

Fractional dynamics of fermion generations

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Abstract

The dynamics of fermion generations is treated in the framework of the fractional Lagrangian and Hamiltonian in the compact extra dimension. The resulting spectra for u, d, e generations are described by the only mass parameter and slightly different fractional numbers, while the neutrino spectra need another mass parameter and a single fractional number for all neutrino species. New definitions of CKM and PMNS mixing matrices with standard data are given as the interference integrals in the extra dimension. A possibility of higher fermion states as dark matter candidates is shortly discussed.

1 Introduction

Fermion masses display a highly hierarchical order, where the mass ratios in the up sector can be larger than 10^3 , while in the down sector of the order of 10^2 , and less in the neutrino sector.

At the same time all attempts to find any sign of the internal fermion structure have failed and yielded only the lower limit of the internal scale of the order of several TeV [1].

The standard flavor theory [2] (see also the recent reviews [3, 4, 5]) contains more than 20 parameters, which are adjusted to explain the experimental data, and to suppress the unobserved processes.

One example is the absence of the terms yielding the FCNC type reactions (e.g. $\bar{\psi}_u \hat{Z} \psi_c$), in the leading approximation.

It is the main purpose of this paper to address three important points of the flavor theory: 1) what is the basic dynamics of the flavor spectrum, which makes it so high hierarchical, 2) why FCNC events are absent in the leading approximation, 3) what is the dynamics, which generates CKM matrix. As will be seen, our approach using the fractional dynamics in an extra dimension, provides unexpected results with interesting possibilities.

One of those is a possible explanation of the dark matter as the higher states of the extra dimension.

These features, as well as different patterns of fermion mixings in the quark and neutrino sectors are subjects of numerous investigations in the framework of the 4d Standard Model (SM) (see [6, 7] and references therein).

Following the idea of the compact extra dimension [8], more recently the focus of the analysis turned to theories in extra dimensions [9, 10, 11, 12, 13, 14, 15, 16] e.g. in the five dimensional anti-de Sitter (*AdS*) geometry, as well as in the modified warped extra dimensions [16, 17, 18].

One of the main motivations of these studies was the idea to connect the Planck and SM scales in one general approach, using the exponential factors differing the Plank scale from the SM scales.

At the same time the presence of the new dimension yields a possibility of a new dynamical mechanism for the creation of new states – the Kaluza-Klein (KK) states [8], and a possible danger, since an infinite tower of KK states may appear in the sector under investigation.

At the moment there are successful models, generating fermions and realistic mixing at the price of introduction additional parameters [16, 17, 18, 19].

It is the purpose of the present paper to introduce a simplest possible model, exploiting extra dimension with the first goal to reproduce the observed fermion masses. As will be seen, with the introduction of one mass scale parameter (μ), common to charged leptons, up and down quarks, and one dimensionless fractal parameter γ , one is able to reproduce fermion masses in all three sectors, slightly varying γ in these sectors. The neutrino sector requires a different pair of value (μ, γ) for all three species.

To achieve this goal we introduce the path integral formalism in $5d$, where the extra (fifth) dimension is dynamically independent from the 4d, and serves only to generate fermion masses. This is similar to the KK mechanism, but differs in the metric and boundary conditions, which produce a specific hierarchical spectrum.

In our approach no connection to gravitation and Planck scale is contained, but the main emphasis is done on the need of an extra mechanism for the mass generation, in addition to the existing ones in 4 d: Higgs-type mechanism, using vacuum condensate and the confinement mechanism, using vacuum correlators [20]. The latter ensures almost 99% of the mass in the visible part of the Universe, since it gives mass to all hadrons, including baryons. The rest of the mass, the 1% of it, is due to fermions, and this is the goal of the paper to find the source of it, different from the possible Higgs mechanism, unable at present to predict masses explicitly. At the same time the invisible (dark) part of matter is roughly 6 times larger and we tentatively envisage for it the same source of the extra (fifth) dimension, where excited generations lack 4d charges and participate only in gravitation.

There are several reasons to search for external sources of mass generation. Indeed, the high masses of heaviest quarks would imply a strong interaction to be created within 4d. As an example, the mass generation theory based on the rainbow version of the Dyson-Schwinger equation [19], where all SM fields participate, would require in all sectors much stronger couplings than available in theory. In simple words, to be as massive as 175 GeV, as the t quark is, and interacting only via QCD and reasonable (mass-independent) Higgs couplings seems to be not selfconsistent.

On the other hand, with the extraneous mass generation mechanism, fermions can be considered in our 3d space as elementary objects, with no internal structure, connected with the internal mass distribution, which might explain the failure of all attempts to find that structure [1]. Therefore it seems necessary to separate the 4d dynamics with all SM interacting fields, and the mass generation process, which may occur via other degrees of freedom, and in this paper we take the 5-th dimension to be responsible for the mass generation alone, with no interaction between the 5-th and 4d coordinates (however with SM interaction in 4d of the mass, created in the 5-th coordinate).

At this point one meets with an unexpected difficulty, of the growth of the mass eigenvalue spectrum with the number n in the ordinary dynamics as $m_n \sim n^\alpha$ with $\alpha \sim 4 \div 8$. It seems to be impossible with all known types of interaction in the fifth dimension (unless one uses different parts of space-time with different metrics). Therefore one has to leave the familiar formalism of field theory in D dimensions and look elsewhere.

We shall use below the so-called fractional dynamics, which is well-developed in many fields, see [21, 22, 23] for books and reviews. In this formalism the

kinetic terms (in our case in the fifth dimension) acquire a non-integer power form, which creates a much more versatile character of dynamics. Below in the next sections we demonstrate the new form of the fermion spectrum, generated by the fractional kinetic term, which seems to agree well with the experimental data.

The main objective of this paper is to suggest a mechanism of external mass creation, which ensures the relativistic fermion spectrum with a minimal theoretical input. As a second step we investigate the mechanism of flavor mixing and obtain the forms of resulting CKM and PNMS matrices. In doing this we introduce the new form of those matrices as the overlap integrals in the 5th dimension of the generation eigenfunctions. Correspondingly in the standard term of the Lagrangian, e.g. $\bar{q}'_u \hat{W} q_d$ the 4d integral is extended to 5d. It is shown, that the resulting CKM and PNMS matrices can be made realistic.

The paper is organized as follows. In the next section we derive the dynamical equation for the fermion mass eigenvalues, leaving details of derivation to the Appendix. In section 3 we formulate a simple model, which imitates the same results for the fermion masses, as in the fractional dynamics approach. In section 4 the results for the fermion masses are compared with the data and the values of few parameters are fixed. In section 5 we discuss the problem of the FCNC and show, how it is resolved when the violating terms are defined via the wave functions in the fifth dimension. The concluding section is devoted to the discussion of results.

2 Fractional dynamics

The standard dynamics, based of the quadratic time derivative of coordinates or the field variable in the quantum mechanics and quantum field theory respectively, was an object of modifications for different purposes.

In one case the aim was to extend the dynamics to the cases, when the system experiences chaotic motion, diffusion processes etc., see e.g. [21, 22, 23] and the references therein. The typical for this case is the resulting kinetic term of the type $\left(\frac{\partial z}{\partial t}\right)^\alpha$ or $(\dot{\varphi})^\alpha$ with $0 \leq \alpha \leq 2$.

At the same time the problem of nonstandard power α of time derivatives, and consequently of the same power α in the field propagator p^α , discussed below in the paper, was studied in the framework of the quantization of quantum gravitation, see e.g. [24] and the recent review [25]. In this case α

was considered to be equal to 4, and the main problem was to circumvent the Ostrogradsky analysis, which in the standard Hamiltonian approach yields the spectrum instability [26].

The way out of this problem was recently suggested in [27], where it was shown, that using the path integral approach instead of the classical Hamiltonian formalism, one ends up with the results, which do not show fatal instabilities, and display the reasonable spectrum.

In what follows we shall use the relativistic path integral formalism in 5d, where in the fifth dimension the time derivative can have a fractal character and the power α may be noninteger and larger than 2.

We shall be using the Fock-Feynman-Schwinger path integral suggested in [20, 28, 29] For another forms see e.g. [30, 31].

Following this method, the Green's function of a scalar particle $\varphi(x)$ in the N dimensional Euclidean space-time can be written as

$$G(1, 2) = \left\langle 1 \left| \frac{1}{m_4^2 - \partial_\mu^2} \right| 1 \right\rangle = \int_0^\infty ds (Dz)_{xy} e^{-K}, \quad (1)$$

where K is the kinetic factor playing the role of path integral Lagrangian, and (Dz) is the path measure

$$K_N = \int_0^s d\tau \left(m_4^2 + \sum_{\mu=1}^N \frac{1}{4} \left(\frac{\partial z_\mu}{\partial \tau} \right)^2 \right) \quad (2)$$

$$(Dz)_{12} = \int \frac{dp^{N-1}}{(2\pi)^{N-1}} \prod \frac{d^{N-1} \Delta z(n)}{(4\pi\varepsilon)^{N/2}} e^{ip(\sum_n \Delta z(n) - (x(1) - x(2)))} \quad (3)$$

Here $x(1) = \mathbf{x}_1; x(2) = \mathbf{x}_2$.

In the case of one fractal dimension, e.g. $N = 4 + 1$, and

$$m^2 - \partial_\mu^2 \rightarrow m^2 - \sum_{\nu=1}^4 \partial_\nu^2 - (\partial_5^2)^\alpha \mu^{2-2\alpha}, \quad (4)$$

where μ is a new scale parameter, the form K_N changes, as shown in the Appendix, to

$$K_4 + K_5 = \int_0^s d\tau \left(m_4^2 + \frac{1}{4} \sum_{\mu=1}^4 \left(\frac{\partial z_\mu}{\partial \tau} \right)^2 + \text{const} \left(\frac{dy}{d\tau} \right)^\beta \right), \quad \beta = \frac{2\alpha}{2\alpha - 1}. \quad (5)$$

The basic transformation, which is needed to find the spectrum in the 5-th dimension, yields the Hamiltonian form, which in the leading Fock amplitude can be written as [28, 29]

$$G(1, 2) = \sqrt{\frac{T_4}{8\pi}} \int_0^\infty \frac{d\omega}{\omega^{3/4}} \langle 1 | e^{-H(\omega)T_4} | 2 \rangle \quad (6)$$

where

$$H(\omega) = \frac{\mathbf{p}^2 + \mu^{2-2\alpha} p_5^{2\alpha} + \omega^2 + m_4^2}{2\omega} 2\omega, \quad p_5 = \frac{\partial}{i\partial x_5}, \quad x_5 \equiv y. \quad (7)$$

We assume, that the y dependence is always separated (factorized) in the resulting wave functions,

$$\psi(\mathbf{x}, y) = \chi(\mathbf{x})\varphi(y), \quad (8)$$

so that $\varphi(x_5)$ satisfies equation

$$p_5^2 \varphi(y) = -\frac{\partial^2}{\partial y^2} \varphi(y) = m_5^2 \varphi(y), \quad \mu^{2-2\alpha} p_5^{2\alpha} \varphi = \mu^2 \left(\frac{m_5}{\mu}\right)^{2\alpha} \varphi(y). \quad (9)$$

Note, that in (6) one can write following [28, 29] for 4d

$$\langle \mathbf{x} | e^{-H(\omega)T} | \mathbf{x}' \rangle = \sum_k \langle \mathbf{x} | k \rangle e^{-E_k(\omega)T} \langle k | \mathbf{x}' \rangle \quad (10)$$

where k runs over discrete and continuous levels of $H(\omega)$. In the simplest case of no 4d interaction one has

$$\langle \mathbf{x} | k \rangle = \chi_k(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \exp(i\mathbf{k}\mathbf{x}). \quad (11)$$

In the 5d case we add in notations (10) the

$$\langle \bar{x} | e^{-H(\omega)T} | \bar{x}' \rangle = \sum_{k,n} \chi_k(\mathbf{x}) \psi_n(y) e^{-E_{kn}(\omega)T} \chi_k^*(\mathbf{x}') \psi_n^*(y') \quad (12)$$

and the Green's function acquires the form

$$G(1, 2) = \sqrt{\frac{T_4}{8\pi}} \int_0^\infty \frac{d\omega}{\omega^{3/4}} \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \sum_{n=1,2,3} \psi_n^+(1) e^{-E_n(\omega)T_4 + i\mathbf{p}(\mathbf{x}_1 - \mathbf{x}_2)} \psi_n(2) \quad (13)$$

e.g. $\psi_n(1) = \psi_n(x_5)$ is the eigenfunction of the Hamiltonian

$$H_5 = \mu^{2-2\alpha} p_5^{2\alpha} + U(x_5), \quad (14)$$

where $U(x_5)$ is a possible interaction in x_5 , which we disregard below.

To go on-shell, one finds the extremum of $H(\omega)$ as a function of ω , and assuming

$$(p_5^2)^\alpha \psi_n = (m_5^2(n))^\alpha \psi_n \quad (15)$$

one obtains

$$E_n = \sqrt{\mathbf{p}^2 + m_4^2 + \mu^{2-2\alpha} m_5^{2\alpha}(n)} \quad (16)$$

In this way each fermion obtains its individual wave function ψ_n , which accompanies it from the beginning to the end of the process without change, unless in the total Lagrangian appear the terms, which contain integration over dx_5 , as will be shown in the next chapter.

At this point we take into account the spinor character of the fermion and write $S(1, 2)$ instead of $G(1, 2)$ following the formalism of [29].

$$S(x, x') = \left(\frac{1}{m_4 + \hat{\partial}} \right)_{xx'} = (m_4 - \hat{\partial})_x \sqrt{\frac{T}{8\pi}} \int_0^\infty \frac{d\omega}{\omega^{3/2}} \langle \bar{x} | e^{-H(\omega)T} | \bar{x}' \rangle. \quad (17)$$

One can define the covariant derivative in 5d as

$$\hat{\partial} = \sum_{\nu=1}^4 \partial_\nu \gamma_\mu + \Gamma_5 (\partial_y)^\alpha, \quad \Gamma_5 = \left(\frac{1}{i} \right)^\alpha \mu^{1-\alpha}, \quad (18)$$

where μ is a mass dimension in the fifth coordinate. Here \mathbf{x}, \mathbf{x}' include only 3d spacial coordinates and the fifth coordinate is y namely, $\bar{x} = \mathbf{x}, y$, $\bar{x}' = \mathbf{x}', y$, and m_4 implies the mass generated in 4d (if any).

Finally to define the spectrum due to the 5 dimension, one can write, assuming periodic boundary conditions

$$\mu^{2-2\alpha} p_5^{2\alpha} \psi_n(y) = (\pi n)^{2\alpha} \mu^2 \psi_n(y). \quad (19)$$

which obtain with the wave functions

$$\psi_n(y) = \sqrt{\frac{2}{\pi}} \sin(n\pi y \mu), \quad \mu y \in [0, 1] \quad (20)$$

which yields the spectrum

$$m_5(n) = \mu(\pi n)^\alpha, \quad n = 1, 2, 3, \dots \quad (21)$$

As will be seen in the next section this type of spectrum can be useful in comparison with experimental fermion masses.

3 An equivalent simple model

As a simple alternative, one can consider the interval in the $x_5 \equiv y$ space, which depends on the energy (mass), which is generated on this interval, namely

$$0 \leq y \leq y_+(m); \quad y_+(m) = \frac{y_0}{\left(\frac{m}{\mu}\right)^\gamma}. \quad (22)$$

Again assuming the mass equation, but now with the standard p^2 term,

$$p^2\psi_n = m_n^2\psi_n, \quad \psi_n = \sqrt{\frac{2}{\pi}} \sin\left(\frac{\pi n y}{y_+(m_n)}\right), \quad (23)$$

one arrives at the mass relation

$$m_n^2 = \frac{\pi n}{y_0} \left(\left(\frac{m_n}{\mu} \right)^\gamma \right)^2 \quad (24)$$

and choosing $y_0 = \mu^{-1}$, one has

$$m_n = \mu(\pi n)^{\frac{1}{1-\gamma}}, \quad n = 1, 2, 3. \quad (25)$$

Comparing this result with (21) in the previous section, obtained within the fractional dynamics approach, one realizes, that

$$\alpha = \frac{1}{1-\gamma}. \quad (26)$$

This derivation of course does not tell anything about the possible values of α and γ , however one cannot exclude here the values of $\gamma > 1$, which imitate not growing, but rather decreasing with n spectrum. The latter can be appropriate e.g. for neutrinos, where one can have $m_{\nu_1} > m_{\nu_2} > m_{\nu_3} > \dots$

To conclude this section, it is interesting to check the uncertainty principle $\Delta p \Delta x \gtrsim 1$, which in our case looks like

$$\Delta m \cdot \Delta y \rightarrow m_n y_+(m_n) = \left(\frac{m_n}{\mu} \right)^{1-\gamma} = \pi n, \quad (27)$$

which is strangely reminiscent of the space quantisation.

4 The 5d fermion spectrum

From (19) one has $m_5(n) = m_n = \mu(\pi n)^\alpha$, $n = 1, 2, 3, \dots$ and writing α in terms of the power γ in (25), one has

$$m_n = \mu \cdot (n\pi)^{\frac{1}{1-\gamma}}, \quad n = 1, 2, 3, \dots \quad (28)$$

To simplify matter, we choose $\mu = 10^{-4}$ MeV, and find for each fermion the corresponding γ_n and α_n from (26), (28).

We start with the charged leptons e, μ, τ with masses 0.511 MeV, 105.65 MeV and 1776.82 MeV respectively and taking for them $n = 1, 2, 3$ in (28), we obtain the values shown in Table 1.

Table 1: The slope parameters α_n, γ_n for leptons e, μ, τ

	e	μ	τ
n	1	2	3
exp. mass(MeV)	0.511	105.65	1776.82
α_n	7.4595	7.547	7.441
γ_n	0.8659	0.8675	0.8656

Note, that if one chooses the universal slope, ($\bar{\alpha} = 7.45410$), $\bar{\gamma} = 0.8658$ then the masses e, τ are the same within 1% while the mass of lepton μ drops by 18% $m_\mu \cong 88.65$ MeV.

Here one can expect the danger of appearing the fourth generation charged lepton for $n = 4, m(n = 4) = 15.5$ GeV. We shall discuss this issue below in the last section, where we assume that this lepton can not acquire 4d charges and hence may play the role of the dark matter particle (as well as higher excited states).

We turn now to the quark sector (d, s, b), and keep the same $\mu = 10^{-4}$ MeV. We keep in mind, that the quark masses are scale dependent and usually defined at different scales, e.g. the light quarks at the scale 2 GeV, while c, b and t quark masses are given in the $\overline{\text{MS}}$ scheme. This yields some mass shifts as compared to a unique scale definition, which are smaller than produced by differences in α_n, γ_n . Again from (28) we obtain the slope values $\alpha_n, \gamma_n, n = 1, 2, 3$, shown in Table 2.

One can see, that the d quark and to minor extent the s quark decline from the general slope value. If one chooses ($\bar{\alpha} = 7.896450$, $\bar{\gamma} = 0.8733$),

Table 2: The slope parameters α_n, γ_n for quarks d, s, b

	d	s	b
n	1	2	3
exp. mass(MeV)	4.1 ÷ 5.8	101 $^{+29}_{-21}$	4190 $^{+180}_{-60}$
α_n	9.4520	7.5226	7.8236
γ_n	0.894	0.867	0.872

then b quark stays within errors, while masses of d and s become 0.842 MeV and 200.6 MeV. At this point one should mention, that light quark masses are not fixed experimentally with good accuracy, since are subject to strong interactions, and in particular the s quark mass inside strange mesons may be taken within an interval of about 100 MeV.

The same procedure for u, c, t quarks, as in previous case leads to the values shown in Table 3.

Table 3: The slope parameters for quarks u, c, t

	u	c	t
n	1	2	3
exp. mass(MeV)	1.7 ÷ 3.3	1270 $^{+70}_{-90}$	172000 \pm 900 \pm 1300
α_n	8.8465	8.90	9.4796
γ_n	0.8869	0.8876	0.8945

Again, if one fixes α_n, γ_n at the c quark value ($\bar{\alpha} = 8.903, \bar{\gamma} = 0.8876$), then the u quark mass stays within experimental limits, while the t quark mass becomes 3.5 times less, $m_t = 47.4$ GeV. This means, that effectively the “trajectory $\alpha(n)$ ” tends to larger values for larger n , and this has an important consequence for the fate of the fourth (or subsequent) generation.

In an alternative approach one can consider the b and t quarks in the framework of the strong integration of the third generation with the Higgs field, which yields the additional mass to both quarks. Indeed, if one keeps the same α for the 3d generation, as for the second, then masses of b and t would be 2.133 GeV and 4.687 GeV respectively and the rest should be supplied by the Higgs (or else by the new TeV scale physics).

In this case the total mass according to (16) is $m_t^2 = m_5^2 + m_4^2 = (172 \text{ GeV})^2$ with $m_5 = 46.87 \text{ GeV}$ and $m_4 = 165.49 \text{ GeV}$. This latter mass easily fits in the uncertainty intervals of the $Ht\bar{t}$ coupling, found experimentally at LHC [1]. Therefore with the unique slope parameter $\alpha_{u,c,t} = 8.90$ and $\mu = 10^{-4} \text{ GeV}$ one obtains experimental mass values, $m_u = 2.657 \text{ MeV}$, $m_c = 1.27 \text{ GeV}$ and $m_t = \sqrt{(46.87)^2 + (165.49)^2} = 172 \text{ GeV}$. In a similar way keeping the same slope $\alpha_{d,s,b} = 7.5226$ for all d, s, b quarks and the Higgs supported mass of the b quark $m_4(b) = 3.60 \text{ GeV}$ one obtains $m_s = 101 \text{ MeV}$ and $m_b = m_b(\text{exp}) = \sqrt{m_4^2(b) + m_5^2(b)} = 4.19 \text{ GeV}$. Note, that the resulting uncertainty in the $Hb\bar{b}$ coupling is again within experimental limits.

Now we come to the neutrino sector and we must first of all change our universal scale μ from 10^{-4} MeV to a smaller value. Below we list in Table 4 the resulting values of slope parameters and neutrino masses.

Table 4: The neutrino slope parameters and resulting masses for the parameter μ_ν in (28), $\mu_\nu = 3.01 \cdot 10^{-6} \text{ eV}$.

	ν_1	ν_2	ν_3
n	1	2	3
resulting masses (10^{-3} eV)	0.434	8.84	51.5
α_n	4.346	4.346	4.346
γ_n	0.77	0.77	0.77

The resulting overall $\bar{\alpha}_\nu$ is $\bar{\alpha}_\nu = 4.346$, i.e. approximately is twice as small compared to three previous generations.

One can check, that the resulting masses satisfy very well the experimental relations [2, 7]

$$|m_3^2 - m_2^2| = (2.44 \pm 0.08)10^{-3} \text{ eV}^2$$

$$m_2^2 - m_1^2 = (7.53 \pm 0.18) \cdot 10^{-5} \text{ eV}^2.$$

In this way the model is able to adjust all fermions with a few parameters.

One can conclude, that the reasonable approximation for the quark and charged leptons occurs with the only scale parameter $\mu = 10^{-4} \text{ MeV}$ and one

fixed slope parameter for each taste (e, d, u) , while only the t quark requires 0.3% larger slope γ_τ . Also remarkably the neutrinos fit in perfectly well in this scheme.

5 CKM matrix and PMNS matrix from the fifth dimension

In the Standard Model (SM) the flavor Lagrangian for quarks has the following form in the mass basis.

$$L^{FL} = \bar{q}_i \hat{D} q_i + \frac{g}{\sqrt{2}} \bar{u}_L^i \hat{W}^+ V_{CKM}^{ij} d_L^j + \bar{u}_L^i \lambda_u^{ij} u_R^j \left(\frac{(v+h)}{\sqrt{2}} \right) + \bar{d}_L^i \lambda_d^{ij} d_R^j \left(\frac{(v+h)}{\sqrt{2}} \right) + h.c. \quad (29)$$

where V_{CKM}^{ij} is the CKM matrix.

It is assumed, that the W interaction violates the original structure of \hat{u}, \hat{d} vectors, corresponding to the u, d mass eigenvalues, generated by two last terms in (29), and this fact leads to the nondiagonal matrix structure of V_{CKM} .

The last two terms in (29) are basically important for the mechanism of fermion masses, since fixation of the constants $\lambda_{ij}^u, \lambda_{ij}^d$ by the corresponding values of masses, e.g. of b or c quarks allows to predict uniquely the yield of $b\bar{b}$ and $c\bar{c}$ quarks in the Higgs decay [3, 4]. However the accuracy of the LHC data in PDG [2] is not enough to justify fully this mechanism. Therefore we have suggested above in this paper another mechanism of mass generation – the fractional dynamics mechanism and allow the Higgs terms in (29), but consider them as additional terms with undefined couplings λ^u, λ^d , which can be also vanishing, or provide additional mass values e.g. for the third generation (b, t) .

In what follows we shall leave this formalism of SM and instead exploit the fermion wave function in the fifth coordinate.

Indeed, we have 4 sets of eigenfunctions for ν, e, d and u generations, which depend on the same fifth coordinate y , defined in the fixed region, for example, $y_{\min} \leq y \leq y_{\max}$. These sets we associate with current fermion states, which may not coincide with mass eigenstates described in the previous section.

These four sets can be denoted as $\chi_i^{(f)}(y)$, $f = e, \nu, d, u$.

Since the fermion masses are given by $m_5^{(f)}(n)$, one can neglect in the first approximation any Higgs generated contribution and define the $V_{CKM}^{(5)}$ as

$$V_{ik}^{CKM} = \int_{y_{\min}}^{y_{\max}} \chi_i^{+(f)}(y) \chi_k^{(f')}(y) dy. \quad (30)$$

In the quark case $(f, f') = (u, d)$, and in the lepton case $(f, f') = (e, \nu)$.

It is important, that we assign in this way the 5d integral to the matrix elements containing W vertices with fermions.

Instead all terms with γ, Z , i.e. the neutral currents yield the diagonal matrix in i, k , i.e. the FCNC contribution is zero in this approximation and higher order (e.g. box) diagrams are needed to ensure nonzero results, as it is supported by experimental data [5, 6].

One may ask, which properties of the flavor sets produce the observed structure of the CKM matrix with its almost diagonal form for $(f, f') = (u, d)$ and the TBM form for $(e\nu)$.

It is clear, that these flavors differ in their 4d interactions; not so strongly in the case of (u, d) pair, and significantly in the case of $(e\nu)$. Thus one must deduce, that $\chi_i^{(f)}(y)$ may be influenced by the 4d interactions and flavor symmetries.

One may compare the matrix V_{CKM} with angles between two orthogonal 3d systems of coordinates, which are only slightly rotated with respect to each other in the case of (u, d) and strongly rotated in the $(e\nu)$ case.

We start with the CKM mixing matrix, which has the following structure in the Wolfenstein parametrization

$$V_{CKM}^{ij} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} = \begin{pmatrix} 1 - \frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1 - \frac{\lambda^2}{2} & A\lambda^2 \\ A\lambda^3(1 - \rho - i\eta), & -A\lambda^2 & 1 \end{pmatrix} \quad (31)$$

with the estimates [2]

$$\begin{aligned} |V_{ud}| \simeq |V_{cs}| \simeq |V_{tb}| \simeq 1, \quad |V_{us}| \simeq |V_{cd}| \cong 0.22 \\ |V_{cb}| \simeq |V_{ts}| \simeq 4 \cdot 10^{-2}, \quad |V_{ub}| \simeq |V_{td}| \simeq 5 \cdot 10^{-3} \end{aligned} \quad (32)$$

and we are choosing the set of orthonormal functions $\chi_i^{(d)}, \chi_k^{(u)}$ denoting $\chi_i^{(d)} (i = d, s, b), \chi_k^{(u)} (k = u, c, t)$ $\chi_i^{(d)} = U_{in}^{(d)} \psi_n(\varphi); \chi_k^{(u)} = U_{kn'}^{(u)} \psi_{n'}(\varphi)$, one has

$$V_{CKM}^{ij} = \int_0^\pi \chi_i^{*(u)} \chi_j^{(d)} d\varphi = (U^{(u)})_{in}^* U_{jn}^{(d)} = U^{(d)} U^{(u)+}{}_{ji}. \quad (33)$$

Assuming, that $U^{(d)}, U^{(u)}$ are unitary matrices, one ensures the unitarity of the resulting CKM matrix V_{CKM} e.g. in the simplest example one takes

$$\psi_n(\varphi) = \sqrt{\frac{2}{\pi}} \sin n\varphi, \quad n = 1, 2, 3, \dots, 0 \leq \varphi \leq \pi, \varphi = \pi\mu y. \quad (34)$$

To produce the mixing one can use two different orthonormal sets, which can be constructed from (34). To simplify the matter we choose for the (u, c, t) the original set (34) with $n = 1, 2, 3$, while for (d, s, b) one can assign as a first approximation the set

$$\begin{aligned} \chi_1^{(d)} &= \sqrt{\frac{2}{\pi}} (V_{11} \sin \varphi + V_{12} \sin 2\varphi + V_{13} \sin 3\varphi) \\ \chi_2^{(d)} &= \sqrt{\frac{2}{\pi}} (V_{21} \sin \varphi + V_{22} \sin 2\varphi + V_{23} \sin 3\varphi) \\ \chi_3^{(d)} &= \sqrt{\frac{2}{\pi}} (V_{31} \sin \varphi + V_{32} \sin 2\varphi + V_{33} \sin 3\varphi), \end{aligned} \quad (35)$$

where V_{ij} are the same as in (31).

Here the, orthonormality of the set $\chi_i^{(f)}$ is restored, when explicit unitary values of V_{ik} are used in (35), so that the unitarity of V_{CKM} is supported, when two orthonormal sets of $\chi_{u,c,t}$ and $\chi_{d,s,b}$ are used for the definition of V_{CKM} as in (4).

It is clear, that the resulting CKM matrix given in(31) will be exactly reproduced by the integrals

$$V_{ij} = \int_0^\pi \chi_i^{(u)+}(\varphi) \chi_j^{(d)}(\varphi) d\varphi. \quad (36)$$

It is clear, that the sets of $\chi_i(\varphi)$ can be chosen in many different ways, should depend on 4d SM quantum numbers.

We now turn to the PMNS matrix U_{PMNS} , which we also define in the fifth dimension,

$$\chi_i^{(\nu)} = U_{in}^{(\nu)} \psi_n(\varphi); \quad \chi_k^{(l)} = U_{kn}^{(l)} \psi_n(\varphi) \quad (37)$$

with the result

$$U_{ik}^{PMNS} = \int_0^\pi d\varphi \chi_i^{(l)+}(\varphi) \chi_k^{(\nu)}(\varphi) = U_{nk}^{(l)+} U_{in}^{(\nu)} = (U^{(\nu)} U^{(l)+})_{ik} \quad (38)$$

Again, as in the case of U^{CKM} , the unitarity of U^{PMNS} is ensured by the unitary matrices $U^{(\nu)}, U^{(l)}$.

We shall be using the standard parametrisation of U^{PMNS} with the convention of the smallest mixing angle in $s_{13} = \sin \theta_{13}$, [2, 7], also assuming the Dirac neutrino nature.

$$U^{PMNS} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \quad (39)$$

with the best fit values [2, 7]

$$\begin{aligned} \sin^2 \theta_{12} &= 0.308 \pm 0.017; & \sin^2 \theta_{23} (\Delta m^2 > 0) &= 0.437_{-0.023}^{+0.033} \\ \sin^2 \theta_{13} (\Delta m^2 > 0) &= 0.0234_{-0.0019}^{+0.0020}, & \delta/\pi &= 1.39_{-0.27}^{+0.38}. \end{aligned} \quad (40)$$

In what follows we choose, as in the CKM matrix case, the more massive lepton matrix $U^{(l)}$ to be diagonal, $U_{ij}^{(l)} = \delta_{ij}$. As a consequence one has $U^{PMNS} = U^{(\nu)}$. One can see, that even with the simplistic choice of the matrices $U^{(\nu)}, U^{(l)}$ and the orthonormal set of functions $\{\sqrt{\frac{2}{\pi}} \sin(n\varphi)\}$ allows to reproduce U^{PMNS} . In this way the choice of the fifth dimension as the source of the generation dynamics, and at the same time, of the dynamics behind the flavor mixing, might be of interest for the future development. For instance, the matrices $U^{(\nu)}$ and $U^{(l)}$ depend on the symmetry and interaction in the $\{\nu\}$ and $\{l\}$ generations, or, rather, on the ‘‘projections’’ of these symmetries in the x_5 space.

6 Conclusion and discussion

We have studied above the possible way to obtain flavor hierachical masses from one scale μ , and the high power kinetic term p^α in the extra dimension. Surprisingly the resulting masses for charged leptons and quarks are described well with one scale μ and slightly different slope parameters, while neutrinos need another scale μ and unique slope.

We have extended this analysis to calculate the CKM matrix via the overlap integrals of extra dimensional fermion wave functions and obtain reasonable results when the full current sets of (u, c, t) and (d, s, b) as functions of x_5 are slightly different.

It is important, that we assign in this way the 5d integral to the matrix elements containing γ , Z , and W vertices with fermions.

As a result all terms with γ , Z , i.e. the neutral currents yield the diagonal matrix in n, n' i.e. the FCNC contribution is zero in this approximation and higher order (e.g. box) diagrams are needed to ensure nonzero results, as it is supported by experimental data (see e.g. [5] and [6]).

In this picture one obtains, however, many excited fermionic states with $n > 3$, and one can associate these states with dark matter, if those do not acquire nonzero charges in weak, strong and electromagnetic interactions, as may be dictated by symmetry requirements. Indeed one may argue, that three lowest generations are subject to some kind of the three-fold (replica) symmetry, as it was assumed before in [34, 35, 36], also using extra dimensions in [37, 38, 39, 40, 41], and recently in the discrete flavor symmetry approach [7, 42, 43] or else in the three-site gauge model of flavor hierarchy [44], where the Pati-Salam symmetry model is considered as a product $(PS)^3$. In this framework one can treat the higher generations as not connected to the standard SM symmetries and interactions. The fermions of these higher generations would not have any charges, except for the mass – the gravitational charge, and hence behave as particles of the dark matter. The resulting high mass of these states ($m_u^* \geq 2.60$ TeV, $m_d^* \geq 47.8$ GeV, $m_e^* \geq 15.61$ GeV, $m_\nu^* \geq 0.18$ eV) might explain the mass dominance of dark matter over the visible one. The fundamental ground of the presented approach, which would allow to calculate α for different fermions, is still missing and waits for additional study.

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Appendix 1

From fractional Lagrangian to fractional path-integral Hamiltonian

I. We start with a quantum mechanical example and consider the 1d Lagrangian

$$L = \eta(\dot{x}^2)^{\beta/2} - V; \quad p = \frac{\partial L}{\partial \dot{x}} = \eta\beta\dot{x}(\dot{x}^2)^{\frac{\beta}{2}-1} \quad (\text{A1.1})$$

and the resulting Hamiltonian assumes the form

$$H = p\dot{x} - L = \eta(\beta - 1)(\dot{x}^2)^{\beta/2} + V = \eta(\beta - 1) \left(\frac{p^2}{\eta^2\beta^2} \right)^{\frac{\beta}{2(\beta-1)}} + V. \quad (\text{A1.2})$$

Writing (A1.2) in the form $H = \mu^{1-\alpha}p^\alpha$, one can see that $\alpha = \frac{\beta}{\beta-1}$ may be however large, when β tends to 1, implying a strong hierarchy of eigenvalues.

II. We now turn to the field theory and define the structure of the kinetic part of Lagrangian for the scalar field $\varphi(x, y \equiv x_5)$.

$$L \rightarrow \left(\frac{\partial\varphi^+}{\partial x_\mu} \right) \left(\frac{\partial\varphi}{\partial x_\mu} \right) + \mu^{2-2\alpha} L\varphi^+ (\partial_5^2)^{\alpha-1/2} \varphi + (\partial_5^2)^{\alpha-1/2} \varphi^+ \partial_5 \varphi. \quad (\text{A1.3})$$

This gives for the field propagator

$$D_\varphi = \frac{1}{m_q^2 - \partial_\mu^2 - \mu^{2-2\alpha} \partial_5^{2\alpha}}. \quad (\text{A1.4})$$

In the same way one obtains the propagator for the fermion, where the form (A1.4) appears in the denominator.

III. We now turn to the path integral form of the propagator D_φ , doing the same transformations as in [29], but now with $\partial^{2\alpha}$ instead of ∂^2 in the denominator. One has using [23]

$$\begin{aligned} D_\varphi &= \langle 1 | \int_0^\infty ds e^{-s(m_4^2 - \partial_\mu^2 - \mu^{2-2\alpha} \partial_5^{2\alpha})} | 2 \rangle = \\ &= \langle 1 | \int_0^\infty ds (D^5 z)_{12} e^{-K} \Phi_{4d}(x, y) | 2 \rangle. \end{aligned} \quad (\text{A1.5})$$

Here $K = K_4 + K_5$ with

$$K_4 = \int_0^s d\tau \left(m_4^2 + \frac{1}{4} \left(\frac{dz_\mu}{dt} \right)^2 \right). \quad (\text{A1.6})$$

To arrive at K_5 one should use the procedure, described in [29] for the standard path integral, see also [21] in the case of the fractional derivatives. To this end consider one step of the path integral in the x_5 coordinate

$$\langle y_{k+1} | e^{-\Delta\tau(-\partial_5^2)^\alpha \mu^{2-2\alpha}} | y_k \rangle = \int dq e^{iq(y_{k+1}-y_k) - \Delta\tau q^{2\alpha} \mu^{2-2\alpha}}. \quad (\text{A1.7})$$

Finding the extremum of the integral in dq one obtains the exponent

$$\exp\left(-\int_0^s d\tau \mu^2 \left[-\frac{1}{4\alpha^2} \left(\frac{dy}{d\tau}\right)^2\right]^{\frac{\alpha}{2\alpha-1}}\right) = \exp(-K_5). \quad (\text{A1.8})$$

One can see, that K_5 tends to the form K_4 in the limit $\alpha \rightarrow 1$ (with $m_4 \equiv 0$).

To obtain the path integral Hamiltonian one can use the same relations as in the part I of the Appendix with $\frac{\alpha}{2\alpha-1} = \frac{\beta}{2}$ yielding finally

$$H_5 = Ap_5^{2\alpha} \quad (\text{A1.9})$$

where A is made of constants α and μ . Writing finally as in [29] $d\tau = \frac{dt_4}{2\omega}$ one arrives at the form given in Eq. (3). A more accurate form, is obtained in the same way as in section (3.1) of [21], in the form of the Fox H-function.

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