Disformal invariance of continuous media with linear equation of state

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We show that the effective theory describing single component continuous media with a linear and constant equation of state of the form $p = w\rho$ is invariant under a 1-parameter family of continuous disformal transformations. In the special case of $w = 1/3$ (ultrarelativistic gas), such a family reduces to conformal transformations. As examples, perfect fluids, homogeneous and isotropic solids are discussed.

I. INTRODUCTION

Disformal transformations have been considered in the framework of large-distance modification of gravity [1], and have become a very active field of research in modern cosmology and modified gravity, finding applications, for example, in Bekenstein's TeVeS formalism [2], bimetric theories of gravity [3], scalar-tensor theories [4, 5] and disformal inflation [6]. Introduced in the context of Finsler geometry [1], these transformations describe the most general relation between two geometries in one and the same gravitational theory, preserving the causal structure and the weak equivalence principle.

More recently, it was discovered that the structure of Horndeski Lagrangian is invariant under a particular class of disformal transformations [7].

As noted in Ref.[8], the Einstein field equations are invariant under invertible disformal transformations, as well as more general scalar-tensor theories [9][1]. This fact has been used to show that some extensions of Horndeski theory, obtained by performing a general disformal transformation to the Horndeski Lagrangian, lead to the same equations of motion of the original theory and are thus free of ghost instabilities [10].

On the other hand, if General Relativity (or a more general scalar-tensor theory) is reformulated in terms of an auxiliary metric, which is related with the original "physical" metric by a non-invertible disformal transformation, a new degree of freedom of gravity is switched on [8, 9, 11]. This scalar degree of freedom behaves as an irrotational pressureless perfect fluid, i.e. it can mimic a cold dark matter component, or, more generally, irrotational dust. The resulting theory, called mimetic dark matter, was first proposed in [12] and explored in [13] (see also [14–16] and [17] for the relation with Hořava-Lifshitz gravity).

Remarkably, disformal transformations have found other applications, for example they have been used in the context of primordial tensor modes during inflation [18].

Furthermore, in a general fixed background even a free massless scalar field is invariant under a particular class of local disformal transformations [19] and disformal invariance of Maxwell's field equations was proved in [20] while invariance of the Dirac equation was studied in [21].

The goal of this paper is to analyze the general consequences of invariance under a disformal transformation for a rather large class of systems, namely continuous media, that are relevant for cosmology, modified gravity or analog gravity.

Our analysis relies on the observation that the physics of fluids or solids can be derived by an unique Lagrangian [22–24]. This remarkable result has been investigated using the language of the effective field theory [25–30] using the pull-back formalism, which had already been used to describe the dynamics of continuous media [31–34]. Basically, the symmetries of the theory allow to extract all the dynamical information and the thermodynamics concerning the medium from an action principle.

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[1] This is actually true for a generic non-singular field redefinition

The paper is organized as follows. In Section II we show that any Lagrangian invariant under this class of disformal transformations describes a medium with a linear equation of state. Then, as an example, in Section III we show that the action for a perfect fluid with a linear equation of state is disformal invariant, the special case of an irrotational perfect fluid is discussed in Section IV. In Section V we generalize this result to homogeneous and isotropic solids. Finally, in Section VI, we summarize and discuss our results.

Notation. Our signature is $(-+++)$; greek indices run from 0 to 3. Units are such that $\hbar = c = k_B = 1$.

II. WEYL AND DISFORMAL TRANSFORMATIONS

Invariance under Weyl (or conformal) transformations is characterized by a vanishing trace of the energy-momentum tensor (EMT), disformal ones constitute a generalization.

Let us start by considering the rather general action

$$S = \int d^4x \sqrt{-g}\, \mathcal{L}(g_{\mu\nu}, \phi^I, \partial\phi^I, \partial_1 \ldots \partial_n \phi^I)\,; \tag{1}$$

where $I = 1, \ldots, N$ and ϕ^I are generic matter fields. A Weyl transformation is defined by

$$g_{\mu\nu} \to \bar{g}_{\mu\nu} = e^{\sigma(x)} g_{\mu\nu}\,; \tag{2}$$

or considering infinitesimal transformations connected with identity, corresponding to $\sigma_0 = 0$, we get

$$\delta g_{\mu\nu} = \bar{g}_{\mu\nu} - g_{\mu\nu} = g_{\mu\nu}\delta\sigma + \mathcal{O}(\delta\sigma^2)\,. \tag{3}$$

If the action (1) is invariant under (2), then

$$\delta S = \frac{1}{2}\int d^4x \sqrt{-g} T^{\mu\nu} g_{\mu\nu}\delta\sigma = 0\,; \tag{4}$$

we then conclude that the trace of the EMT vanishes

$$T^{\mu}{}_{\mu} = 0\,. \tag{5}$$

The extension to one parameter group of disformal transformations connected to the identity crucially depends on the the matter content. In particular, a 4-vector emerging from the matter's action is needed. The simplest option is to suppose that the energy-momentum tensor describes a single-component medium of the following form [34]

$$T_{\mu\nu} = \rho\, u_\mu\, u_\nu + p\, h_{\mu\nu} + \pi_{\mu\nu}\,; \tag{6}$$

where u_μ is the timelike eigenvector of $T_{\mu\nu}$ and $h_{\mu\nu} = g_{\mu\nu} + u_\mu\, u_\nu$ is the projector orthogonal to u_μ. The energy density ρ, the isotropic pressure p and the anisotropic stress $\pi_{\mu\nu}$ are respectively defined as the following projections of the energy-momentum tensor $T_{\mu\nu}$

$$\rho = T_{\mu\nu}u^\mu u^\nu, \qquad p = \frac{1}{3}T_{\mu\nu}h^{\mu\nu}, \qquad \pi_{\mu\nu} = h^\alpha{}_\mu h^\sigma{}_\nu T_{\alpha\sigma} - \frac{1}{3}h_{\mu\nu}h_{\alpha\sigma}T^{\alpha\sigma}. \tag{7}$$

We have also supposed that the EMT is in the so-called Landau-Lifshitz frame [35, 36]. By construction the anisotropic tensor is traceless, $\pi^\mu{}_\mu = 0$. The key property needed to define disformal transformation is the existence of of a single timelike eigevenvector u_μ.

Consider now the one-parameter group of disformal transformations defined by

$$\begin{aligned} g_{\mu\nu} \to \bar{g}_{\mu\nu} &= e^\sigma g_{\mu\nu} + \left[e^\sigma - e^{3w\sigma}\right] u_\mu u_\nu \\ &= e^\sigma h_{\mu\nu} - e^{3w\sigma} u_\mu u_\nu \\ &\equiv A h_{\mu\nu} - C u_\mu u_\nu \end{aligned} \tag{8}$$

so that for $\sigma = 0$ we obtain the identity transformation, and the factor 3 is included for later convenience. Expanding around the identity, we get the variation of the metric

$$\delta g_{\mu\nu} = \bar{g}_{\mu\nu} - g_{\mu\nu} = \left[g_{\mu\nu} + (1 - 3w)\, u_\mu u_\nu\right]\delta\sigma\,. \tag{9}$$

Suppose that the action (1), when the EMT has the form (6), is invariant under the transformation (8)

$$\delta S = \frac{1}{2}\int d^4x\sqrt{-g}T^{\mu\nu}\delta g_{\mu\nu} = \frac{1}{2}\int d^4x\,\left[T^{\mu}{}_{\mu} + (1-3w)\,\rho\right]\delta\sigma = 0\,. \tag{10}$$

Therefore, we have

$$3\,p - \rho = (3w-1)\,\rho\,; \tag{11}$$

and we conclude that one-parameter disformal transformation (8) can be defined for one-component fluids and invariance implies

- a linear equation of state $p(\rho) = w\rho$;
- a constant speed of sound $c_s^2 = w$.

Note that for $w = \frac{1}{3}$, the disformal transformation (8) reduces to a Weyl transformation.

III. PERFECT FLUIDS AND DISFORMAL TRANSFORMATIONS

The results of the previous section are very general, here we focus on the case of perfect fluids. The most general class of perfect fluids can be effectively described by using four scalar fields Φ^A representing Eulerian coordinates on the fluid's world-volume, or, equivalently, as Stueckelberg fields associated to broken space-time translations. At leading order (LO) in a derivative expansion the action can be written as [26, 37, 38]

$$S = \int d^4x\sqrt{-g}\,U(b, Y) \tag{12}$$

where U is a generic function and

$$B^{mn} = g^{\mu\nu}\partial_\mu\Phi^m(x,\tau)\partial_\nu\Phi^n(x,\tau), \qquad m,n = 1,2,3, \qquad b = \sqrt{\det(\boldsymbol{B})}\,, \qquad Y = u^\mu\partial_\mu\Phi^0(x,\tau); \tag{13}$$

the fluid's 4-velocity satisfies

$$u^\mu\partial_\mu\Phi^a = 0\,, \qquad u^\mu u_\mu = -1\,. \tag{14}$$

The perfect fluid Lagrangian is invariant under $\Phi^0 \to \Phi^0 + f(\Phi^a)$ and internal volume-preserving diffeomorphisms $\Phi^a \longrightarrow f^a(\Phi)$ with $\det\left(\frac{\partial f^a}{\partial\Phi^b}\right) = 1$. The energy-momentum tensor of the system is of course the one of perfect fluid with [26, 38]

$$\rho = Y\,U_Y - U\,, \qquad p = U - b\,U_b\,. \tag{15}$$

Notice that in general the fluid equation of state is not barotropic, unless a special form for U is chosen. For instance, a barotropic equation of state for $U(Y,b)$ with $p/\rho = w$ =const. can be obtained by choosing [37] $U \propto b^{1+w}\mathcal{U}(b^{-w}Y)$ where $\mathcal{U}$ is an arbitrary function or alternatively $U \propto (b^{1+w} + Y^{1+1/w})$. Dust (pressureless matter) $p = 0$ is obtained when $U \propto \mathcal{U}(Y)b$, while radiation $p = \frac{1}{3}\rho$ when $U \propto b^{4/3}\mathcal{U}(b^{-1/3}Y)$. Finally, an equation of state $w = -1$ can be obtained by choosing $U \propto \mathcal{U}(bY)$.

From the general discussion in section II, the action describing a perfect fluid with a linear equation of state $p = w\rho$ is invariant under a disformal transformation. Indeed, under (8) we have

$$B^{ab} \to \bar{B}^{ab} = B^{ab}\,A^{-1}\,, \qquad b \to \bar{b} = b\,A^{-3/2}\,, \qquad Y \to \bar{Y} = Y\,C^{-1/2}\,. \tag{16}$$

The determinant of the metric transforms as

$$\sqrt{-g} \;\to\; \sqrt{-\bar{g}} \;=\; \sqrt{-g}\,A^{3/2}\sqrt{C} \tag{17}$$

(see Appendix C of [2] for a derivation of the above expression), so we conclude that the action transforms as

$$S = \int d^4x\sqrt{-g}\,U(b,Y) \to \int d^4x\sqrt{-g}\,A^{3/2}\sqrt{C}\,U(b\,A^{-3/2}, Y\,C^{-1/2})\,. \tag{18}$$

By considering an infinitesimal transformation, $|\delta\sigma| \ll 1$, it is easy to show that the action is invariant if

$$U - b\,U_{,b} = w\,(Y\,U_Y - U)\ . \tag{19}$$

As a result

$$p = w\rho \tag{20}$$

i.e. a barotropic perfect fluid characterized by a linear equation of state with w constant. The case $w = -1$, i.e. a cosmological constant, and $w = 0$, i.e. non relativistic matter (dust), are subcases of a perfect fluid with $p = -\rho$ and $p = 0$, respectively.

IV. IRROTATIONAL PERFECT FLUIDS

At leading order in a derivative expansion the most general action describing a fluid can be written as [26, 27, 37, 38]

$$S = \int d^4x\sqrt{-g}\,U(b, Y, X)\,; \tag{21}$$

with

$$X = g^{\mu\nu}\partial_\mu\Phi^0\partial_\nu\Phi^0\,. \tag{22}$$

Actually, the EMT derived from the above action describes a two-component fluid system or a superfluid. An interesting subcase is when only X is present, namely $U(X)$ and it was used [39] as one of the first examples of the effective description of fluid dynamics in terms of scalar fields. Here, we present the effective description of irrotational perfect fluids and irrotational dust, and we show that the invariance under disformal transformation implies a linear equation of state. The EMT reads

$$T_{\mu\nu} = 2\,U'\,X\,v_\mu\,v_\nu + U\,g_{\mu\nu}\,, \qquad v^\mu = \frac{\partial^\mu\Phi^0}{\sqrt{-X}}\,; \tag{23}$$

which has a perfect fluid form with

$$\rho = 2\,U'\,X - U, \qquad p = U\,. \tag{24}$$

In addition being the 4-velocity v_μ proportional to an exact 1-form, the fluid is irrotational.

By performing an infinitesimal disformal transformation, it is possible to show that the action is invariant if [2]

$$U(X) \sim X^{\frac{1}{2}\left(1+\frac{1}{w}\right)} \tag{25}$$

corresponding to an irrotational perfect fluid characterized by a linear equation of state, $p = w\rho$.
Note that the effective Lagrangian is singular for $w = 0$. In this case, we consider the mimetic action [12, 40]

$$S = \int d^4x\sqrt{-g}\,\rho(x)\,(X + 1) \tag{26}$$

which describes irrotational fluid-like dust with the energy density given by the Lagrange multiplier ρ and velocity potential Φ^0. The energy-momentum tensor is

$$T^{on-shell}_{\mu\nu} = \rho\,\partial_\mu\Phi^0\partial_\nu\Phi^0\,; \tag{27}$$

where we identify ρ as the on-shell energy density, e.g. when the constraint $X = -1$ is enforced.

Moreover, the theory is invariant under the shift symmetry $\Phi^0 \to \Phi^0 + \lambda$, with $\lambda = const.$. The associated conserved Noether current is

$$J_\mu = \rho\,\partial_\mu\Phi^0\,. \tag{28}$$

The mimetic action is invariant under (8) with $w = 0$, i.e. a conformal transformation of the spatial metric

$$\begin{cases} h_{\mu\nu} & \to \bar{h}_{\mu\nu} = A h_{\mu\nu} \\ \rho & \to \bar{\rho} = A^{-\frac{3}{2}}\rho \end{cases}\,. \tag{29}$$

where the rescaling of the Lagrange multiplier ρ is required by the conservation of the Noether current under (8), $\nabla_\mu J^\mu = 0 \to \bar{\nabla}_\mu \bar{J}^\mu = 0$.

[2] More generally, we can have $X^\alpha f(\Phi^0)$, with f an arbitrary function.

V. SOLIDS AND DISFORMAL TRANSFORMATIONS

Finally, let us consider the case of isotropic and homogeneous solids, characterized by the presence of a non vanishing anisotropic tensor, and described at LO by the effective action [37, 41]

$$S = \int d^4x\sqrt{-g}\,U(\tau_1,\tau_2,\tau_3)\,; \tag{30}$$

where

$$\tau_n = \mathrm{Tr}(\boldsymbol{B}^n)\,, \quad n = 1,2,3\ , \qquad [\boldsymbol{B}]^{ab} = B^{ab}\,. \tag{31}$$

The EMT has the following form

$$T_{\mu\nu} = U\,g_{\mu\nu} - 2\left(U_{\tau_1}\,\delta^{ab} + 2\,U_{\tau_2}\,B^{ab} + 3\,U_{\tau_3}\,B^{ac}\,B^{cb}\right)\partial_\mu\Phi^a\partial_\nu\Phi^b\,, \tag{32}$$

and represents the relativistic generalization of the energy-momentum tensor of an elastic material. The stress state is given by B^{ab} and the isotropic solid or jelly is not stressed when $B^{ab} = \delta^{ab}$. In particular

$$\rho = -U\,, \quad p = U - 2\sum_{n=1}^{3} n\,U_{\tau_n}\tau_n\,. \tag{33}$$

In the previous sections, we have seen that a perfect fluid, characterized by a linear equation of state $p = w\rho$, is invariant under the disformal transformation (8). Conversely, a perfect fluid, invariant under this disformal transformation, is necessarily described by a linear equation of state. Here we generalize this invariance property to solids with linear equation of state. From (16) we have that $\tau_n \to A^{-n}\,\tau_n$, $n = 1,2,3$. Thus the solid action transforms as

$$S = \int d^4x\sqrt{-g}\,U(\tau_1,\tau_2,\tau_3) \to \bar{S} = \int d^4x\sqrt{-g}\,A^{3/2}\sqrt{C}\,U(A^{-1}\tau_1,\,A^{-2}\tau_2,A^{-3}\tau_3)\,. \tag{34}$$

Under an infinitesimal transformation $|\delta\sigma| \ll 1$, the action is invariant if

$$3(1+w)\,U - \sum_{n=1}^{3} n\,U_{\tau_n} = 0\,. \tag{35}$$

The general solution of (35) can be written as

$$U = \tau_1^{3(1+w)/2}\,\mathcal{F}\left(\frac{\tau_2}{\tau_1^2}\,,\frac{\tau_3}{\tau_1^3}\right)\,, \tag{36}$$

which leads to a barotropic equation of state of the form: $p = (2+3\,w)\rho$. Thus, the action for a solid, characterized by a linear equation of state, is invariant under (8).

VI. DISCUSSION

In this work we have studied the invariance properties under a 1-parameter family class of disformal transformations (8), that can be considered as a deformation of Weyl transformations.
Each family is characterized by a constant w that has a manifest physical interpretation. Indeed, we have shown that every Lagrangian invariant under these disformal transformations is associated with an energy-momentum tensor characterized by a linear and equation of state of the form $p = w\rho$.
When $w = 1/3$ the family of disformal transformations reduces to conformal ones and we recover the well-known conformal invariance of a Lagrangian describing ultra-relativistic matter. As explicit examples of our general analysis we have considered perfect fluids and solids, generalizing the analysis for a free scalar field given in [19].

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