The Game-Theoretic Formation of Interconnections Between Networks

Ebrahim Moradi Shahrivar and Shreyas Sundaram

Abstract

We introduce a network design game where the objective of the players is to design the interconnections between the nodes of two different networks G_1 and G_2 in order to maximize certain local utility functions. In this setting, each player is associated with a node in G_1 and has functional dependencies on certain nodes in G_2. We use a distance-based utility for the players in which the goal of each player is to purchase a set of edges (incident to its associated node) such that the sum of the distances between its associated node and the nodes it depends on in G_2 is minimized. We consider a heterogeneous set of players (i.e., players have their own costs and benefits for constructing edges). We show that finding a best response of a player in this game is NP-hard. Despite this, we characterize some properties of the best response actions which are helpful in determining a Nash equilibrium for certain instances of this game. In particular, we prove existence of pure Nash equilibria in this game when G_2 contains a star subgraph, and provide an algorithm that outputs such an equilibrium for any set of players. Finally, we show that the price of anarchy in this game can be arbitrarily large.

Index Terms

Interconnected Networks, Network Design, NP-hardness, Nash Equilibria, Price of Anarchy, Hub-and-Spoke.

I. Introduction

There is a growing realization that many large scale networks consist of interconnected subnetworks [1]–[3]. Examples include coupled energy infrastructure and communication networks [4], cyber-physical systems [5], and transportation networks (such as the flight networks of different airlines) [6]. Understanding the structure of large-scale networks and the implications of this structure for the effective functioning of the network has been the subject of many studies throughout the past decade [7], [8]. One approach to investigate this problem is through the framework of *random graphs* where each subnetwork is drawn from a certain probability distribution [9], [10]. For (single) random networks, properties such as connectivity, robustness against structural and dynamical failures, and edge expansion have been widely investigated in the literature [11]–[14], with recent extensions to interconnected networks [4], [15]–[18].

An alternative perspective on understanding the structure of networks is to view the edges as being optimally placed (either by a central designer, or by different decision makers) in order to maximize some given utility function(s) [19]–[22]. The classical literature on optimal network design has predominantly focused on the construction of a single network [7], [23]. In [24], we proposed a multi-layer network formation setting in which, given a network $G = (V, E)$, the network designer aims to find a network $G_1 = (V, E_1)$ such that distances between nodes that are neighbors in G are minimized in G_1; in this context, networks G and G_1 represent different types of relationships between the set of nodes V. We then exploited this setting to formulate a multi-layer network formation game

Ebrahim Moradi Shahrivar is with the Department of Mechanical and Mechatronics Engineering at the University of Waterloo. E-mail: emoradis@uwaterloo.ca.

Shreyas Sundaram is with the School of Electrical and Computer Engineering at Purdue University. E-mail: sundara2@purdue.edu.

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where each layer corresponds to a player that is optimally choosing its edge set in response to the edge sets of the other players.

In this paper, we consider the game-theoretic formation of edges *between* two given networks $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ on two different sets of nodes V_1 and V_2. We assume that there are dependencies between nodes in V_1 and V_2, i.e., some of the nodes in V_1 require connections to (or information from) some of the nodes in V_2 in order to function. These dependencies are captured by a bipartite network $G_I = (V_1 \cup V_2, E_I)$ where $E_I \subseteq V_1 \times V_2$, and an edge $(v_i, v_j) \in E_I$ indicates that $v_i \in V_1$ is dependent on $v_j \in V_2$. We assume that each node in V_1 is a player and builds a set of edges between itself and nodes in V_2 in order to maximize a distance-based utility function. As a motivating abstraction for this problem, consider a cyber-physical system where G_1 is a power network (with the nodes representing substations) and G_2 is a sensor network. Suppose that the sensor nodes are responsible for gathering critical information (e.g., power usage, line failures, etc.) from different geographical regions and this information is required by the power stations that supply electricity to those regions. This setting has been investigated from different perspectives over the past few years [1], [4], [5], [15], [25]–[27] and our results add to this literature by studying the game-theoretic allocation of interconnecting edges between the power and sensor networks. We model dependencies between the substations and sensors (which correspond to nodes in G_1 and G_2, respectively) by the network G_I. Suppose that neighboring nodes in each network are capable of exchanging information with each other. The substation operators wish to construct connections to the sensor network in such a way that they minimize the number of hops required to gather data from their interdependent nodes (where the number of hops is measured with respect to the connections within G_1 and G_2 and the edges constructed between the networks). This leads to an *interconnection network design game* (INDG) with distance utilities where the utility of each player (operator) depends on its own set of edges as well as the set of edges constructed by other players.

The INDG setting also matches the framework studied in [28] for merging two social networks where the goal is to construct a set of edges between the networks such that the integrated network has diameter no more than a fixed value. Besides considering a cost for constructing edges and having a different utility function, the *nodes* are the decision makers in our setting, whereas [28] assumes a central network designer. Distance-based utilities have also been used to study computer networks (where nodes represent the computers and edges are the communication links) [29], [30]. In this case, network G_I models the virtual dependencies among the computers in cluster G_1 and cluster G_2, indicating the set of pairs of nodes that wish to exchange information. The designed interconnection network represents the physical communication network between the two clusters. Yet another application of the INDG with distance utilities arises in studying interconnections between the transportation networks of two countries. We will elaborate on this example in Section IV.

We start our investigation of the INDG by showing that it is NP-hard to find a best response for each player. Despite the NP-hardness of the problem, we characterize some useful properties of the best response which consequently enable us to determine a Nash equilibrium instance for certain cases of the game. Specifically, we study the existence of Nash equilibria in an INDG with distance utilities when network G_2 has a star subgraph (similar to the "hub-and-spoke" structure seen in various transportation networks [31], [32] or in sensor networks with fusion centers [33], [34]) and there is full interdependency between nodes in G_1 and G_2. We show that this setting possesses a Nash equilibrium for any set of players with arbitrary benefit functions and edge costs. We partition the set of players into two sets consisting of high and low edge cost players and show that in any Nash equilibrium, all of the high-cost players that have a low-cost player in their vicinity "free ride" and choose not to construct any interconnections to G_2. At the end, we provide some insights about our results via a simulation involving random network models that have been previously used to capture interdependencies between power and sensor/communication networks [4], [17], [26], [27]. Our simulations suggest that the social welfare of the constructed networks is higher when all of the players have equal cost of constructing edges, compared to the case where they have heterogeneous edge costs.

II. Definitions

An undirected network (or graph) is denoted by $G = (V, E)$ where $V = \{v_1, v_2, \ldots, v_n\}$ is the set of nodes (or vertices) and $E \subseteq \{(v_i, v_j) | v_i, v_j \in V, v_i \neq v_j\}$ denotes the set of edges. If there is an edge between two nodes, they are said to be neighbors. The number of neighbors of a node $v_i \in V$ in graph G is called its degree and is denoted by $\deg_i(G)$. A path from node v_1 to v_k in graph G is a sequence of distinct nodes $v_1 v_2 \cdots v_k$ where there is an edge between each pair of consecutive nodes of the sequence. The length of a path is the number of edges in the sequence. We denote the shortest distance between nodes v_i and v_j in graph G by $d_G(v_i, v_j)$. If there is no path from v_i to v_j, we take $d_G(v_i, v_j) = \infty$. The diameter of the graph G is $\max_{v_i, v_j \in V, v_i \neq v_j} d_G(v_i, v_j)$. A cycle is a path of length two or more from a node to itself. A graph $G' = (V', E')$ is called a subgraph of $G = (V, E)$, denoted as $G' \subseteq G$, if $V' \subseteq V$ and $E' \subseteq E \cap \{V' \times V'\}$. A graph is connected if there is a path from every node to every other node. A subgraph $G' = (V', E')$ of G is a component if G' is connected and there are no edges in G between nodes in V' and nodes in $V \setminus V'$. A graph $G = (V, E)$ is called bipartite if there exist two *disjoint* subsets $V_1, V_2 \subseteq V$ such that $V_1 \cup V_2 = V$ and $E \subseteq V_1 \times V_2$, i.e., G does not have any edge with both endpoints in V_1 or V_2. The set of all possible bipartite graphs with two partitions V_1 and V_2 is denoted by $G^{V_1 \times V_2}$.

III. Distance-Based Utility

Jackson and Wolinsky introduced a canonical problem in network formation which involves distance-based utilities [7]. In their formulation, each node is a decision maker, and chooses its connections to other nodes in the network. In any formed network, each node receives a benefit of $b(k)$ from nodes that are k hops away, where $b : \{1, 2, \cdots, n-1, \infty\} \rightarrow \mathbb{R}_{\geq 0}$ is a real-valued, nonincreasing, nonnegative function (i.e., nodes that are further away provide smaller benefits) and $b(\infty) = 0$. Furthermore, constructing the edge (v_i, v_j) incurs a cost of c to both endpoints v_i and v_j. The total utility that node v_i receives from the constructed network $G = (V, E)$ is

$$u_i(G) = \left(\sum_{v_j \in V : v_i \neq v_j} b(d_G(v_i, v_j)) \right) - c \deg_i(G). \tag{1}$$

Thus, the nodes have to compromise between adding more links (which provides a larger benefit by reducing the distances between nodes) and decreasing the cost by using fewer edges. When $b(\cdot)$ is a strictly decreasing function, there are only a few different kinds of socially optimal (or efficient) networks, depending on the relative values of the link costs and connection benefits: the empty network (for high edge costs), star (for medium edge costs) and the complete network (for low edge costs) [23].

IV. Interconnection Network Design Game

Assume that we are given two arbitrary networks $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. In this paper, we consider a setting in which each node in V_1 constructs a set of edges to nodes in V_2 such that some utility function is maximized. This leads to a game with the nodes of G_1 as the players.

Definition 1: Consider two arbitrary networks $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with $V_1 = \{x_1, \cdots, x_n\}$ and $V_2 = \{y_1, \cdots, y_m\}$. An instance of the *interconnection network design game* (INDG) $\mathcal{G} = (P, (S_i)_{P_i \in P}, (\Psi_i)_{P_i \in P}, G_1, G_2)$ has a set of n players $P = \{P_1, P_2, \cdots, P_n\}$ where player P_i is associated with node $x_i \in V_1$ for $1 \leq i \leq n$. The strategy space of player P_i is $S_i = 2^{\{x_i\} \times V_2}$, i.e., all possible subsets of edges from x_i to nodes in V_2. The action of player P_i is an element of S_i and is denoted by W_i, i.e., W_i is a set of edges from x_i to a certain subset of V_2. By an abuse of notation, we take $B = \cup_{j=1}^n W_j$ to indicate the bipartite graph $B = (V_1 \cup V_2, \cup_{j=1}^n W_j)$. The utility of player P_i is given by a function $\Psi_i : S_1 \times S_2 \times \cdots \times S_n \to \mathbb{R}$, where the j^{th} argument[1] is the action of the j^{th} player for $1 \leq j \leq n$. □

The characteristics of the game and the optimal strategies for each player will depend on the form of the utility functions Ψ_i. In this paper, we consider a modified version of the distance utility function in (1) as the payoff to the players. Specifically, we assume that there are dependencies between nodes in the graphs G_1 and G_2 which is represented by a bipartite network $G_I = (V_1 \cup V_2, E_I)$ with two partitions V_1 and V_2 and $E_I \subseteq V_1 \times V_2$. Let $I_i \subseteq V_2,\ 1 \leq i \leq n$, denote the set of neighbors of $x_i \in V_1$ in the network G_I. Then the objective of player P_i is to find the optimal set of edges to construct to V_2 such that distance between its associated node x_i and the set of nodes in I_i is minimized. In addition to the technological applications that we mentioned in Section I, the INDG can be utilized to model problems in transportation. For instance consider a modified version of the problem studied in [16] where we are given the traffic flow between cities of two different countries C_1 and C_2. Each of these countries has a domestic transportation service which connects its cities and is modeled by networks G_1 and G_2. A city in C_1 and a city in C_2 are said to be interdependent if the traffic flow between them is higher than some threshold, and this interdependency is represented by an edge between them in the network G_I. The players of the game correspond to transportation service planners at each node in C_1, who are faced with the problem of finding the optimal set of routes to establish from their associated city to cities of the country C_2 such that distance between the interdependent cities is minimized. It is clear that the structure of the interconnection between cities inside the countries C_1 and C_2 (modeled as G_1 and G_2) affects the optimal decisions made by the players.

Definition 2: An instance

$$\mathcal{G} = (P, (S_i)_{P_i \in P}, (u_i)_{P_i \in P}, G_1, G_2, G_I)$$

of the game in Definition 1 is said to be an *interconnection network design game with distance utilities* if the utility function of player P_i, $1 \leq i \leq n$, with action $W_i \in S_i$ has the form

$$\begin{aligned} \Psi_i(W_1, \cdots, W_n) &= u_i(\cup_{j=1}^n W_j | G_1, G_2, G_I) \\ &= \left(\sum_{y \in I_i} b_i(d_G(x_i, y)) \right) - c_i |W_i|, \end{aligned} \tag{2}$$

where $G = (V_1 \cup V_2, E_1 \cup E_2 \cup (\cup_{j=1}^n W_j))$. □

As we can see in the utility function $u_i(\cdot)$, only the distances between node x_i and the set of nodes I_i matter. Furthermore, each player has to pay only for his/her constructed edges. The benefit functions $b_i(\cdot)$ are nonnegative, nonincreasing and satisfy $b_i(\infty) = 0$, and all costs c_i are positive, and can be different across players.

We will use W_{-i} to denote the vector of actions of all players except player P_i, and use $\Psi_i(W_i, W_{-i})$ to denote the utility of player P_i with respect to the given vector $(W_1, W_2, \ldots, W_n)$. Based on the definition of the game, we say that a vector of actions $(W_1, W_2, \ldots, W_n)$ is a *Nash equilibrium* if and only if $W_i \in \arg\max_{W \in S_i} \Psi_i(W, W_{-i})$ for all $i \in \{1, 2, \ldots, n\}$. In this case, W_i is said to be a *best response* action to W_{-i} with respect to the utility function Ψ_i. For the rest of this paper, whenever we say INDG, by default we mean an interconnection network design game with distance utilities.

Remark 1: The benefit function $b_i(\cdot)$ can be chosen to capture how quickly (in terms of number of hops) node $v_i \in V_1$ needs to communicate with its interdependent nodes in V_2. For example, consider

[1]The utility function Ψ_i is also a function of G_1 and G_2 which will be omitted from the argument list as long as it is clear from the context.

again the cyber-physical system abstraction described in Section I, where the nodes in V_1 are power substations and nodes in V_2 are sensors that measure certain quantities of interest. If substation $v_i \in V$ is able to tolerate a routing delay of up to k hops from each of the sensors it depends on, but higher routing delays are useless, then the associated benefit function can be chosen as $b_i(1) = \cdots = b_i(k) > 0$, and $b_i(d) = 0$ for $d > k$. Alternatively, if node v_i is able to tolerate any routing delay, but would prefer shorter delays, $b_i(\cdot)$ can be chosen to be an appropriate strictly decreasing function. Our formulation allows different nodes to have different benefit functions and edge costs, encoding heterogeneity in the players of the game. □

V. Characteristics of the Best Responses

In this section, we characterize some important properties of the best response actions for the players. We start by determining the complexity of finding a best response action for the players in the INDG.

A. Complexity

In order to characterize the complexity of finding the best response actions for the papers, we first formulate the *decision problem* corresponding to optimizing the utility (2), as follows.[2]

Definition 3: **Best Response Interconnection (BRI).**
INSTANCE: A given instance

$$\mathcal{G} = (P, (S_i)_{P_i \in P}, (u_i)_{P_i \in P}, G_1, G_2, G_I),$$

of INDG, a player $P_j \in P$, a joint strategy by all other players $W_{-j} = \cup_{i \neq j} W_i$ and a threshold $r \in \mathbb{R}_{>0}$.

QUESTION: Does there exist an action $W_j \in S_j$ for the player P_j such that

$$u_j(W_j \cup W_{-j} | G_1, G_2, G_I) = \left(\sum_{y \in I_j} b_j(d_G(x_j, y)) \right) - c_j |W_j| \geq r,$$

where $G = (V_1 \cup V_2, E_1 \cup E_2 \cup W_j \cup W_{-j})$? □

We now provide the following theorem showing that finding a best response for the players, given arbitrary networks G_1, G_2, G_I, and arbitrary non-increasing benefit functions $b_i(\cdot)$ and edge costs $c_i > 0$ for the players, is impossible in polynomial-time (unless P = NP).

Theorem 1: The Best Response Interconnection problem is NP-hard. □

To prove this theorem, we provide a reduction from the NP-complete Dominating Set Problem [35]. A dominating set of the network $G_d = (V_d, E_d)$ is a subset $S \subseteq V_d$ such that for all $u \in V_d \setminus S$, u has a neighbor in the set S.

Definition 4: **Dominating Set Problem.**
INSTANCE: Network $G_d = (V_d, E_d)$ and positive integer $k \leq |V_d|$.
QUESTION: Does the network G_d have a dominating set S with $|S| \leq k$? □

We are now in place to prove Theorem 1.

Proof of Theorem 1: Given an instance of the dominating set problem with $G_d = (V_d, E_d)$ and k, define an instance of the BRI problem with $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$ and $G_I = (V_1 \cup V_2, E_I)$ as follows

$$\begin{aligned} V_1 = \{x_1\},\ V_2 = V_d,\ E_1 = \phi,\ E_2 = E_d,\ E_I = V_1 \times V_2 \\ b_1(3) < b_1(1) - c_1 < b_1(2) \\ r = k(b_1(1) - c_1) + (|V_2| - k) b_1(2). \end{aligned} \quad (3)$$

[2]Decision problems are those with "yes" or "no" answers, and form the basis of the complexity classes P and NP. Since optimization problems can be solved by repeatedly solving a corresponding decision problem (e.g., by determining whether there is a solution that provides a utility larger than a certain threshold), showing that the decision problem is NP-hard is sufficient to show NP-hardness of the optimization problem. We refer to standard textbooks such as [35] for more details and background on complexity theory.

For example $c_1 = 2, b_1(1) = 4, b_1(2) = 3, b_1(3) = 1$ and $b_1(k) = 0$ for all $k \geq 4$ satisfies the above conditions. In the above instance of the BRI, there is only one node in V_1 (with associated player P_1), and this player is fully dependent on all nodes in V_2 (i.e., we have $I_1 = V_2$ in (2)). Hence, the BRI problem is to determine whether P_1 has an action W_1 such that $u_1(W_1|G_1, G_2, G_I) \geq r$.

The above instance of the BRI problem can be constructed in polynomial time. In the rest of the proof, we show that the answer to the above instance of the BRI problem is "yes" if and only if the answer to the given instance of the Dominating Set Problem is "yes".

Suppose that the graph $G_2 = G_d$ has a dominating set $S \subset V_2$ with $|S| \leq k$ and thus the answer to the given instance of the Dominating set problem is "yes". Then by defining $W_1 = \{(x_1, v)|v \in S\}$, the distance between node x_1 and any node in V_2 is at most 2. Since $|W_1| \leq k$,

$$
\begin{aligned}
u_1(W_1|G_1, G_2, G_I) &= |W_1|(b_1(1) - c_1) + (|V_2| - |W_1|)b_1(2) \\
&= |W_1|(b_1(1) - c_1) + (|V_2| - k)b_1(2) + (k - |W_1|)b_1(2) \\
&\geq |W_1|(b_1(1) - c_1) + (|V_2| - k)b_1(2) + (k - |W_1|)(b_1(1) - c_1) \\
&= r.
\end{aligned}
$$

Therefore, the answer to the constructed instance of the BRI problem in (3) is "yes" as well.

Next suppose that the answer to the defined instance of BRI in (3) is "yes", i.e., there exists a $W_1 \in S_1$ such that $u_1(W_1|G_1, G_2, G_I) \geq r$. If there is a node $v \in V_2$ such that $d_G(x_1, v) \geq 3$, we can add the edge (x_1, v) to W_1; this would increase the benefit of the network by at least $b_1(1) - b_1(3)$ and incur a cost of c_1. Since $b_1(1) - b_1(3) > c_1$, this would increase the utility of P_1. Thus without loss of generality we can take the distance between node x_1 and any node in V_2 to be at most 2 under the constructed edge set W_1.

Consider the set of nodes $S \subseteq V_2$ that are incident to at least one edge in W_1, i.e., $S = \{v \in V_2|(x_1, v) \in W_1\}$. All of the nodes in $V_2 \setminus S$ are connected to at least one of the nodes in S due to the assumption that the distance between any node in V_2 and node x_1 is at most 2. Thus S is a dominating set of the network G_2. On the other hand, the assumption that $u_1(W_1|G_1, G_2, G_I) \geq r$ yields

$$
\begin{aligned}
0 &\leq u_1(W_1|G_1, G_2, G_I) - r \\
&= |W_1|(b_1(1) - c_1) + (|V_2| - |W_1|)b_1(2) - r \\
&= (|W_1| - k)(b_1(1) - c_1) + (k - |W_1|)b_1(2) \\
&= (|W_1| - k)(b_1(1) - c_1 - b_1(2)).
\end{aligned}
$$

Since $b_1(1) - c_1 < b_1(2)$, we must have that $|W_1| \leq k$. Hence, $|S| = |W_1| \leq k$. This means that network G_2 has a dominating set of size less than k. Thus the answer to the given instance of the Dominating Set Problem is "yes". ■

Given that BRI is a NP-hard problem, finding best response actions in the INDG with distance utilities is nontrivial in general. In the next section, we provide some properties of the best response actions that will be helpful in characterizing the best responses of the players in certain cases.

B. Properties of the Best Response

Lemma 1: Let W_j be a best response to W_{-j} in the INDG

$$\mathcal{G} = (P, (S_i)_{P_i \in P}, (u_i)_{P_i \in P}, G_1, G_2, G_I).$$

Then we have that

1) $|W_j| \leq |I_j|$.
2) If $b_j(1) > b_j(2)$, then $|W_j| = |I_j|$ if and only if $W_j = \{(x_j, y)|y \in I_j\}$.

□

Proof: Let $G = (V_1 \cup V_2, E_1 \cup E_2 \cup W_j \cup W_{-j})$. We use contradiction to prove the first statement. Assume that $|W_j| > |I_j|$, then

$$\begin{aligned} u_j(W_j \cup W_{-j}|G_1, G_2, G_I) &= \left(\sum_{y \in I_j} b_j(d_G(x_j, y)) \right) - c_j|W_j| \\ &\leq |I_j| b_j(1) - c_j|W_j| \\ &< |I_j|(b_j(1) - c_j) \\ &= u_j(W'_j \cup W_{-j}|G_1, G_2, G_I), \end{aligned}$$

where $W'_j = \{(x_j, y)|y \in I_j\}$. Thus W_j is not a best response to W_{-j} which is a contradiction to the assumption of the lemma.

To prove the second statement, note that if $W_j = \{(x_j, y)|y \in I_j\}$, then $|I_j| = |W_j|$. Thus we only have to show that when $b_j(1) > b_j(2)$, if $|I_j| = |W_j|$, then $W_j = \{(x_j, y)|y \in I_j\}$. Assume by way of contradiction that there exists $y^* \in I_j$ such that $(x_j, y^*) \notin W_j$. This means that $d_G(x_j, y^*) \geq 2$ and thus

$$\begin{aligned} u_j(W_j \cup W_{-j}|G_1, G_2, G_I) &< |I_j| b_j(1) - c_j|I_j| \\ &= u_j(W'_j \cup W_{-j}|G_1, G_2, G_I), \end{aligned}$$

where, again, $W'_j = \{(x_j, y)|y \in I_j\}$. This is a contradiction and thus we must have $\{(x_j, y)|y \in I_j\} \subseteq W_j$. We also know that $|W_j| \leq |I_j|$ and therefore, have the required result. ■

The next lemma characterizes a best response action of the players when the cost of constructing edges is less than a certain threshold. The proof follows the same reasoning as the proof in [23] for the formation of (single) networks under low edge costs.

Lemma 2: Let W_j be a best response to W_{-j} in the INDG

$$\mathcal{G} = (P, (S_i)_{P_i \in P}, (u_i)_{P_i \in P}, G_1, G_2, G_I).$$

If $c_j < b_j(1) - b_j(2)$, then $W_j = \{(x_j, y)|y \in I_j\}$. Furthermore, if $c_j = b_j(1) - b_j(2)$, then $W_j = \{(x_j, y)|y \in I_j\}$ is a best response action for player P_j. □

Proof: Suppose that $y^* \in I_j$ and $(x_j, y^*) \notin W_j$. Then $b_j(d_G(x_j, y^*)) \leq b_j(2)$ where $G = (V_1 \cup V_2, E_1 \cup E_2 \cup W_j \cup W_{-j})$. Adding the edge (x_j, y^*) to W_j increases the utility of W_j by at least $b_j(1) - c_j - b_j(2) > 0$ which contradicts the assumption that W_j is a best response and thus $(x_j, y^*) \in W_j$. Hence $\{(x_j, y)|y \in I_j\} \subseteq W_j$. By Lemma 1, we know that $|W_j| \leq |I_j|$ and therefore, $W_j = \{(x_j, y)|y \in I_j\}$.

For the case that $c_j = b_j(1) - b_j(2)$, note that adding the edge (x_j, y^*) to W_j does not decrease the utility of W_j and thus as in the above argument, $W_j = \{(x_j, y)|y \in I_j\}$ is a best response action for P_j. ■

The next result gives an upper-bound on the maximum number of edges that a player P_j with $c_j > b_j(1) - b_j(2)$ will form in a Nash equilibrium.

Lemma 3: Let W_j be a best response to W_{-j} in the INDG

$$\mathcal{G} = (P, (S_i)_{P_i \in P}, (u_i)_{P_i \in P}, G_1, G_2, G_I).$$

If $b_j(1) - b_j(2) < c_j$, then $|W_j| \leq |D|$, where D denotes the smallest dominating set of the network G_2. □

Proof: If $|I_j| \leq |D|$, we have the result by the first statement of Lemma 1. Thus consider the case that $|I_j| > |D|$. Assume by way of contradiction that $|W_j| > |D|$. Let $G = (V_1 \cup V_2, E_1 \cup E_2 \cup W_j \cup W_{-j})$. Then

$$\begin{aligned} u_j(W_j \cup W_{-j}|G_1, G_2, G_I) &\leq |W_j|(b_j(1) - c_j) + (|I_j| - |W_j|)b_j(2) \\ &= |D|(b_j(1) - c_j) + (|W_j| - |D|)(b_j(1) - c_j) + (|I_j| - |W_j|)b_j(2) \\ &< |D|(b_j(1) - c_j) + (|I_j| - |D|)b_j(2) \\ &= u_j(W'_j \cup W_{-j}|G_1, G_2, G_I), \end{aligned}$$

where $W'_j = \{(x_j, y)|y \in D\}$. Thus W'_j produces more utility than W_j for player P_j which is a contradiction to the assumption that W_j is a best response to W_{-j}. ■

We will apply Lemma 3 later in Section VI to determine a Nash equilibrium instance of the INDG when G_2 has a star subgraph. The next lemma provides a threshold on the edge costs of the players in order for them to have nonempty actions.

Lemma 4: Let W_j be a best response to W_{-j} in the INDG

$$\mathcal{G} = (P, (S_i)_{P_i \in P}, (u_i)_{P_i \in P}, G_1, G_2, G_I).$$

If $c_j > b_j(1) + (|I_j| - 1)b_j(2)$, then $W_j = \phi$, i.e., it is not beneficial for the player P_j to construct any edges incident to its associated node x_j. □

Proof: Assume by way of contradiction that $|W_j| \geq 1$. Given $G = (V_1 \cup V_2, E_1 \cup E_2 \cup W_j \cup W_{-j})$, we have

$$\begin{aligned} u_j(W_j \cup W_{-j}|G_1, G_2, G_I) &\leq |W_j|(b_j(1) - c_j) + (|I_j| - |W_j|)b_j(2) \\ &= b_j(1) - c_j + (|W_j| - 1)(b_j(1) - c_j) + (|I_j| - |W_j|)b_j(2) \\ &\leq b_j(1) - c_j + (|I_j| - 1)b_j(2) < 0, \end{aligned}$$

where in the above, we are using the fact that $b_j(1) - c_j < b_j(2)$ by the assumption of the lemma. Therefore, we must have that $|W_j| = 0$ which yields the required result. ■

In the next result, we propose a condition under which a player disregards the network constructed by another player when considering the best response. We define the *R-radius* of a player $P_i \in P$ with $b_i(1) - c_i > 0$ as the minimum integer $R_i > 0$ (or ∞) such that $b_i(1) - c_i > b_i(R_i + 1)$.

Lemma 5: Consider two players $P_i, P_j \in P$ with *R-radii* R_i and R_j, respectively. For a given instance of INDG

$$\mathcal{G} = (P, (S_i)_{P_i \in P}, (u_i)_{P_i \in P}, G_1, G_2, G_I),$$

assume that W_i and W_j are best response actions to W_{-i} and W_{-j}, respectively. If $d_{G_1}(x_i, x_j) \geq R_i + R_j - 1$, then the actions of the players P_i and P_j are such that shortest paths from nodes x_i and x_j to the nodes that they depend on in V_2 are node disjoint in G_1. □

Proof: The idea behind the proof stems from the fact that for any two nodes $x_i, x_j \in V_1$ with $d_{G_1}(x_i, x_j) \geq R_i + R_j - 1$, there does not exist any node $x_k \in V_1$ that simultaneously has distance less than R_i to x_i and less than R_j to x_j. To formally prove the lemma, consider $\{(x_i, y_i), (x_j, y_j)\} \subseteq E_I$. By way of contradiction, assume that the shortest paths from x_i to y_i and x_j to y_j intersect at a node $x_k \in V_1$. Without loss of generality, let $d_{G_1}(x_i, x_k) \geq R_i$. This means that $d_G(x_i, y_i) \geq R_i + 1$ where $G = (V_1 \cup V_2, E_1 \cup E_2 \cup W_i \cup W_{-i})$. Now consider $W'_i = W_i \cup \{(x_i, y_i)\}$ as a modified action of player P_i. This new action will increase the utility of player P_i by at least $b_i(1) - c_i - b_i(R_i + 1) > 0$, which is a contradiction to the assumption that W_i is a best response to W_{-i}. ■

The following example illustrates the application of Lemma 5 in determining a Nash equilibrium of the INDG.

Example 1: Consider networks $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ depicted in Fig. 1a with the given dependency network G_I between them (shown by dashed edges). Assume that $b_i(3) > b_i(1) - c_i > b_i(4)$

for $i \in \{1,6\}$ which yields $R_1 = R_6 = 3$. Nodes x_1 and x_6 correspond to the players P_1 and P_6, respectively. Note that since all of the other nodes $x_i \in V_1 \setminus \{x_1, x_2\}$ have $I_i = \emptyset$, their associated players do not construct any edges in any Nash equilibrium by Lemma 1. Both x_1 and x_2 are dependent on all nodes in G_2, as illustrated for x_1 in Fig. 1b. The distance between nodes x_1 and x_6 in G_1 is

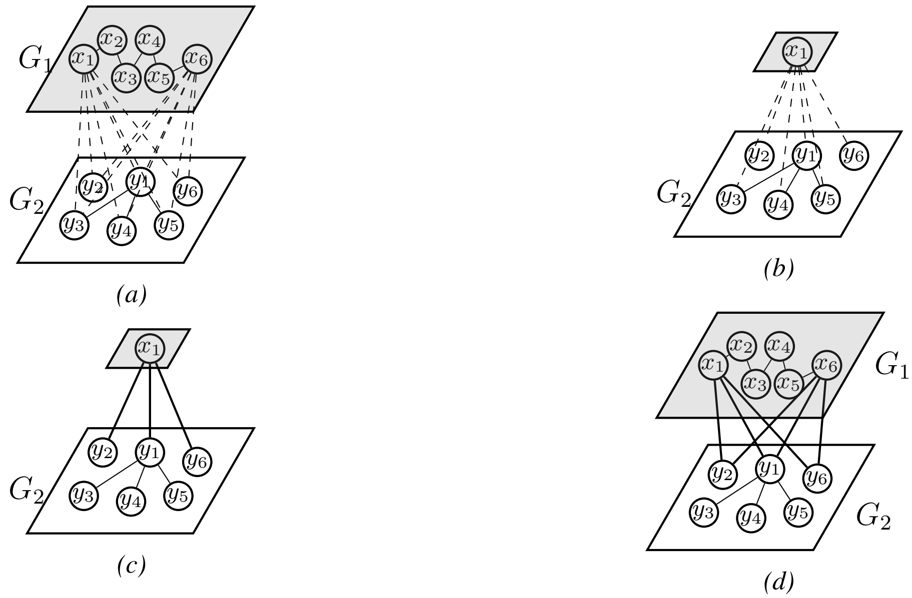


Fig. 1: (a) Networks G_1 and G_2 with interdependencies shown by dashed edges. (b) Interdependencies of player P_1 with nodes in G_2. (c) Best response action of P_1 (d) A Nash equilibrium instance.

5 and thus the networks constructed by players P_1 and P_6 will be such that the shortest paths from x_1 to the nodes in G_2 are node disjoint (in G_1) from the shortest paths from x_6 to the nodes in G_2, by Lemma 5. Fig. 1c demonstrates a best response for player P_1. Using the optimal action of P_1 and Lemma 5, we can determine a Nash equilibrium as shown in Fig. 1d.

VI. Nash Equilibrium of INDG for Networks Containing Star Subgraphs

With our results on best responses in hand, we now turn our attention to proving the existence of a Nash equilibrium. While it is challenging to show this for general G_1, G_2 and G_I, we will prove that the INDG always has a Nash equilibrium when G_2 contains a star subgraph,[3] and $G_I = (V_1 \cup V_2, E_I)$ is the complete bipartite network, i.e., $E_I = V_1 \times V_2$. We allow G_1 to be arbitrary. Without loss of generality, let $y_1 \in V_2$ be a hub node in $G_2 = (V_2, E_2)$, i.e., $(y_1, y) \in E_2 \; \forall y \in V_2 \setminus \{y_1\}$. As we illustrate later, the presence of heterogeneous players (captured by individual benefit functions and edge costs) along with the arbitrary structure of G_1 leads to non-trivial interconnection networks in equilibrium, even under the above assumptions on G_2 and G_I.

To develop our results, we partition the set of players P into two sets: *high-cost players*

$$S_H = \{P_i \in P | b_i(1) - b_i(2) < c_i\}, \tag{4}$$

and *low-cost players*

$$S_L = \{P_i \in P | b_i(1) - b_i(2) \geq c_i\}. \tag{5}$$

Recall that we assumed $V_1 = \{x_1, \cdots, x_n\}$ and $V_2 = \{y_1, \cdots, y_m\}$. For the rest of this section, we denote the number of players $|P|$ by $|V_1| = n$ and the number of nodes in $|V_2|$ by m.

[3]Such networks can be used to represent, for example, sensor networks that have a fusion center, or transportation networks that have a "hub-and-spoke" structure [6], [31]–[33].

We begin our analysis in this section with the following useful corollary of Lemma 2, which determines a best response action for the low-cost players.

Corollary 1: Assume that $P_i \in S_L$. Then $W_i = \{(x_i, y) | y \in V_2\}$ is a best response action for player P_i regardless of the actions of the other players.

For the rest of this section, we assume that low-cost players always set their action according to the best response given by Corollary 1. In the next proposition, we discuss the best responses of high-cost players when there is a low-cost player in their neighborhood. We define the *L-radius* of a player $P_i \in S_H$ as the maximum nonnegative integer L_i such that[4]

$$b_i(1) - c_i + (m-1)b_i(2) \leq m b_i(L_i + 1). \tag{6}$$

Proposition 1: Let P_i be a high-cost player. Suppose that there exists a low-cost player $P_j \in S_L$ such that the distance between x_j and x_i is less than $L_i + 1$ (i.e., $d_{G_1}(x_i, x_j) < L_i + 1$), where L_i is the L-radius of player P_i. Then, if P_j has constructed edges to all nodes in V_2, the empty network is a best response action for player P_i.

Proof: Let W_i denote a best response action of the player $P_i \in S_H$ with respect to W_{-i}. Node x_i has distance $d \leq L_i$ to x_j which is associated with a low-cost player $P_j \in S_L$ that is connected to all of the nodes in V_2. Now assume that $W_i \neq \phi$. Then we have

$$\begin{aligned}
\Psi_i(W_1, \cdots, W_n) &= u_i(\cup_{j=1}^n W_j | G_1, G_2, G_I) \\
&= \left(\sum_{y_j \in I_i} b_i(d_G(x_i, y_j)) \right) - c_i |W_i| \\
&\leq |W_i|(b_i(1) - c_i) + (m - |W_i|) b_i(2) \\
&= b_i(1) - c_i + (|W_i| - 1)(b_i(1) - c_i) + (m - |W_i|) b_i(2) \\
&\leq b_i(1) - c_i + (|W_i| - 1) b_i(2) + (m - |W_i|) b_i(2) \\
&= b_i(1) - c_i + (m-1) b_i(2) \\
&\leq m b_i(L_i + 1) \leq m b_i(d+1).
\end{aligned} \tag{7}$$

Therefore, player P_i can increase its utility by changing its action to be the empty network and connecting to the nodes it depends on in G_2 via edges constructed by the low-cost player P_j. ■

The above result shows that the existence of a low-cost player in the proximity of a high-cost player will make the high-cost player a *free rider* in any Nash equilibrium, i.e., the high-cost player does not construct any edges, and instead benefits from the low-cost player's edges.

Remark 2: Note that Corollary 1 and Proposition 1 do not rely on G_2 having a star subgraph, and hold whenever the low-cost players have dependencies on all nodes in G_2. □

Corollary 2: Assume that $P_i \in S_H$. Then for any best response action of the player P_i, node x_i is either connected to only the center of a star subgraph in G_2 (e.g., node $y_1 \in V_2$) or it does not have any edges.

Proof: Since G_2 has a star subgraph, the size of its smallest dominating set is 1 (e.g., the center of the star, y_1). Therefore, by Lemma 3, we must have that $|W_i| \leq 1$. Furthermore, the proof of Lemma 3 shows that with a single edge, $W_i = \{(x_i, y_1)\}$ produces the highest possible utility for P_i. Proposition 1 gives an instance of the situation when $W_i = \phi$. ■

Although Corollary 2 limits the set of best response actions of a high-cost player to two actions (namely, connect to a hub in G_1 or not), it is not clear whether this game has a pure strategy Nash equilibrium for any set of players with arbitrary network G_1, edge cost c_i and benefit function $b_i(\cdot)$.

[4] A player P_i with $b_i(1) - c_i + (m-1)b_i(2) < 0$ is defined to have $L_i = \infty$, and his/her best response action is always the empty network by Lemma 4.

We prove existence of a pure Nash equilibrium in this game by providing an algorithm that outputs such an equilibrium. To do this, we first define an index r_i for each high-cost player $P_i \in S_H$, called the *r-radius*. The r-radius of player P_i with benefit function $b_i(\cdot)$ and edge cost c_i is defined as the maximum nonnegative integer r_i such that[5]

$$b_i(1) - c_i + (m-1)b_i(2) \leq b_i(r_i+1) + (m-1)b_i(r_i+2). \tag{8}$$

Note that by the above definition, $L_i \geq r_i$ where L_i was defined in (6). For a given r-radius r_i, we define the r_i-neighborhood of node x_i as

$$N_i = \{x_j | P_j \in S_H \text{ and } d_{G_1}(x_i, x_j) \leq r_i\}. \tag{9}$$

If a high-cost player P_i has another high-cost player P_j with a single edge to a hub node in V_2 such that $x_j \in N_i$, then player P_i is better off with no edge to V_2. This statement is also true if P_j is a low-cost player by Proposition 1 and the fact that $r_i \leq L_i$. The following proposition formally states these ideas.

Proposition 2: Let P_i be a high-cost player with r-radius r_i. Suppose that there exists a player P_j such that x_j is connected to a hub node in V_2 and $d_{G_1}(x_i, x_j) \leq r_i$. Then the empty network is a best response action for the player P_i with respect to W_{-i}.

The results that we provided in this section enable us to give an algorithm that outputs a Nash equilibrium instance of the interconnection network design game with distance utilities for an arbitrary network G_1 and arbitrary benefit function and cost of edges.

Theorem 2: Assume that network G_2 has a star subgraph and G_I is a complete bipartite graph with partitions V_1 and V_2. Then the interconnection network design game with distance utilities in Definition 2 always has a pure strategy Nash equilibrium. □

Proof: We prove this theorem by providing an algorithm that outputs a Nash equilibrium instance of the game given by a set of actions $(W_1, W_2, \cdots, W_n)$ for the players. The steps of the algorithm are as follows:

1) Connect nodes associated to the low-cost players to all of the nodes in V_2.
2) Take S_H^∞ as the set of all high-cost players with $r_i = \infty$ (which includes all of the players with $L_i = \infty$). Set the actions of all players $P_i \in S_H^\infty$ to be the empty network, i.e., $W_i = \emptyset$.
3) Determine the set S_H^L which consists of all high-cost players that have a low-cost player in their L_i-neighborhood where L_i denotes the L-radius, i.e.,

$$S_H^L = \{P_i \in S_H | \exists P_j \in S_L \text{ such that } d_{G_1}(x_i, x_j) \leq L_i\}.$$

   Set the actions of these players to be the empty network (by Proposition 1).
4) Let $Q \subseteq S_H \setminus (S_H^L \cup S_H^\infty)$ be the set of players whose actions have not been determined yet. If the set Q is empty, exit the algorithm. Otherwise, let $P_i \in Q$ be the player with the lowest r-radius. Connect x_i (i.e., the node associated to P_i) via a single edge to a central node in G_2. Remove P_i from Q.
5) Set the action of all high-cost players $P_j \in Q$ with $x_i \in N_j$ to the empty network and remove them from the set Q. Recall that N_j is the r_j-neighborhood of player P_j and was defined in (9).
6) Return to step 4.

We now argue that the output of the above algorithm is in fact a Nash equilibrium. Since the actions of low-cost players are in accordance with Corollary 1, they are best responses to the other actions. Next, note that if a high cost player $P_i \in S_H^\infty$ wants to take a best response action with respect to the actions of the other players, he/she has to choose the empty network by Lemma 4; this is the action

[5] A player P_i with $b_i(1) - c_i + (m-1)b_i(2) < 0$ is defined to have $r_i = \infty$, and his/her best response action is always the empty network by Lemma 4.

that our proposed algorithm assigns to these players. The same is true for all high cost players with a low-cost player in their L_i-neighborhood, according to Proposition 1. Thus we only need to prove optimality of the actions of the remaining players which are determined through steps 4 to 6. Note that all of the remaining players have $r_i < \infty$.

Consider the set $S_H \setminus (S_H^L \cup S_H^\infty) = \{P_{i_1}, \cdots, P_{i_t}\}$ and assume without loss of generality that $r_{i_1} \leq r_{i_2} \leq \cdots \leq r_{i_t}$. Under the algorithm, the action of P_{i_1} is $W_{i_1} = \{(x_{i_1}, y_1)\}$. We know that there is no low-cost player in the L_{i_1}-neighborhood of P_{i_1}, since $P_{i_1} \in S_H \setminus (S_H^L \cup S_H^\infty)$. Similarly, there is no high-cost player in $S_H^L \cup S_H^\infty$ with a nonempty action in the N_{i_1} neighborhood of P_{i_1}. Now assume that there exists a player $P_{i_j} \in S_H \setminus (S_H^L \cup S_H^\infty)$ with $j > 1$ and $|W_{i_j}| = 1$. We have to show that $x_{i_j} \notin N_{i_1}$, since otherwise the action of player P_{i_1} will not be optimal. In step 5 of the algorithm, we set the actions of all players P_{i_q} such that $x_{i_1} \in N_{i_q}$ to the empty network and remove them from the set Q. Hence, we must have that $x_{i_1} \notin N_{i_j}$, i.e., $d_{G_1}(x_{i_1}, x_{i_j}) > r_{i_j} \geq r_{i_1}$. Therefore, $x_{i_j} \notin N_{i_1}$ and thus the action of P_{i_1} is optimal.

The actions of all players that construct the empty network in step 5 are optimal, by Proposition 2.

Finally, consider any player P_{i_j} with $|W_{i_j}| = 1$ and $j > 1$. We know that $x_{i_k} \notin N_{i_j}$ for any $k < j$ with $|W_{i_k}| = 1$; otherwise the action of player P_{i_j} would have been set to the empty network in step 5 of the algorithm after assigning the action of player P_{i_k} in step 4. Moreover, by a reasoning similar to the argument for optimality of P_{i_1}'s action, we can show that for any player P_{i_t} with $t > j$ and $|W_{i_t}| = 1$, we have $x_{i_t} \notin N_{i_j}$. Therefore, the action of player P_{i_j} is a best response.

Thus, each player is playing their best response given the actions of the rest of the players, which implies that the given vector of actions is a Nash equilibrium. ■

Remark 3: Note that the algorithm provided in the proof of the above theorem corresponds to *sequential best response dynamics* by the players, where they move in the order specified by the algorithm. □

The following example illustrates the steps of the algorithm, and the corresponding Nash equilibrium.

Fig. 2: (a) Network G_1 (b) Network G_2.

Example 2: Consider two networks $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ depicted in Figures 2a and 2b with complete dependencies between nodes in G_1 and G_2. Assume that the cost of constructing edges is equal to 1 for all of the players, i.e., $c_i = 1, 1 \leq i \leq 9$. Suppose the benefit functions for the players take the values given in Table I. Based on these values, player 7 is a low-cost player (since $c_7 < b_7(1) - b_7(2)$) and the rest of the players have high edge costs, i.e.,

$$S_L = \{P_7\},$$
$$S_H = \{P_1, P_2, \cdots, P_6, P_8, P_9\}.$$

The corresponding values of the radii r_i and L_i (given by inequalities (8) and (6), respectively) are shown in the table. We now follow the algorithm prescribed in the proof of Theorem 2.

1) P_7 is the only low-cost player, and thus we connect x_7 to all of the nodes in G_2, i.e., $W_7 = \{(x_7, y_i) | 1 \leq i \leq 7\}$.

	$b_i(1)$	$b_i(2)$	$b_i(3)$	$b_i(4)$	$b_i(5)$	L_i	r_i
P_1	1.5	1.3	1.2	1.1	0.2	2	1
P_2	1.2	0.8	0.5	0.2	0	1	0
P_3	1.1	0.9	0.1	0	0	1	0
P_4	0.9	0.8	0.7	0.5	0.2	2	1
P_5	1.2	1.1	0.9	0.2	0.1	1	0
P_6	1.3	1	0.5	0.4	0.3	1	0
P_7	3	1	0.5	0.5	0.4	NA	NA
P_8	1.2	0.8	0.7	0.5	0.4	1	1
P_9	1.2	1.1	1.1	1	0.2	3	2

TABLE I: Benefit function, r-radius and L-radius of the players in Example 2.

2) For each node v_i whose distance to the low-cost player x_7 is at most L_i, we set that player's action to be empty. These nodes are given by $\{P_1, P_3, P_8, P_9\}$, and thus $W_1 = W_3 = W_8 = W_9 = \emptyset$.
3) The second player has the lowest r-radius among the remaining players and thus we set its action to $W_2 = \{(x_2, y_1)\}$. Since $\nexists P_j, j \in \{4, 5, 6\}$ such that $x_2 \in N_j$, we must choose the next player with the lowest r_i. Recall that N_j was defined in (9).
4) Player P_5 with $r_5 = 0$ has the lowest r-radius among the remaining players. Thus we set $W_5 = \{(x_5, y_1)\}$. Again since $\nexists P_j, j \in \{4, 6\}$ such that $x_5 \in N_j$, we must choose the next player with the lowest r_i.
5) Finally, we choose player P_6 with $r_6 = 0$ and set its action to $W_6 = \{(x_6, y_1)\}$. Due to the fact that $x_6 \in N_4$, we set the action of player P_4 to the empty network, i.e., $W_4 = \phi$.

Fig. 3 demonstrates the output of the algorithm given in the proof of Theorem 2 when networks G_1 and G_2 depicted in Fig. 2b are given as input.

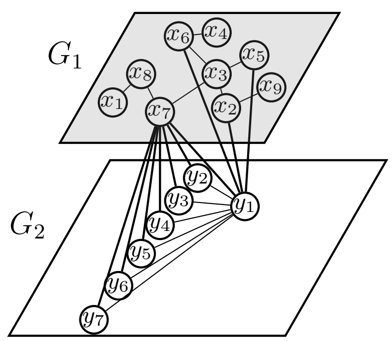


Fig. 3: Networks G_1 and G_2 with the Nash equilibrium interconnection network G_p connecting them. Network G_p was produced by the algorithm given in the proof of Theorem 2.

As one can see, the role of the hub nodes is crucial in the structure of the Nash equilibrium interconnection networks. While low edge cost players connect their associated nodes in G_1 to all of the nodes in the network G_2 (and thus themselves become hubs), the remaining high-cost players either choose (I) the empty network and connect via edges constructed by other players or (II) they directly connect to the hub node in network G_2.

A. Price of Anarchy

The concept of "price of anarchy" (PoA) was introduced in [36] to measure how selfish behavior of the individual players degrades the efficiency of the output in a non-cooperative game. Given a strategy $W = (W_1, W_2, \ldots, W_n)$ taken by the players and $T(W) = \sum_{i=1}^{n} u_i(W_i \cup W_{-i} | G_1, G_2, G_I)$ as the social welfare function, PoA is defined as

$$PoA = \frac{\max_{W \in S} T(W)}{\min_{W \in E} T(W)},$$

where S denotes the joint strategy space and $E \subseteq S$ is the set of strategies in Nash equilibrium.

We show via the following example that the PoA can be arbitrarily large in the INDG.

Example 3: Consider two networks $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, each containing star subgraphs centered on nodes $x_1 \in V_1$ and $y_1 \in V_2$, respectively, i.e.,

$$\{(x_1, x_i)|x_i \in V_1 \setminus \{x_1\}\} \subseteq E_1$$
$$\{(y_1, y_i)|y_i \in V_2 \setminus \{x_2\}\} \subseteq E_2.$$

Suppose that we have full dependencies between nodes in V_1 and V_2, i.e., $G_I = (V_1 \cup V_2, V_1 \times V_2)$. Assume that all of the players $P_i, 1 \le i \le |V_1|$ in the INDG have $c_i = 2.1, b_i(1) = 1$ and $b_i(2) = b_i(3) = 1/(|V_2|-1)$, where $|V_1| > |V_2| > 1$. This means that $b_i(1) - c_i + (|V_2|-1)b_i(2) = -0.1 < 0$ for all of the players $P_i \in P$ and thus by Lemma 4, none of the players constructs any edges. Therefore, the social welfare value is $T(W) = 0$ for all strategies in Nash equilibrium.

Now consider the socially optimal interconnection strategy, i.e., the strategy that maximizes $T(\cdot)$. For the strategy $W^\star = (W_1^\star, W_2^\star, \dots, W_n^\star)$ where $W_1^\star = \{(x_1, y_1)\}$ and $W_i^\star = \phi, i \neq 1$, we have that

$$\begin{aligned} T(W^\star) &= \sum_{i=1}^{n} u_i(W_i^* \cup W_{-i}^* | G_1, G_2, G_I) \\ &= b_1(1) - c_1 + (|V_2| - 1)b_1(2) + \sum_{j=2}^{|V_1|} (b_j(2) + (|V_2| - 1)b_j(3)) \\ &= -0.1 + \frac{(|V_1| - 1)|V_2|}{|V_2| - 1} > 0. \end{aligned}$$

Therefore, the network that maximizes the social welfare function has a nonzero utility and thus PoA is trivially infinite.

Remark 4: The network $G_{SocOpt} = \cup_{i=1}^{n} W_i$ that maximizes the social utility function $T(W)$ is called the socially optimal network. Similar to the proof of the Theorem 1, one can show that finding the socially optimal network is an NP-hard problem. □

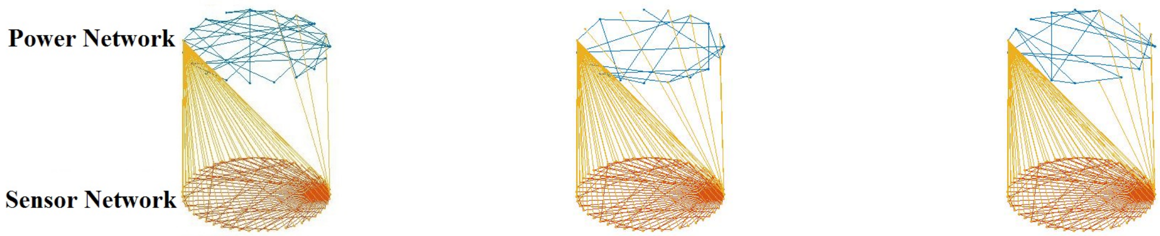
Remark 5: When all players have low costs for constructing edges, one can show that the Nash equilibrium networks are also socially optimal. □

VII. Simulation and Numerical Analysis

In this section, we use our algorithm from the previous section to investigate the Nash equilibria that arise in large interconnected cyber-physical systems consisting of a power network and a sensor/communication network (as described in Section I). Here, we use the same experimental setup as [26], where networks G_1 and G_2 have a synthetic scale-free (SF) structure. Such scale-free networks have also been used to model power and communication networks in many of the other works in this area [1], [3], [27].

In order to provide comparisons and insights, we also consider the cases that the power network is an Erdos-Renyi (ER) random network, or a geometric random network. In ER random networks, each edge is placed independently with a fixed probability [9]. Although ER networks are not typically representative of real-world networks, they are a common baseline model for studying large scale networks [9], [23]. Geometric random (GR) networks consist of a set of spatially distributed nodes, and there is an edge between two nodes if their distance (in some metric) is less than a given threshold [37].

We consider the case that there are 500 power substations that supply electricity to 5000 regions. Therefore, there are 500 and 5000 nodes in the power (G_1) and the sensor (G_2) networks, respectively. We create a hub node in the sensor network by connecting an arbitrary node in V_2 to all of the other



(a) Scale-Free Network *(b) Erdos-Renyi Network* *(c) Geometric Random Network*

Fig. 4: Output of the proposed algorithm in Theorem 2 when the communication network has SF structure and the power network has (a) SF, (b) ER and (c) GR structure. Note that in order to keep the figures clean and the structures visible, we have decreased the number of nodes in the power and communication networks to 20 and 50 nodes, respectively.

nodes in V_2. Furthermore, we assume that the interdependency network G_I is a full bipartite network, i.e., $E_I = V_1 \times V_2$. We consider the same benefit function for all of the players in the game, i.e., $b_i(\cdot) = b(\cdot)$ for all $1 \leq i \leq 500$. In our simulation, we set $b(1) = 1.2, b(2) = 0.7, b(3) = 0.6, b(4) = 0.5, b(5) = 0.3, b(6) = 0.2$ and $b(k) = 0$ for $k \geq 7$. In order to model the situation that we have a wide range of players, we choose the cost of constructing edges for the players uniformly at random between 0.01 and 2500. We set the parameters of the SF, ER and GR networks such that they all have approximately the same number of edges. We consider three scenarios:

1) The power network is a SF network constructed by preferential attachment [38], with 5 initial nodes. Each newly added node connects to six existing nodes.
2) The power network is an ER network with edge formation probability of 0.024.
3) The power network is a GR network in which nodes are uniformly distributed in a 2×2 square, and the threshold to form an edge is set at a distance of 0.18.

In all of the above cases, we assume that the sensor network has a fixed SF structure constructed by preferential attachment with 5 initial nodes, and each newly added node connects to one of the existing nodes. Figure 4 demonstrates the power and sensor networks and the set of Nash equilibrium interconnection edges between them (we have reduced the number of nodes to keep the structures visible). These edges are constructed according to the algorithm in the proof of Theorem 2. In Figure 4, there is only one low cost player; by Corollary 1, we know that such players always construct edges to all of the nodes in G_2 and this is independent of the actions of the other players or structures of the networks G_1 and G_2. We have summarized some of the salient features of the Nash equilibrium interconnected networks in Table II. These results are produced by averaging over 100 instances of the random networks and player edge costs. As the data in the table indicates, the number of constructed interconnection edges (and the corresponding social welfare and distances between nodes) is approximately the same in all of the three forms of the power networks, namely SF, ER and GR. This suggests that for the setting considered here (involving random power networks and uniformly distributed edge costs across the nodes), the topology of the power network does not significantly impact the characteristics of the Nash equilibrium interconnection network.

Next, we investigate the impact of heterogeneity in edge costs on the social welfare and the Nash equilibrium networks. In the previous scenario we assumed that each player's edge cost was chosen from a uniform distribution on the interval $[0.01, 2500]$. Here, we consider the case where the cost of constructing edges is equal to the mean value of the previous costs, i.e., $c_i = 1250$ for $1 \leq i \leq 500$. Given the benefit function for the players specified in the previous scenario, all of the players have high edge costs (i.e., $|S_L| = 0$). The results are shown in Table III.

As the data in Table III suggests, when players have homogeneous edge costs (equal to the mean

	SF	ER	GR
$\lvert E_1 \rvert$	2970.7	2980.8	2945.1
Diameter of G_1	4	4.42	∞
Total Number of Interconnection Edges Constructed	5106.7	5103.5	5104.7
Interconnection Edges Constructed by High Cost Players	106.7	103.5	104.7
Average Distance Between Interdependent Nodes	4.00	4.00	4.00
Social Welfare of the Interconnected Network	1221790	1222388	1222214

TABLE II: Features exhibited by the interconnected networks formed in 3 different scenarios, based on the algorithm given in the proof of Theorem 2. These results are produced by averaging over 100 instances of random networks and edge costs. The GR network is disconnected in some of the instances, which we indicate with an average diameter of ∞.

	SF	ER	GR
$\lvert E_1 \rvert$	2970.7	2980.8	2945.1
Diameter of G_1	4	4.42	∞
Interconnection Edges Constructed by High Cost Players	12.7	18.8	30.8
Average Distance Between Interdependent Nodes	3.95	3.92	3.87
Social Welfare of the Interconnected Network	1246825	1245300	1242250

TABLE III: Features exhibited by the interconnected networks formed in 3 different scenarios when all of the players have the same cost $c = 1250$ *of constructing edges. These results are produced by averaging over 100 instances of random networks.*

of the costs assigned to the players previously), there is a significant decrease in the number of interconnection edges built by the players. Interestingly, this is accompanied by a *decrease* in the average distances between the players and their interdependent nodes (in all of the three different scenarios), compared to the situation with heterogeneous edge costs considered previously. This leads to an *increase* in the social welfare under homogeneous edge costs, as shown by the last rows of Tables II and III. This phenomenon can be explained via Proposition 2 and the r-radius of the players (defined in (8)). Specifically, when all players have the same edge cost $c = 1250$, their corresponding r-radius is $r_i = 2$ for all $1 \leq i \leq 500$. If a player P_i (associated with node x_i) constructs an edge to the center of the sensor network G_2, by Proposition 2, no player P_j with $d_{G_1}(x_i, x_j) \leq 2$ will construct an interconnection edge. A player P_j with distance $d_{G_1}(x_i, x_j) > 2$ will construct an edge, however. Thus, the set of players that construct interconnection edges under homogeneous edge costs are spaced apart fairly regularly (i.e., every node is at most distance 2 in G_1 from a node that has constructed an edge). In the case of heterogeneous edge costs, however, the distances (in G_1) between nodes that have constructed edges is no longer bounded by 2. Thus, even though there are more nodes that construct edges (due to relatively low edge costs), the average distance between nodes and their interdependent nodes increases. This also leads to a decrease in social welfare.

VIII. Related network design problems

There are many instances of network design problems that have been studied in the computer science and algorithms literature. The interconnection network design problem that we investigated in this paper has similarities to the Island-Connection (IC) model that was studied in [39] where nodes (as network designers) have distance-based utilities. In this model, there are clusters of geographically close nodes (called islands) and it is assumed that the price of intra-island edge construction is less than that of inter-island edge construction. While the IC model considers a homogeneous set of players, the INDG model includes the case that players have different cost and benefit functions. Furthermore, the topologies of networks G_1 and G_2 (which correspond to islands in the IC model) in INDG are fixed, whereas in IC the structure of the islands depends on the cost of intra-island edge construction. When the cost of intra-island and inter-island edge formation are lower than certain thresholds, [39] shows that there are complete connections inside the islands. In addition, while the distance between all pairs of inter-island nodes are taken into account in the IC model, our INDG model allows the interdependency network G_I to characterize the set of important inter-island distances.

Another related work is the best response network problem (BRN) [30], which directly generalizes the classical distance-utility network formation problem given by (1). Specifically, in BRN, there is a central network designer with distance-based utility and a set of pairs of nodes that wish to communicate via short hops. These pairs are encoded as a network $G_1 = (N, E_1)$, where the presence of an edge $(x_1, x_2) \in E_1$ indicates that x_1 and x_2 wish to be close together in the constructed network. The utility produced by a constructed network $G = (N, E)$ is

$$u(G|G_1) = \left(\sum_{(v_i, v_j) \in E_1} b(d_G(v_i, v_j)) \right) - c|E|. \tag{10}$$

The network that maximizes this utility function is called the best response network to G_1. In [30], we showed that finding a best response network with respect to an arbitrary network G_1 is NP-hard. A key difference between BRN and INDG is that in INDG, each node acts as a network designer (i.e., no central network designer) and builds edges to nodes in a different network. In fact, networks G_1 and G_2 in the definition of the INDG problem have no equivalent correspondence in BRN.

IX. Conclusion

We introduced the interconnection network design game between two networks G_1 and G_2. In this game, there is a heterogeneous (in terms of utility function) set of network designers, each associated with a node in the network G_1. Each node in G_1 is dependent on certain nodes in G_2, and these dependencies are captured by a network G_I. The utility of the players is defined based on the distance-utility function where the objective of each player is to build a set of edges from its associated node to nodes in the network G_2 such that distances between its associated node and the nodes it depends on in G_2 are minimized. We showed that finding a best response action of a player is NP-hard. Nevertheless, we showed certain important properties of the best response networks, which enabled us to find a Nash equilibrium for certain instances of the game. Finally, we applied our framework to model the interdependencies between communication and power networks. Our simulations suggest that the social welfare is larger when players are homogeneous in terms of their edge construction costs, compared to players with heterogeneous edge costs.

One interesting avenue for future research is to consider other classes of utility functions for the players. Another important topic for further research on this problem is to address the scenario where players can build different *types* of edges (e.g., representing different types of relationships between the nodes). Defining appropriate utility functions to capture this scenario, along with a characterization of the resulting Nash equilibria, would be of interest. Finally, proving existence of Nash equilibria when G_2 has an arbitrary structure would be of value.

References

[1] J. Gao, S. V. Buldyrev, H. E. Stanley, and S. Havlin, "Networks formed from interdependent networks," *Nature Physics*, vol. 8, pp. 40–48, 2012.

[2] F. Radicchi and A. Arenas, "Abrupt transition in the structural formation of interconnected networks," *Nature Physics*, vol. 9, pp. 717–720, 2013.

[3] M. Kivela, A. Arenas, M. Barthelemy, J. P. Gleeson, Y. Moreno, and M. A. Porter, "Multilayer networks," *Journal of Complex Networks*, pp. 1–69, 2014.

[4] C. M. Schneider, N. Yazdani, N. A. M. Arajo, S. Havlin, and H. J. Herrmann, "Towards designing robust coupled networks," *Scientific Report*, vol. 3, no. 1969, pp. 1–7, 2013.

[5] M. Beccuti, G. Franceschinis, M. Kaâniche, and K. Kanoun, "Multi-level dependability modeling of interdependencies between the electricity and information infrastructures," in *Critical Information Infrastructure Security*, 2009, pp. 48–59.

[6] A. Cardillo, M. Zanin, J. Gmez-Gardees, M. Romance, A. J. G. del Amo, and S. Boccaletti, "Modeling the multi-layer nature of the European air transport network: Resilience and passengers re-scheduling under random failures," *The European Physical Journal Special Topics*, vol. 215, no. 1, pp. 23–33, 2013.
[7] M. O. Jackson, *Social and Economic Networks*. Princeton University Press, 2008.
[8] S. Goyal, *Connections: An Introduction to the Economic of Networks*. Princeton University Press, 2009.
[9] P. Erdos and A. Renyi, "On random graphs," *Publicationes of Mathematicae Debrecen*, vol. 6, pp. 290–297, 1959.
[10] D. Watts and S. Strogatz, "Collective dynamics of small-world networks," *Nature*, vol. 393, no. 4, pp. 440–442, 1998.
[11] H. Zhang, E. Fata, and S. Sundaram, "A notion of robustness in complex networks," *IEEE Transactions on Control of Network Systems*, vol. 2, no. 3, pp. 310–320, 2015.
[12] M. Pirani, E. Moradi Shahrivar, and S. Sundaram, "Coherence and convergence rate in networked dynamical systems," in *IEEE 54nd Annual Conference on Decision and Control (CDC)*, 2015, pp. 968–973.
[13] J. Ghaderi and R. Srikant, "Opinion dynamics in social networks: A local interaction game with stubborn agents," *Automatica*, vol. 50, pp. 3209 – 3215, 2014.
[14] L. Blume, D. Easley, J. Kleinberg, R. Kleinberg, and E. Tardos, "Network formation in the presence of contagious risk," *ACM Trans. Econ. Comput.*, vol. 1, no. 2, pp. 6:1–6:20, 2013.
[15] O. Yagan, D. Qian, J. Zhang, and D. Cochran, "Optimal allocation of interconnecting links in cyber-physical systems: Interdependence, cascading failures, and robustness," *IEEE Transactions on Parallel and Distributed Systems*, vol. 23, no. 9, pp. 1708–1720, 2012.
[16] M. Kurant and P. Thiran, "Layered complex networks," *Phys. Rev. Lett.*, vol. 96, p. 138701, 2006.
[17] J. Gao, S. V. Buldyrev, S. Havlin, and H. E. Stanley, "Robustness of a network formed by n interdependent networks with a one-to-one correspondence of dependent nodes," *Phys. Rev. E*, vol. 85, p. 066134, 2012.
[18] E. M. Shahrivar, M. Pirani, and S. Sundaram, "Robustness and algebraic connectivity of random interdependent networks," in *5th IFAC Workshop on Distributed Estimation and Control in Networked Systems*, vol. 48, no. 22, 2015, pp. 252 – 257.
[19] M. Brautbar and M. Kearns, "A clustering coefficient network formation game," in *Proc. 4th International Conference on Algorithmic Game Theory*, 2011, pp. 224–235.
[20] S. Goyal and A. Vigier, "Robust networks," *Working Paper*, 2011.
[21] M. Kearns, S. Judd, and Y. Vorobeychik, "Behavioral experiments on a network formation game," in *Proc. 13th ACM Conference on Electronic Commerce*, 2012, pp. 690–704.
[22] W. Saad, Z. Han, T. Basar, M. Debbah, and A. Hjorungnes, "Network formation games among relay stations in next generation wireless networks," *IEEE Transactions on Communications*, vol. 59, no. 9, pp. 2528–2542, September 2011.
[23] M. O. Jackson and A. Wolinsky, "A strategic model of social and economic networks," *Journal of Economic Theory*, vol. 71, no. 0108, pp. 44–74, 1996.
[24] E. M. Shahrivar and S. Sundaram, "Strategic multi-layer network formation," in *IEEE 52nd Annual Conference on Decision and Control (CDC)*, 2013, pp. 582–587.
[25] M. Parandehgheibi and E. Modiano, "Robustness of interdependent networks: The case of communication networks and the power grid," in *IEEE Global Communications Conference (GLOBECOM)*, 2013, pp. 2164–2169.
[26] D. T. Nguyen, Y. Shen, and M. T. Thai, "Detecting critical nodes in interdependent power networks for vulnerability assessment," *IEEE Transactions on Smart Grid*, vol. 4, no. 1, pp. 151–159, 2013.
[27] S. V. Buldyrev, R. Parshani, G. Paul, H. E. Stanley, and S. Havlin, "Catastrophic cascade of failures in interdependent networks," *Nature*, vol. 464, no. 7291, pp. 1025–1028, 2010.
[28] A. Moskvina and J. Liu, "Integrating networks of equipotent nodes," in *Computational Social Networks: 5th International Conference, CSoNet Proceedings*. Springer International Publishing, 2016, pp. 39–50.
[29] A. Fabrikant, A. Luthra, E. Maneva, C. H. Papadimitriou, and S. Shenker, "On a network creation game," in *Proc. Twenty-second Annual Symposium on Principles of Distributed Computing*, ser. PODC '03, 2003, pp. 347–351.
[30] E. M. Shahrivar and S. Sundaram, "The strategic formation of multi-layer networks," *IEEE Transactions on Network Science and Engineering*, vol. 2, no. 4, pp. 164–178, 2015.
[31] J.-P. Rodrigue, C. Comtois, and B. Slack, *The Geography of Transport Systems*, 3rd ed. Routledge, 2013.
[32] "A strategy for a future mail processing and transportation network," U.S. Postal Service Office of Inspector General, Tech. Rep. RARC-WP-11-006, 07 2011.
[33] J. Yick, B. Mukherjee, and D. Ghosal, "Wireless sensor network survey," *Computer networks*, vol. 52, no. 12, pp. 2292–2330, 2008.
[34] D. L. Hall and J. Llinas, "An introduction to multisensor data fusion," *Proceedings of the IEEE*, vol. 85, no. 1, pp. 6–23, 1997.
[35] T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein, *Introduction to Algorithms*, 3rd ed. The MIT Press, 2009.
[36] E. Koutsoupias and C. Papadimitriou, "Worst-case equilibria," in *Proceedings of the 16th Annual Conference on Theoretical Aspects of Computer Science, STACS*, 1999, pp. 404–413.
[37] M. Nekovee, "Worm epidemics in wireless ad hoc networks," *New Journal of Physics*, vol. 9, no. 6, p. 189, 2007.
[38] A.-L. Barabasi and R. Albert, "Emergence of scaling in random networks," *Science*, vol. 286, no. 5439, pp. 509–512, 1999.
[39] M. O. Jackson and B. W. Rogers, "The economics of small worlds," *Journal of the European Economic Association*, vol. 3, pp. 617–627, 2005.