behavior of K_{ccc}^1 and similar integrals: Let us split Eq. (168) into two parts:

$$K_{ccc}^1 = \frac{1}{\pi(n-3)} \int_0^{L_s} q(L) L e^{-L^2/(n-3)} dL + \int_{L_s}^\infty q(L) L e^{-L^2/(n-3)} dL \quad (172)$$

where L_s is a fixed number, $L_s \gg L_c$. In the first term, the exponential can be replaced by one for large $n \gg L_s^2$. In the second term, we can safely substitute the asymptotic expression for $q(L)$ from Eq. (170), and $n - 3$ can be replaced by n .

$$K_{ccc}^1 = \frac{1}{\pi n} \int_0^{L_s} q(L) L dL + \int_{L_s}^\infty \frac{\phi_0}{L^2} e^{-L^2/(n-3)} dL \quad (173)$$

with $\phi_0 = 3\pi/32$. Now, the first integral can be written as

$$\begin{aligned} \int_0^{L_s} q(L)L dL &= \int_0^\infty q(L)L - \int_{L_s}^\infty q(L)L dL \\ &= - \int_{L_s}^\infty q(L)L dL = -\phi_0 \int_{L_s}^\infty \frac{dL}{L^2} \end{aligned} \quad (174)$$

where I made use of property (171) and inserted the asymptotic expression for $q(L)$, which is justified for the range of large L in this integral. Combining Eqs. (173,174), one obtains,

$$K_{ccc}^1 = \frac{\phi_0}{\pi n} \int_{L_s}^\infty \frac{e^{-L^2/n} - 1}{L^2} dL \quad (175)$$

At this point, the substitution $x = L/\sqrt{n}$ is made,

$$K_{ccc}^1 = \frac{\phi_0}{\pi n^{3/2}} \int_{L_s/\sqrt{n}}^\infty \frac{e^{-x^2} - 1}{x^2} dx \quad (176)$$

Since L_s is a fixed number, the lower limit of this integral goes to zero for $n \rightarrow \infty$ and one has

$$K_{ccc}^1 = -\frac{\phi_0}{\pi n^{3/2}} \int_0^\infty \frac{1 - e^{-x^2}}{x^2} dx \quad (177)$$

The remaining integral can be solved exactly. The final asymptotic result is

$$K_{ccc}^1 = -\frac{3\sqrt{\pi}}{32} \frac{1}{n^{3/2}} \quad (178)$$

5.3. A special case at $n = 3$

It is possible to obtain an analytical solution for certain K -integrals at $n = 3$, for example for K_{2c}^{11} . First, I use the decomposition from Eq. (150), $\mathbf{L}_3 = \mathbf{L}_2 + \mathbf{e}_1$. The probability density $p(\beta, L_2)$ to find the vector \mathbf{L}_2 at certain angle β and certain length L is now far from being exponential and actually is zero for all $L > 2v_0$. However, \mathbf{L}_2 is only composed of two segments, each of length v_0 ; and for given angle β and length L_2 , there is only two ways to add up two segments to create the vector. If the angles of the corresponding segments are given by α_2 and α_3 , respectively, these two solutions are given by: $(\alpha_2, \alpha_3) = (\beta + \gamma, \beta - \gamma)$ or $(\beta - \gamma, \beta + \gamma)$ where γ involves the inverse cosine taken in the first quadrant, $\gamma = \cos^{-1}(L/2v_0)$. Calculating the Jacobian for the transformation from the variables (α_2, α_3) to the new variables, (β, L) and taking into account that there is two possible ways to construct the same vector, the integral over the two segment orientations can be written as:

$$\int_0^{2\pi} d\alpha_2 \int_0^{2\pi} d\alpha_3 \dots = \int_0^{2\pi} d\beta \int_0^{2v_0} \frac{2}{v_0 \sqrt{1 - L^2/(4v_0^2)}} dL_2 \dots \quad (179)$$

That means the probability density to find vector \mathbf{L}_2 at lengths between L and $L + dL$ and angles between β and $\beta + d\beta$ is given by

$$p(\beta, L_2) = \frac{1}{2\pi^2 v_0 \sqrt{1 - L^2/(4v_0^2)}} \quad (180)$$

for $0 \leq L_2 < 2v_0$ and is zero otherwise. Now, I can rewrite expression (152) in terms of the new probability density: Using the dimensionless variable $L = L_2/v_0$, I find,

$$\begin{aligned} K_{2c}^{11} &= \frac{1}{(2\pi)^3} \int_0^{2\pi} d\alpha_1 \int_0^{2\pi} d\alpha_2 \int_0^{2\pi} d\alpha_3 \cos^2 \Phi_1 \cos 2\alpha_1 \\ &= \frac{1}{2\pi^2} \int_0^2 \frac{h(L) dL}{\sqrt{1 - L^2/4}}. \end{aligned} \quad (181)$$

Note that $h(L) = \pi(1 - L^2)/2$, thus this integral can be solved analytically,

$$K_{2c}^{11}(n=3) = \frac{1}{4\pi} \int_0^1 \frac{1 - L^2}{\sqrt{1 - \frac{L^2}{4}}} dL = \frac{\sqrt{3}}{4\pi} - \frac{1}{12} = 0.05449889... \quad (182)$$

This result agrees perfectly with the numerical result from Table 1.

6. Consistency checks via Green-Kubo relations

In this section, velocity and stress auto-correlation functions will be evaluated analytically in the Molecular Chaos approximation. These expressions can be useful for consistency checks in agent-based simulations of the VM. In addition, these auto-correlation functions can be used in Green-Kubo relations. These relations make neither use of the main kinetic equation (15) nor of the Chapman-Enskog expansion. Therefore, they provide a fast, alternative route to important transport coefficients such as the shear viscosity of a VM-fluid. Green-Kubo relations are usually used for fluids in thermal equilibrium. It will be shown below that they also give accurate and consistent results for a non-equilibrium “fluid” such as the Vicsek-model.

Consider the velocity autocorrelation of the focal particle $i = 1$,

$$G(n\tau) = \langle v_{1x}(0)v_{1x}(n\tau) \rangle \quad (183)$$

Hence, $G(0) = \langle v_{1x}^2 \rangle = v_0^2/2$. For a time lag of one time step, $n = 1$, we have

$$G(\tau) = v_0^2 \langle \cos \alpha_1 \cos (\Phi_1 + \xi_1) \rangle \quad (184)$$

where ξ_1 is a random angle which is equally distributed in the interval $[-\eta/2, +\eta/2]$, and $\Phi_1(\{\alpha_j\})$ is the average angle. The precollisional angles are denoted by α_j . Applying trigonometric identities and averaging over the noise term ξ_1 gives

$$\hat{G}(\tau) \equiv \frac{G(\tau)}{k_B T} = 4 \frac{\sin(\eta/2)}{\eta} \langle \cos \alpha_1 \cos \Phi_1 \rangle_c \quad (185)$$

Here, I divided by $v_0^2/2 = k_B T$ because the kinetic temperature in a two-dimensional system is defined by $\langle \vec{v}^2 \rangle = 2k_B T$ for particles with mass $m = 1$. This temperature definition is also consistent (up to terms of order ϵ^2) with the pressure derived in the Chapman-Enskog expansion, see discussion in Sec. 4.8. The brackets with the subscript “c” in Eq. (185) denote averages not only over the angles of all n particles which are

contained in the collision circle of particle 1 but also over the number n of these particles itself (including particle 1),

$$\langle \dots \rangle_c \equiv \sum_{n=1}^N \frac{p_n}{(2\pi)^n} \prod_{i=1}^n \int_0^{2\pi} d\alpha_i \dots \quad (186)$$

Assuming Molecular Chaos, the number of particles in a circle around a focal particle is Poisson-distributed with $p_n = M_R^{n-1}/(n-1)!$. Thus, one obtains

$$\hat{G}(\tau) = \frac{4}{\eta} \sin \frac{\eta}{2} \langle \cos \alpha_i \cos \Phi_i \rangle_c = \frac{4}{\eta} \sin \frac{\eta}{2} \sum_{n=1}^N e^{-M_R} \frac{M_R^{n-1}}{(n-1)!} K_C^1(n) \quad (187)$$

where the integrals $K_C^1(n)$ are given in Table 1.

The asymptotic expression for large M_R can be obtained by the procedure outlined in Eqs. (40–42) with the result

$$\hat{G}(\tau) \approx \frac{4}{\eta} \sin \frac{\eta}{2} K_C^1(M+1) \approx \frac{1}{\eta} \sin \frac{\eta}{2} \sqrt{\frac{\pi}{M+1}} \quad (188)$$

For $\eta = 2\pi$ the velocity correlation is exactly zero. This is expected because the new velocity directions are chosen completely random in this limit. The correlation is also predicted to vanish in the limit $M_R \rightarrow \infty$. This makes sense because in this limit the focal particle has a vanishing impact on the average angle Φ_1 : the focal particle is “overpowered” by the infinitely many other particles in its collision circle.

The Green-Kubo relation for the self-diffusion coefficient in a system with discrete time step is given by

$$D = \tau \sum_{n=0}^{\infty}{}' G(n\tau) \quad (189)$$

where the prime on the sum indicates that the $n = 0$ term is weighted by the factor $1/2$, see Ref. [57]. Assuming molecular chaos leads to a geometric series for the higher order time correlations function, $\hat{G}(n\tau) = (\hat{G}(\tau))^n$, and the self-diffusion coefficient follows as

$$D = \frac{\tau v_0^2}{4} \frac{1 + \hat{G}(\tau)}{1 - \hat{G}(\tau)} \quad (190)$$

This formula is consistent with the expectation for a completely uncorrelated random walk in two dimensions with fixed step size $l = v\tau$ which is realized for $\eta = 2\pi$,

$$D_{random} = \frac{\tau v^2}{4} \quad (191)$$

To derive the shear viscosity, the autocorrelation function of the xy -component of the kinetic stress tensor $\sigma_{xy}^{kin} = \sum_{i=1}^N v_{ix} v_{iy}$,

$$G^{kin}(n\tau) = \langle \sigma_{xy}^{kin}(0) \sigma_{xy}^{kin}(n\tau) \rangle \quad (192)$$

is evaluated in a similar way. Already at the equal time level, $n = 0$, there is a difference to a regular fluid. This is because in the VM the x - and y -component of a given particle's

velocity are strongly correlated, $v_y = \sqrt{v_0^2 - v_x^2}$. As a result, $G^{kin}(0)$ is only half as big as in a regular fluid,

$$\begin{aligned} G^{kin}(0) &= \sum_{i=1}^N \sum_{j=1}^N \langle v_{ix}(0)v_{iy}(0)v_{jx}(0)v_{jy}(0) \rangle \\ &= N \langle v_{1x}^2(0)v_{1y}^2(0) \rangle = N (\langle v_{1x}^2 \rangle - \langle v_{1x}^4 \rangle) \\ &= \frac{Nv_0^4}{8} = \frac{N(k_B T)^2}{2} \end{aligned} \quad (193)$$

For $n = 1$ one has,

$$G^{kin}(\tau) = \sum_{i=1}^N \sum_{j=1}^N \langle v_{ix}(0)v_{iy}(0)v_{jx}(\tau)v_{jy}(\tau) \rangle \quad (194)$$

$$= N \sum_{j=1}^N v_0^4 \langle \cos \alpha_1 \sin \alpha_1 \cos(\Phi_j + \xi_j) \sin(\Phi_j + \xi_j) \rangle \quad (195)$$

If molecular chaos is assumed, only those n particles that are in the collision circle around the focal particle $i = 1$, contribute to the sum over j on the r.h.s. of Eq. (195). All those particles contribute to the average angle Φ_1 , i.e. $\Phi_j = \Phi_1$, and one finds

$$G^{kin}(\tau) = Nv_0^4 \langle n \cos \alpha_1 \sin \alpha_1 \cos \Phi_1 \sin \Phi_1 \rangle_c \langle \cos 2\xi_1 \rangle \quad (196)$$

The noise average gives $\langle \cos 2\xi_1 \rangle = \sin(\eta)/\eta$. Averaging over the Poisson-distributed particle number fluctuations and the pre-collisional angles α_j yields,

$$\langle n \cos \alpha_1 \sin \alpha_1 \cos \Phi_1 \sin \Phi_1 \rangle_c = \frac{1}{2} \langle n \sin(2\alpha_1) \cos \Phi_1 \sin \Phi_1 \rangle_c \quad (197)$$

$$= \sum_{n=1}^N e^{-M_R} \frac{n M_R^{n-1}}{2(n-1)!} K_{2s}^{12}(n) \quad (198)$$

According to Eq. (147), $K_{2s}^{12} = K_{2c}^{11}$, and it follows that

$$G^{kin}(\tau) = \frac{Nv_0^4}{2} \sum_{n=1}^N e^{-M_R} \frac{n^2 M_R^{n-1}}{n!} K_{2c}^{11}(n) \quad (199)$$

Replacing $v_0^2/2$ by $k_B T$ and using the auxiliary variable p from Eq. (47) gives the final result for the kinetic stress correlations,

$$G^{kin}(\tau) = \frac{1}{2} (k_B T)^2 N p \quad (200)$$

Due to Molecular Chaos, the temporal correlations decay as a geometric series,

$$\frac{G^{kin}(\tau)}{G^{kin}(0)} = \frac{G^{kin}([n+1]\tau)}{G^{kin}(n\tau)} = const = p \quad (201)$$

where the constant ratio was found by using Eqs. (193) and (200). The kinetic part of the shear viscosity is then obtained by means of the usual Green-Kubo relation for a system with discrete time dynamics, see for example Ref.[57, 58],

$$\begin{aligned} \nu^{kin} &= \frac{\tau}{Nk_B T} \sum_{n=0}^{\infty} G^{kin}(n\tau) = \frac{\tau}{Nk_B T} G^{kin}(0) \left(\frac{1}{2} + \sum_{k=1}^{\infty} p^k \right) \\ &= \frac{\tau k_B T}{4} \frac{1+p}{1-p} = \frac{v_0^2 \tau}{8} \frac{1+p}{1-p} \end{aligned} \quad (202)$$

The integral K_{2c}^{11} is given in Table 1, and the asymptotic expressions for p are provided in Eqs. (49) and (53). The result for ν^{kin} , Eq. (202), agrees perfectly with the shear viscosity $-v_0^2\tau h_1$, Eqs. (127) and (133), which was derived by a completely different approach – a kinetic theory based on an Enskog-like equation and a subsequent Chapman-Enskog expansion. This agreement provides a non-trivial consistency test for the kinetic theory presented in this paper.

7. Conclusion

Macroscopic evolution equations for interacting many-body systems do not just “emerge”; they follow from microscopic laws. However, it is often difficult to quantitatively establish this link. This is especially the case for open systems which cannot be described by a Hamiltonian and which might have genuine multi-particle interactions that are not pairwise additive. Therefore, the general form of the macroscopic equations is often obtained by symmetry arguments. This can lead to hydrodynamic equations with many unknown parameters. The Vicsek-model (VM) [10] is a well-known example for this kind of open systems. It is one of the simplest models to study collective motion of self-driven particles. Here, I show how the macroscopic transport equations can be systematically derived from the microscopic interaction rules of the standard VM. Whereas most of the results of this derivation have already been published and briefly discussed [13, 15, 16], in this paper, the details of the used analytical techniques together with additional insights are presented. I set up the exact evolution equation for the N -particle probability distribution and show how it can be reduced to an Enskog-like kinetic equation for the one-particle density by means of the molecular chaos approximation. No linearization or single-relaxation time approximation of the collision operator are needed, and the particle density does not have to be small. A non-standard Chapman-Enskog expansion of the kinetic equation in the formal ordering parameter ϵ , and a self-consistent closure of the infinite hierarchy of moment equations is proposed. This procedure involves an expansion in spatial and temporal gradients, and contains a fast time scale. By means of the Chapman-Enskog expansion, hydrodynamic equations for the density and momentum density are derived and all transport coefficients that are relevant up to third order in ϵ are given. The transport coefficients depend on special n -dimensional integrals which I call K -integrals. It is shown, how these integrals can be analytically evaluated for $n = 1, 2, 3$ and for $n \rightarrow \infty$.

Apart from explaining the elaborate analytical techniques, the main results of this paper are the following: (i) the hydrodynamic equation for the momentum density, Eq. (130) and the corresponding transport coefficients in Eqs. (127–129), (ii) the insight that even in a nonequilibrium model with correlated components of the particle velocity such as the VM, the expression for the shear viscosity, Eqs. (133,202), can be obtained by either a Green-Kubo relation or the Chapman-Enskog expansion, (iii) the analytical and numerical results for the angular integrals in Table 1, and (iv) a formula for the

self-diffusion coefficient of the VM, Eq. (190).

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Appendix A. Linear stability analysis

It is interesting to investigate the stability of a homogeneous, globally ordered state against small perturbations of density and order. The results of such an analysis for the standard Vicsek-model have been briefly presented in Refs. [13, 15], and in Fig. 10 of Ref. [54] were compared to another version of the VM. In Ref. [18] a similar analysis has been performed for a Vicsek-like model with binary interactions and continuous time dynamics. Furthermore, in Ref. [16] a detailed calculation was given for the special case of pure longitudinal perturbations in the large density limit, $M \gg 1$. The main insight was, that a long wave length instability occurs right at the onset of collective order. The growth of the perturbations finally leads to the formation of large density waves that show hysteresis, and provide a mean-field mechanism to modify the character of the order-disorder transition from continuous to discontinuous, see Ref. [14]. This seems to be a generic result which occurs in VM-like models with a coupling between local density and order, and is consistent with earlier results on a Vicsek-like model with binary interactions, Refs. [17, 18]. VM-like models without a coupling between density and order, such as the Vicsek-model with topological interactions, Refs. [54, 59, 60], do not show this instability. In this Appendix, the mathematical details of the stability analysis for the standard VM will be given.

Density ρ and momentum density \vec{w} are expanded around a homogeneous ordered state, $\rho_0, \vec{w}_0 = w_0 \hat{\mathbf{n}}$ as

$$\begin{aligned} \rho(\mathbf{x}, t) &= \rho_0 + \delta\rho e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \\ \vec{w}(\mathbf{x}, t) &= \vec{w}_0 + (\hat{\mathbf{t}} \delta u + \hat{\mathbf{n}} \delta v) e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \\ \mathbf{k} &= k_n \hat{\mathbf{n}} + k_t \hat{\mathbf{t}}, \end{aligned} \tag{A.1}$$

where $\hat{\mathbf{n}}$ is the unit vector in the normal direction, defined by the direction of the unperturbed flying direction; $\hat{\mathbf{t}}$ is the unit vector in the transverse direction with $\hat{\mathbf{n}}\cdot\hat{\mathbf{t}} = 0$. Consequently, k_n and k_t are the normal and the transverse components of the wave vector, respectively. $\delta\rho$ is the amplitude of the density perturbation, δv and δu are the amplitudes of the normal and transverse perturbations of the momentum, respectively. w_0 , the amplitude of the unperturbed momentum is given by

$$w_0 = \sqrt{\frac{1-\lambda}{q_3}} \tag{A.2}$$

and is nonzero in the ordered phase where $\lambda > 1$ and $q_3 < 0$. Inserting the Ansatz, Eq. (A.1), into the macroscopic equations, Eq. (94) and (130) and neglecting terms of higher than linear order in the perturbations gives three coupled equations for $\delta\rho$, δv and δu , which can be written in form of a homogeneous system of linear equations, $M \cdot \mathbf{F} = \mathbf{0}$ where F is a vector with $\mathbf{F}^T = (\delta\rho, \delta v, \delta u)$ and M is a nonsymmetric 3×3 matrix given by

$$\begin{aligned}
M_{11} &= -\omega \\
M_{12} &= k_n \\
M_{13} &= k_t \\
M_{21} &= ih_2k^2k_n + h_4w_0k^2 - ih'_3k_nw_0^2 - i\alpha k_n + q_2w_0(k_t^2 - k_n^2) \\
&\quad + iq_4w_0^2k_n + ik_3w_0^2k_n + q'_3w_0^3 + \lambda'w_0^2 \\
M_{22} &= i\omega + h_1k^2 - i(2h_3 - q_1)w_0k_n + 3q_3w_0^2 + \lambda - 1 \\
M_{23} &= -i(2h_3 + q_1)w_0k_t \\
M_{31} &= ih_2k^2k_t + ih'_3w_0^2k_t - i\alpha k_t - 2q_2k_nk_tw_0 + iq_4w_0^2k_t - ik_3w_0^2k_t \\
M_{32} &= i(2h_3 + q_1)w_0k_t \\
M_{33} &= i\omega + h_1k^2 - i(2h_3 - q_1)w_0k_n + q_3w_0^2 + \lambda - 1
\end{aligned} \tag{A.3}$$

Setting the determinant of this matrix to zero, gives the three branches of the dispersion relation $\omega(k)$, which were obtained using Mathematica. The real parts of two of these branches are always negative but one of the branches shows a long wave length instability in a small window $\eta_L < \eta < \eta_C$ below the threshold to collective (homogeneous) motion. In this window, the real part of omega is positive for wavenumbers $0 \leq k \leq k_{max}$. More details on these results can be found in Refs. [13, 54, 16].

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