Vacua on the Brink of Decay

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Abstract

We consider free massive matter fields in static scalar, electric and gravitational backgrounds. Tuning these backgrounds to the brink of vacuum decay, we identify a term in their effective action that is singular. This singular term is universal, being independent of the features of the background configuration. In the case of gravitational backgrounds, it can be interpreted as a quantum mechanical analog of Choptuik scaling. If the background is tuned slightly above the instability threshold, this singular term gives the leading contribution to the vacuum decay rate.

Dedicated to the memory of Ludwig Faddeev

Contents

1 Introduction

In this article, we study strong background fields which may be able to destroy their own environment. This happens when the mass gap of the Quantum Field Theory (QFT) in question, due to the external field, tends to zero and eventually becomes negative. We identify a universal singularity in the effective action of the background field, which signals instability of the vacuum, as the mass gap vanishes.

Background field configurations which lead to particle production are associated to the formation of a "horizon", i.e., a length scale in which it becomes energetically more favorable to produce particles than to sustain the field configuration. This definition of the horizon is more general than the one usually discussed in the literature. For example, it implies the existence of electromagnetic horizons. As a typical example of an electric horizon, consider electrically charged particles of mass m in a background electrostatic potential $a_t(x)$. It is clear that if the "voltage" $A \equiv a_t(+\infty) - a_t(-\infty)$ satisfies $A > 2m$ particles will be produced and not much will remain of the vacuum[1]. For a gravitational background, the location of the horizon defined in this new way agrees with that of the causal horizon (see e.g. [1]).

Technically, the phenomenom of particle production can be diagnosed by calculating the vacuum decay rate, given by the imaginary part of the one-loop effective action, obtained after integrating out massive matter fields. Many results are available for the effective action for a constant background field; for an incomplete list of references, see [2–8] and [9, 10] for reviews. In these examples, the horizon is always present[2]. In order to study particle production for field configurations near the threshold, we must consider a gapped matter sector coupled to background fields. The mass gap acts as a barrier, preventing particle production for weak backgrounds. We would like to find singular terms in the effective action as we approach the particle production threshold. In this regime, the effective action is real. If we dial the background field strength above the threshold, the effective action acquires an imaginary piece coming from the singular term. This imaginary part gives the vacuum decay rate.

In this article, we consider different scenarios of strong background fields. We consider background scalar, electric and gravitational fields. We couple these backgrounds to free massive scalar matter, and determine the singular terms in the one-loop determinant of the matter fields in the background geometry tuned to the vicinity of the threshold. The fixed backgrounds are not necessarily a solution of the source-free equations of motion; we assume that there are suitable sources that sustain the static background configuration, and focus on the quantum mechanical response of matter fields to the background.

The electromagnetic threshold singularity might be experimentally testable in the near future by producing strong electric pulses with lasers. The gravitational threshold singularity is a quantum analogue of Choptuik scaling [11]. Choptuik numerically simulated the gravitational collapse of a distribution of dust particles. If the initial data is tuned above a critical value, the final state has a black hole. Choptuik recognized a remarkable scaling law in the mass of the

[1] Similar considerations apply to magnetically charged particles in a background magnetostatic field.

[2] The critical field strength associated to Schwinger e^+e^- pair production, $E_c = m_e^2/e$, gives the electric field value for which the pair production rate becomes non-exponentially suppressed. The pair production rate is, however, nonzero for any value of the background constant electric field, $\Gamma \sim (e^2 E)\exp(-E_c/E)$.

black hole, as a function of how much above criticality the initial data is. For various setups of initial data, he found scaling laws for the black hole mass with the same exponent. Our result shares the same robustness to the shape of the gravitational potential.

While the exact result for the effective action depends on the details of the background field, we argue that the threshold singularity is universal. The physical reason for that is the following: right above threshold, the first pair production event will be very soft, and the pair will have a very long wavelength. As the wavelength of the excitation is very long, it is not sensitive to the fine features of the background.

For an early discussion of these ideas, see [12]. For recent work that partially overlaps with the results presented here, see [13], where universality of the particle production rate is found for electric fields slightly above threshold. Our below-threshold singularity, when extrapolated above the threshold, agrees with the result reported in [13].

Outline In section 2, we compute the threshold singularity for three different backgrounds – scalar, electric and gravitational fields. In section 3, we present our conclusions. In appendix A, we derive in detail a formula for the gravitational effective action in terms of the transmission coefficient; all other cases follow a similar derivation. In appendix B, we discuss the behavior of the transmission coefficient when the mass gap is small. Finally, in appendix C, we quote the exact transmission coefficients for some electric and gravitational backgrounds. The exact results agree with our general considerations in the main text.

2 Threshold Singularities

Having set the stage, let us study the threshold singularity for various quantum field theories in background fields. In this section, we determine the piece in the effective action which becomes singular as a parameter in the external field configuration reaches the threshold. At the threshold, a very long wavelength pair is produced, which can only probe the rough features of the external field configuration. This allows us to find a universal answer for the singularity, regardless of the precise shape of the external field. The nature of the singularity is slightly different for scalar, vector and gravitational external fields.

2.1 Scalar Fields

We first consider a $1+1$ quantum field theory of a free massive scalar field in a static, position dependent background $U(x)$. We want to compute the one-loop effective action

$$\langle \text{out}|\text{in}\rangle = e^{iS_{\text{eff}}(U)} = \int D\phi \, \exp\left(\frac{i}{2}\int \mathrm{d}t\mathrm{d}x \left((\partial_t\phi)^2 - (\partial_x\phi)^2 - (m^2+U(x))\phi^2\right)\right) , \tag{2.1}$$

where we take $U(x)$ to be an arbitrary smooth function with asymptotic values $U(\pm\infty)\to 0$ (see figure 1). A very similar model was studied from a different point of view in [14].

If we consider a family of potentials controlled by some parameters, and tune these parameters in $U(x)$ to a certain threshold value, the effective action acquires an imaginary part. In this case there is a simple way to argue that the threshold singularity will be of square root type and

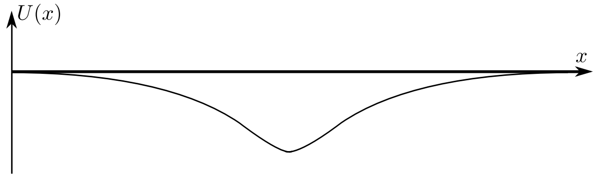

Figure 1: Plot of a schematic form of the potential $U(x)$. As its parameters are tuned, the potential becomes deeper. At threshold, the background is too strong, and a bound state of the quantum field is spontaneously produced.

related to the lowest bound state in the potential $U(x)$. The effective action is proportional to the logarithm of the determinant of the Schrödinger operator

$$iS_{\text{eff}}(U) = -\frac{T}{2}\text{Tr}\log\left(\partial_x^2 + \omega^2 - m^2 - U(x)\right) , \tag{2.2}$$

where we used the Fourier representation for the time coordinate and, as we work in the approximation that the background is static, we obtain a factor of T from the amount of time that the background has been switched on. Assuming that $E_n(U)$ is the spectrum of the operator $-\partial_x^2 + U(x)$, we find

$$iS_{\text{eff}}(U) = -\frac{T}{2}\sum_n \int_{\mathcal{C}} \frac{d\omega}{2\pi} \log\left(\omega^2 - m^2 - E_n(U)\right) , \tag{2.3}$$

where the index n labels discrete and continuous eigenstates. The contour $\mathcal{C}$ in the complex ω-plane is chosen according to the Feynman $i\epsilon$-prescription $m \to m - i\epsilon$, $\epsilon > 0$. On the real axis we have multiple branch points $\omega = \pm\sqrt{m^2 + E_n(U)}$, where we assume that the lowest bound state $E_0(U) > -m^2$ and the contour $\mathcal{C}$ goes above the branch cuts for $\omega > 0$ and below the branch cuts for $\omega < 0$ as shown in fig. 2. Because we assumed that $E_0(U) > -m^2$ we can Wick rotate

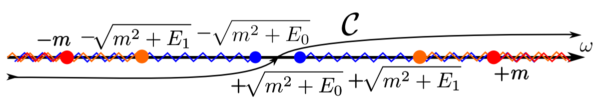

Figure 2: The integration contour $\mathcal{C}$ in the complex ω-plane. At the threshold when $E_0(U) \to -m^2$, the contour $\mathcal{C}$ is pinched by two branch points $\omega = \pm\sqrt{m^2 + E_0}$.

the contour $\mathcal{C}$ along the complex axis, so $\omega \to i\omega$ and we obtain a manifestly real expression for the effective action

$$iS_{\text{eff}}(U) = -i\frac{T}{2}\sum_n \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \log\left(\omega^2 + m^2 + E_n(U)\right) . \tag{2.4}$$

We see that the possibility to Wick rotate is related to the vacuum stability. When the lowest bound state $E_0(U)$ approaches $-m^2$ the branch points start pinching the contour $\mathcal{C}$, and this leads to the appearance of the singular terms in the effective action. So one can easily compute

$$S_{\text{eff}}(U) = -\frac{T}{2}\sqrt{m^2 + E_0(U)} + \ldots , \tag{2.5}$$

where we omitted less singular and non-singular terms.

It is instructive to see how this singularity arises when we express the effective action through the scattering data related to the potential $U(x)$. Namely, below we are going to show that the effective action can be expressed in terms of a logarithm of the transmission coefficient of a wave passing the potential $U(x)$. For the electric and gravitational cases this method will be more convenient.

We begin by differentiating the effective action with respect to the mass, and obtain

$$i\frac{\partial S_{\rm eff}(U)}{\partial m^2} = -\frac{i}{2}\int \mathrm{d}t\,\mathrm{d}x\, G_F(t,x;t,x)\,, \tag{2.6}$$

where $G_F(t,x;t',x') \equiv \langle\phi(x,t)\phi(x',t')\rangle$ is the Feynman Green's function[3]. We can use the Fourier representation for the time components of the Green's function, which then satisfies

$$\left(\partial_x^2 + \omega^2 - m^2 - U(x)\right) G_F(\omega;x,x') = i\delta(x-x')\,. \tag{2.7}$$

By defining mode functions $f_{\rm in}(x)$ and $f_{\rm out}(x)$, which are annihilated by the Schrödinger operator

$$\left(\partial_x^2 + \omega^2 - m^2 - U(x)\right) f_{\rm in/out}(x) = 0 \tag{2.8}$$

and satisfy the following boundary conditions

$$f_{\rm in}(x) \xrightarrow{x\to-\infty} \frac{e^{-ipx}}{\sqrt{2p}},\quad f_{\rm out}(x) \xrightarrow{x\to+\infty} \frac{e^{-ipx}}{\sqrt{2p}}\,,\quad p=\sqrt{\omega^2-m^2}\,, \tag{2.9}$$

we can express the Green's function as

$$G_F(\omega;x,x') = i\frac{f_{\rm in}(x)f^*_{\rm out}(x')\theta(x'-x) + (x\leftrightarrow x')}{W(f_{\rm in},f^*_{\rm out})}\,, \tag{2.10}$$

where the Wronskian is $W(f_{\rm in},f^*_{\rm out}) \equiv f_{\rm in}(x)\partial_x f^*_{\rm out}(x) - f^*_{\rm out}(x)\partial_x f_{\rm in}(x)$. The functions $f_{\rm in}$ and $f_{\rm out}$ are related by Bogoliubov coefficients α and β

$$f_{\rm in}(x) = \alpha f_{\rm out}(x) + \beta f^*_{\rm out}(x)\,, \tag{2.11}$$

where $|\alpha|^2 - |\beta|^2 = 1$, and a simple computation gives $W(f_{\rm in},f^*_{\rm out}) = i\alpha$. We see that $1/\alpha$ is the transmission coefficient, which depends on ω, m and $U(x)$; it can be obtained by solving the quantum mechanical scattering problem (2.8). It is possible to show that the effective action $S_{\rm eff}(U)$ is controlled entirely by the coefficient α [15] (see also [8, 16]).

In order to evaluate the effective action $S_{\rm eff}(U)$, we see from (2.6) and (2.10) that we must compute $\int \mathrm{d}x\, f_{\rm in}(x)f^*_{\rm out}(x)$. It is possible to express this integral through the coefficient α. For this we write the left-hand side of (2.8) for $f_{\rm in}(x)$ with m^2, and multiply the equation by $f^*_{\rm out}(x)$ which is solution of the same equation but with $m^2+\delta m^2$. Analogously we multiply the equation for $f^*_{\rm out}(x)$ with $m^2+\delta m^2$ by $f_{\rm in}(x)$ with m^2. We subtract both expressions, integrate the result

[3] We omit the time ordering symbol of the Feynman Green's function to avoid confusion with the time T that the background is switched on.

over x, and keep the first nontrivial terms in δm^2. This gives $\int \mathrm{d}x f_{\mathrm{in}}(x) f^*_{\mathrm{out}}(x) = -i\partial\alpha/\partial m^2$, where we used the Feynman $i\epsilon$-prescription $m \to m - i\epsilon$, $\epsilon > 0$. We finally obtain

$$S_{\mathrm{eff}}(U) = \frac{1}{2} iT \int_{\mathcal{C}} \frac{\mathrm{d}\omega}{2\pi} \log \alpha(\omega)\,, \tag{2.12}$$

where the choice of the contour $\mathcal{C}$ is explained above and shown in figure 2. In appendix A, we give a detailed derivation of this formula for the gravitational case, which is technically the most complicated.

As we see from (2.12), finding $S_{\mathrm{eff}}(U)$ has now been reduced to a 1-D scattering problem. Again the singularity arises in the integral (2.12) when the contour $\mathcal{C}$ is pinched by branch points. The result (2.12) is not so surprising, as indeed in the scattering theory it is well-known that the transmission coefficient $1/\alpha$ is an analytic function of energy E on the physical sheet $\sqrt{E}$ ($\mathrm{Im}\sqrt{E} > 0$), except for the points of discrete spectrum $E = E_n$, in which the amplitude has simple poles. Thus the coefficient $\alpha \sim (p - i\sqrt{|E_0|})/(p + i\sqrt{|E_0|})$ near the pinching branch points and computing the integral (2.12) for $E_0(U) \to -m^2$ one recovers the result (2.5). So we see that in the scattering approach the singularity mechanism is similar. More generally, the relation between scattering data and the determinant of the Schrödinger operator is well-known and has been thoroughly investigated [17].

2.2 Electric Fields

Let us now consider the case of a free massive complex scalar in a strong electric field. We work in $1+1$ dimensions, but some of our results can be generalized to higher dimensions. The one-loop effective action is given by

$$\langle \mathrm{out}|\mathrm{in}\rangle = e^{iS_{\mathrm{eff}}(a)} = \int D\phi D\bar{\phi}\, \exp\left(i\int \mathrm{d}t\mathrm{d}x \left(|\partial_t\phi + i a_t \phi|^2 - |\partial_x \phi|^2 - m^2|\phi|^2\right)\right), \tag{2.13}$$

where we picked the static gauge $a = a_t(x)\mathrm{d}t$ for our background configuration, and chose $a_t(x)$ to be a smooth and monotonic function with asymptotic values $a_t(-\infty) = -A/2$ and $a_t(+\infty) = +A/2$ (see fig. 3). The asymptotic values $\pm A/2$ are symmetric without loss of generality, by

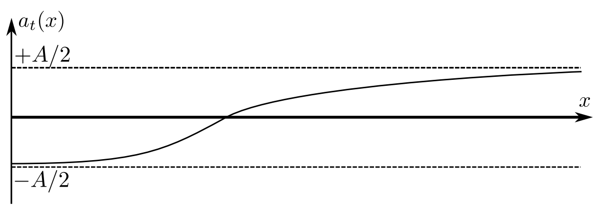

Figure 3: Plot of a schematic form of the potential $a_t(x)$. We assume that $A < 2m$.

a simple shift of the potential. Other than monotonicity, we do not require the curve $a_t(x)$ to have any special property. We will see that particle production becomes favorable if $A > 2m$. If $A < 2m$, the effective action will be purely real and the vacuum is stable. As $A \to 2m$ from below, we will show that the effective action acquires a logarithmic singularity.

To gain some intuition of the pair production threshold in the electric case, we analyze the classical equations of motion, using band theory, in the asymptotic regimes $x \to \pm\infty$. The energies of excitations are given by

$$\omega_\pm = a_t(x) \pm \sqrt{p^2 + m^2}\,. \tag{2.14}$$

In figure 4, we see that the maximum and minimum points of the energy move as one goes from $x = -\infty$ to $x = +\infty$. When the bottom of the valence band comes up to the top of the conduction band, it becomes energetically favorable to disrupt the vacuum by pair production. This implies that threshold is reached when:

$$\text{min}\ (\omega_+) - \text{max}\ (\omega_-) = A - 2m > 0\,. \tag{2.15}$$

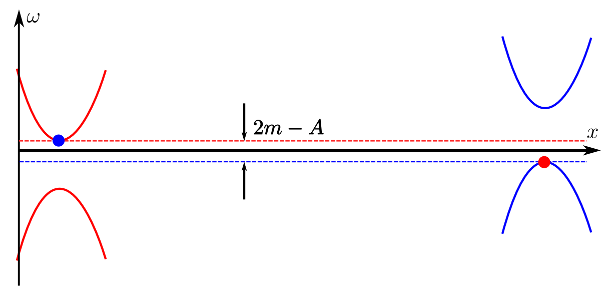

Figure 4: Plot of the “bands” of the matter field. When the background is too strong the bottom of the valence band comes up to the top of the conduction band and it becomes energetically favorable to trigger the tunneling and disrupt the vacuum by pair production.

Now we proceed with the calculation of the effective action. Once again, it is convenient to differentiate it by mass

$$i\frac{\partial S_{\text{eff}}(a)}{\partial m^2} = -i\int \mathrm{d}t\,\mathrm{d}x\, G_F(t,x;t,x), \tag{2.16}$$

where $G_F(t,x;t',x') \equiv \langle \phi^*(x,t)\phi(x',t')\rangle$ is the Feynman Green's function. We can use the Fourier representation for the time components of the Green's function, which then satisfies

$$\left(\partial_x^2 + (\omega - a_t(x))^2 - m^2\right) G_F(\omega;x,x') = i\delta(x-x')\,. \tag{2.17}$$

Finding $S_{\text{eff}}(a)$ has now been reduced to a 1-D scattering problem similar to the scalar case we treated above. So we define mode functions f_{in} and f_{out} which are annihilated by the operator in the left hand side of (2.17). In terms of f_{in} and f_{out}, the Green's function is given by

$$G_F(\omega;x,x') = i\frac{f_{\text{in}}(x)f^*_{\text{out}}(x')\theta(x'-x) + (x \leftrightarrow x')}{W(f_{\text{in}}, f^*_{\text{out}})}\,. \tag{2.18}$$

The functions f_{in} and f_{out} satisfy the following boundary conditions

$$f_{\text{in}}(x) \xrightarrow{x\to-\infty} \frac{e^{-ip_- x}}{\sqrt{2p_-}}, \quad f_{\text{out}}(x) \xrightarrow{x\to+\infty} \frac{e^{-ip_+ x}}{\sqrt{2p_+}}\,, \tag{2.19}$$

where $p_\pm = \sqrt{(\omega \mp A/2)^2 - m^2}$. The two solutions are related by Bogoliubov coefficients, $f_{\text{in}}(x) = \alpha f_{\text{out}}(x) + \beta f^*_{\text{out}}(x)$. Using the same method as in the scalar case, we obtain for the effective action

$$S_{\text{eff}}(a) = iT\int_{\mathcal{C}} \frac{\mathrm{d}\omega}{2\pi}\, \log\alpha(\omega)\,. \tag{2.20}$$

The contour $\mathcal{C}$ must be chosen according to the Feynman $i\epsilon$-prescription $m \to m - i\epsilon$, $\epsilon > 0$, which gives $p_\pm \to p_\pm + i\epsilon$. There are multiple branch cuts on the real ω-axis. They start at the points corresponding to zeros of p_- and p_+ and also α. Therefore the contour $\mathcal{C}$ should go below the branch cuts for $\omega \to -\infty$ and above the branch cuts for $\omega \to +\infty$ and pass between the left and right branch cuts near $\omega = 0$ (see fig. 5). In general we may have a branch point which corresponds to $\alpha = 0$ but when A is very close to $2m$ the branch cuts corresponding to $p_- = p_+ = 0$ will pinch the contour $\mathcal{C}$ first. We see that this mechanism is different from the scalar case, where the effect is due to branch cuts corresponding to $\alpha = 0$.

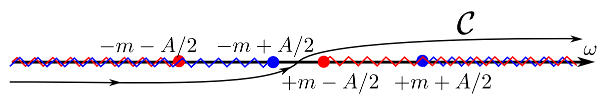

Figure 5: The integration contour $\mathcal{C}$ in the complex ω-plane. The branch cuts here correspond to points where $p_- = p_+ = 0$. When the electric background near the threshold $A \to 2m$, the branch points $\omega = \pm(m - A/2)$ pinch the contour $\mathcal{C}$.

So as $A \to 2m$ the threshold is reached when the contour $\mathcal{C}$ is pinched by the branch points at $\omega = +m - A/2$ and $\omega = -m + A/2$. This already hints at some universality, meaning that the most singular piece in the effective action will be largely agnostic about the particular shape of $a_t(x)$, but only depend on how close to threshold its maximum value is. Since the singularity appears as the branch points pinch the contour $\mathcal{C}$, we need to look at $\alpha(\omega \approx \pm(m - A/2))$. As the band gap closes, we can use an argument which gives a general form of the coefficient α. Leaving the details to appendix B, when $\omega \approx \pm(m - A/2)$ and $A \approx 2m$, the coefficient α is given by an infinite series in small p_+ and p_- and has the following form

$$\alpha = \frac{-ic_0 + c_- p_- + c_+ p_+ + ic_{+-} p_+ p_- + \dots}{2\sqrt{p_+ p_-}}, \tag{2.21}$$

with $p_\pm \equiv \sqrt{(\omega \mp A/2)^2 - m^2}$, and (c_0, c_-, c_+, c_{+-}) being shape-of-$a_t(x)$-dependent, but mass and frequency-independent real numbers. Other coefficients in this expansion are not important for the singular terms in the effective action. The conservation of current implies $c_+ c_- - c_0 c_{+-} = 1$.

Having an expression for α, we can evaluate the effective action. It is convenient to differentiate $\log \alpha(\omega)$ once by m^2, thus obtaining

$$\begin{aligned}\frac{\partial \log \alpha(\omega)}{\partial m^2} = &-\frac{1}{2}\left(\frac{c_+ + ic_{+-} p_-}{p_+} + \frac{c_- + ic_{+-} p_+}{p_-} + \dots\right) \\ &\times \frac{1}{-ic_0 + c_- p_- + c_+ p_+ + \dots} + \frac{1}{4}\left(\frac{1}{p_-^2} + \frac{1}{p_+^2}\right).\end{aligned} \tag{2.22}$$

At this point the integral $\int_\mathcal{C} \mathrm{d}\omega \frac{\partial}{\partial m^2} \log \alpha(\omega)$ is convergent and well defined. The last term in the right-hand side of (2.22) has no branch cuts and can be evaluated in closed form; it is an uninteresting, non-singular piece of $S_{\text{eff}}(a)$. So let us consider the first term in (2.22). We expect to obtain singular terms from vicinity of the points $\omega \approx \pm(m - A/2)$. It is possible to extract the non-analytic part from various integrals contributing to $\partial S_{\text{eff}}(a)/\partial m^2$. For instance it is not difficult to show that

$$\int_{m-A/2}^{2m} \frac{d\omega}{p_-(-ic_0 + c_+p_+ + c_-p_- + ic_{+-}p_+p_- + \dots)} =$$

$$= k_0 - \frac{c_+}{c_0^2}(2m - A)\log\left(\frac{2m-A}{2m}\right) + k_1(2m - A) + \dots\,, \tag{2.23}$$

where the coefficients k_0 and k_1 depend on $c_0, c_+, c_-, c_{+-}, \dots$ and on the upper limit of the integral, but the singular term depends only on c_0 and c_+. Analyzing various types of integrals arising from (2.22) and similar to (2.23) we finally obtain the most-singular non-analytic term of the effective action

$$\frac{\partial S_{\text{eff}}(a)}{\partial m^2} = -\frac{T}{2\pi}\frac{c_+c_- - c_0c_{+-}}{2c_0^2}\left(\frac{2m-A}{2m}\right)\log\left(\frac{2m-A}{2m}\right) + \dots\,, \tag{2.24}$$

and so it follows that

$$S_{\text{eff}}(a) = \frac{Tm^3}{2\pi c_0^2}\left(\frac{2m-A}{2m}\right)^2\log\left(\frac{2m-A}{2m}\right) + \dots\,, \tag{2.25}$$

where we have omitted less singular and non-singular terms.

Let us make a few comments about (2.25):

- The term $1/c_0^2$ is proportional to the transmission amplitude of the effective potential, thus for long smooth gauge fields it is exponentially damped.

- This term in the effective action is neither local in space (as in the usual derivative expansion) or in momentum space (as in the Euler-Heisenberg effective action). Neither of these representations can capture the threshold singularity, as we are always below threshold in the former case, and always above threshold in the latter.

- Despite depending on A, the effective action is gauge invariant, as $A = \int_{-\infty}^{+\infty} \mathrm{d}x\, E(x)$.

- $\operatorname{Im} S_{\text{eff}}$ can be reliably obtained by analytic continuation from (2.25), once we go slightly above the threshold, with $(A - 2m) \ll m$. The amount of phase space available to pair produce depends on the dimension of the spacetime. A quick estimate gives

  $$\operatorname{Im} S_{\text{eff}}(a) \sim \int_0^{k_{\max}} \mathrm{d}^{d-2}k\ (A - 2\,m_{\text{eff}}(k))^2 \sim (A - 2m)^{\frac{d+2}{2}}\,, \tag{2.26}$$

  where $m_{\text{eff}}(k) \equiv \sqrt{m^2 + k^2}$ is the effective mass of the produced particles, and the integral over transverse momenta runs over a finite range, determined by the condition $A - 2m_{\text{eff}}(k_{\max}) = 0$. For $d = 4$, $\operatorname{Im} S_{\text{eff}}(a) \sim (A - 2m)^3$, as argued in [13]. Notice that $S_{\text{eff}}(a)$ will contain a factor of V_{d-2}, the volume of the transverse directions, in higher dimensions.

- The expression (2.25) is clearly invalid if $c_0 = 0$. In the regime $c_0 \ll (2m - A) \ll m$ one finds a different type of singularity

  $$S_{\text{eff}}(a) = -\frac{Tm}{2\pi}\frac{c_+c_-}{c_+^2 + c_-^2}\left(\frac{2m-A}{2m}\right)\log\left(\frac{2m-A}{2m}\right) + \dots\,. \tag{2.27}$$

We recover the expression above for a "quenching" electric field, with $a_t(x) = A/2\,\mathrm{sgn}(x)$, where one has exactly $c_0 = 0$.

As a particular example of the formulas presented above, we can determine the precise form of the Bogoliubov coefficient for a family of potentials $a_t(x) = A/2\tanh(x/l)$, parametrized by the width of the potential l. The result is presented in appendix C. This family of potentials includes the quenching example when $l \to 0$, and we can show that the singularity is different in that case, confirming our last bullet point above. Expanding the Bogoliubov coefficient around small p_+ and p_-, we find the same structure argued for in this section, given by (2.21).

2.3 Gravitational Fields

Finally we consider the case of a massive free scalar field in a strong gravitational background. Again we work in $1+1$ dimensions and we would like to determine the one-loop effective action

$$\langle \mathrm{out}|\mathrm{in}\rangle = e^{iS_{\mathrm{eff}}(v)} = \int D\phi\, \exp\left(-\frac{i}{2}\int \mathrm{d}t\mathrm{d}x\sqrt{-g}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\right)\right)\,. \tag{2.28}$$

For our purposes it is convenient to describe the metric $g_{\mu\nu}$ in Painlevé-Gullstrand coordinates,

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = dt^2 - (dx - v(x)dt)^2\,, \tag{2.29}$$

where we choose $v(-\infty) = 0$ and let $v(x)$ increase smoothly for growing x up to some value $v(+\infty) = V$, similarly to the electrostatic potential $a_t(x)$ (see figure 6). This geometry would

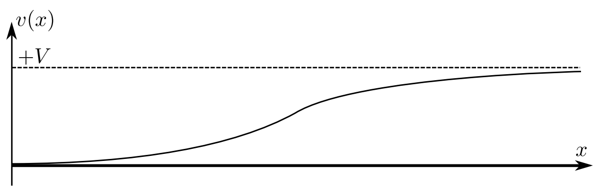

Figure 6: Plot of a schematic form of the gravitational potential $v(x)$. We assume that $V < 1$.

have a horizon at x_h if $v(x_h) = 1$ and thus $g_{tt} = 0$. As we will show later, the vanishing of g_{tt} coincides with the criterion for vacuum decay. For x at which $v(x) < 1$, we can interpret $v(x)$ as the escape velocity from the position x [18]. So our criterion for vacuum stability is that $v(x) < 1$ for all x. Therefore we assume that $V < 1$ but we tune V to the threshold value, i.e. $V \to 1$. This case is mathematically closer to the equipotential planes with fixed asymptotics, in the electric case of the previous section. We will show that the effective action acquires a square root singularity when $V \to 1$.

Let us consider the semiclassical analysis for the gravitational case. The nature of the gap is slightly different than in the electric case. This is due to the different structure of the single-particle Hamiltonian [12, 19, 20]. The energies of excitations are given by

$$\omega_\pm(p,x) = p\,v(x) \pm \sqrt{p^2 + m^2}\,, \tag{2.30}$$

and threshold corresponds to $V \to 1$, as

$$\min(\omega_+) - \max(\omega_-) = 2m\sqrt{1 - V^2}\,. \tag{2.31}$$

The purpose of the mass term is just to open a gap between positive and negative energy bands (see fig. 7), as particle production can occur for gravitational fields without a horizon when the matter sector is gapless [21].

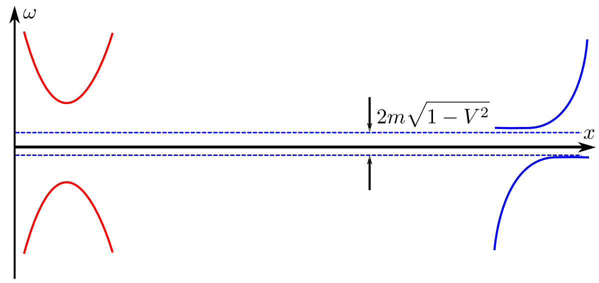

Figure 7: Here we show the bands and the dominant tunneling event, which comes from the top of the lower blue band touching the bottom of the upper blue band.

Notice that when $v(x) = 1$, $g_{tt} = 0$, which, in these coordinates, is the usual definition of the horizon. In other words, our criterion for the location of the horizon being the distance at which it becomes energetically favorable to pair produce coincides with the definition coming from the causal structure of spacetime. Also, notice that a shift $v \to v + C$ is not unphysical, for the case of a static metric. In order to remove the constant C, one could use a Galilean transformation, but this would imply a redefinition of the time coordinate. In other words, the criterion for vacuum instability depends not just on the difference between the asymptotic values of the velocity field, like in the electric field example, but also on its absolute values as $x \to \pm\infty$.

As usual to compute the effective action we differentiate it by the mass term

$$i\frac{\partial S_{\text{eff}}(v)}{\partial m^2} = -\frac{i}{2}\int \mathrm{d}t\,\mathrm{d}x\, G_F(x,t;x,t)\,, \tag{2.32}$$

where $G_F(x,t;x',t') \equiv \langle \phi(x,t)\phi(x',t')\rangle$ is the Feynman Green's function and we used that for our metric $\sqrt{-g} = 1$. The Green's function obeys the equation

$$\left(\partial_x((1-v^2)\partial_x) - 2v\partial^2_{xt} - (\partial_x v)\partial_t - \partial_t^2 - m^2\right) G_F(x,t;x',t') = i\delta(x-x')\delta(t-t')\,. \tag{2.33}$$

This equation contains a term with a first order derivative in x, which naively makes it difficult to apply the previous strategy of expressing the effective action as an integral over the logarithm of the transmission coefficient. Nevertheless, by properly changing variables, we are able to obtain a similar $\log\alpha$ formula for the effective action.

To proceed one can check that the Green's function $G_F(x,t;x',t')$ can be written in the form

$$G_F(x,t;x',t') = \int \frac{\mathrm{d}\omega}{2\pi}\frac{e^{i\omega(t-t')}e^{i(\chi(x)-\chi(x'))}}{\sqrt{(1-v^2(x))(1-v^2(x'))}} G_\omega(x,x')\,, \tag{2.34}$$

where the new Green's function $G_\omega(x,x')$ obeys a Schrödinger-like equation

$$\left(\partial_x^2 + \frac{\omega^2 - m^2(1-v^2) + (\partial_x v)^2}{(1-v^2)^2} + \frac{v\,\partial_x^2 v}{1-v^2}\right) G_\omega(x,x') = i\delta(x-x')\,. \tag{2.35}$$

and the function χ is defined as $\partial_x \chi = \omega v/(1-v^2)$. It is easy to check that (2.34) indeed satisfies (2.33). Therefore for the effective action we obtain

$$\frac{\partial S_{\text{eff}}(v)}{\partial m^2} = -\frac{T}{2}\int \frac{\mathrm{d}\omega}{2\pi}\int \mathrm{d}x \frac{G_\omega(x,x)}{1-v^2(x)} \,. \tag{2.36}$$

The Green's function $G_\omega(x,x')$ as usual can be expressed through f_{in} and f_{out} functions

$$G_\omega(x,x') = i\frac{f_{\text{in}}(x)f^*_{\text{out}}(x')\theta(x'-x) + (x \leftrightarrow x')}{W(f_{\text{in}}, f^*_{\text{out}})} \,, \tag{2.37}$$

where f_{in} and f_{out} are anihilated by the Schrödinger-like operator in (2.35) and have asymptotics

$$f_{\text{in}}(x) \xrightarrow{x\to-\infty} \frac{1}{\sqrt{2p_-}}e^{-ip_- x}, \quad f_{\text{out}}(x) \xrightarrow{x\to+\infty} \frac{1}{\sqrt{2p_+}}e^{-ip_+ x} \,, \tag{2.38}$$

where we denoted

$$p_- = \sqrt{\omega^2 - m^2}, \quad p_+ = \frac{1}{1-V^2}\sqrt{\omega^2 - m^2(1-V^2)} \,, \tag{2.39}$$

and, as usual, we define α and β as $f_{\text{in}}(x) = \alpha f_{\text{out}}(x) + \beta f^*_{\text{out}}(x)$. As shown in appendix A we can bring the formula (2.36) to our usual form

$$S_{\text{eff}}(v) = \frac{1}{2} iT \int_{\mathcal{C}} \frac{d\omega}{2\pi} \log \alpha(\omega) \,. \tag{2.40}$$

The pinching singularity comes again from very small frequencies near the points where $p_+ = 0$, and we need to determine α for $\omega \sim m\sqrt{1-V^2}$ (see figure 8). Notice that the relevant feature

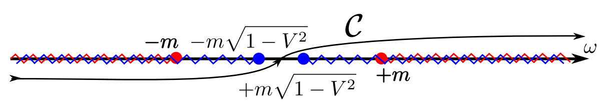

Figure 8: The integration contour $\mathcal{C}$ in the complex ω-plane. Branch points $\omega = \pm m\sqrt{1-V^2}$ corresponding to $p_+ = 0$ pinch the contour $\mathcal{C}$ near $\omega \approx 0$ when $V \to 1$. This "pinching" region determines the singular piece of the effective action.

is the behavior of the mode functions as $x \to \infty$, and for $\omega \sim m\sqrt{1-V^2}$, the mode function has very small modulation with ω. This means that the transmission coefficient behaves like

$$\alpha \approx \frac{-id_0 + d_+ p_+ + \dots}{2\sqrt{p_+}} \,, \tag{2.41}$$

as $p_- \approx m$, and the model dependence is encoded in the coefficients (d_0, d_+). We can once again take derivatives of the effective action to isolate its singular piece. It turns out that differentiating once with respect to m^2 is enough to isolate the singular term. Following the same steps as in the electric case, we arrive at

$$S_{\text{eff}}(v) = mT\sqrt{1-V^2} + \dots \,. \tag{2.42}$$

In the appendix C we find an exact $\alpha(\omega)$ for the step potential $v(x) = V\theta(x)$. In this case one can calculate the integral over ω in (2.40) exactly and obtain the result (2.42).

This singular part of the effective action can be written in a "local" form[4] due to the different tunneling pattern when the vacuum breaks down – pairs are produced at large x, rather than at both small and large x. There is also an analogous term to the electric threshold result, $\sim (1-V^2)\log(1-V^2)$, but it is subleading to (2.42). Notice that even in the special case $d_0 = 0$ we obtain the same singularity, albeit with different overall coefficient. Another interesting thing is that the leading term (2.42) does not care about the detailed coefficients $d_{0,+}$ – as long as they are nonzero, the only relevant data from the metric is the value of V. This is unlike the electric case, where the leading singular term depends on $2m - A$ but also on c_0.

This threshold singularity is a quantum analog of Choptuik scaling. Choptuik considered a family of initial data labeled by a parameter p. Under time evolution using Einstein's equations, he found [11] that the final state had a black hole of mass $M \sim (p - p_{\rm cr})^\gamma$, for $p > p_{\rm cr}$. The exponent γ is largely independent on the details of the family of initial data.

Above criticality, the formation of a black hole indicates the appearance of a horizon. Our critical exponent is entirely analogous, but is a quantum diagnostic of the appearance of the horizon. In our case, we look at $S_{\rm eff}(v)$ rather than M, criticality is reached when $V = 1$, and the critical exponent is $\gamma = 1/2$ [5].

3 Conclusions

In this paper, we argued that the crossover between the quantum mechanical stability and instability of background fields has certain universal features. This is largely due to the first unstable process triggered right above threshold having very long wavelength and low energy. This soft emission process probes only the roughest features of the external background, and the threshold singularity can be easily expressed in terms of rough background data. There are many avenues for further investigation:

- Our analysis was restricted to gaussian matter fields. How would interactions in the matter sector change the critical exponents in the threshold singularity?
- Can we connect our results to existing methods for treating backreaction in black holes [23, 24]? It would be nice to incorporate our threshold singularity to the problem of formation of a black hole, in order to see if vacuum polarization delays its formation, or prevents formation whatsoever for initial data close enough to threshold.
- Finally, it would also be interesting to find the threshold singularity for more realistic field configurations: for example, a spherically symmetric configuration, like a star, where we take a mass shell to be very close to its Schwarzschild radius.

We leave such fascinating problems to the near future.

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[4] The volume form on the timelike surface $x \to +\infty$ is given by $\sqrt{1-V^2} = \sqrt{g_{\rm ind}}$.

[5] Interestingly, an analysis of black hole formation in a different context gives the same critical exponent [22].

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A Derivation of $S_{\text{eff}} = i\int d\omega \log\alpha$

In this appendix, we derive the formula for the gravitational effective action in terms of the Bogoliubov coefficient α in detail. The electromagnetic and scalar cases simply follow from this.

Using the expression (2.37) for $G_\omega(x,x')$ and formula (2.36) we obtain for the effective action

$$\frac{\partial S_{\text{eff}}(v)}{\partial m^2} = -i\frac{T}{2}\int\frac{\mathrm{d}\omega}{2\pi}\int\mathrm{d}x\frac{f_{\text{in}}(x)f^*_{\text{out}}(x)}{(1-v^2(x))W(f_{\text{in}},f^*_{\text{out}})}\,, \tag{A.1}$$

where one can easily calculate $W(f_{\text{in}},f^*_{\text{out}}) = i\alpha$. We must now calculate the integral over x in (A.1). In order to proceed, we do the following: consider the equations

$$\begin{aligned}
&\partial_x^2 f_{\text{in},m^2}(x) + \left(\frac{\omega^2 - m^2(1-v^2)+(\partial_x v)^2}{(1-v^2)^2} + \frac{v\,\partial_x^2 v}{1-v^2}\right) f_{\text{in},m^2}(x) = 0\,,\\
&\partial_x^2 f^*_{\text{out},m^2+\delta m^2}(x) + \left(\frac{\omega^2 - (m^2+\delta m^2)(1-v^2)+(\partial_x v)^2}{(1-v^2)^2} + \frac{v\,\partial_x^2 v}{1-v^2}\right) f^*_{\text{out},m^2+\delta m^2}(x) = 0\,.
\end{aligned} \tag{A.2}$$

Multiplying the first equation by $f^*_{\text{out},m^2+\delta m^2}(x)$ and the second by $f_{\text{in},m^2}(x)$ and subtracting them we obtain

$$\partial_x(f_{\text{in},m^2}\partial_x f^*_{\text{out},m^2+\delta m^2} - f^*_{\text{out},m^2+\delta m^2}\partial_x f_{\text{in},m^2}) = \delta m^2\frac{f_{\text{in},m^2}(x)f^*_{\text{out},m^2+\delta m^2}(x)}{1-v^2(x)}\,. \tag{A.3}$$

Integrating over x the left and the right parts from $-L$ to L, where $L\to+\infty$ we get

$$\delta m^2\int_{-L}^{+L}\mathrm{d}x\frac{f_{\text{in},m^2}(x)f^*_{\text{out},m^2+\delta m^2}(x)}{1-v^2(x)} = (f_{\text{in},m^2}\partial_x f^*_{\text{out},m^2+\delta m^2} - f^*_{\text{out},m^2+\delta m^2}\partial_x f_{\text{in},m^2})|_{-L}^{+L}\,. \tag{A.4}$$

Because we take $L\to\infty$, we can use the asymptotic expressions for $f_{\text{in,out}}$ (2.38) and find

$$\begin{aligned}
&(f_{\text{in},m^2}\partial_x f^*_{\text{out},m^2+\delta m^2} - f^*_{\text{out},m^2+\delta m^2}\partial_x f_{\text{in},m^2})|_{-L}^{+L} =\\
&= \delta m^2\left(-i\frac{\partial\alpha}{\partial m^2} - L\alpha\frac{\partial(p_-+p_+)}{\partial m^2} - \frac{1}{2}i\Big(\beta^*\frac{\partial\log p_-}{\partial m^2}e^{2iLp_-} - \beta\frac{\partial\log p_+}{\partial m^2}e^{2iLp_+}\Big)\right) + \dots\,.
\end{aligned} \tag{A.5}$$

We use the Feynman $i\epsilon$-prescription $p_\pm\to p_\pm + i\epsilon$ with infinitesimal $\epsilon>0$, so the oscillating terms above are zero for large L. Finally we obtain

$$(f_{\text{in},m^2}\partial_x f^*_{\text{out},m^2+\delta m^2} - f^*_{\text{out},m^2+\delta m^2}\partial_x f_{\text{in},m^2})|_{-L}^{+L} = -i\delta m^2\alpha\frac{\partial}{\partial m^2}\Big(\log\alpha + L(p_-+p_+)\Big)\,. \tag{A.6}$$

Putting this together, we find

$$\frac{\partial S_{\text{eff}}(v)}{\partial m^2} = i\frac{T}{2}\int_{\mathcal{C}} \frac{\mathrm{d}\omega}{2\pi}\frac{\partial}{\partial m^2}\Big(\log\alpha + L(p_- + p_+)\Big). \tag{A.7}$$

Our last task is to argue that the terms proportional to L are unimportant. By that we mean that they only carry uninteresting dependence on the background. The term proportional to Lp_- is harmless, depending only on m, but the term proportional to Lp_+ seems to have nontrivial dependence on V. Let us write it more explicitly

$$\int_{\mathcal{C}} \mathrm{d}\omega\, Lp_+ = \int_{\mathcal{C}} \mathrm{d}\omega \frac{1}{1-V^2}(\omega^2 - m^2(1-V^2))^{1/2}\,. \tag{A.8}$$

If we change variables $\omega = \omega'(1-V^2)^{1/2}$ then the V dependence drops out of the integral and it is exactly equal to the Lp_- integral. This argument is too fast, as the integral is UV divergent. The correct argument is that the cutoff is background dependent. For the p_- integral, we are at $x \to -\infty$ so we choose some hard cutoff Λ in frequency space. At $x \to +\infty$, the metric is $\mathrm{d}t^2(1-V^2)$ so we must choose the cutoff $\Lambda/(1-V^2)^{1/2}$ in frequency space, to take into account the warping of time intervals. This renders the Lp_+ integral to have no interesting dependence on V. In summary, up to non-important terms, we find

$$S_{\text{eff}}(v) = \frac{1}{2} iT\int_{\mathcal{C}} \frac{d\omega}{2\pi}\log\alpha(\omega)\,. \tag{A.9}$$

B Behavior of α Near Threshold

In this appendix we derive the behavior of the Bogoliubov coefficient α when the effective mass gap is very small. In other words, we find the first few terms in an expansion for α around vanishing mass gap. To start, let us consider the Schrödinger equation

$$(\partial_x^2 + U(x) - m^2)f = 0\,, \tag{B.1}$$

where $U(x)$ either switches off or asymptotes to some fixed values $U(\pm\infty)$ in a smooth way. We are interested in the cases where the effective mass gap at infinity

$$p_\pm^2 \equiv U(\pm\infty) - m^2 \tag{B.2}$$

is very small, namely $p_\pm^2 \ll m^2$. We consider here the case in which both $p_\pm^2$ are small. In the main text, the gravitational background is such that only in one extreme the mass gap vanishes. Applying our formulas to that example is straightforward.

In the region outside of which $U(x)$ is varying, the mass term is either p_+^2 or p_-^2, which we assume are small. Neglecting those terms, we get

$$\partial_x^2 f = 0\,, \tag{B.3}$$

therefore the solutions of the equations of motion are

$$f = a_1 + b_1 x \quad x \ll 0, \qquad f = a_2 + b_2 x, \quad x \gg 0 \tag{B.4}$$

where the potential varies significantly close to $x = 0$. The coefficients (a_1, b_1) and (a_2, b_2) are linearly dependent

$$a_1 = c_- a_2 + c_{+-} b_2 , \quad b_1 = c_0 a_2 + c_+ b_2 \, , \tag{B.5}$$

where the coefficients (c_0, c_+, c_-, c_{+-}) are independent of $p_\pm$ (as $p_\pm$ do not appear in the differential equation with linear functions as solutions), and, from the conservation of current, it follows that we can choose (c_0, c_+, c_-, c_{+-}) to be real, with $c_+c_- - c_0c_{+-} = 1$. Then matching the solutions (B.4) with the asymptotic solutions in terms of plane waves, we obtain

$$a_1 = \frac{1}{\sqrt{2p_-}} , \quad b_1 = \frac{-ip_-}{\sqrt{2p_-}} , \quad a_2 = \frac{\alpha + \beta}{\sqrt{2p_+}} , \quad b_2 = \frac{-ip_+(\alpha - \beta)}{\sqrt{2p_+}} \, , \tag{B.6}$$

and solving the equations (B.5) we get

$$\alpha = \frac{-ic_0 + c_-p_- + c_+p_+ + ic_{+-}p_-p_+}{2\sqrt{p_-p_+}} \, , \quad \beta = \frac{ic_0 - c_-p_- + c_+p_+ + ic_{+-}p_-p_+}{2\sqrt{p_-p_+}} \, . \tag{B.7}$$

Now having α we can evaluate the effective action. These expressions are only valid for $|p_\pm| \ll m$.

C Exact Solutions

C.1 Electric example

The exact solution in the electric case is available for the gauge field profile $a_t(x) = A/2 \tanh(x/l)$. In this case one can obtain an exact Bogoliubov coefficient α; it is given by

$$\alpha = \frac{i/l}{\sqrt{p_+p_-}} \frac{\Gamma(1 - ip_-l)\Gamma(1 - ip_+l)}{\Gamma(\rho - \frac{i}{2}l(p_- + p_+))\Gamma(1 - \rho - \frac{i}{2}l(p_- + p_+))} \, , \tag{C.1}$$

where $\rho = \frac{1}{2} + \frac{1}{2}\sqrt{1 - A^2l^2}$ and p_-, p_+ are defined below (2.19). If we first tune $l \to 0$ we obtain the step potential $a_t(x) = A/2 \, \mathrm{sgn}(x)$ and the Bogoliubov coefficient (C.1) simplifies to $\alpha = (p_- + p_+)/2\sqrt{p_+p_-}$. On the other hand, if l is fixed and we are in the regime where p_+ and p_- are small, we find

$$\alpha = \frac{\sin \pi\rho}{\pi l} \frac{(2i - l(\psi(\rho) + \psi(1 - \rho) + 2\gamma_E)(p_- + p_+) + \dots)}{2\sqrt{p_+p_-}} \, , \tag{C.2}$$

where $\psi(x)$ is the digamma function, and γ_E is the Euler-Mascheroni constant. This form of the α coefficient agrees with (B.7).

C.2 Gravitational example

In this subsection we are going to find the coefficient α in the case of a step potential $v(x) = V\theta(x)$. To proceed it is convenient to write the Schrödinger equation for the operator (2.35) as

$$\partial_x \left((1 - v^2)\partial_x \Big(\frac{f_{\rm in}(x)}{\sqrt{1 - v^2}} \Big) \right) + \frac{\omega^2 - m^2(1 - v^2)}{(1 - v^2)^{3/2}} f_{\rm in}(x) = 0 \, . \tag{C.3}$$

Now integrating this equation from $x = -\delta$ to $x = \delta$ with $\delta \to 0$ we find the boundary conditions for $f_{\rm in}(x)$ and $\partial_x f_{\rm in}(x)$ at $x = 0$:

$$f_{\rm in}(0^+) = \sqrt{1-V^2} f_{\rm in}(0^-), \quad \sqrt{1-V^2}\partial_x f_{\rm in}(0^+) = \partial_x f_{\rm in}(0^-) \,. \tag{C.4}$$

Using these boundary conditions one can find

$$\alpha = \frac{p_- + (1-V^2)p_+}{2\sqrt{(1-V^2)p_- p_+}} \,, \tag{C.5}$$

where $p_- = \sqrt{\omega^2 - m^2}$ and $p_+ = \sqrt{\omega^2 - m^2(1-V^2)}/(1-V^2)$.

References

[1] R. M. Wald, *General Relativity*. The University of Chicago Press, 1984.

[2] J. S. Schwinger, "On gauge invariance and vacuum polarization," *Phys. Rev.* **82** (1951) 664–679.

[3] L. Parker, "Quantized fields and particle creation in expanding universes. 1.," *Phys. Rev.* **183** (1969) 1057–1068.

[4] S. W. Hawking, "Particle Creation by Black Holes," *Commun. Math. Phys.* **43** (1975) 199–220.

[5] G. W. Gibbons and S. W. Hawking, "Cosmological Event Horizons, Thermodynamics, and Particle Creation," *Phys. Rev.* **D15** (1977) 2738–2751.

[6] G. K. Savvidy, "Infrared Instability of the Vacuum State of Gauge Theories and Asymptotic Freedom," *Phys. Lett.* **71B** (1977) 133–134.

[7] N. K. Nielsen and P. Olesen, "An Unstable Yang-Mills Field Mode," *Nucl. Phys.* **B144** (1978) 376–396.

[8] A. M. Polyakov, "De Sitter space and eternity," *Nucl. Phys.* **B797** (2008) 199–217, `arXiv:0709.2899 [hep-th]`.

[9] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space*. Cambridge Monographs on Mathematical Physics. Cambridge Univ. Press, Cambridge, UK, 1984.

[10] G. V. Dunne, "Heisenberg-Euler effective Lagrangians: Basics and extensions," *From fields to strings* **1** (2004) 445–522, `arXiv:hep-th/0406216 [hep-th]`.

[11] M. W. Choptuik, "Universality and scaling in gravitational collapse of a massless scalar field," *Phys. Rev. Lett.* **70** (1993) 9–12.

[12] A. M. Polyakov, "Self-destructive manifolds and the cosmological constant," *PiTP 2014, Institute for Advanced Study. https://static.ias.edu/pitp/2014/schedule.html* (2014) .

[13] H. Gies and G. Torgrimsson, "Critical Schwinger pair production II - universality in the deeply critical regime," *Phys. Rev.* **D95** no. 1, (2017) 016001, `arXiv:1612.00635 [hep-th]`.

[14] A. Migdal, *Fermions and bosons in strong fields*. Nauka, Moscow, 1978.

[15] A. I. Nikishov, "On vacuum vacuum amplitude and Bogolyubov coefficients," *J. Exp. Theor. Phys.* **96** no. 2, (2003) 180–192, `arXiv:hep-th/0207085 [hep-th]`. [Zh. Eksp. Teor. Fiz.123,no.2,211(2003)].

[16] D. Krotov and A. M. Polyakov, "Infrared Sensitivity of Unstable Vacua," *Nucl. Phys.* **B849** (2011) 410–432, `arXiv:1012.2107 [hep-th]`.

[17] L. D. Faddeev, "The Inverse problem in the quantum theory of scattering," *J. Math. Phys.* **4** (1963) 72–104. [Usp. Mat. Nauk14,57(1959)].

[18] A. J. S. Hamilton and J. P. Lisle, "The River model of black holes," *Am. J. Phys.* **76** (2008) 519–532, `arXiv:gr-qc/0411060 [gr-qc]`.

[19] G. E. Volovik, "On de Sitter radiation via quantum tunneling," *Int. J. Mod. Phys.* **D18** (2009) 1227, `arXiv:0803.3367 [gr-qc]`.

[20] G. E. Volovik, "Particle decay in de Sitter spacetime via quantum tunneling," *JETP Lett.* **90** (2009) 1–4, `arXiv:0905.4639 [gr-qc]`.

[21] G. L. Pimentel, A. M. Polyakov, and G. M. Tarnopolsky, "Vacuum decay in CFT and the Riemann–Hilbert problem," *Nucl. Phys.* **B907** (2016) 617–632, `arXiv:1512.06721 [hep-th]`.

[22] A. Strominger and L. Thorlacius, "Universality and scaling at the onset of quantum black hole formation," *Phys. Rev. Lett.* **72** (1994) 1584–1587, arXiv:hep-th/9312017 [hep-th].

[23] G. E. Volovik, "Simulation of Painleve-Gullstrand black hole in thin He-3 - A film," *JETP Lett.* **69** (1999) 705–713, arXiv:gr-qc/9901077 [gr-qc]. [Pisma Zh. Eksp. Teor. Fiz.69,662(1999)].

[24] M. K. Parikh and F. Wilczek, "Hawking radiation as tunneling," *Phys. Rev. Lett.* **85** (2000) 5042–5045, arXiv:hep-th/9907001 [hep-th].