Around Logical Perfection

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Abstract

In this article we present and describe a notion of "logical perfection". We extract the notion of "perfection" from the contemporary logical concept of categoricity. Categoricity (in power) has become in the past half century a main driver of ideas in model theory, both mathematically (stability theory may be regarded as a way of approximating categoricity) and philosophically. In the past two decades, categoricity notions have started to overlap with more classical notions of robustness and smoothness. These have been crucial in various parts of mathematics since the nineteenth century.

We postulate and present the category of logical perfection. We draw on various notions of perfection from mathematics of the 19th and 20th centuries and then trace the relation to the concept of categoricity in power as a logical notion of what a "mathematically perfect" structure is.

This essay is an attempt to present the idea of **logical perfection** to a philosophical audience. This expression is often used informally in mathematical practice and sometimes also in more formal discussion around mathematics. This happens sometimes in the form of an aesthetic criterion, and is one of the strongest drivers of mathematical activity and one of the main tests for its relevance. Since the advent of the discipline of mathematical logic it has become possible to investigate a potentially adequate formal notion by mathematical means. Roughly, before giving more detail, *a mathematical* *object of a certain "size" is logically perfect if in a certain formal language it allows a "concise" description fully determining the object.*[1]

This notion, in particular, is central in the third author's paper [14] and has been implicitly present, mainly as a motivating factor, in a number of other research papers in various branches of mathematics.

Of course, writing for a wider audience means we skip many subtle mathematical details and avoid as much as possible using technical terms.

We draw on various analogies to show that logical perfection has strong versions outside of mathematics. Moreover, we will argue that logically perfect structures can be used for the study of the physical world, making the idea relevant not just in mathematics but in the realm of physics too. The question whether logical perfection manifests itself in other areas of the human activity (such as art) is left open here; we may only hope it will raise the interest of some of our readers.

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1 Why logical perfection?

The interest in looking for some kind of *perfection* in mathematical structures is not new. In the history of their discipline, mathematicians have been driven to think about this kind of perfection, albeit for different sorts of motivations, and have tried to capture this idea by means of mathematical tools. Let us mention a few of these attempts in the work of Galois, Riemann and Grothendieck and build up a first collection of examples for our discussion of logical perfection.

Galois made a bold switch from the classical perspective of looking directly for solutions to algebraic equations to a study of the *symmetry* of possible solutions: a move toward a *completion* of the set of *possible solu-*

[1]The essay has arisen from many conversations and collaboration the authors have had during the past few years and it is originally based on two talks the first author gave, one in Paris and the other one in Bogotá, about the work of the third author. The second author then brought some additional perspective. The first author wants to mention in particular the talk given at Bogotá during the workshop *Mapping traces: Representation from Categoricity to Definability* organised by the second author and María Clara Cortés at Universidad Nacional de Colombia in 2014. This was very helpful since the audience consisting of philosophers, mathematicians and artists made the idea of writing about logical perfection for a general audience possible.

tions by means of the study of the group of symmetries of all the solutions that could exist, and by filtering out the interaction between enlarging the field where these solutions could appear and the group of such symmetries. The resulting theory (aptly named much later "Galois Theory") goes way beyond the initial quest for solutions to algebraic equations and changed the ground to an idea (the symmetry of possible solutions in extensions of fields, and the duality between the groups of these symmetries and the field extensions) that is still central after two centuries, and whose scope goes much beyond the wildest dreams Galois could have had. This is a first form of "perfection" for us: **completeness of possible solutions,** and **register of emerging symmetry.** More importantly even, underlying these two aspects of perfection arising in Galois' work, there is a kind of *uniqueness*, only reached once all possible solutions (and their symmetries) are considered. This wholeness, this uniqueness, albeit implicit in the work of Galois, is a component of the main tenets of the notion of *logical* perfection we propose.

Our second example is Riemann's work on the foundations of geometry. In a move parallel to Galois', he went beyond understanding geometry in terms of global axioms and laid the ground for a "local" approach, driven by a "metric" (a way of measuring distances) that could change, twist, curve *itself*. Instead of placing objects such as curves, planes, surfaces inside a "global space" (as had been done for aeons in mathematics), Riemann put the *twisting* itself, so to speak, at centerstage: instead of placing twisted, curved objects "in" a space, the space itself became the twisting. Here, the notion of logical perfection is of a different kind from what we had in the Galois example. There, global symmetry (and the connection between symmetry and extensions where solutions live) was the expression of that perfection. Here the perfection is rather the new *flexibility* Riemann's construction offers us, when compared with earlier incarnations of "space", of geometry[2].

Our third example, much more contemporary and of a different kind, is Grothendieck's new foundations of algebraic geometry. Very roughly, the concept of a general notion of "space" is again at stake. But here Grothendieck essentially first "disassembles" the surfaces or curves (called

[2]Suffice it to say that half a century later, Einstein would base his General Relativity Theory on Riemann's work: the mathematical content of Einstein's theory is essentially present in Riemann's approach to geometry. Here, the "perfection" aspect has more the flavor of a way to construct *many* possible geometries, one for each "Riemann metric" —one for each way to "twist" space, so to speak— and a global treatment of all of these geometries (and moreover, mathematical ways of classifying and comparing them).

more generally "varieties" by mathematicians) by putting all the weight of the analysis into one single aspect (localisation) of the space and then finding a system for placing these localisations in a coherent way. By doing this, Grothendieck creates a version of "space" (called *affine scheme*) that embodies two movements: first, the localisation (and the possibility of treating only *one aspect* of the space) and second, the coherence. This highlights yet a different aspect of logical "perfection": the possibility of regarding space as a coherent way of pasting localised versions of itself.

What is new in the approach presented here is that we claim the existence of a relevant rigorous mathematical concept which allowed an amazingly deep theory, and has led to a new understanding of a number of specific structures central to modern mathematics. This rigorous concept is defined within the discipline called **model theory**, a subdiscipline of mathematical logic, which deals with formal languages and their semantics. One can confidently claim that the central concept of present-day model theory is that of **stability** of formal theories and one key notion of stability theory (from which it started in the 1960s) is that of **uncountably categorical** theories. Through the efforts of many people, and most prominently by contributions of S. Shelah (see [7]), we now have a rather comprehensive **classification theory** which establishes an effective hierarchy in the "universe" of mathematical structures (or their theories)[3]. The hierarchy is effectively based on the complexity of the system of invariants which ultimately describe a given structure, a model of a formal theory[4]. The highest level of the hierarchy corresponds to the simplest system of invariants. This corresponds, in some sense, to a highest level of perfection.

The previous emphasis on the stability hierarchy, and in particular the region near its "top" (uncountably categorical theories) *describes* a mathematically rigorous (and completely abstract) approach to a notion relevant to a working definition of logical perfection. We still have to address the issue of how adequate and useful this notion is, which dividing lines it draws and which important mathematical structures satisfy the criteria.

[3]An interesting interactive visualisation of "a map of the universe" can be seen online at http://www.forkinganddividing.com . Unfortunately, the graphics present it as a flat landscape although there is a natural feel that the "more stable" structures should be at higher levels of the landscape.

[4]Shelah also uses the criteria of whether the given first-order theory has a *structure theorem*, that is if the isomorphism types of models of the theory can be classified in terms of a simple combinatorial structure.

An interesting observation from the cumulated experience with the study of the stability hierarchy would establish (very roughly) that *the higher a structure is in the hierarchy, the closer it is to a "centre of mathematical universe".* We may take this center to be algebraic geometry in the broadest sense[5]. In some (limited) sense we may define the most general form of geometry to be the structures populating the top levels of stability hierarchy[6].

2 Logical perfection and the issue of uniqueness

In the previous section we posited one reason why we may consider *categoricity* (in uncountable cardinals) as a center of classification theory: the observation that many "central mathematical structures" (those from algebraic geometry or those corresponding to linear phenomena) seem to hover close to that region[7].

The notion of categoricity concretises the meaning of *uniqueness.* One says that a collection of statements in a formal language *(set of axioms)* is *categorical* if it has just one model, up to isomorphism. This expression "up to isomorphism" means that we do not want to distinguish two structures if they differ only by the way their elements are presented.

The choice of the formal language is very essential. Usually it is meant to

[5]Definining exactly what algebraic geometry "in the broadest sense" means is not immediate, but we may take Grothendieck's ideas as the main guide.

[6]Working on this presumption one arrives at a meaningful notion of non-classical geometric spaces (see [1], [13], [15] and the discussion in section 3) which in a more conventional mathematical setting are treated via the formalism of *non-commutative (or quantum) geometry.* The latter approach is essentially a syntactic algebraic analysis avoiding geometric semantics.

[7]There are important exceptions to this reason. The first one is obvious: real numbers are far from being categorical yet are also clearly central mathematical structures in many senses. However, aside from the infinite order that is the reason for their non-categoricity, the exhibit a rather simple structure of *definable sets*: each one of them is really a finite union of intervals. This notion, called **o-minimality,** provides reasons to place them in a region where some of the good properties of uncountably categorical structures still work, albeit in a different way. The role of interactions between complex analysis and real analysis is mimicked by this correspondence. The second exception is subtler: classification theory provides many other regions that, while not corresponding to the "supremely perfect" uncountably categorical region, they exhibit very strong regularity and smoothness properties.

be a first-order language, that is one which allows only finite length formulas and quantifiers "for all" and "there exists" which refer to elements of the structure in question (but not to relations or functions). However, as the research in the last three decades has shown, much of what will be said below about categoricity in the first-order context holds in a more general setting.

The notion of categoricity has existed for as long as logic has been formalised. But in the context of first order languages one realises very quickly, from basic facts of the theory, that the above *absolute* categoricity can only hold for descriptions of finite structures. For infinite structures $\mathbf{M}$ it is possible to have uniqueness in some cases if we add to the first order description the (non-first-order) statement fixing the cardinality κ of the structure $\mathbf{M}$. This relative categoricity is called **categoricity in cardinality (in power)** κ or κ-categoricity.

One has to distinguish two types of cardinalities in the context of categoricity, namely, uncountable (large) and countable (the minimal infinite) categoricity. We are interested in uncountable categorically describable structures which entails that the structure is much bigger than the size of its description. A remarkable fact was proved by Michael Morley in 1964, namely, that categoricity in one uncountable cardinality implies the categoricity in all uncountable cardinalities: the actual value of the uncountable cardinal is irrelevant[8].

The study of this kind of structures has been in the focus of research in model theory for at least 60 last years. The amazing conclusion derived from the research is that among the huge diversity of mathematical structures there are very few which satisfy the (slightly narrower) definition of categoricity, and those can be classified. These certainly seem to corresponding to an ideal of *logical perfection*, in the following sense: categorical structures M determine a first order theory $Th(M)$ (the set of all sentences that are true in M) and then comes the reason why we call them "logically perfect": **all**

[8]This is in sharp contrast with countable categoricity. Countably categorical structures might also in some sense be candidates to a kind of perfection, probably - but all the geometric features of uncountably categorical structures are lost in that case. This dependence on the cardinality might be regarded as non-logical in some sense, but the case of uncountable categoricity has strong logical properties as well as strong geometric properties.

other structures that satisfy the theory $Th(M)$ and are of the same cardinality as M are isomorphic to M. In other words, uncountably categorical structures are inextricably linked to their logical description; the description $T = Th(M)$ completely determines the structure M (with the usual caveat of "up to isomorphism" and because of limitations in the expressive power of first order logic[9] provided also one considers only structure of the same cardinality as M).

It is not that surprising that a remarkable example of such theory is the theory of the field of complex numbers $\mathbb{C}$ in the language based on algebraic operations $+$ and $\times$. Note that this is the language where algebraic geometry is naturally done[10] but we can not, e.g. distinguish the real part of a complex number, so we can not speak about the real numbers when working over $\mathbb{C}$. Recall that the theory of the field $\mathbb{R}$ of real numbers is not categorical[11].

Complex numbers are present everywhere in mathematics as are the reals. However, there is a significant difference in the theories and in fact complex geometry and the geometry of real manifolds are two different specialisations within mathematics. Classification theory detects the difference and following the above logic in effect claims a certain "priority" of complex geometry.

Of course, for a mathematician the choice of an area of research is a personal matter and is usually made on either historic or aesthetic grounds. Both complex and real geometry are equally respected fields of mathematical research although from our point of view the first is fundamental while the second is auxiliary. We stress again the fact that it was Bernhard Riemann who first understood how real and complex geometries interact with one another and how the study to the latter introduces a whole new range of powerful methods of algebraic geometry into the field.

Different criteria work in the studies of real world. Here the wrong choice of mathematical setting can have adverse effect on the understanding of reality. The mathematical model of Newtonian physics was based on real analytic geometry. This tradition continued into the new physics with the model enriched by more and more uses of complex numbers, seen rather as convenient auxiliary tools. One of the first who pointed to the importance of reversing

[9]Namely, the Löwenheim-Skolem theorem.

[10]In algebraic geometry classical objects are solution sets of algebraic expressions, that is, polynomials written with $+$ and $\cdot$

[11]And is not even stable!

this perspective was Roger Penrose in his 1978 address at the International Congress of Mathematicians under the title "The complex geometry of the natural world", [8]. In more recent decades, with the arrival of string theory, the priority or at least the centrality of complex geometry is undeniable.

To summarise the *logicality* of our notion of perfection: we started with various notions of perfection as we did in Section 1, coming from differing examples in the history of mathematics but then in this section we narrowed our focus to the notion of uniqueness and its logical expression, (uncountable) **categoricity**. Then we remarked that a whole classification theory that encompasses *all first order theories*[12] on the one hand grew up out of the attempts to prove the Morley theorem and its generalisations and on the other hand ended up providing ways of callibrating *exactly how far* from categoricity one is, in terms of smoothness/regularity properties that slowly vanish as we go further and further away from categoricity. It is in this very sense that categoricity has been playing the role of a logical form of perfection. A posteriori we realise that a major part (although not all) of *central* mathematics actually happens to be one of the theories that are uncountably categorical.

3 Logically perfect structures: the role of geometry

Perhaps the most remarkable feature of model-theoretic classification theory is that it exposes a geometric nature of some "perfect" structures. The geometric features of those structures arise from their logical definition, albeit in a highly non-trivial and initially unforeseen way. These were discovered in the course of proving the original ground-breaking categoricity theorem of Michael Morley (see previous section) as the key technical instruments of the proof: Morley rank, homogeneity and, added in later versions of the proof, **dimension** (Baldwin and Lachlan), and associated combinatorial geometries (Marsh, Zilber). It took a while to realise the *geometric character* of the technical definitions and to develop a new geometric intuition around the notions. In particular, Morley rank is a very good analogue of dimension in algebraic and analytic geometry and thus we can think of "curves", "surfaces" and so on in the very general context of categorical and even stable theories.

[12]and in more recent decades much more

This stage of the theory is summarised in the monograph [6] by A. Pillay.

In the 1980s the third author formulated a Trichotomy Conjecture (see [11]) which, based on the above intuition, suggested that any uncountably categorical structure is "reducible" to either an object of algebraic geometry, or linear algebra, or to a simple combinatorial structure. Although in many special classes the conjecture has been confirmed, the general case was refuted by Ehud Hrushovski who found remarkable counter-examples opening fascinating new perspectives on the nature of model theory (its interactions with geometry) and its links with the analytic world.

Around the same time, a way to fix the Trichotomy conjecture was found. This required narrowing the class of structures subject to the conjecture —in some sense, this amounted to refining the notion of logical perfection. This was done by being more careful in choosing the logic in question. Namely, our logical language must be able to distinguish *positively formulated* statements from their negations. The axioms of a good (perfect) theory must be "equational" just like laws of physics and objects of geometry are given by equations (and never by negating equations). And this is already the principle on which algebraic geometry is built on! It studies curves, surfaces, shapes given as solution sets for systems of algebraic equations. Algebraic geometry treats such sets as *closed in the Zariski topology.* The corresponding generalisation of this notion in the context of categorical and stable structures led to the notion of a **Zariski structure** (or *Zariski geometry*) introduced by Hrushovski and the third author.

This improvement in the notion led to a desired Classification Theorem (Hrushovski, Zilber 1993, see [12]):

The class of Zariski geometries satisfies the Trichotomy Principle and therefore Zariski geometries are reducible to[13] classical structures such as the field of complex numbers and vector spaces.

[13]Here "reducible to" can be taken in a first reading as a technical nuisance not requiring much explanation. The typical example of Zariski geometry is a (complex) algebraic variety (glued from affine charts) with possibly a vector bundle over it, a description of which can require quite a lot of technical detail. Such a description eventually reduces to the structure of the complex field itself. However, the constructions described by the theorem can go beyond the technicalities of this example, so beyond algebraic and complex geometry. Ten years after the classification theorem, a closer analysis of what "reducible to" could mean led to the discovery that a huge source of new Zariski structures is noncommutative (or quantum) algebraic geometry, see [13].

It is hard to describe what exactly the subject of geometry as practised by mathematicians is, but non-commutative geometry is a much bigger mystery. It is best identified as the study of algebraic structures, non-commutative *co-ordinate rings*, that supposedly correspond to hypothetical geometric spaces which are not necessarily visualisable. Historically, these were physicists who, starting from the famous "magic paper" of Heisenberg of 1925 [3], have given up to the attempts to describe the physics of micro-world in classical terms and instead used a purely formal algebraic calculus (algebraic quantum mechanics) to successfully explain the behaviour of elementary particles. One can say that the physics of the micro-world lives in an unusual, previously unknown, geometric space which requires a non-commutative algebra to describe. Paralleling this the very centre of the logical universe is occupied by structures which mathematically stem from the same source.

The fusion of geometry with other branches of mathematics, for instance, number theory and representation theory, was one of the biggest programs in the mathematics of the 20th century[14]. We would like to believe that the fusion of logic (model theory) with other branches of mathematics is one of the biggest and ambitious programs of the mathematical research for the 21st century. In particular, the "new geometry" arising from model theoretical considerations has the potential to become an important area of research in mathematics and beyond. And the study of logically perfect structures gives a crucial insight.

Summarizing, the search of logically perfect structures leads to consider geometric/topological ingredients in logic which has as a consequence that a refinement of the idea of logical perfection is obtained. During this process the idea of Zariski structures arises from purely logical considerations but with a geometrical flavor and motivation. So far, our discussion has not left the realm of mathematics but as our previous discussions (and the title of the essay suggests) we want to go beyond mathematics, entering the "real world". A question arises: Are logically perfect structures helpful for understanding the "real world"? We answer this question in the positive and now provide some insights for exploring that possibility.

[14]The figure of Grothendieck was essential in formulating and developing this program in the broadest generality.

4 Logical perfection and physics

We now focus on a different kind of problem: programs for new foundations of quantum gravity, and the issue of tackling an appropriate notion of geometric space for physics. On the face of it, this problem would seem quite remote from our notion of logical perfection. There is however a deep link, as we will describe.

Let us quote again Roger Penrose (his ICM address [8]):

> *"Even at the most elementary level, there are still severe conceptual problems in providing a satisfactory interpretation of quantum mechanical observations in a way compatible with the tenets of special relativity. And quantum field theory, which represents the fully special-relativistic version of quantum theory, though it has had some very remarkable and significant successes, remains beset with inconsistencies and divergent integrals whose illeffects have been only partially circumvented. Moreover, the present status of the unification of general relativity with quantum mechanics remains merely a collection of hopes, ingenious ideas and massive but inconclusive calculations.*
> *In view of this situation it is perhaps not unreasonable to search for a different viewpoint concerning the role of geometry in basic physics. Broadly speaking, "geometry", after all, means any branch of mathematics in which pictorial representations provide powerful aids to one's mathematical intuition. It is by no means necessary that these "pictures" should refer just to a spatio-temporal ordering of physical events in the familiar way..."*

Penrose continues to discuss structures of complex geometry as new geometric tools in quantum physics. However, today this seems to be far from enough. From a similar reasoning the physicist C. Isham and the philosopher of physics J. Butterfield reached a bold program for building a new foundation of quantum gravity physics, based on Grothendieck toposes as the most general form of geometric space (see [5]).

Naturally, Isham-Butterfield is not the only program to tackle the problem (see e,g, the non-commutative geometry approach [2] by A. Connes and M. Marcolli, which however does not reveal a geometric space as such) but it seems to be the most ambitious and general[15].

A project, which may be seen as similar in spirit is suggested and started in [15] and in shorter form in [1]. Like other such programs the key is the respective notion of the geometric space for physics. Our suggestion is based on the philosophy of logical perfection; after all it is reasonable to expect that the geometric structure of the universe should be as perfect as it goes. Correspondingly, the geometric space of quantum mechanics as suggested in [15] emerges from a Zariski structure (see page 9) or rather, from a sheaf of Zariski structures[16].

It is equally important to note that the logical analysis inherent in our method clarifies the correspondence between (possibly noncommutative) algebras as they emerge in physics and geometry and the respective geometric spaces. In essence the algebras present us with the syntactic tools allowing to check in calculations what can be seen graphically and dealt with geometrically. The geometric space is thus a semantic interpretation of the syntactically given data. In classical cases, such as commutative finitely generated algebras, this corresponds to the well-known duality at the foundation of algebraic geometry. For commutative C^*-algebras we have the Gel'fand-Naimark duality linking those to locally compact Hausdorff spaces. In non-commutative cases the situation becomes much more complex but model theory is in the best position to deal with the challenge.

Another, different, line of collusion between categoricity and physics has been explored by D. Howard and I. Toader in the past two decades (see [4, 9]). Their take on categoricity is more akin to the original Veblen formulation than to the role categoricity has acquired in contemporary model theory.

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[15]Maybe too general as to the best of our knowledge there is no interesting calculation produced out of it.

[16]The following three facts clarify the connection between Zariski structures and the Isham-Butterfield topos:

1. The sheaf of Zariski structures, the model of quantum mechanics, can be interpreted as a concrete realisation of an Isham-Butterfield topos.
2. The construction essentially generalises [13] building a Zariski structure corresponding to the non-commutative algebra represented by the canonical commutation relation $\mathrm{QP} - \mathrm{PQ} = i\hbar$.
3. The analysis of the language and definability issues in the structure draws a clear line between notions which are *observable* (in the sense of physics) and which are not.

We finish this section with the conclusion that the principle of logical perfection, as unconventional as it may sound to some, does not disagree with other modern approaches to the mathematical foundations of physics.

5 Concluding remarks

Our concept of logically perfect structures emerges as the result of a fifty year classification project in logic. The theory is deep and technical but the concept can be expressed in simple intuitive terms.

The defining property of logical perfection is uniqueness, or technically uncountable categoricity. This property implies certain internal harmony: homogeneity and the presence of a notion of dimension. This harmony is a manifestation of a certain kind of geometricity, which itself is a consequence of the infusion of geometric/topological ingredients in logic that brings forth the flexibility and generality of logically perfect structures. Finally, since logical structures are at the top of the classification hierarchy, they are suitable as background structures for physics and represent a good idea of geometric space in a very broad sense.

An additional feature to support our notion of logically perfect structures is the "filtration" of perfection provided by classification theory. As mentioned above, classification theory not only places all first order theories in a sort of map *with respect to* categorical theories but provides a kind of measure of going away from perfection. It provides technical ways to measure, for arbitrary theories, what features of perfection might have been lost and which ones remain. The second author's forthcoming interview with Saharon Shelah explores further several peculiarities of this connection[10].

The features described above (uniqueness, geometricity, representability) have concrete mathematical formulations, as we have briefly mentioned. In addition, they help us to understand the role of those structures in the wider program of studying the syntax/semantics duality. As we have tried to show, logically perfect structures can be seen as located in the geometric/semantical side of the mentioned duality, giving a new approach to the notion of noncommutative (or quantum) geometric space, which traditionally has been treated by means of syntactic/algebraic tools. Pursuing this program of interpreting the duality between algebraic and geometric objects as an extension of the duality between syntax and semantics appears to us as one of the most interesting lines of research for the future, not only in mathematics. The idea of representing one object by another (in this case its dual) can certainly be extrapolated beyond mathematics. This idea deserves more investigation.

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