Phase velocity and light bending in a gravitational potential

José-Philippe Pérez and Brahim Lamine

Institut de Recherche en Astrophysique et Planétologie (IRAP)-CNRS-Université de Toulouse- 14 avenue Édouard Belin, 31400 Toulouse, France

jose-philippe.perez@irap.omp.eu brahim.lamine@irap.omp.eu

Abstract

In this paper we review the derivation of light bending obtained before the discovery of General Relativity (GR). It is intended for students learning GR or specialist that will find new lights and connexions on these historic derivations. Since 1915, it is well known that the observed light bending stems from two contributions : the first one is directly deduced from the equivalence principle alone and was obtained by Einstein in 1911; the second one comes from the spatial curvature of spacetime. In GR, those two components are equal, but other relativistic theories of gravitation can give different values to those contributions. In this paper, we give a simple explanation, based on the wave-particle picture of why the first term, which relies on the equivalence principle, is identical to the one obtained by a purely Newtonian analysis. In this context of wave analysis, we emphasize that the dependency of the velocity of light with the gravitational potential, as deduced by Einstein concerns the phase velocity. Then, we wonder whether Einstein could have envisaged already in 1911 the second contribution, and therefore the correct result. We argue that considering a length contraction in the radial direction, along with the time dilation implied by the equivalence principle, could have led Einstein to the correct result.

Introduction

The Newtonian theory of the deviation of light bending was published in 1801 by the German physicist J. Soldner [1, 2]. The author develops Kepler's classical motion of a particle of light, of mass m, submitted to the gravitational force exerted by a mass M with spherical symmetry. He obtained the usual hyperbolic motion and computed the deflection angle χ_N of the trajectory in the Newtonian approximation. By applying this analysis to a particle of light grazing the Sun, he found the value $\chi_N \approx 0.87\,\text{as}$, which is exactly half of the experimental value measured in 1919 [3]. In the first section, we review the computation of Soldner, with modern notations.

In 1911 [4], Einstein proposed a new analysis of light bending, based on the equivalence principle alone. He was led to the conclusion that a dilation of duration is produced by a gravitational potential. This leads to the conclusion that a certain velocity of light should depend on the gravitational potential Φ,

$$c_{p,\Phi} = c\left(1 + \frac{\Phi}{c^2}\right) \tag{1}$$

This velocity is smaller than c, the value in the absence of potential ($\Phi = -GM/r < 0$). In his original paper of 1911 [4], Einstein does not give a real physical interpretation of this velocity, but simply speaks of « speed of light ». Using the principle of Huygens-Fresnel, he deduced the trajectory of a light ray by requiring that they are normal to wave front. Curiously, he found the same expression as the Newtonian result of Soldner. In the second section, we review the Einstein argument in a slightly different way, which shed new light on the Einstein derivation. In particular, we show that the velocity obtained by Einstein has to be interpreted as a phase velocity, and not the light speed (that remain a fundamental constant). We then argue that the de Broglie wave transposition of Soldner's analysis explain the identical result obtained by Einstein.

Only a few years later, as part of the complete theory of general relativity [5, 6], Einstein obtained the correct value of this deviation, i.e. the double of the previous result. Many authors have discussed the reason of the doubling of the Newtonian result in GR [7]. In the third part of this paper, we propose a new light to interpret this doubling. For this, we propose a generalization of the physical analysis of Einstein, accompanying the time dilation due to a gravitational potential, by a concomitant contraction of the radial lengths (see [8] where this idea has already been proposed, though with a different approach as we do). This derivation is an intuition that could have had Einstein, more than a formal proof, because it is already known that the correct result cannot be recovered simply from the equivalence principle and the Newton's limit alone [9].

0.1 Newton theory of Soldner

In this section, we briefly summarize how Soldner computed light bending by a massive body from a Newtonian approach. For a complete historical perspective about the Newtonian influence of gravitation on light, see [10]. For this, he hypothetized that light is made of material particles, for which it is possible to apply Newton's laws in order to obtain the trajectory. To justify his hypothesis, he added, in the part related to the objections which might be opposed to him, that light should be considered as matter :

« Hopefully, no one would find it objectionable that I treat a light ray as a heavy body. That light rays have all the absolute [basic] properties of matter one can see from the phenomenon of aberration which is possible only because light rays are truly material. And furthermore, one cannot think of a thing which exists and works on our senses that would not have the property of matter. »

The computation of Soldner is prior to Maxwell's theory, in which the speed of light is a constant [1]. In Soldner's perspective, the speed of a particle of light is not a constant, but varies along the path around the massive body, just like an ordinary material particle. In his publication, there is therefore a free parameter, which he took as being the speed of light measured at the level of perihelion P; in the following, we will note this velocity as v_P.

The trajectory of a particle of light A can be deduced from the conservation of the massic mechanical energy $e_m = v^2/2 - GM/r$ and the massic angular

1. In particular, it is independent of the gravitational potential, because gravitation and electromagnetism are not coupled in this theory.

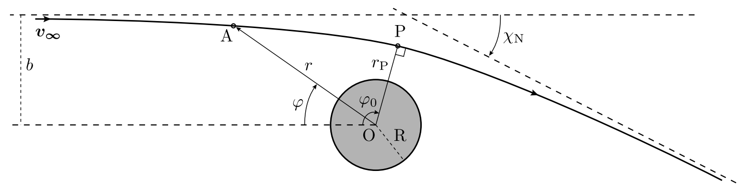

FIG. 1 – *Diagram of deviation of a light ray by a mass with spherical symmetry. Notations are defined in the text.*

momentum $\boldsymbol{\ell} = r^2\dot{\varphi}\,\boldsymbol{e}_z$. In these expressions, r and φ are the polar coordinates of A, in the plane of motion defined by O and the normal vector $\boldsymbol{\ell}$ (Fig. 1). Combining these two expressions give :

$$\dot{r}^2 + \frac{\ell^2}{r^2} - 2\frac{GM}{r} = 2e_m \tag{2}$$

The mass m of the particle of light, which was unknown to Soldner in 1801, does not appear in this equation. This observation is simply a reformulation, in the case of light, of the underlying hypothesis of the equality of the gravitational mass, which appears in gravitational energy, and of the inertial mass, present in the angular momentum. This hypothesis was early postulated by Galileo and then tested experimentally, with a relative precision of 10^{-3}, by Newton using pendulums made of different materials.

Following Soldner, the constant ℓ can be expressed with respect to the speed of light v_P at perihelion, $v_P = r_P\dot{\varphi}$. Likewise, one can also introduce the impact parameter b, so that $\ell = r_P v_P = b\,v_\infty$ (see Fig. 1 for notation). Equation (2) can be rewritten using dimensionless quantities. Introducing $\rho \equiv r/r_P$ (as Soldner did), expression (2) is written more conveniently, if we introduce the gravitational potential at perihelion $\Phi_P \equiv -GM/r_P$, as

$$r_P^2\left(\frac{\mathrm{d}\rho}{v_P\mathrm{d}t}\right)^2 + \frac{1}{\rho^2} + \frac{2}{\rho}\frac{\Phi_P}{v_P^2} = \frac{2e_m}{v_P^2} = \left(\frac{v_\infty}{v_P}\right)^2 \tag{3}$$

This equation is identical to the one obtained by Soldner. He solved the equation (3) with lenghty calculations, because the usual Binet change of variable was not yet known. Using the reduced Binet variable $u \equiv 1/\rho = r_P/r$, one finds from (3) :

$$\frac{\mathrm{d}^2u}{\mathrm{d}\varphi^2} + u = -\frac{\Phi_P}{v_P^2} \equiv \frac{r_P}{p} \tag{4}$$

If the light is grazing on the surface of the attractive body, $r_P = R$, with R the radius of the massive body. The dimensionless quantity $-\Phi_P/v_P^2$ is positive and reduces to the compactness $\mathcal{C} \equiv GM/(Rv_P^2)$ of the object, which physically represent the ratio between the gravitational energy and the mass energy. For objects like planets or stars, the compactness is very small compared to unity, so that the right-hand side of equation (4) is very small and the solution is nearly the usual Newton solution. For the Sun and the Earth, we find respectively (taking $v_P \approx c$) :

$$\mathcal{C}_\odot \approx 2 \times 10^{-6} \qquad \text{and} \qquad \mathcal{C}_\oplus \approx 7 \times 10^{-10} \tag{5}$$

The solution of equation (4) is given by $u(\varphi) = A\cos(\varphi - \varphi_0) + r_P/p$. We then determine the constant A using the condition on perihelion, $u = 1$ when $\varphi = \varphi_0$, which gives $1 = A - \Phi_P/v_P^2$. Thus, the solution for r is a conic of parameter p and eccentricity e :

$$r = \frac{p}{1 + e\,\cos(\varphi - \varphi_0)} \quad \text{with} \quad p = r_P\left(\frac{v_P^2}{-\Phi_P}\right) \quad \text{and} \quad e = \left(\frac{v_P^2}{-\Phi_P}\right) - 1 \tag{6}$$

We can also relate e with the massic mechanical energy :

$$e_m = \frac{v_P^2}{2}\left(1 + \frac{2\Phi_P}{v_P^2}\right) = \frac{v_P^2}{2}\left(\frac{e-1}{e+1}\right) = \frac{v_P^2}{2}\left(\frac{\Phi_P}{v_P^2}\right)^2 (e^2 - 1) \tag{7}$$

The previous expression allows to study the type of trajectories as a function of the value of the eccentricity : hyperbolic motion for $e_m > 0$ ($e > 1$) parabolic motion for $e_m = 0$ ($e = 1$) and elliptic motion for $e_m < 0$ ($e < 1$). Soldner found that in pratice $e_m > 0$, because the condition $-\Phi_P/v_P^2 \ll 1$ was satisfied for the stars known at that time. Therefore $e \gg 1$ according to eq. (6) and the corresponding trajectories of the particles of light are hyperbolic ones.

Soldner briefly evoked the existence of bounded solutions, characterized by $e_m < 0$, i.e. $GM/(r_P v_P^2) > 1/2$. He added, however, that this condition was not realistic, or in any case it did not correspond to any known object at that time[2]. Indeed, the stars seen in the sky were already considered as sun-like, whose mass and radius were known with sufficient precision. The compactness should be of the same order of magnitude than $\mathcal{C}_\odot$, and therefore very small (see equation (5)).

The Newtonian deviation angle χ_N is easily obtained by writing the asymptotic condition $r \to \infty$, i.e. $\cos(\varphi_{\rm in} - \varphi_0) = -1/e$. By choosing $\varphi_{\rm in} = 0$ for the direction of the incident ray, the ray emerges asymptotically in $\varphi = \pi + \chi_N$, so that $\cos\varphi_0 = \cos(\pi + \chi_N - \varphi_0) = -1/e$. Hence $\chi_N = 2\varphi_0 - \pi$ and therefore $\tan\varphi_0 = \tan(\chi_N/2 + \pi/2) = -\tan^{-1}(\chi_N/2)$. Since $\cos\varphi_0 = -1/e$, $\tan\varphi_0 = -(e^2-1)^{1/2}$, and we find the following result of Soldner :

$$\tan\left(\frac{\chi_N}{2}\right) = \frac{1}{(e^2-1)^{1/2}} = \frac{-\Phi_P/v_P^2}{(1+2\Phi_P/v_P^2)^{1/2}} \tag{8}$$

This Newtonian result is an exact result, which does not rely on any assumption. In the limit $-\Phi_P/v_P^2 \ll 1$, it gives $\chi_N \approx 2GM/(r_P v_P^2)$. Or, since $r_P\, v_P = b\, v_\infty$ and $r_P \approx b$ (at lowest order in $-\Phi_P/v_P^2$) :

$$\chi_N \approx \frac{2GM}{b\, v_\infty^2} = \frac{r_S}{b}\left(\frac{c}{v_\infty}\right)^2 \quad \text{where} \quad r_S = \frac{2GM}{c^2} \tag{9}$$

is the Schwarzchild radius. In order to estimate the orders of magnitude, Soldner used the speed of light measured by Bradley in 1729, using the aberration of stars [11][3]. The result obtained by Soldner is half the one predicted by general relativity in 1915 [5]. Moreover, its expression (8) is not universal, because it involves the speed of light at perihelion (or equivalently v_∞), the latter being not considered, at the time of Soldner, as a universal constant. However, Soldner seems to suppose that this speed, which is much greater than the speed

2. Soldner wrote *« Since it does not matter how much mass it would be so great that it could produce such an acceleration gravity, a light ray describes, in the world known to us, always hyperbola. »* We will discover much later that such objects, for which the trajectory of light realizes $e_m < 0$, do exist in nature, for example black holes. Note that Michell already considered bounded trajectory of light, but in a rather different situation : he considered radial trajectory of light from massive objects, from which the escape velocity would be greater than the speed of light [10].

3. Note that Bradley obtained this speed, in unit of speed of the Earth around the Sun, the latter being poorly known at the time.

of celestial objects (planets, stars), must be, according to the law of Galilean composition of velocities, quite close to the value which he used in its numerical applications (see also the discussion in [10] who takes up the argument of Michell about the variation of the speed of light in a gravitation field). Assuming that $v_\infty \approx c$, one obtains, if the light is grazing, for the Sun and the Earth respectively :

$$\chi_{N,\odot} \approx 0.87\,\text{as} \quad \text{and} \quad \chi_{N,\oplus} \approx 0.28 \times 10^{-3}\,\text{as}$$

Soldner deduced from these numerical results that the deviation of light near the Sun was too small to be measured at his time[4]. He (unknowingly) announced a result that will be tested experimentally more than a century later [3]. It is interesting to note that he publishes the result of his analysis, even if the conclusion of this one is that the effect is not observable[5].

0.2 Einstein relativistic theory of 1911

Einstein already noticed in 1907, in his review article on special relativity, that, according to the principle of equivalence, a light ray has to be bent by gravitation [12]. In 1911 he carefully studied the influence of a gravitational potential Φ on the propagation of light in vacuum. For a review of the original derivation, see [13]. He based its arguments on two pillars :

– special relativity, including Maxwell theory of electromagnetism. It contains in particular the universal character of the speed of light in vacuum and the Doppler-Fizeau effect.

– the equivalence principle he developed to build the theory of general relativity; this principle affirms the equivalence between an observer at rest in

4. He concludes with this sentence : *« So it is clear that nothing is necessary, at least in the present state of practical astronomy, that one should take into account the disturbance of light rays by attracting celestial bodies. »*

5. He even adds in its conclusion: *« At any rate, I do not believe that there is any need on my part to apologize for having published the present essay just because the result is that all perturbations are unobservable. »*

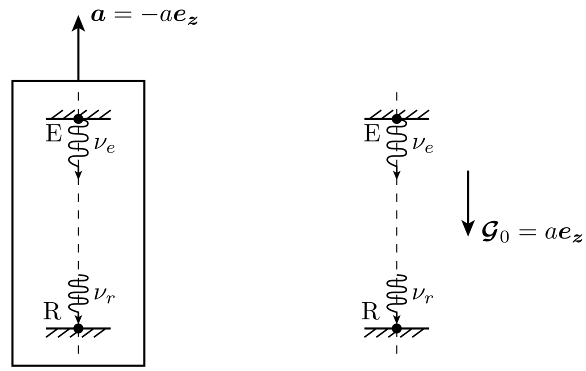

FIG. 2 – *Diagram of the experience of Pound and Rebka and illustration of the equivalence principle. Notations are defined in the text.*

a uniform gravitational field and an observer uniformly accelerated in the absence of gravitation (see [14] for philosophical considerations concerning the principle of equivalence).

Inspired by Einstein's reasoning let us consider two observers, each one having a clock of the same manufacture. These two observers are assumed to have a uniform acceleration a, for example by being both in the same rocket subjected to this acceleration. These two observers exchange photons, from the emitter E to the receiver R located at a distance H (Fig. 2 on the left). Due to the Doppler-Fizeau effect, the frequency ν_r of the electromagnetic wave received by R differs from the frequency ν_e of the wave emitted by E. At lowest order (ignoring relativistic corrections which would produce a negligible second-order effect here), the photon is received by R after a time interval H/c. The velocity of E is then $v = aH/c$. As a result, according to the Doppler-Fizeau effect, the relation between ν_r and ν_e is (still at lowest order) :

$$\nu_r = \nu_e \left(1 + \frac{v}{c}\right) = \nu_e \left(1 + \frac{aH}{c^2}\right) \tag{10}$$

Because of the equivalence principle, the situation in an accelerated rocket is physically equivalent to the one of rest observers in a uniform gravitational field $\boldsymbol{\mathcal{G}} = \mathcal{G}_0\, \boldsymbol{e}_z$, such that $\mathcal{G}_0 = a$ (Fig. 2 on the right). We remind that $\mathcal{G}$ is such that the newtonian gravitational force F exerted on a mass m submitted to the gravitational field is $F = m\mathcal{G}$. Introducing now the gravitational potential Φ, one has $\Phi_e - \Phi_r = \mathcal{G}_0 H > 0$. The gravitational potential is related, up to a constant, to the gravitational potential energy of a mass m in the gravitational field by the relation $\mathcal{E}_p = m\Phi$. Thus :

$$\nu_r = \nu_e \left(1 + \frac{\Phi_e - \Phi_r}{c^2}\right) \quad \text{or} \quad \nu_r \left(1 + \frac{\Phi_r}{c^2}\right) = \nu_e \left(1 + \frac{\Phi_e}{c^2}\right) \tag{11}$$

to first order [15].

This theoretical prediction of Einstein has been tested experimentally for the first time by Pound and Rebka in 1960 [16]. In his article written in 1911, Einstein proposed to measure this effect using the shift of the spectral lines of the Sun, while emphasizing that the effect was very small since $\mathcal{C}_\odot \approx 2 \times 10^{-6}$.

According to Einstein, equation (11) does not express just a simple Doppler-Fizeau effect on an electromagnetic wave, but more fundamentally an influence of the gravitational potential on time. To reach this conclusion, one can argue that the number of oscillation cycles in a wave packet exchanged between E and R must be preserved [6]. Therefore, introducing the proper durations τ_e and τ_r measured by clocks in E and R, one has $\nu_r \, \mathrm{d}\tau_r = \nu_e \, \mathrm{d}\tau_e$, that is to say $\nu_\Phi \, \mathrm{d}\tau_\Phi = \mathrm{Cte}$ or equivalently :

$$\frac{\mathrm{d}\tau_\Phi}{1 + \Phi/c^2} = \mathrm{d}\tau_0 \tag{12}$$

τ_0 being the proper duration measured by a distant observer, located at a point for which $\Phi \approx 0$ (typically at infinity). What is true for the photon frequency must be true for all other fields : in other words it is the proper duration τ_Φ that flows differently for E and for R.

The dependency of τ_Φ with the gravitational potential has of course to remain compatible with the foundations of the special relativity and the equivalence principle. It implies, in particular, that the speed of light, as measured

6. Likewise, Einstein argued that the number of nodes and antinodes between E and R, when a standing wave is established between the transmitter and the receiver, has to be constant, otherwise we would be in the presence of a non-stationary process, which is excluded.

by a observer at the point where he stands (this precision is important), has to stay equal to c,

$$\frac{\mathrm{d}r}{\mathrm{d}\tau_\Phi} = c \quad \text{which implies} \quad \frac{1}{1+\Phi/c^2}\frac{\mathrm{d}r}{\mathrm{d}\tau_0} = c \tag{13}$$

Hence the speed of light $c_{p,\Phi}$ measured by a *distant* observer (with proper time τ_0), who observes the propagation of the latter in the vicinity of a massive star, will be [7]

$$c_{p,\Phi} = \frac{\mathrm{d}r}{\mathrm{d}\tau_0} = c\left(1+\frac{\Phi}{c^2}\right) \tag{14}$$

Einstein obtained this expression in 1911 [4] with a different argument. Nevertheless, in his paper, he was not clear about the physical interpretation of this velocity. In particular, he was a little bit embarassed with the fact that special relativity and the equivalence principle has to imply a constancy of the speed of light, while its result shows in the contrary a dependency with the gravitational potential. He even wrote that « the principle of the constancy of the speed of light is not valid in the sense that serves as a basis for the usual theory of relativity ». In fact, there is no inconsistency with special relativity and the key point here is that this velocity $c_{p,\Phi}$ is relevant only for a distant observer. An observer at the level of the perihelion would indeed measure that the speed of light is equal to c at this point, and this is not in contradiction with equation (14). The second key ingredient is that this velocity is in fact a *phase* velocity. Einstein did not mention this term in his paper of 1911, where he used the generic term « speed of light » without distinguishing between phase or group velocity. If this velocity is interpreted as a group velocity, it would imply that light would be bend in the opposite direction, that is to say outwards instead of towards the central body!

Hopefully, Einstein used a wave analysis of the bending, and therefore arrived to a bending towards the central mass. To do this, he considered the propagation of a wave front propagating at velocity $c_{p,\Phi}$ in a non uniform gravitational

7. Note that this relation, relativistic in essence, supposes that the gravitational potential Φ is defined without additive constant; in Newtonian mechanics, the effect of the constant is neutralized by the infinite value of the speed of propagation of light.

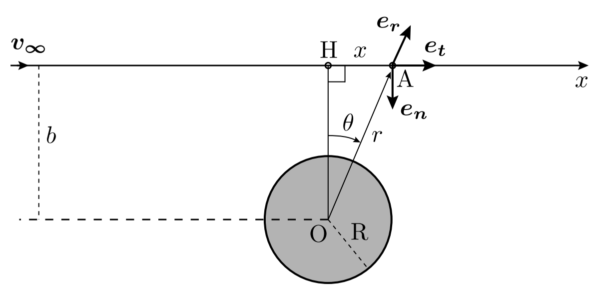

FIG. 3 – *Deflection of a light beam by a spherical mass distribution, according to Einstein in 1911. The straight line is the unperturbed trajectory.*

potential, and deduced the trajectory of light through the Malus theorem. We adopt here another approach, based on the eikonal equation.

Indeed, the dependency of the phase velocity of light with a gravitational potential Φ can also be interpreted in terms of an effective refraction index n_Φ of the (empty) medium in which light propagates, according to :

$$n_\Phi = \frac{c}{c_{p,\Phi}} = \frac{1}{1+\Phi/c^2} \approx 1 - \frac{\Phi}{c^2} > 1 \tag{15}$$

In order to determine the trajectory, one can now use the eikonal equation in the (approximation of the geometrical optics). Introducing the Frenet base $(\boldsymbol{e}_t, \boldsymbol{e}_n)$ and the curvilinear abscisse s along the trajectory, the equation of the light ray is given by [17] :

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(n_\Phi\, \boldsymbol{e}_t\right) = \mathbf{grad}\, n_\Phi \tag{16}$$

Multiplying this equation by $\boldsymbol{e}_n$ and introducing the elementary deflection angle of the path, $\mathrm{d}\chi = -\mathrm{d}\boldsymbol{e}_n \cdot \boldsymbol{e}_t$, one gets :

$$n_\Phi \frac{\mathrm{d}\chi}{\mathrm{d}s} = -\frac{1}{c^2}\, \mathbf{grad}\, \Phi \cdot \boldsymbol{e}_n$$

Since $\mathrm{d}\chi/\mathrm{d}s$ is of order 1, we can take $n_\Phi \approx 1$ at zeroth order. We then find the integral expression of the deflection angle $\chi_{E,11}$ obtained by Einstein in 1911 :

$$\chi_{E,11} \approx -\frac{1}{c^2} \int \mathbf{grad}\, \Phi \cdot \boldsymbol{e}_n\, \mathrm{d}s \tag{17}$$

The minus sign indicates a deviation towards the massive object. Treating $\mathbf{grad}\,\Phi$ as a small perturbation, the previous integral (17) can be computed on a straight line rather than the actual curved trajectory. If we denote by x the coordinate of the current point A on the trajectory, one gets, using $\Phi = -GM/r$ (Fig. 3) :

$$\chi_{E,11} \approx \frac{GM}{c^2} \int \frac{\cos\theta}{r^2}\, \mathrm{d}x \tag{18}$$

where $\boldsymbol{e}_r \cdot \boldsymbol{e}_n = -\cos\theta$; the angle θ varies from $-\pi/2$ to $\pi/2$ when x moves between $-\infty$ to ∞. Since $r = b/\cos\theta$ and $x = b\tan\theta$, with b the impact parameter, $\mathrm{d}x = b\,\mathrm{d}\theta/\cos^2\theta$ and therefore :

$$\chi_{E,11} = \frac{GM}{bc^2} \int_{-\pi/2}^{\pi/2} \cos\theta\, \mathrm{d}\theta = \frac{r_S}{b} \tag{19}$$

This result is identical to the one of Soldner, although the approaches adopted are substantially different. To understand the reason, let us use the wave aspect of any physical object, based on the Hamilton-Jacobi formalism and the link between the action S associated to a particle and the phase $\varphi = S/\hbar$ of the associated wave [18]. The velocity of the particle of light is given by $v^2 = v_\infty^2 - 2\Phi$, so that

$$v = v_\infty \left(1 - \frac{2\Phi}{v_\infty^2}\right)^{1/2} \approx v_\infty \left(1 - \frac{\Phi}{v_\infty^2}\right) > v_\infty \tag{20}$$

As already mentioned, should this velocity be interpreted as a phase velocity, it would give an effective refractive index $n_\Phi < 1$ (see equation (15)), and therefore an opposite light bending compared to observations. In order to determine the phase velocity $c_{p,\Phi}$, we can consider the displacement of a wavefront ($\varphi = \mathrm{cte}$) between t and $t + \mathrm{d}t$.

$$0 = \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\partial\varphi}{\partial t} + \boldsymbol{c_{p,\Phi}} \cdot \mathbf{grad}\,\varphi \tag{21}$$

The displacement being perpendicular to the wavefront, $\boldsymbol{c_{p,\Phi}}$ is colinear to $\mathbf{grad}\,\varphi$. Finally, replacing φ with $S/\hbar$, we deduce :

$$\frac{\partial S}{\partial t} + c_{p,\Phi}\, |\mathbf{grad}\, S| = 0 \tag{22}$$

This equation is analogous to the Hamilton-Jacobi equation [19], provided that $c_{p,\Phi}$ is expressed as a function of the generalized momentum. Then, one can use the fact that the time derivative of the action is equal to the opposite of the Hamiltonian, $\partial S/\partial t = -H$. And because the Hamiltonian does not depend explicitly on time, it is a constant $H = \mathcal{E}$ so that :

$$c_{p,\Phi} = \frac{\mathcal{E}}{|\mathbf{grad}\, S|} = \frac{\mathcal{E}}{p} \tag{23}$$

where the generalized momentum $p = \gamma m v$ [20] is identified with $\mathbf{grad}\, S$ in Hamilton-Jacobi formalism ($p_i = \partial S/\partial q_i$ [19]). Combining the previous equations gives finally [8] :

$$v \times c_{p,\Phi} = \frac{\mathcal{E}}{\gamma m} \approx c^2 \tag{24}$$

since $\mathcal{E} = \gamma m c^2 + m\Phi \approx \gamma m c^2$. It can be seen that the mass of the particle disappears and that this last relation is also valid for relativistic particles. It leads to the following relation between the phase velocity in the presence of a gravitational potential, and the phase velocity in its absence:

$$c_{p,\Phi} \approx \frac{c^2}{v_\Phi} \approx \frac{c^2}{v_\infty(1 - \Phi/v_\infty^2)} \approx c\left(1 + \frac{\Phi}{c^2}\right) < c \tag{25}$$

where we used $\Phi/v_\infty^2 \ll 1$ and $v_\infty \approx c$. This expression of the phase velocity is exactly the same as the one obtained by Einstein in 1911. As shown above, the wave associated to the particle of light is the fundamental ingredient to understand the identical results obtained by Soldner and Einstein. Note nevetheless that the Einstein result is more universal because the speed of light c is a real constant of nature (v_∞ is not).

0.3 Einstein relativistic theory of 1915

In 1915, Einstein re-analyzed, in the framework of his theory of general relativity, the deviation of a light ray by a mass distribution with spherical

8. The velocity v_Φ can be interpreted as the group velocity of the electromagnetic light wave, which allows to recover the well-known relation (24) on the product between the group and phase velocity.

symmetry. He obtained a result which is the double of what he initially published in 1911. In this new result, a first contribution is attributed to the influence of the gravitational potential on time (it is exactely the effect computed in 1911), and a second contribution, of the same magnitude, is related to the deformation of space (spatial curvature). As already mentioned, this new result was confirmed experimentally in 1919 [3].

In this last section, we wonder whether Einstein could have come to the right answer already in 1911. We first explain why the formal answer is no, and then propose a guess that could have lead Einstein to the track of general relativity before 1915.

To begin with, Einstein could not have established rigorously the correct expression until he had completed the theory of general relativity. The reason is that there are several possible relativistic theories of gravitation, which are all in agreement with the equivalence principle (see [21] for a review), but differ from Einstein's GR. Also, different attempts have been made to simply recover the Schwarzschild metric from the equivalence principle and the Newtonian limit alone, but none succeeded [9]. Only experiments finally made it possible to decide in favor of Einstein theory. All these relativistic theories of gravitation predict a first contribution identical to the one obtained by the Newton approach (cf equation (19)). In GR, as already shown, this contribution is understood as stemming from a curvature of time. The difference lies in the second contribution, which physically depends on the way space is curved by energy. For example, in Nordström's theory of gravitation of 1913 [22], the two previous contributions precisely cancel each other and give a deviation of light which is identically zero [9], in contradiction with the experiment of 1919 [3]. Nevertheless, Nordström's theory is theoretically viable, fully relativistic and in accordance with the equivalence principle.

9. In a modern point of view, this is due to the fact that the Nördstrom theory is a scalar theory ϕ, and that the coupling Lagrangian should be ϕT with T the trace of the energy-momentum tensor. For an electromagnetic field, this trace is zero, and therefore ligh cannot be coupled to a scalar.

However, one of the lessons of special relativity is that space and time are profoundally linked into a spacetime concept. Therefore, it seems natural to apply to space what has been observed with time: if duration depends on the gravitational field, length should also depends on gravitational field. The question is to know what modification should be done on length. Going back to equation (12), we can write the relation between $\mathrm{d}\tau$ and $\mathrm{d}\tau_\Phi$ as a time dilation relation. Indeed, by posing $\Phi = -2v_G^2$, one has:

$$\mathrm{d}\tau_0 = \gamma_\Phi \mathrm{d}\tau_\Phi \quad \text{with} \quad \gamma_\Phi = \left(1 + \frac{2\Phi}{c^2}\right)^{-1/2} = \left(1 - \frac{v_G^2}{c^2}\right)^{-1/2} \geq 1 \qquad (26)$$

The duration in a distant observer is dilated. One can try a contraction of length in the radial direction, that is to say in the direction in which the gravitational potential varies. We would then have:

$$\mathrm{d}r_0 = \frac{\mathrm{d}r_\Phi}{\gamma_\Phi} \qquad (27)$$

with $\mathrm{d}r_\Phi$ the length travelled during time $\mathrm{d}\tau_\Phi$ at the level of the particle of light, while $\mathrm{d}r_0$ is the length as seen by a distant observer. Then, instead of starting from (13), we have to require, because of the equivalence principle,

$$c = \frac{\mathrm{d}r_\Phi}{\mathrm{d}\tau_\Phi} \qquad (28)$$

So that the phase velocity would be given by

$$c_{p,\Phi} = \frac{\mathrm{d}r_0}{\mathrm{d}\tau_0} = \frac{1}{\gamma_\Phi^2}\frac{\mathrm{d}r_\Phi}{\mathrm{d}\tau_\Phi} = \frac{c}{\gamma_\Phi^2} = c\left(1 + \frac{2\Phi}{c^2}\right) \qquad (29)$$

This is the new phase velocity measured by a distant observer. We obtain the same relation as the equation (14), simply replacing Φ with 2Φ. It is worth noting that the radial contraction of equation (27) is nothing else that a space curvature. This contraction also define the right direction of the parallel transport of the photon [23].

The previous result is retrieved, in a more modern way, by considering the following modification of the square of the interval:

$$\mathrm{d}s^2 = \left(1 - \frac{r_S}{r}\right) c^2\mathrm{d}t^2 - \frac{1}{1 - r_S/r}\,\mathrm{d}r^2 \quad \text{with} \quad \frac{r_S}{r} = \frac{2GM}{rc^2} = -\frac{2\Phi}{c^2} \qquad (30)$$

We recover the space-time interval of the Schwarzchild metric proposed by the latter in 1916 [24]. The trajectory of the light can be obtained according to $\mathrm{d}s^2$ and therefore

$$c_\Phi = \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{c}{\gamma_\Phi^2} = c\left(1 + \frac{2\Phi}{c^2}\right) = c\left(1 - \frac{r_S}{r}\right) \tag{31}$$

This expression of c_Φ looks like the one obtained initially by Einstein in 1911, with the factor 2 which affects the gravitational potential. It is then sufficient to use Einstein's wave reasoning to obtain a double deviation angle, in accordance with the observations [3].

Let us notice that other choices were *a priori* admissible. For example, in the Nordström theory, this choice would be not to contract the radial lengths, but on the contrary to expand them, $\mathrm{d}r_0 = \gamma_\Phi \mathrm{d}r_\Phi$. This amount to treat space and time with the same factor. In modern langage, it means that the metric is conformally flat, that is to say $\mathrm{d}s^2 = f(\Phi)(c^2\mathrm{d}t^2 - \mathrm{d}r^2)$. This would give $c_{p,\Phi} = c$. Therefore, in this theory, because the phase velocity is a constant, light is not bended. Physically, there is a perfect compensation between the effect on time and the effect on space.

Conclusion

Let's remember the two essential points.

i) From Newtonian perspective, Soldner showed as early as 1801 that light should be deflected by a spherical mass. This deviation is identical (at lowest order) to the one obtained by Einstein in 1911, although their approaches differ substantially. Equality of both results comes from the principle of equivalence and the link between the velocity of a physical object and the velocity of the associated de Broglie wave.

ii) An intuitive reasoning based on the effect of a gravitational potential on radial lengths and thus on the curvature of space could have put Einstein on the track of general relativity as early as 1911; he would then have found the right result for light deviation, and at the same time the Schwarzschild metric (see also [8]).

Bibliographie

[1] Soldner J., *On the deviation of a light ray from its motion along the line of the attraction of a celestial body*, 1974, Phys. Review Lett. (33) pp 1237-1239

[2] Hadley H.G., *Predictions of the gravitational bending of light before einstein*, American Journal of Physics, 1966, 34, 162

[3] Dyson F., Eddington A., Davidson C., *A Determination of Defection of Light by the Sun's Gravitational Field from Observations Made at the Total Eclipse of May 29, 1919*, 1920, Phil. Trans A, vol. CCXX-A 579, pp 291-332

[4] Einstein A., *of the influence of gravity on the propagation of light*, Annalen der Physik, 1911, XXXV, pp 898-908

[5] Einstein A., *Preussiche Academie der Wissenshaften, Sitzungberichte*, Annalen der Physik, 1915, p 831

[6] Einstein A., *The foundations of the theory of general relativity*, Annalen der Physik, 1916, XLIX, pp 769-882, Hawking S., Dunod-Paris, pp 857-912

[7] Lerner L., *A simple calculation of the deflection of light in a Schwarzschild gravitational field*, American Journal of Physics, 1997, 65, 1194

[8] Provost, J.-P. and Bracco, C., *The 1895 Lorentz transformations: historical issues and present teaching*, European Journal of Physics, 2016, 37(4), 045601–11

[9] Gruber R.P. et al., *The impossibility of a simple derivation of the Schwarzschild metric*, American Journal of Physics, 1988, 56, 265

[10] Eisenstaedt, J., *De l'influence de la gravitation sur la propagation de la lumière en théorie newtonienne. L'archéologie des trous noirs*, Arch. Hist. Exact Sci., 1991, Vol. 42, No. 4, p. 315 - 386

[11] Bradley J., *A letter to E. Halley giving an account of a new discovered Motion of the Fix'd Stars*, Physical Transaction of the Royal Society, 1729, 35, pp 637-661

[12] Pais A., *Albert Einstein, Life and Work*, 1993, InterEditions, 539 pages, p 177

[13] Darrigol, O., *Mesh and measure in early general relativity*, Studies in History and Philosophy of Modern Physics, 2015, 52 (Part B), 163–187

[14] Brown, H.R. and Read J. *Clarifying possible misconceptions in the foundations of general relativity*, American Journal of Physics, 2016, 84(5), 327-334

[15] Pérez J-Ph., *Redshift Formulas and the Doppler-Fizeau Effect*, Eur. J. Phys., 2016, Vol. 37, 015604 (8 pp)

[16] Pound R. and G. Rebka, *Gravitational red-shift in nuclear resonance*, Physical Review Letters, 1959, Vol. 3, Number 9, pp 439-441

[17] Pérez J-Ph., *Optique, fondements et applications*, 2013, Dunod 7th edition, 698 pages, p 204 and p 262

[18] Pérez J-Ph., Carles R. and Pujol O., *Quantique, fondements et applications*, 2013, by de Boeck, 1078 pages, p 175

[19] Landau L. and Lifchitz E., *Mechanics*, 1969, Mir, 228 pages, p 191

[20] Pérez J-Ph., *Relativité, fondements et applications*, 2016, Dunod 3rd Edition, 439 pages, p 51

[21] Norton J.D., *Einstein, Nordström and the early demise of scalar, Lorentz-covariant theories of gravitation*, Arch. Hist. Exact Sci., 1992, 45:17. https://doi.org/10.1007/BF00375886

[22] Nordström G., *Zur Theorie der Gravitation vom Standpunkt des Relativistätsprinzips*, 1913, Ann. der Phys. 42, p 533

[23] Ferraro R., *The equivalence principle and the bending of light*, American Journal of Physics, 2003, 71, 168

[24] Schwarzchild K., *Uber das Gravitationsfeld eines Masspunktes nach der Einsteinschen Theorie*, Sitzber. Deut. Akad. Wiss. Berlin Maths-Phys., 1916, Tech. K1, pp 189-196