Quantum Correction from Super-Weyl Transformation in Supergravity

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Abstract

Einstein-frame supergravity is accompanied by quantum correction terms because of super-Weyl transformation. The correction term consists of the field strength terms that respectively originate from gauge, gravitational, and Kähler anomalies. In this paper, it is shown how the full-kind one-loop Jacobian correction term is calculated and represented in $\mathcal{N} = 1, D = 4$ superspace formalism.

1

1 Introduction

Various supersymmetric theories have been subject to researches and successful as quantum field theories till today. For example, path integral calculation of supersymmetric Yang-Mills theory has already been deeply researched [1–5]. However, supergravity theory has not been understood so clearly as super-Yang-Mills, due to the complexity of its Lagrangian and the difficulty in its quantisation. In this paper, as a foothold for a better understanding of quantum supergravity, we aim to calculate the qunatum correction term from super-Weyl transformation in $\mathcal{N} = 1, D = 4$ supergravity.

Matter-coupled supergravity Lagrangian is usually constructed via the Brans-Dicke form, where the scalar curvature (i.e. the kinetic term of the graviton) is unconventionally coupled with the matter scalar:

$$\mathcal{L}_{\mathrm{BD}}/e = -\frac{1}{2}e^{-K/3}\mathcal{R} + \cdots \tag{1.1}$$

and thereafter transformed into the Einstein form by super-Weyl transformation:

$$\mathcal{L}_{\mathrm{E}}/e = -\frac{1}{2}\mathcal{R} + \cdots . \tag{1.2}$$

Here e is the vielbein determinant, $\mathcal{R}$ the (gravitational) scalar curvature, and $K = K(A, A^\dagger)$, an Hermitian function, the Kähler potential for the target manifold of the nonlinear sigma model. Since the transformation from (1.1) to (1.2) includes Weyl rescaling and chiral rotation of the fermions, the full Einstein-frame Lagrangian in quantum theory requires the anomaly correction term:

$$\mathcal{L}_{\mathrm{E}}^{\mathrm{full}} = \mathcal{L}_{\mathrm{E}} + \Delta\mathcal{L}. \tag{1.3}$$

In this paper we calculate $\Delta\mathcal{L}$ at one loop. A previous research [6] has already given a similar consideration and calculation, but it is incomplete in that

- they calculate only the gauge anomaly and ignore the gravitational one and the Kähler one

- they ignore the Kähler-gauge coupling in the covariant derivative of the chiral fermion in the case where the target manifold has a nontrivial Kähler metric.

Hence this paper gives the full-kind, one-loop $\Delta\mathcal{L}$ in the correct form, i.e. with all kinds of (gauge, gravitational, and Kähler) anomalies and with the correct gauge anomaly term where the Kälher metric is nontrivial. In order to do this, we also construct the anomaly-calculation method (Fujikawa method) for the gauge bosons and the Rarita-Schwinger fermion, not only for spin-1/2 fermions, which is already well known. In a previous research [7] they treat the path-integral measure in a prudent but complicated way in order to investigate how the anomaly mediated gaugino mass is determined and expressed in terms of

superfield, while we simply calculate the anomaly terms caused by super-Weyl transformation.

This paper is organised as follows. In Section 2 we review the general super-Weyl transformation and the specific parameter required when the Brans-Dicke form is transformed into the Einstein form. In Section 3 we review the Fujikawa method for spin-1/2 fields and apply it to construct the similar method of spin-1 and spin-3/2 versions. In Section 4, combining them, we calculate the anomaly correction term from super-Weyl transformation and describe it in the terms of the superfield strengths. Section 5 is the summary.

2 Super-Weyl Transformation

There are roughly two ways to construct the supergravity Lagrangian. The one is superspace formalism [8–10], and the other is superconformal tensor calculus[1] [11–18]. This paper is mainly described in superspace formalism, and the notation follows [10] unless otherwise specified.

Matter-coupled supergravity Lagrangian by superspace formalism is constructed inevitably via the kinetic terms of the form [10]

$$
\mathcal{L}_{\text{BD}} = \int d^2\Theta \, 2\mathcal{E} \left[\frac{3}{8} \left(\bar{\mathcal{D}}^2 - 8R \right) e^{-(K+\Gamma)/3} \right.
$$

$$
\left. + \frac{1}{16} H_{(ab)}(\Phi) W^{(a)\alpha} W^{(b)}_\alpha + P(\Phi) \right] + \text{h.c.} \tag{2.1}
$$

$$
= e \left[-\frac{1}{2} e^{-K/3} \mathcal{R} - \frac{1}{4} e^{-K/3} \varepsilon^{klmn} \left(\bar{\psi}_k \bar{\sigma}_l \psi_{mn} - \psi_k \sigma_l \bar{\psi}_{mn} \right) \right.
$$

$$
\left. - \frac{i}{2} \left(\chi^i \sigma^m \mathcal{D}_m \bar{\chi}_i + \bar{\chi}^i \bar{\sigma}^m \mathcal{D}_m \chi_i \right) - \partial^m A^i \partial_m A_i^* + \cdots \right], \tag{2.2}
$$

which has unconventionally matter-coupled kinetic terms (Brans-Dicke form) of the gravitational multiplet and non-Kähler kinetic terms of the chiral matter multiplet. Here $H_{(ab)}$ is the gauge kinetic function, $W^{(a)\alpha}$ the gauge field strength[2], where (a) is the index for the adjoint representation, P the superpotential, K the Kähler potential[3], and $\Gamma = \Gamma(\Phi, \Phi^\dagger, V)$ the gauge counterterm, which renders the Lagrangian gauge invariant. In order to transform Lagrangian into the canonical Einstein (and Kähler) form, Weyl transformations and some other transformations are required and they are united as super-Weyl transformation. Thus in this section we resummarise the general super-Weyl transformation in the superfield level and its special parameter for the transformation from Brans-Dicke form to Einstein form. The results in this section are quoted from [10] and [6] as a whole.

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[1] Actually there is a construction method integrating both methods, which is called superconformal superspace formalism. Refer to [19–23], for example.

[2] $W^{(a)\alpha}$ should not be confused with $W^{\alpha\beta\gamma}$, one of the gravitational field strengths.

[3] Throughout this paper, K denotes $K(\Phi, \Phi^\dagger)$ in the context of superfields, while it denotes $K(A, A^*)$ in the context of component fields.

2.1 General Super-Weyl Transformation

Since supergravity in superspace has a large number of redundant degrees of freedom, torsion constraints are required to reduce them. Super-Weyl transformation is the transformation that includes supervielbein rescaling and preserves the torsion constraints. A general infinitesimal super-Weyl transformation is parametrised by a chiral superfield Σ as[4]

$$\delta_{\rm SW}\mathcal{E} = 6\Sigma\mathcal{E} + \frac{\partial}{\partial\Theta^\alpha}(S^\alpha\mathcal{E})$$

$$\delta_{\rm SW}\Phi = -S^\alpha\frac{\partial}{\partial\Theta^\alpha}\Phi$$

$$\delta_{\rm SW}\left[\left(\bar{\mathcal{D}}^2 - 8R\right)U\right] = -(\bar{\mathcal{D}}^2 - 8R)\left[(4\Sigma - 2\Sigma^\dagger)U\right]$$
$$- S^\alpha\frac{\partial}{\partial\Theta^\alpha}\left[\left(\bar{\mathcal{D}}^2 - 8R\right)U\right]$$

$$\delta_{\rm SW}W_\alpha = -3\Sigma W_\alpha - S^\beta\frac{\partial}{\partial\Theta^\beta}W_\alpha, \tag{2.3}$$

where

$$S^\alpha = \Theta^\alpha\left(2\Sigma^\dagger - \Sigma\right)\big| + \Theta^2\left(\mathcal{D}^\alpha\Sigma\right)\big| \tag{2.4}$$

and U is an arbitrary real superfield of super-Weyl weight zero[5]. These transformations induce an infinitesimal variation of the Lagrangian:

$$\delta\mathcal{L} = \int d^2\Theta\, 2\mathcal{E}\left\{\frac{3}{8}\left(\bar{\mathcal{D}}^2 - 8R\right)\left[2\left(\Sigma + \Sigma^\dagger\right)e^{-(K+\Gamma)/3}\right] + 6\Sigma P\right\} + \text{h.c.} \tag{2.5}$$

Thus a general finite transformation is

$$\mathcal{L} \mapsto \mathcal{L}' = \int d^2\Theta\, 2\mathcal{E}\left\{\frac{3}{8}\left(\bar{\mathcal{D}}^2 - 8R\right)\exp\left[-\frac{1}{3}\left(K + \Gamma - 6\Sigma - 6\Sigma^\dagger\right)\right]\right.$$
$$\left. + \frac{1}{4}H_{(ab)}W^{(a)}W^{(b)} + e^{6\Sigma}P\right\} + \text{h.c.} \tag{2.6}$$

2.2 Parameters from Brans-Dicke to Einstein

By transformation (2.6), the Lagrangian can be transformed into the form without undesirable couplings: without Brans-Dicke couplings of the gravitational multiplet or non-Kähler kinetic terms of the matter multiplet. In order to operate this transformation, it is necessary that Σ should satisfy

$$K| = 6(\Sigma + \Sigma^\dagger)|, \quad (\mathcal{D}_\alpha K)| = 6(\mathcal{D}_\alpha\Sigma)|, \quad (\mathcal{D}^2 K)| = 6(\mathcal{D}^2\Sigma)|, \tag{2.7}$$

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[4]These transformation rules should be understood in Θ expansion: for example, in general χ in Φ can be transformed even though Φ is of Weyl weight zero, since the supervielbein E_M^A is transformed and Θ expansion is dependent on the supervielbein via the covariant derivative.

[5]Here we require only the case of weight zero, but in more general the field U may have nonzero super-Weyl weight. See [10] for that case.

or $K - 6\Sigma - 6\Sigma^\dagger$ has no lowest, Θ, and Θ^2 components [6]. The first condition corresponds to the canonical kinetic term of the graviton, the second to that of the gravitino, and the third to those of the matter multiplets[6]. Note that Γ has no contribution to (2.7), since Γ can be assumed to be in the Wess-Zumino gauge, where Γ has no lowest, Θ, Θ^2 coefficients [10]. These conditions determine the chiral parameter Σ required for the transition to Einstein frame:

$$\Sigma = B + \sqrt{2}\Theta\varphi + \Theta^2 C, \tag{2.8}$$

with

$$B = \frac{1}{12}K + i\phi, \quad \varphi = \frac{1}{6}K_i\chi^i, \quad C = \frac{1}{6}K_iF^i - \frac{1}{12}K_{ij}\chi^i\chi^j, \tag{2.9}$$

where

$$K_i \equiv \frac{\partial K}{\partial A^i}, \quad K_{ij} \equiv \frac{\partial^2 K}{\partial A^i \partial A^j}, \tag{2.10}$$

and F^i is the Θ^2 component of Φ^i. Note that ϕ, the imaginary part of the lowest component, is left arbitrary. From the point of view that the Lagrangian should be of the Einstein and Kähler form, ϕ can be set to 0 by the condition (2.7). However, the degree of freedom of ϕ plays a significant role in the anomaly calculation, as shown in Chapter 4.

3 Fujikawa Method

The transformation (2.3) includes Weyl rescaling of each field, and furthermore chiral rotations of the fermions. Since these transformations are anomalous, a correction term that originates in the variation of the path integral measure should be added to the Lagrangian in the quantum theory[7]. In order to calculate this correction term, an anomaly calculation technique, which is called Fujikawa method, can be utilised. Originally Fujikawa method was devised for the calculation of symmetry breaking (i.e. how much different the divergence of the conserved current is from zero), but the functional measure variation from the field redifinition can also be calculated in almost the same way. Although Fujikawa method for spin-1/2 fields is already well known [24], the similar methods for spin-1 and spin-3/2 fields are yet to be clearly constructed. Thus in this section we review Fujikawa method and construct the similar anomaly calculation methods for the gauge vector bosons and the Rarita-Schwinger field.

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[6]Refer to Appendix A for a detailed explanation.

[7]The anomaly correction term cannot be avoided even if one adopt the Einstein-frame Lagrangian as the starting point of the quantum theory, since the Einstein-frame SUSY transformation contains Weyl rescaling.

3.1 Generalities

In general, the functional measure $\mathcal{D}\phi$, of a field ϕ (not necessarily a scalar), for the (Wick-rotated and Euclideanised) path integral

$$\int \mathcal{D}\phi\, e^{-S[\phi]} = \int \mathcal{D}\phi\, \exp\left(-\int d^4x\, \mathcal{L}[\phi]\right) \tag{3.1}$$

is defined as

$$\mathcal{D}\phi \equiv \prod_n da_n, \tag{3.2}$$

where a_n is the expansion coefficients by the orthonormal basis:

$$\phi(x) = \sum_n a_n \phi_n(x) \tag{3.3}$$

$$\int d^4x\, \phi_m^\dagger(x)\phi_n(x) = \delta_{mn}, \tag{3.4}$$

and the basis functions $\phi_n(x)$ are the eigenfunctions of the "Hermite, gauge-covariant differential operator" $\mathcal{O}$:

$$\mathcal{O}\phi_n = \lambda_n \phi_n. \tag{3.5}$$

Note that ϕ_n and a_n satisfy

$$a_n = \int d^4x\, \phi_n^\dagger(x)\phi(x) \tag{3.6}$$

$$\sum_n \phi_n(x)\phi_n^\dagger(y) = \delta(x-y). \tag{3.7}$$

Then for an infinitesimal transformation $\phi \mapsto \phi' = \phi + \alpha\phi$ with a parameter α,[8] which satisfies $|\alpha| \ll 1$,

$$\begin{aligned}
a_n' &= \int d^4x\, \phi_n^\dagger(x)\phi'(x) \\
&= \sum_m a_m \left(\delta_{mn} + \int d^4x\, \phi_n^\dagger(x)\alpha(x)\phi_m(x)\right) \\
&\equiv \sum_m a_m \left(\delta_{mn} + M_{mn}\right),
\end{aligned} \tag{3.8}$$

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[8]Note that in general the parameter α is not necessarily a scalar, but may have spinor or Lorentz indices if the field ϕ is a spinor or a vector.

and the functional measure varies as

$$\mathcal{D}\phi' = \mathcal{D}\phi \, \det\left(\frac{\partial a'}{\partial a}\right) \tag{3.9}$$

$$\det\left(\frac{\partial a'}{\partial a}\right) = \exp \mathrm{tr} \log\left(\delta_{mn} + M_{mn}\right)$$

$$\sim \exp \mathrm{tr} M$$

$$= \exp\left[\sum_n \int d^4x \, \phi_n^\dagger(x)\alpha(x)\phi_n(x)\right]. \tag{3.10}$$

However, the sum $\sum_n \phi_n(x)\phi_n(x)$ is ill-defined because of (3.7). Therefore this sum should be regularised in a gauge-invariant way:

$$\sum_n \int d^4x \, \phi_n^\dagger(x)\alpha(x)\phi_n(x)$$

$$\to \int d^4x \, \mathrm{Tr}\left[\alpha(x)\sum_n e^{t^2\lambda_n^2}\phi_n(x)\phi_n^\dagger(x)\right]\Bigg|_{t=0}, \tag{3.11}$$

where Tr denotes the full trace, i.e. the trace with respect to all indices such as spinor or coordinate ones[9]. Furthermore, the internal sum is

$$\sum_n e^{t^2\lambda_n^2}\phi_n(x)\phi_n^\dagger(x) = \sum_n e^{t^2\lambda_n^2}\phi_n(y)\phi_n^\dagger(x)\Bigg|_{y=x}$$

$$= e^{t^2\mathcal{O}_x^2}\sum_n \phi_n(y)\phi_n^\dagger(x)\Bigg|_{y=x}$$

$$= e^{t^2\mathcal{O}_x^2}\delta(x-y)\Bigg|_{y=x}$$

$$= \int \frac{d^4k}{(2\pi)^4} e^{-ikx} e^{t^2\mathcal{O}^2} e^{ikx}. \tag{3.12}$$

Thus it suffices to define appropriate $\mathcal{O}$'s and calculate the integral (3.12) for spinors, vectors, and Rarita-Schwinger fields.

The generalities above can be applied also to theories that contains gravity, but note that the transformation parameter should be for the weight-corrected field $\tilde{\phi} \equiv \sqrt{e}\phi$ with no diffeomorphism indices[10]. For example, the parameter for the gravitino should be understood as the one for $\tilde{\psi}_a \equiv \sqrt{e}\psi_a$, not for ψ_m or ψ_a. Here shown are the coefficients for the infinitesimal transformation (2.3)

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[9]Remember that in general ϕ and α can have spinor or coordinate indices.

[10]In the argument below we use fields with diffeomorphism indices for simplicity in the calculation of the spacetime trace, but the results are not affected. Only the transformation parameters are relevant.

with $\Sigma| = \zeta + i\xi$:

$$\delta e_m^a = 2\zeta e_m^a$$
$$\delta \tilde{\psi}_a = 3(\zeta - i\xi)\tilde{\psi}_a$$
$$\delta \tilde{A} = 0$$
$$\delta \tilde{\chi} = 3(\zeta + i\xi)\tilde{\chi}$$
$$\delta \tilde{F} = 2(\zeta + 3i\xi)\tilde{F}$$
$$\delta \tilde{\lambda} = (\zeta - 3i\xi)\tilde{\lambda}$$
$$\delta \tilde{v}_a = -2\zeta \tilde{v}_a$$
$$\delta \tilde{D} = (\zeta - 3i\xi)\tilde{D}. \tag{3.13}$$

3.2 Spinor Field: Review

The path integral approach to the quantum anomaly of a spinor has long been established [24]. The operator $\mathcal{O}$ should be taken to be $\tilde{\slashed{\mathcal{D}}} \equiv \gamma^m \tilde{\mathcal{D}}_m$, where $\tilde{\mathcal{D}}_m$ is the full covariant derivative, including the gravitational, gauge, and Kähler connections. Then the integral (3.12) is estimated as

$$\int \frac{d^4 k}{(2\pi)^4} e^{-ikx} \exp\left[t^2 \tilde{\slashed{\mathcal{D}}}^2\right] e^{ikx}$$
$$= \int \frac{d^4 k}{(2\pi)^4} \exp\left[t^2 \left(\tilde{\slashed{\mathcal{D}}} + i\slashed{k}\right)^2\right]$$
$$= \frac{1}{t^4} \int \frac{d^4 k}{(2\pi)^4} \exp\left(-k^2 + 2it\slashed{k}\tilde{\slashed{\mathcal{D}}} + t^2\tilde{\slashed{\mathcal{D}}}^2\right) \qquad (kt \mapsto k) \tag{3.14}$$
$$\xrightarrow{t \to 0} \frac{1}{384\pi^2} \left[2\Omega^{mn}\Omega_{mn} + 2\left(\Box \gamma^{mn}\Omega_{mn}\right) + 3\left(\gamma^{mn}\Omega_{mn}\right)^2\right], \tag{3.15}$$

where $\Omega_{mn} = [\tilde{\mathcal{D}}_m, \tilde{\mathcal{D}}_n]$, the field strength of the covariant derivative[11].

3.3 Vector Field

The path integral measure for a vector field can be defined basically in a similar way to that for a spinor, but the gauge degree of freedom should be treated carefully. A candidate for the differential operator $\mathcal{O}$ is $\mathcal{O}^2 A_k = \nabla^l \left(\nabla_l A_k - \nabla_k A_l\right)$. However, this $\mathcal{O}^2$ is not elliptic, and therefore the integral (3.12) does not converge. This is because A_k has the gauge freedom, and therefore the integrand of (3.12) is constant in the direction of this freedom. Thus it is necessary to fix the gauge and suppress the divergence. In order to do that, the diffeomorphism-invariant Lorentz gauge condition $\nabla_k A^k = 0$ can be adopted. Then the second-

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[11]Actually (3.14) contains terms that diverge as $1/t^4$ and $1/t^2$, but they cancel out those from the functional measure variation of the bosonic field. Refer to Appendix B for a detailed calculation.

order differential operator above

$$\nabla^l \left(\nabla_l A_k - \nabla_k A_l \right) = \Box A_k + \mathcal{R}_k{}^l A_l$$
$$\sim \left(\delta_k{}^l \Box + \mathcal{R}_k{}^l \right) A_l$$
$$\equiv (\Box + \hat{\mathcal{R}}) A_k \tag{3.16}$$

is now elliptic. The plane-wave trace (3.12) is evaluated as

$$\int \frac{d^4 k}{(2\pi)^4} e^{-ikx} e^{t^2(\Box + \hat{\mathcal{R}})} e^{ikx}$$
$$\xrightarrow{t \to 0} \frac{1}{192\pi^2} \left[2 \left(\Box \mathcal{R} \right) + 6 \mathcal{R}^{mn} \mathcal{R}_{mn} - \mathcal{R}^{klmn} \mathcal{R}_{klmn} \right] \tag{3.17}$$

where $\mathcal{R}_{klmn}$ and $\mathcal{R}_{mn}$ are the (gravitational) Riemann curvature tensor and Ricci tensor respectively. Note that in supergravity theory A_m has only the gravitational connection in its covariant derivative, and therefore the result is written in only terms of the curvature tensor from the metric. Since A_m is transformed only by Weyl rescaling (i.e. scalar-like transformation), we have already taken the coordinate trace in (3.17).

3.4   Rarita-Schwinger Field

A candidate for the gauge-covariant operator is $\mathcal{O}\psi^k = (1/\sqrt{2})\gamma^{klm}\nabla_l\psi_m$, but the gauge condition $\nabla^m \psi_m = 0$ needs to be imposed because of the same reason as the case for vector fields. Then $\mathcal{O}^2$ is modified into an elliptic operator:

$$\mathcal{O}^2 \psi^k = \gamma^{klm} \nabla_l \gamma_{mnp} \nabla^n \psi^p$$
$$\sim \left[\delta_l^k \Box + X^k{}_l \right] \psi^l$$
$$\equiv (\Box + \hat{X}) \psi^k \tag{3.18}$$

where

$$X^k{}_l = \frac{1}{4} \delta_l^k \gamma^{mn} \Omega_{mn} + \frac{1}{2} \left(\gamma^{km} \Omega_{ml} + \gamma_{lm} \Omega^{mk} \right). \tag{3.19}$$

The spacetime trace (3.12) is calculated as

$$\int \frac{d^4 k}{(2\pi)^4} e^{-ikx} e^{t^2(\Box + \hat{X})} e^{ikx}$$
$$\xrightarrow{t \to 0} \frac{1}{384\pi^2} \left[-3 \left(\gamma^{mn} \Omega_{mn} \right)^2 + 6 \gamma^{kl} \Omega_{lm} \gamma^{mn} \Omega_{nk} - 16 \Omega^{mn} \Omega_{mn} \right]$$
$$= \frac{1}{3072\pi^2} \left(6 \mathcal{R}^2 - 12 \mathcal{R}^{mn} \mathcal{R}_{mn} + 22 \mathcal{R}^{klmn} \mathcal{R}_{klmn} - 11 \gamma^{abcd} \mathcal{R}^{mn}{}_{ab} \mathcal{R}_{mncd} \right). \tag{3.20}$$

Similarly to (3.17), we have already taken the coordinate trace. The spinor-index trace is yet to be summed.

9

4 Jacobian Correction Term

In this section we finally calculate the Jacobian correction term $\Delta\mathcal{L}$ in one-loop. First, as a preparation, we in advance write down the square terms of the Yang-Mills and gravitational field strengths in superfield notation. Then we compute only the terms in $\Delta\mathcal{L}$ that can be written as the squares of the bosonic field strengths and describe them in terms of superfield by coefficient comparison.

Yang-Mills field strength is

$$\int d^2\Theta\, 2\mathcal{E}\Sigma W^{(a)\alpha}W_\alpha^{(a)} + \text{h.c.} = -e\xi V^{(a)mn}\tilde{V}_{mn}^{(a)} + \cdots, \tag{4.1}$$

where Σ is a chiral superfield and $\Sigma| = \zeta + i\xi$ and the ellipses denote the terms including fermions.

In Poincaré supergravity there are three gravitational field strengths [25][12]:

$$\int d^2\Theta\, 2\mathcal{E}\Sigma W^{\alpha\beta\gamma}W_{\alpha\beta\gamma} + \text{h.c.}$$
$$= e\left[-\frac{1}{24}\zeta\left(\mathcal{R}^2 - 2\mathcal{R}^{mn}\mathcal{R}_{mn} + \mathcal{R}^{klmn}\mathcal{R}_{klmn}\right) + \frac{1}{48}\xi\epsilon^{abcd}\mathcal{R}^{mn}{}_{ab}\mathcal{R}_{mncd} + \cdots\right] \tag{4.2}$$

$$\int d^2\Theta\, 2\mathcal{E}\Sigma\left(\bar{\mathcal{D}}^2 - 8R\right)\left(G^{\alpha\dot\alpha}G_{\alpha\dot\alpha}\right) + \text{h.c.} = -e\zeta\left(2\mathcal{R}^{mn}\mathcal{R}_{mn} - \frac{4}{9}\mathcal{R}^2 + \cdots\right) \tag{4.3}$$

$$\int d^2\Theta\, 2\mathcal{E}\Sigma\left(\bar{\mathcal{D}}^2 - 8R\right)\left(R^\dagger R\right) + \text{h.c.} = -e\zeta\cdot\frac{1}{18}\mathcal{R}^2 + \cdots. \tag{4.4}$$

Furthermore, there is a Kähler-curvature field strength of chiral superfield:

$$\int d^2\Theta\, 2\mathcal{E}\Sigma\left(\bar{\mathcal{D}}^2 - 8R\right)\Big[R^{ij^*}{}_{kl^*}(\Phi)R_{ij^*mn^*}(\Phi)$$
$$\times (\mathcal{D}^\alpha\Phi^k)(\bar{\mathcal{D}}_{\dot\alpha}\Phi^{\dagger l})(\mathcal{D}_\alpha\Phi^m)(\bar{\mathcal{D}}^{\dot\alpha}\Phi^{\dagger n})\Big] + \text{h.c.}$$
$$= e\zeta\cdot 8R^{ij^*}{}_{kl^*}(A)R_{ij^*mn^*}(A)(\mathcal{D}^p A^k)(\mathcal{D}^q A^{\dagger l})(\mathcal{D}_p A^m)(\mathcal{D}_q A^{\dagger n}) + \cdots, \tag{4.5}$$

where $R_{ij^*kl^*}(\Phi) = R_{ij^*kl^*}(\Phi,\Phi^\dagger)$ is the curvature tensor of the target Kähler manifold in terms of superfield, and $R_{ij^*kl^*}(A) = R_{ij^*kl^*}(A,A^\dagger)$ similarly. From here, we operate the calculation of the Jacobians and describe them in terms of the superfield strengths above, for each of chiral, vector, and gravitational multiplets.

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[12]Note that the reference [25] is incomplete in that they have not calculated the parity-odd term $\epsilon\mathcal{R}\mathcal{R}$ of (4.2), which derives from the imaginary part of $W^{\alpha\beta\gamma}W_{\alpha\beta\gamma}$.

4.1 Chiral Multiplet

The curvature of covariant derivative (i.e. the commutator of covariant derivative) for the chiral spinor is

$$[\mathcal{D}_m, \mathcal{D}_n]\chi^i = M_{mn}{}^i{}_j\chi^j + \frac{1}{4}\mathcal{R}_{mnab}\gamma^{ab}\chi^i, \tag{4.6}$$

where

$$M_{mn}{}^i{}_j = V_{mn}^{(a)}\nabla_j X^{(a)i} + R^i{}_{jkl^*}(A)(\mathcal{D}_m A^k)(\mathcal{D}_n A^{\dagger l}), \tag{4.7}$$

and $X^{(a)i}$ is the Killing vectors of the target Kähler manifold. Thus the plane-wave trace (3.15) is estimated as

$$
\begin{aligned}
&2\Omega^{mn}\Omega_{mn} + 2\left(\Box\gamma^{mn}\Omega_{mn}\right) + 3\left(\gamma^{mn}\Omega_{mn}\right)^2 \\
=&N\left(-\Box\mathcal{R} + \frac{3}{4}\mathcal{R}^2 - \frac{1}{4}\mathcal{R}^{klmn}\mathcal{R}_{klmn} + \frac{1}{8}\gamma^{abcd}\mathcal{R}^{mn}{}_{ab}\mathcal{R}_{mncd}\right) \\
&+ 3\gamma^{klmn}V_{kl}^{(a)}V_{mn}^{(b)}(\nabla_i X^{(a)j})(\nabla_j X^{(b)i}) \\
&+ 4R^{ij^*}{}_{kl^*}(A)R_{ij^*mn^*}(A)(\mathcal{D}^p A^k)(\mathcal{D}^q A^{\dagger l})(\mathcal{D}_p A^m)(\mathcal{D}_q A^{\dagger n}) + \cdots, \tag{4.8}
\end{aligned}
$$

where N is the number of chiral multiplets. Finally, using (3.13), the one-loop correction term from the chiral multiplets is[13]

$$\Delta\mathcal{L}^1_{\text{chiral}} = \Delta\mathcal{L}^1_{\text{chiral;K}} + \Delta\mathcal{L}^1_{\text{chiral;V}} + \Delta\mathcal{L}^1_{\text{chiral;G}}, \tag{4.9}$$

where

$$
\begin{aligned}
&\Delta\mathcal{L}^1_{\text{chiral;K}} \\
=\;&\frac{1}{64\pi^2}\int d^2\Theta\, 2\mathcal{E}\Sigma\left(\bar{\mathcal{D}}^2 - 8R\right)\Big[R^{ij^*}{}_{kl^*}(\Phi)R_{ij^*mn^*}(\Phi) \\
&\qquad\qquad\qquad\qquad\qquad \times (\mathcal{D}^\alpha\Phi^k)(\bar{\mathcal{D}}_{\dot{\alpha}}\Phi^{\dagger l})(\mathcal{D}_\alpha\Phi^m)(\bar{\mathcal{D}}^{\dot{\alpha}}\Phi^{\dagger n})\Big] + \text{h.c.} \tag{4.10}
\end{aligned}
$$

$$\Delta\mathcal{L}^1_{\text{chiral;V}} = -\frac{3}{16\pi^2}\int d^2\Theta\, 2\mathcal{E}\Sigma W^{(a)\alpha}W_\alpha^{(b)}\nabla_i X^{(a)j}\nabla_j X^{(b)i} + \text{h.c.} \tag{4.11}$$

$$
\begin{aligned}
&\Delta\mathcal{L}^1_{\text{chiral;G}} \\
=\;&\frac{N}{128\pi^2}\int d^2\Theta\, 2\mathcal{E}\Sigma\Big[24W^{\alpha\beta\gamma}W_{\alpha\beta\gamma} \\
&\qquad\qquad\qquad + (\bar{\mathcal{D}}^2 - 8R)(G^{\alpha\dot{\alpha}}G_{\alpha\dot{\alpha}} - 64R^\dagger R + 6\mathcal{D}^2 R)\Big] + \text{h.c.} \tag{4.12}
\end{aligned}
$$

---

[13] Note that (4.8) is yet to be summed in spinor index.

4.2 Vector Multiplet

The curvature of covariant derivative for the spinor in the vector multiplet is

$$[\mathcal{D}_m, \mathcal{D}_n]\lambda^{(a)} = -f^{(abc)}V_{mn}^{(b)}\lambda^{(c)} + \frac{1}{4}\mathcal{R}_{mnab}\gamma^{ab}\lambda^{(a)},\qquad(4.13)$$

where $f^{(abc)}$ is the structure constant of the gauge group. Then, together with (3.17) and using (3.13), the correction term from the vector multiplet is evaluated as

$$\Delta\mathcal{L}_{\text{vector}}^1 = \Delta\mathcal{L}_{\text{vector;V}}^1 + \Delta\mathcal{L}_{\text{vector;G}}^1,\qquad(4.14)$$

where

$$\Delta\mathcal{L}_{\text{vector;V}}^1 = \frac{3}{16\pi^2}N_G \int d^2\Theta\, 2\mathcal{E}\Sigma W^{(a)\alpha}W_\alpha^{(a)} + \text{h.c.}\qquad(4.15)$$

$$
\begin{aligned}
&\Delta\mathcal{L}_{\text{vector;G}}^1 \\
=\ &\frac{3}{128\pi^2}N_G \int d^2\Theta\, 2\mathcal{E}\Sigma\Big[-8W^{\alpha\beta\gamma}W_{\alpha\beta\gamma} \\
&\qquad\qquad + (\bar{\mathcal{D}}^2 - 8R)(G^{\alpha\dot\alpha}G_{\alpha\dot\alpha} + 8R^\dagger R + 2\mathcal{D}^2 R)\Big] + \text{h.c.}
\end{aligned}
$$
$$(4.16)$$

Here N_G is the Dynkin index of gauge multiplet, normalised to n for $SU(n)$.

4.3 Gravitational Multiplet

The correction term from the gravitational multiplet is directly calculated from (3.20) and (3.13):

$$
\begin{aligned}
\Delta\mathcal{L}_{\text{gravity}}^1 = \frac{1}{16\pi^2}\int d^2\Theta\, 2\mathcal{E}\Sigma\Big[&-33W^{\alpha\beta\gamma}W_{\alpha\beta\gamma} \\
&+ (\bar{\mathcal{D}}^2 - 8R)(-G^{\alpha\dot\alpha}G_{\alpha\dot\alpha} + 10R^\dagger R)\Big] + \text{h.c.}
\end{aligned}
$$
$$(4.17)$$

5 Summary

In this paper we have calculated the one-loop quantum correction term to the supergravity Lagrangian from super-Weyl field redefinition, including the transformation from the Brans-Dicke form to the Einstein-frame Lagrangian. This correction term is essentially inevitable since the Einstein-frame supersymmetry transformation includes Weyl rescaling and chiral rotation of the fields. The correction term is in total described as

$$
\begin{aligned}
\Delta\mathcal{L}^1 =\ &\Delta\mathcal{L}_{\text{chiral;K}}^1 + \Delta\mathcal{L}_{\text{chiral;V}}^1 + \Delta\mathcal{L}_{\text{chiral;G}}^1 \\
&+ \Delta\mathcal{L}_{\text{vector;V}}^1 + \Delta\mathcal{L}_{\text{vector;G}}^1 + \Delta\mathcal{L}_{\text{gravity}}^1,
\end{aligned}
$$
$$(5.1)$$

where each term is evaluated as (4.10), (4.11), (4.12), (4.15), (4.16), and (4.17) in terms of superfield.

In order to calculate the Jacobian correction term, we first expanded the anomaly-calculation method, which is already well known as Fujikawa method for a spinor, to the ones for a vector boson and a Rarita-Schwinger field. Thereafter, by coefficient comparison, we adjusted the numerical factors in front of superfield strengths and finally determined the entire form of the correction term.

The entire correction term is in a complicated form with many terms and miscellaneous coefficients. This is because we are considering Poincaré supergravity. Poincaré supergravity is a theory that is obtained by gauge fixing from conformal supergravity, and therefore Poincaré supergravity is a "mathematically halfway" theory in a sense. On the other hand one could consider an anomaly calculation similar to this paper in conformal supergravity. Since anomaly calculation is a "purely mathematical" operation, it might be expected that the anomaly term could be simpler in conformal supergravity, where all the field strengths, including the Kähler curvature, are put in a single supermultiplet. Further researches are expected for this topic.

Acknowledgements

A   Determination of Super-Weyl Parameter

In this section we derive the conditions (2.7), which finally determine the parameters (2.9), for the super-Weyl transformation from Brans-Dicke form to Einstein form. For the detail of Θ expansion of each superfield, refer to [10].

First, out of the full Lagrangian

$$\int d^2\Theta \, 2\mathcal{E} \left(\mathcal{D}^2 - 8R\right) \exp\left[-\frac{1}{3}\left(L + \Gamma\right)\right], \tag{A.1}$$

where $L = K - 6\Sigma - 6\Sigma^\dagger$, the kinetic terms of each fields are included in the term

$$\int d^2\Theta \, 2\mathcal{E} \left(\mathcal{D}^2 - 8R\right) L. \tag{A.2}$$

Since

$$\left.(2\mathcal{E}R)\right|_{\Theta=0} = \frac{1}{6}e\mathcal{R} + \cdots, \tag{A.3}$$

it is required that $L| = 0$ for the Einstein canonical kinetic term of the graviton.

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Next, since

$$(\mathcal{D}_\alpha R)| = -\frac{1}{6}\sigma_a\bar{\sigma}_b\psi_\alpha^{ab} + \cdots, \tag{A.4}$$

with $\psi_{mn} = \mathcal{D}_m\psi_n - \mathcal{D}_n\psi_m$, it should be imposed that $(\mathcal{D}_\alpha L)| = 0$ in order to remove differential coupling like $\chi\sigma^{ab}\psi_{ab}$.

Finally, since the term

$$\bar{\mathcal{D}}^2 K(\Phi, \Phi^\dagger) = -\Theta^2 K_{i^*j^*}\mathcal{D}^m A^{\dagger i}\mathcal{D}_m A^{\dagger j} + \cdots \tag{A.5}$$

spoils the Kähler form of the matter kinetic term, it is necessary to adjust with Σ so that $(\mathcal{D}^2 L)| = 0$.

B Calculation of Heat Kernel Expansion

In this section we derive a formula in order to calculate the spacetime trace (3.15), (3.17), and (3.20). The methods and results in this section are in reference to [26] as a whole.

First,

$$\int \frac{d^4k}{(2\pi)^4} e^{-ikx} e^{t^2(\Box+\hat{X})} e^{ikx}$$

$$= \int \frac{d^4k}{(2\pi)^4} \exp\left\{t^2\left[(\mathcal{D}+ik)^2 + \hat{X}\right]\right\} \tag{B.1}$$

$$= \frac{1}{t^4} \int \frac{d^4k}{(2\pi)^4} \exp\left[t^2\left(\Box+\hat{X}\right) + 2itk\cdot\mathcal{D} - k^2\right] \tag{B.2}$$

$$\sim \int \frac{d^4k}{(2\pi)^4} \frac{1}{24} e^{-k^2} \times$$

$$: \left[16(k\cdot\mathcal{D})^4 + 12(-4)\left(\Box+\hat{X}\right)(k\cdot\mathcal{D})^2 + 12\left(\Box+\hat{X}\right)^2\right] :, \tag{B.3}$$

where $\sim$ denotes extracting the constant term of the Laurent expansion in t, the colon $: X :$ denotes Weyl-ordered product of X, i.e.

$$: AB^2 : = \frac{1}{3}\left(AB^2 + BAB + B^2A\right), \tag{B.4}$$

and we have used

$$\frac{1}{t^4} e^{At^2+Bt+C} \sim \frac{1}{24} : e^C\left(B^4 + 12AB^2 + 12A^2\right) : . \tag{B.5}$$

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Continue to calculate (B.3) to obtain

$$
\int \frac{d^4k}{(2\pi)^4} e^{-ikx} e^{t^2(\Box + \hat{X})} e^{ikx}
$$

$$
=\frac{1}{96\pi^2}\Big\{\Box^2 + \mathcal{D}^\mu\Box\mathcal{D}_\mu + \mathcal{D}^\mu\mathcal{D}^\nu\mathcal{D}_\mu\mathcal{D}_\nu
$$
$$
-2\left[\left(\Box + \hat{X}\right)\Box + \mathcal{D}^\mu\left(\Box + \hat{X}\right)\mathcal{D}_\mu + \left(\Box + \hat{X}\right)\Box\right]
$$
$$
+3\left(\Box + \hat{X}\right)^2\Big\} \tag{B.6}
$$

$$
=\frac{1}{96\pi^2}\left[\mathcal{D}^\mu\mathcal{D}^\nu\mathcal{D}_\mu\mathcal{D}_\nu - \mathcal{D}^\mu\Box\mathcal{D}_\mu + 3\hat{X}^2 + \Box\hat{X} - 2\mathcal{D}^\mu\hat{X}\mathcal{D}_\mu + \hat{X}\Box\right] \tag{B.7}
$$

$$
=\frac{1}{96\pi^2}\left[\frac{1}{2}\Omega^{\mu\nu}\Omega_{\mu\nu} + 3\hat{X}^2 + \left(\Box\hat{X}\right)\right], \tag{B.8}
$$

[14]where $\Omega_{\mu\nu} \equiv [\mathcal{D}_\mu, \mathcal{D}_\nu]$ and we have used

$$
\int d^4k\, e^{-k^2} = \pi^2 \tag{B.9}
$$

$$
\int d^4k\, e^{-k^2} k_\mu k_\nu = \frac{\pi^2}{2} g_{\mu\nu} \tag{B.10}
$$

$$
\int d^4k\, e^{-k^2} k_\mu k_\nu k_\rho k_\sigma = \frac{\pi^2}{4}\left(g_{\mu\nu}g_{\rho\sigma} + g_{\mu\rho}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\rho}\right). \tag{B.11}
$$

For example, in the case for a spinor,

$$
\Box + \hat{X} = \tilde{\slashed{\mathcal{D}}}^2 = \gamma^\mu\gamma^\nu\tilde{\mathcal{D}}_\mu\tilde{\mathcal{D}}_\nu = \Box + \frac{1}{2}\gamma^{\mu\nu}\Omega_{\mu\nu}, \tag{B.12}
$$

and therefore $\hat{X} \equiv \frac{1}{2}\gamma^{\mu\nu}\Omega_{\mu\nu}$. Substitute this into (B.8) and finally obtain (3.15).

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