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Teleparallel equivalent of general relativity and local Lorentz transformation: Revisited*

Gamal G.L. Nashed[1,2,3] and B. Elkhatib[2]

[1]Centre for theoretical physics, the British University in Egypt, 11837 - P.O. Box 43, Egypt.

[2]Mathematics Department, Faculty of Science, Ain Shams University, Cairo, Egypt.

[3]Egyptian Relativity Group (ERG).

e-mail:nashed@bue.edu.eg

It is well known that the field equations of teleparallel theory which is equivalent to general relativity (TEGR) completely agree with the field equation of general relativity (GR). However, TEGR has six extra degrees of freedom which spoil the true physics. These extra degrees are related to the local Lorentz transformation. In this study, we give three different tetrads of flat horizon space-time that depend only on the radial coordinate. One of these tetrads contains an arbitrary function which comes from local Lorentz transformation. We show by explicate calculations that this arbitrary function spoils the calculations of the conserved charges. We formulate *a skew-symmetric tensor* whose vanishing value put a constraint on the arbitrary function. This constraint makes the conserved charges are free from the arbitrary function.

1 Introduction

In the theory of GR, the gravitational field is gained by the curvature of space-times. Particles are obliged by the curvature of the space-times to move on geodesics. Therefore, the theory of GR is indeed geometrized by the gravitational field. The theory of GR has some constraints on the classical level. The forecasts of GR are in agreement with the experimental data accessible till now. Unification of the main four forces has continued as a favorite

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subject in physics. Because of the unknown attitude of GR at the quantum level, the scientistic community required the unified theory. For this purpose, Einstein did his attempt which was failed. In this research, we study his suggested theory, which is called TEGR as an alternative theory of gravitational field in the absence of unifying it with quantum theory.

The TEGR is established on Weitzenböck geometry [1]. In this theory torsion behaves as a force on objects. Therefore, in TEGR, there is no geodesics but only force equations [2]. The GR theory is characterized mathematically by Einstein's field equations, in which the geometry of the space-time is described by one side and the physics of the space-time is described by the other side. To find a solution in GR is not an easy task, therefore, we demanded certain symmetry constraints on the space-time metric. The significance of symmetries is understandable in GR owing to the laws of conservation of matter in the space-time which can be studied and realized with the aid of the symmetry constraints [3].

The TEGR theory is identical to GR. The GR theory is a geometric theory constructed on the Levi-Civita connection, that has curvature and zero torsion. However, TEGR employs anther connection. This connection is given by Weitzenböck which is defined on a space so that is globally flat, i.e., has zero curvature. This connection is known as a Weitzenböck connection and has a vanishing Ricci tensor but non-trivial torsion. This connection is employed to construct a Lagrangian built on a gravitational scalar named as the torsion scalar, T. The dynamics of this Lagrangian are equal to GR and this follows from the result

$$R = -T - B,$$

with R being Ricci scalar and B is a boundary term linked to the divergence of the torsion. Due to the fact that B is a total derivative, it does not contribute to the equations of motion and therefore, the Lagrangian of TEGR is equal to that of GR.

The aim of this study is to discuss the effect of the extra degrees of TEGR on the true physics and how one can fix these degrees so that they do not contribute to the true physics. In §2, an introduction to TEGR theory is presented. In §3, several tetrad fields having flat horizons are given and application to the field equations of TEGR is explained. New analytic, solutions are derived in §3. In §4, calculation of the conserved quantities of each solution have been carried out and we have shown how the unphysical extra degrees contribute to the true physics. In §5, we gave a skew-symmetric tensor which constrains the extra degree and shows the effect of this tensor on the acceleration components of an observer at infinity. Main results are discussed in final section.

2 Introduction to TEGR theory

Teleparallel theory equivalent to general relativity considers as another construction of GR of Einstein. The main entity of TEGR theory is the tetrad fields[†] (${e^i}_\alpha$). In TEGR theory, the metric can be built using the tetrad: $\mathrm{g}_{\alpha\beta} = \lambda_{ij}{\mathrm{e}^i}_\alpha {\mathrm{e}^j}_\beta$, with $\lambda_{ij} = \mathrm{diag}(1,-1,-1,-1)$ being the Minkowskian metric, thus the symmetric connection , Levi-Civita , ${\overset{\circ}{\Gamma}{}^\alpha}_{\beta\gamma}$ can be constructed. Nevertheless, it is likely to build Weitzenböck non-symmetric connection ${\Gamma^\alpha}_{\beta\gamma} = {e_i}^\alpha \partial_\gamma {e^i}_\beta$ [1]. The Weitzeinböck spacetime is labled as a pair (M, e_μ), whereas M is a D-dimensional manifold and e_β $(\beta = 0, \cdots, 3)$ are D-linear independent vector defined globally on M. The covariant derivative of the tetrad using Weitzenböck connection is vanishing, i.e. $\nabla_\alpha {\mathrm{e}^i}_\beta \equiv 0$. Therefore, the vanishing of the covariant derivative of the tetrad identifies the auto parallelism or absolute parallelism restriction. Actually, the operator ∇_α has a big issue, that is not invariant under local Lorentz transformations (LLT). The issue permits all LLT invariant quantities to rotate freely at each point of the space [4]. Thus, the symmetric metric is not able to guess one form of tetrad field; therefore, the additional degrees of freedom have to be restricted such that one physical frame can be identified. The Weitzenböck connection is has a vanishing curvature however it has a torsion defined by

$$\mathrm{T}^\alpha{}_{\beta\gamma} := \Gamma^\alpha{}_{\beta\gamma} - \Gamma^\alpha{}_{\gamma\beta}. \tag{1}$$

The contortion is defined as

$$\mathrm{K}^{\alpha\beta}{}_\gamma := -\frac{1}{2}\left(\mathrm{T}^{\alpha\beta}{}_\gamma - \mathrm{T}^{\beta\alpha}{}_\gamma - \mathrm{T}_\gamma{}^{\alpha\beta}\right). \tag{2}$$

In the TEGR one can construct three Weitzenböck invariants: $I_1 = T^{\alpha\beta\gamma}T_{\alpha\beta\gamma}$, $I_2 = T^{\alpha\beta\gamma}T_{\beta\alpha\gamma}$ and $\mathrm{I}_3 = \mathrm{T}^\alpha \mathrm{T}_\alpha$, where $\mathrm{T}^\alpha = \mathrm{T}_\beta{}^{\alpha\beta}$. We next define the invariant $\mathrm{T} = \mathrm{A}_1\mathrm{I}_1 + \mathrm{A}_2\mathrm{I}_2 + \mathrm{A}_3\mathrm{I}_3$, where A_k, $k = 1,2,3$ are constants [4]. When $\mathrm{A}_1 = 1/4$, $\mathrm{A}_2 = 1/2$ and $\mathrm{A}_3 = -1$ the invariant T will be identical with Ricci scalar $\mathrm{R}^{(\overset{\circ}{\Gamma})}$, up to divergence term. In this sense, the teleparallel gravity will be identical with GR. The torsion scalar of TEGR is defined as

$$\mathrm{T} := \mathrm{T}^\alpha{}_{\beta\gamma}\mathrm{S}_\alpha^{\beta\gamma}, \tag{3}$$

with $\mathrm{S}_\alpha{}^{\beta\gamma}$ being the superpotential tensor defined as

$$\mathrm{S}_\alpha{}^{\beta\gamma} := \frac{1}{2}\left(\mathrm{K}^{\beta\gamma}{}_\alpha + \delta^\beta_\alpha \mathrm{T}^{\rho\gamma}{}_\rho - \delta^\gamma_\alpha \mathrm{T}^{\rho\beta}{}_\rho\right). \tag{4}$$

The tensor $\mathrm{S}_\alpha{}^{\beta\gamma}$ is skew symmetric tensor in the last two indices.

The identity between torsion scalar and Ricci one is given by

$$\mathrm{R}^{(\overset{\circ}{\Gamma})} = -\mathrm{T}^{(\Gamma)} - 2\,\nabla_\alpha \mathrm{T}^{\beta\alpha}{}_\beta, \tag{5}$$

[†]The Greek symbols indicate the elements of tangent space and Latin components indicate the symbols of the spacetime.

such that the covariant derivative is with regard to the symmetric connection, i.e. Levi-Civita. The total derivative term is the one on the second term on the right-hand side of the above equation. Therefore, there will be no contribution of the divergence term to the variation of the scalar torsion T instead of the Ricci scalar. In this sense, the torsion and Ricci scalars are identical. Despite the quantitative equivalence, they, T an R, are qualitatively not equivalent. For instant, Ricci scalar tensor is invariant under LLT whilst the divergence term $\nabla_\alpha \mathrm{T}^{\beta\alpha}{}_\beta$ is not invariant and therefore the torsion scalar. Therefore, the theory of TEGR *action* is not form invariant with respect to LLT [5, 6, 7].

The action of the gauge gravitational field Lagrangian is given by [8, 9]

$$\mathrm{S} = \frac{\mathrm{M}_{\mathrm{Pl}}^2}{2} \int |e| \left[(\mathrm{T} - 2\Lambda) + \mathrm{L}_M(\Phi_A)\right] \, d^4x, \tag{6}$$

with L_M being the Lagrangian of matter fields Φ_A and M_{Pl} being the mass of Planck, that is connected to gravitational constant G through $\mathrm{M}_{\mathrm{Pl}} = \sqrt{\hbar c/8\pi G}$. In this study we use the units $G = c = \hbar = 1$ and $|e| = \sqrt{-g} = \det\left(e^i{}_\alpha\right)$. Making variation of Eq. (6) regard to the tetrad fields $\mathrm{e}^i{}_\alpha$ give the following field equations [8]

$$\partial_\beta(e\mathrm{S}_i{}^{\alpha\beta}) = 4\pi \mathrm{e}\mathrm{e}_i{}^\beta(\mathrm{t}_\beta{}^\alpha + \Theta_\beta{}^\alpha), \tag{7}$$

with $\mathrm{S}_i{}^{\alpha\beta} = \mathrm{e}_i{}^\rho \mathrm{S}_\rho{}^{\alpha\beta}$, being the (pseudo) tensor $\mathrm{t}_\alpha{}^\beta$ and

$$\mathrm{t}_\alpha{}^\beta = \frac{1}{16\pi}[4|rmT^\rho{}_{\gamma\alpha}\mathrm{S}_\rho{}^{\beta\gamma} - \delta_\alpha^\beta(\mathrm{T} - 2\Lambda)], \tag{8}$$

is the energy-momentum tensor

$$\Theta_\alpha{}^\beta = \mathrm{e}^i{}_\alpha \left(-\frac{1}{e}\frac{\delta \mathrm{L}_M}{\delta \mathrm{e}^i{}_\beta}\right). \tag{9}$$

As the tensor $\mathrm{S}_i{}^{\alpha\beta}$ is anti-symmetric, i.e $\mathrm{S}_i{}^{\alpha\beta} = -\mathrm{S}_i{}^{\beta\alpha}$, this leads to $\partial_\alpha\partial_\beta(\mathrm{eS}_i{}^{\alpha\beta}) \equiv 0$ [4]. Therefore,

$$\partial_\beta \left[\mathrm{e}\mathrm{e}_i{}^\rho(\mathrm{t}_\rho{}^\beta + \Theta_\alpha{}^\beta)\right] = 0.$$

The pseudo-tensor t_α^β is disappeared in the theory of GR. To prob its behavior we see that the previous equation leads us to the conservation

$$\frac{\mathrm{d}}{\mathrm{dt}} \int_V \mathrm{e}\mathrm{e}_i{}^\alpha(\mathrm{t}_{0\,\alpha} + \Theta_{0\,\alpha}) d^3x = -\oint_\Sigma \left[\mathrm{e}\mathrm{e}_i{}^\alpha(\mathrm{t}_{j\alpha} + \Theta_{j\alpha})\right] d\Sigma^j.$$

The integration of the previous equation is carried out on 3-dimensional volume limited by the surface. Therefore, $\mathrm{t}_\alpha{}^\beta$ represent the energy-momentum tensor of the gravitational field [8].

3 Flat transverse solutions

We apply the TEGR field equations (7) to the *first flat transverse section* which gives the following tetrad written in cylindrical coordinate (t, r, ϕ, z) as:

$$\left(e^i{}_\mu\right) = \begin{pmatrix} \mathrm{a(r)} & 0 & 0 & 0 \\ 0 & \mathrm{b(r)} & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r \end{pmatrix}, \tag{10}$$

Substituting from (10) into (3) we calculate torsion scalar as

$$T = -\frac{2(2a'r + a)}{r^2 a e^2}, \tag{11}$$

where $a' := \frac{da(r)}{dr}$. Using (10) in (7) we get

$$\Theta_0{}^0 = \frac{\Lambda r^2 e^3 + 2rb' - b}{2r^2 e^3}, \quad \Theta_1{}^1 = \frac{\Lambda r^2 e^2 a - 2ra' - a}{2r^2 e^2 a},$$
$$Theta_2{}^2 = \Theta_3{}^3 = \frac{\Lambda r e^3 a - rba'' + ra'b' + ab' - ba'}{2re^3 a}. \tag{12}$$

The solution of the above differential equation has the form

$$a(r) = \frac{c_1\sqrt{\Lambda r^3 + 3c_2}}{\sqrt{r}}, \qquad b(r) = \mp\frac{\sqrt{3r}}{\sqrt{\Lambda r^3 + 3c_2}}, \tag{13}$$

The *second flat transverse section* tetrad is given by

$$\left(e_1{}^i{}_\mu\right) = \begin{pmatrix} a(r) & 0 & 0 & 0 \\ 0 & b(r)\cos\phi & -r\sin\phi & 0 \\ 0 & b(r)\sin\phi & r\cos\phi & 0 \\ 0 & 0 & 0 & r \end{pmatrix}. \tag{14}$$

Tetrad (14) is related to (10) by a rotation matrix given by

$$\left(\Lambda^i{}_j\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & r \end{pmatrix}. \tag{15}$$

Substituting from (14) into (3) we evaluate the torsion scalar as

$$T=-\frac{2(rba'+ab-2ra'-a)}{r^2ae^2}. \qquad (16)$$

Using (14) in (7) we get get the same differential equation given by Eq. (12) which have the same solution given by Eq. (13).

The *third flat transverse section* tetrad is given by

$$\left({e_3}^i{}_\mu\right)=\begin{pmatrix} -a(r)\mathfrak{L} & -\mathfrak{H}b(r)\cos\phi & -r\mathfrak{H}\sin\phi & 0\\ a(r)\mathfrak{H}\cos\phi & b(r)(\sin^2\phi+\mathfrak{L}\cos^2\phi) & r\sin\phi\cos\phi(\mathfrak{L}-1) & 0\\ a(r)\mathfrak{H}\sin\phi & b(r)\sin\phi\cos\phi(\mathfrak{L}-1) & r(\cos^2\phi+\mathfrak{L}\sin^2\phi) & 0\\ 0 & 0 & 0 & r \end{pmatrix}, \qquad (17)$$

where $\mathfrak{L}=\sqrt{1+\mathfrak{H}^2}$ and $\mathfrak{H}$ is an arbitrary function of r. This arbitrary function perseveres the flat horizon of tetrad (17). Tetrad (17) is related to (10) by a LLT matrix given by

$$\left({\Lambda_1}^i{}_\mu\right)=\begin{pmatrix} -\mathfrak{L} & -\mathfrak{H}\cos\phi & -\mathfrak{H}\sin\phi & 0\\ \mathfrak{H}\cos\phi & \sin^2\phi+\mathfrak{L}\cos^2\phi & \sin\phi\cos\phi(\mathfrak{L}-1) & 0\\ \mathfrak{H}\sin\phi & \sin\phi\cos\phi(\mathfrak{L}-1) & \cos^2\phi+\mathfrak{L}\sin^2\phi & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}. \qquad (18)$$

Substituting from (17) into (3) we evaluate the torsion scalar as

$$T=\frac{2(ba\mathfrak{H}^2+ba+rb\mathfrak{H}^2a'+rba'-ab\mathfrak{L}-b\mathfrak{L}ra'+rab\mathfrak{H}\mathfrak{H}'-a\mathfrak{L}-2ra'\mathfrak{L}))}{r^2ae^2\mathfrak{L}}. \qquad (19)$$

Using (17) in (7) we get the same differential equation given by Eq. (12) which have the same solution given by Eq. (13). Therefore, the field equations (7) are not able to give a specific form of the arbitrary function $\mathfrak{H}$ as expected due to the fact that the field equations (7) are equivalent to GR. In the next section we are going to discuss the physical relevance of each tetrad.

4 Conserved quantities and intuitive of Einstein-Cartan theory

There are many modifications of GR. In the frame of theoretical physics, the theory of Einstein-Cartan (EC), which is identified as “Einstein-Cartan-Sciama-Kibble theory”, is classic gravitation theory like GR in which its connection has no skew symmetric part. Thus, in EC, the torsion tensor can be accompanied to the spin of the matter, by the same pattern the curvature is joined to the momentum and energy of the matter. Actually, spin of matter using non-flat spacetime demands that the torsion tensor does not vanishing but be a variable, i.e., stationary action. The theory of EC deals with the torsion tensor and metric as independent which give the right extension of conservation law in the existence of the gravitational field. The EC theory constructed by Élie Cartan in [10] and recently it has many application [11]. The Lagrangian of EC has the form [12]:

$$\mathcal{L}(\vartheta^i,\ \Gamma^j{}_k) = -\frac{1}{2\kappa}\left(\mathrm{R}^{ij}\wedge\eta_{ij} - 2\Lambda\eta\right), \tag{20}$$

with ϑ^i being the co-frame one form, $\Gamma^j{}_k$ being the connection, κ and Λ are the gravitational and the cosmological constants. The Lagrangian given by Eq. (20) is a form invariant under diffeomorphism and Lorentz local transformation [12]. Carrying out the principle of least action to equation (20) we get [12, 13]

$$\mathrm{E}_i := -\frac{1}{2\kappa}\left(\mathrm{R}^{jk}\wedge\eta_{ijk} - 2\Lambda\eta_i\right), \qquad\qquad \mathrm{H}_{ij} := \frac{1}{2\kappa}\eta_{ij}, \tag{21}$$

where η_{ij} is a two form given in the Appendix A, E_i is the energy-momentum and H_{ij} is the rotational gauge field momentum. The momentum of translation and the spin take the following form

$$\mathrm{H}_i := -\frac{\partial\mathcal{L}}{\partial \mathrm{T}^i} = 0, \qquad\qquad \mathrm{E}_{ij} := -\vartheta_{[i}\wedge\mathrm{H}_{j]} = 0. \tag{22}$$

The minimally coupling of matter is supposed such that $\frac{\partial \mathrm{L}_{Matter}}{\partial \mathrm{T}^i} = 0$ and $\frac{\partial \mathrm{L}_{matter}}{\partial \mathrm{R}^i_j} = 0$.[‡] The conserved current is given by [12]

$$\jmath[\xi] = \frac{1}{2\kappa}d\left\{{}^*\left[dk + \xi\rfloor\left(\vartheta^i\wedge\mathrm{T}_j\right)\right]\right\}, \qquad \text{where}$$

$$\mathrm{k} = \xi_i\vartheta^i, \qquad \text{and} \qquad \xi^i = \xi\rfloor\vartheta^i, \tag{23}$$

where $*$ is defined as the Hodge duality and ξ is an arbitrary vector field $\xi = \xi^i\partial_i$. ξ^i in this study are four parameters ξ^0, ξ^1, ξ^2 and ξ^3. Because this study is in the frame of theory of TEGR which is equivalent to GR, thus, torsion is nil and conserved charge, Eq. (23), is given by

$$\mathrm{Q}[\xi] = \frac{1}{2\kappa}\int_{\partial\mathrm{S}} {}^*\mathrm{dk}. \tag{24}$$

Expression (24) was given by Komar [14]–[18] and is invariant under diffeomorphism.

The coframe $\vartheta_1{}^i$ of solution (13) using tetrad (10) has the form:

$$\vartheta^{\hat{0}} = adt, \qquad \vartheta_1{}^{\hat{1}} = bdr, \qquad \vartheta_1{}^{\hat{2}} = rd\theta, \qquad \vartheta_1{}^{3} = rd\phi. \tag{25}$$

[‡] The derivative of the coframe vanishes if ξ is a Killing vector field, i.e. $\mathcal{L}_\xi\vartheta^i = 0$ [12].

By using equation (25) in equation (24) we obtain

$$k = a^2\xi_0 dt - b^2\xi_1 dr - r^2[\xi_2 d\theta + \xi_3 d\phi]. \tag{26}$$

Total derivative of Eq. (26) gives

$$d\mathrm{k} = 2[aa'\xi_0(dr \wedge dt) + r\xi_2(d\theta \wedge dr) - r\xi_3(dr \wedge d\phi)]. \tag{27}$$

From the inverse of (25) using (27) in (24) and taking the Hodge-duality to dk, we get

$$\mathrm{Q}[\xi_t] = \mathrm{Q}[\xi_r] = \mathrm{Q}[\xi_\theta] = \mathrm{Q}[\xi_\phi] = 0. \tag{28}$$

Equation (28) indicates in clear way that the total conserved charge of (13) are nil. Carrying out the same procedure to tetrads (14) and (17) we get the same result of Eq. (28). Thus, equation (24) must redefined to get a well defined value, i.e. Eq. (24) needs a regularization.

5 Regularization using relocalization

Expression (24) is form invariant under diffeomorphism and Lorentz local transformation. However, it is demonstrated that plus to diffeomorphism and Lorentz local transformation there exists other defect in the form of conserved quantities. This defect lies in equations of motion which permit a relocalization [12]. Therefore, conserved quantities can be altered using relocalization. Relocalization appeared from amercement of the gravitational Lagrangian through a total derivative term has the form

$$\mathrm{S}' = \mathrm{S} + d\Phi, \qquad \text{where} \qquad \varPhi = \Phi(\vartheta^j, \varGamma_i{}^j, \mathrm{T}^i, R_i{}^j), \tag{29}$$

with $\varGamma_i{}^j$ is a 1-form connection. The second term in Eq. (29), i.e. $d\Phi$, amendments the boundary part of the Lagrangian, premitting the equations of motion to be in a covariant form ([12] and references therein). It has been explained that the total conserved charges could be regularized by applying a relocalization procedure. It has been explained that to solve the odd result derived in Eq. (28), one has to employ relocalization given by the boundary expression that appears in the Lagrangian. We use the relocalization in the form

$$H_{ij} \to H'_{ij} = H_{ij} - 2\beta\eta_{ijkl}R^{kl},$$

that is originated from the modification of the Lagrangian [12]

$$L \to S' = S + \beta d\varPhi,$$

where

$$H'_{ij} = \left(\frac{1}{2\kappa} - \frac{4\beta\Lambda}{3}\right)\eta_{ij} - 2\beta\eta_{ijkl}\left(\mathrm{R}^{kl} - \frac{\Lambda}{3}\vartheta^k\vartheta^l\right). \tag{30}$$

Assuming β, which exists in Eq. (30) has the value $\frac{3}{8\Lambda\kappa}$ in 4-dimension to confirm the cancelation of the vanishing value (that comes from inertia) which exists in Eq. (28). Thus, the conserved charges after regularization has the form [12, 19, 20]

$$J[\xi] = -\frac{3}{4\kappa\Lambda}\int_{\partial S} \eta_{ijkl}\Xi^{ij}\, W^{kl}, \tag{31}$$

with W^{ij} is the Weyl 2-form described by

$$\mathrm{W}^{ij} = \frac{1}{2} C_{kl}{}^{ij}\vartheta^k \wedge \vartheta^l, \tag{32}$$

where $C_{ij}{}^{kl} = e_i{}^\mu e_j{}^\nu e^k{}_\alpha e^l{}_\beta C_{\mu\nu}{}^{\alpha\beta}$ is the Weyl tensor and Ξ^{ij} is denoted by

$$\Xi_{ij} = \frac{1}{2} e_j \rfloor e_i \rfloor dk. \tag{33}$$

The conserved currents $J[\xi]$ given by Eq (31) are form invariant under diffeomorphism and Lorentz local transformation. These currents $J[\xi]$ are linked to the vector field ξ on the spacetime manifold.

calculating the necessary components of Eq. (31) we get

$$\Xi_{01} = -\frac{a'\xi_0}{b}, \qquad \Xi_{13} = -\frac{\xi_3}{b}. \tag{34}$$

Using Eq. (34), we get the value of $\eta_{ijkl}\Xi^{ij}\, W^{kl}$ in 4-dimension in the form

$$\eta_{ijkl}\Xi^{ij}\, W^{kl} = \frac{2\sqrt{3}c_1c_2\xi_0(2\Lambda r^3 - 3c_2)}{3r^3}. \tag{35}$$

Substituting Eq. (35) in (31) we get

$$J[\xi_t] = \frac{\sqrt{3}}{2}\xi_0\pi c_1 c_2 + \left(\frac{1}{r^3}\right), \qquad J[\xi_r] = J[\xi_z] = J[\xi_\phi] = 0. \tag{36}$$

Equation (36) shows that the two constants c_1 and c_2 may take the values $c_2 = \frac{2}{\sqrt{3}\pi}$ and $c_2 = M$ in which total mass and angular momentum takes the form [21, 22]

$$E = M + \left(\frac{1}{r}\right), \qquad J[\xi_r] = J[\xi_z] = J[\xi_\phi] = 0. \tag{37}$$

Repeat the same calculation we get the non-vanishing components Ξ^{ij} of tetrad (14) to have the form

$$\Xi_{01} = \frac{a'\cos\phi\xi_0}{b}, \qquad \Xi_{02} = -\frac{a'\sin\phi\xi_0}{b}, \qquad \Xi_{13} = -\frac{\cos\phi\xi_3}{b}, \qquad \Xi_{23} = -\frac{\sin\phi\xi_3}{b}. \tag{38}$$

Using Eq. (31), we get the value of $\eta_{ijkl}\Xi^{ij}\mathrm{W}^{kl}$ in 4-dimension in the same form of Eq. (35) which gives the same conserved quantities as given by Eq. (37).

The survive components Ξ^{ij} of tetrad (17) are

$$\Xi_{01} = -\frac{\xi_0 a'[(2\mathfrak{H}^2+1)\cos^2\phi+\mathfrak{L}\sin^2\phi]}{b}, \quad \Xi_{02}=-\frac{\xi_0 a'\sin\phi\cos\phi(2\mathfrak{H}^2+1-\mathfrak{L})}{b}, \quad \Xi_{03}=-\frac{\xi_3\cos\phi\mathfrak{H}}{b},$$
$$\Xi_{12} = \frac{\xi_0 a'\mathfrak{H}\sin\phi(2[\mathfrak{L}-1]\cos^2\phi+1}{b}, \quad \Xi_{13}=\frac{\xi_3(\sin^2\phi+\cos^2\phi\mathfrak{L})}{b}, \quad \Xi_{23}=\frac{\xi_3\sin\phi\cos\phi(\mathfrak{L}-1)}{b}. \quad (39)$$

Using Eq. (31), we get the value of $\eta_{ijkl}\Xi^{ij}W^{kl}$ in 4-dimension in the form

$$\eta_{ijkl}\Xi^{ij}W^{kl} = \frac{2\sqrt{3}c_1c_2\xi_0(2\mathfrak{H}^2[\mathfrak{L}-1]\sin^2\phi\cos^2\phi-2\mathfrak{H}^2-1)(2\Lambda r^3-3c_2)}{3r^3}. \quad (40)$$

Substituting Eq. (40) in (31) we get

$$J[\xi_t] = \frac{\sqrt{3}c_1c_2\xi_0(2\mathfrak{H}^2[\mathfrak{L}-1]\sin^2\phi\cos^2\phi-2\mathfrak{H}^2-1)(2\Lambda r^3-3c_2)}{4r^3}, \qquad J[\xi_r]=J[\xi_\theta]=J[\xi_\phi]=0. \quad (41)$$

Equation (41) shows that the arbitrary function $\mathfrak{H}$ which describes inertia contributes to the true physics.

6 Physical constrains on the inertia

As we discussed in the previous sections that we have three tetrad fields reproduce the same metric. The TEGR field equations give the same solution of the two unknown functions however, they can not able to give any specific form of the arbitrary function. Also we have shown that the scalar torsion depend on the tetrad as is known in the literature which means that it is not local Lorentz transformation. In the previous section we try to calculate the conserved quantities and show that they depend on the inertia. In this section, we are going to assume specific form of skew-tensor and see if this tensor will help in solving the above problem or not? It is well known that the field equations of $f(T)$ is non-symmetric [23]–[35][§]

$$S_{(\mu\nu)}{}^{\rho}T_{,\rho}\, f(T)_{TT} - \left[e^{-1}e^a{}_\mu\partial_\rho\left(be_a{}^\alpha S_\alpha{}^{\rho\nu}\right) - T^\alpha{}_{\lambda\mu}S_\alpha{}^{\nu\lambda}\right] f(T)_T - \frac{1}{4}g_{\nu\mu}(f(T)-2\Lambda) = 4\pi\mathcal{T}_{\nu\mu},$$
$$S_{[\mu\nu]}{}^{\rho}T_{,\rho}\, f(T)_{TT} = 0. \quad (42)$$

Therefor, in TEGR the skew-symmetric is satisfied automatic due to the fact that $f_{TT}=0$. Now let us check if the skew symmetric tensor

$$S_{[\mu\nu]}{}^{\rho}T_{,\rho}, \quad (43)$$

is vanishing for the above tetrad fields or not? For the first tetrad given by Eq. (10) Eq. (43) is satisfied automatic. Therefore, tetrad (10) we call it a physical tetrad. For the second tetrad field, given by (14), Eq. (43) is satisfied identically.

[§]We will denote the symmetric part by ( ), for example, $A_{(\mu\nu)}=(1/2)(A_{\mu\nu}+A_{\nu\mu})$ and the antisymmetric part by the square bracket [ ], $A_{[\mu\nu]}=(1/2)(A_{\mu\nu}-A_{\nu\mu})$.

Using Eq. (43), we get for the third tetrad field, Eq. (17), the following non-vanishing components:

$$S_{[10]1}T^{,1} = \frac{\mathfrak{H}\cos\phi}{2\mathfrak{L}^{3/2}abr^4}\Biggl(b\Biggl[r^2ab\mathfrak{L}^4a'' + r^2\mathfrak{H}a^2b\mathfrak{L}^4\mathfrak{H}'' - r^2b\mathfrak{L}^4a'^2 - raa'\mathfrak{L}^2(r\mathfrak{L}^2b' - b[r\mathfrak{H}\mathfrak{H}' - \mathfrak{L}^2])$$
$$-a^2[rb'\mathfrak{L}^2(r\mathfrak{H}\mathfrak{H}' + \mathfrak{L}^2) - b(r^2\mathfrak{H}'^2 - 2\mathfrak{L}^2)]\Biggr] - 2\mathfrak{L}^{3/2}[r^2ab(b+2)a'' - r^2b(b+2)a'^2 - raa'(r(b+4)b' + b^2 + 2b)$$
$$-a^2(rb'(b+2) + 2b^2 + 2b]\Biggr),$$
$$S_{[20]1}T^{,1} = \frac{\mathfrak{H}\sin\phi}{2\mathfrak{L}^{3/2}ab^2r^3}\Biggl(b\Biggl[r^2ab\mathfrak{L}^4a'' + r^2\mathfrak{H}a^2b\mathfrak{L}^4\mathfrak{H}'' - r^2b\mathfrak{L}^4a'^2 - raa'\mathfrak{L}^2(r\mathfrak{L}^2b' - b[r\mathfrak{H}\mathfrak{H}' - \mathfrak{L}^2])$$
$$-a^2[rb'\mathfrak{L}^2(r\mathfrak{H}\mathfrak{H}' + \mathfrak{L}^2) - b(r^2\mathfrak{H}'^2 - 2\mathfrak{L}^2)]\Biggr] - 2\mathfrak{L}^{3/2}[r^2ab(b+2)a'' - r^2b(b+2)a'^2 - raa'(r(b+4)b' + b^2 + 2b)$$
$$-a^2(rb'(b+2) + 2b^2 + 2b]\Biggr). \tag{44}$$

Solution of Eq. (44) has the form

$$\mathfrak{H} = \pm\frac{\sqrt{1 - 9r^4{c_2}^2\Lambda - 27r{c_2}^2c_1}}{3c_2\sqrt{r(r^3\Lambda + 3c_1)}}. \tag{45}$$

Equation (45) is a solution of Eq. (44) and when we use it Eq. (41) we get

$$E = M + \left(\frac{1}{r}\right). \tag{46}$$

which is identical with Eq. (37). This means that the solution of the skew-symmetric tensor removes the inertia from the true physics.

7 Discussion and conclusion

We have discussed the TEGR theory and its 6 extra degrees of freedom. For this purpose we have studied three tetrad fields, with the flat horizon, reproduce the same metric. The first tetrad field has two unknown functions and the field equations of TEGR give an analytic form of these functions. The conserved quantities of this tetrad are calculated and we have got a finite conserved quantity in the temporal coordinate after using the Regularization using relocalization. We coined this tetrad as the physical tetrad. The reason for this name comes from the fact that we have defined a skew-symmetric tensor that is must vanish identically in the framework of TEGR. The first tetrad satisfied this property.

For the second tetrad which is obtained from the first one by multiplied it by a rotation matrix. We have calculated the scalar torsion of this tetrad and have shown that the scalar torsion is not invariant under local Lorentz transformation. We have calculated the field equations and show that they are not different from the first tetrad and consequently the solution of the two unknown functions are the same as of the first tetrad. The discussion of the conserved quantities and the skew-symmetric tensors are the same of the first tetrad.

For the third tetrad, we have shown that it is related to the first tetrad through a local Lorentz transformation that contains an arbitrary function, $\mathfrak{H}$. We have calculated the scalar torsion of this tetrad and have shown that it depends on the arbitrary function. As usual, the field equations of TEGR can not fix any form of the arbitrary function. We have calculated the conserved quantities of this tetrad and have shown that the temporal component of the coordinate depends on the arbitrary function. This shows in a clear way that the inertia contributes to the true physics. We have calculated the skew-symmetric tensor of this tetrad and have shown that some of its components are not vanishing. We have solved these non-vanishing components and have derived a form of the arbitrary function. When we have substituted this solution in the form of the conserved quantities we have shown that the energy will coincide with the value of the physical tetrad.

Appendix A

Notation

The indices i, j, $\cdots$ are the (co)-frame components whilst $\mu, \nu, \cdots$ are the holonomic spacetime coordinates. The hats $\hat{0}, \hat{1}, \cdots$ c indicate special frame components. The exterior product is denoted by $\wedge$. The interior product of ξ and Ψ is described by $\xi \rfloor \Psi$. The vector dual to the 1-forms ϑ^i is labeled by e_i and their inner product satisfy $e_i \rfloor \vartheta^j = \delta_i{}^j$. Employing local coordinates x^μ, we have $\vartheta^i = e^i{}_\mu dx^\mu$ and $e_i = e_i{}^\mu \partial_\mu$ where $e^i{}_\mu$ and $e_i{}^\mu$ are the components covariant and contravariant of the tetrad fields. The volume $\eta := \vartheta^{\hat{0}} \wedge \vartheta^{\hat{1}} \wedge \vartheta^{\hat{2}} \wedge \vartheta^{\hat{3}}$ defines 4-form. Moreover, we can altered

$$\eta_i := e_i \rfloor \eta = \frac{1}{3!} \epsilon_{ijkl} \, \vartheta^j \wedge \vartheta^k \wedge \vartheta^l,$$

where ϵ_{ijkl} is totally antisymmetric with $\epsilon_{0123} = 1$.

$$\eta_{ij} := e_j \rfloor \eta_i = \frac{1}{2!}\epsilon_{ijkl}\ \vartheta^k \wedge \vartheta^l, \qquad \eta_{ijk} := e_k \rfloor \eta_{ij} = \frac{1}{1!}\epsilon_{ijkl}\ \vartheta^l.$$

Finally, we can define

$$\eta_{ijkl} := e_l \rfloor \eta_{ijk} = e_l \rfloor e_k \rfloor e_j \rfloor e_i \rfloor \eta,$$

which is the tensor density of Levi-Civita. The following useful identities

$$\begin{aligned} \vartheta^i \wedge \eta_j &:= \delta^i_j \eta, \qquad \vartheta^i \wedge \eta_{jk} := \delta^i_k \eta_j - \delta^i_j \eta_k, \qquad \vartheta^i \wedge \eta_{jkl} := \delta^i_j \eta_{kl} + \delta^i_k \eta_{lj} + \delta^i_l \eta_{jk}, \\ \vartheta^i \wedge \eta_{jkln} &:= \delta^i_n \eta_{jkl} - \delta^i_l \eta_{jkn} + \delta^i_k \eta_{jln} - \delta^i_j \eta_{kln} \end{aligned}$$

are holds using the η-forms.

The line element $ds^2 := g_{ij}\vartheta^i \bigotimes \vartheta^j$ is defined by the spacetime metric g_{ij} .

Appendix B: Calculations of Weyl tensor and the object $W^{\mu\nu}$

The non-vanishing components of Weyl tensor using solution of tetrad (10) have the form:

$$\begin{aligned} &C_{0101} = -C_{0110} = C_{1010} = -C_{1001} = \frac{-\sqrt{3}c_1c_2}{r^3}, \\ &C_{0202} = -C_{0220} = C_{0303} = -C_{0330} = C_{2020} = -C_{2002} = C_{3030} = -C_{3003} = \frac{-\sqrt{\Lambda r^3 + 3c_2}c_1c_2}{2r^{5/2}}, \\ &C_{1212} = -C_{1221} = C_{1313} = -C_{1331} = -C_{2112} = C_{2121} = -C_{3113} = C_{3131} = \frac{-\sqrt{3}c_1}{2r^{3/2}\sqrt{\Lambda r^3+3c_2}}, \\ &C_{2323} = -C_{2332} = -C_{3223} = C_{3232} = -\frac{c_1}{r}. \end{aligned} \tag{47}$$

The non-vanishing components of $W^{\mu\nu}$ are given by

$$\begin{aligned} &W^{01} = -\frac{\sqrt{3}c_1c_2}{r^3}(dr \wedge dt)], \qquad W^{02} = -\frac{c_1c_2\sqrt{\Lambda r^3 + 3c_2}}{2r^{5/2}}(d\theta \wedge dt), \\ &W^{03} = -\frac{c_1c_2\sqrt{\Lambda r^3 + 3c_2}}{2r^{5/2}}(d\phi \wedge dt), \qquad W^{12} = -\frac{\sqrt{3}c_1}{2r^{3/2}\sqrt{\Lambda r^3 + 3c_2}}(dr \wedge d\theta), \\ &W^{13} = -\frac{\sqrt{3}c_1}{2r^{3/2}\sqrt{\Lambda r^3 + 3c_2}}(dr \wedge d\phi), \qquad W^{23} = -\frac{c_1}{r}(d\theta \wedge d\phi). \end{aligned} \tag{48}$$

By the same method we can calculate the non-vanishing of Weyl tensor and of the object $W^{\mu\nu}$ of the second and third tetrad.

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