Statistical physics of the inflaton decaying in an inhomogeneous random environment

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Abstract

We derive a stochastic wave equation for an inflaton in an environment of an infinite number of fields . We study solutions of the linearized stochastic evolution equation in an expanding universe. The Fokker-Planck equation for the inflaton probability distribution is derived. The relative entropy (free energy) of the stochastic wave is defined. The second law of thermodynamics for the diffusive system is obtained. Gaussian probability distributions are studied in detail.

1 Introduction

The ΛCDM model became the standard cosmological model since the discovery of the universe acceleration [1][2]. It describes very well the large scale structure of the universe. The formation of an early universe is explained by the inflationary models involving scalar fields (inflatons)[3][4]. Such models raise the questions concerning the dynamics from their early stages till the present day. The origin of the cosmological constant as a manifestation of "dark energy" could also be explored. In models of the dark sector we hope to explain the "coincidence problem": why the densities of the dark matter and the dark energy are of the same order today as well as "the cosmological constant problem" [5]: why the cosmological constant is so small. We assume that the dark matter and dark energy consist of some unknown particles and fields. They interact in an unknown way with baryons and the inflaton. The result of the interaction could be seen in a dissipative and diffusive behaviour of the observed luminous matter. The diffusion approximation does not depend on the details of the interaction but only on its strength and "short memory" (Markovian approximation). In this paper we follow an approach appearing in many papers (see

[6][7][8][9] [10] and references quoted there) describing the dark matter and the fields responsible for inflation (inflatons) by scalar fields. In the ΛCDM model the universe originates from the quantum Big Bang. The quantum fluctuations expand forming the observed galactic structure. The transition to classicality requires a decoherence. The decoherence can be obtained through an interaction with an environment. The environment may consist of any unobservable degrees of freedom. In [11][12] these unobservable variables are the high energy modes of the fields present in the initial theory. We assume that the environment consists of an infinite set of scalar fields interacting with the inflaton . The model is built in close analogy to the the well-known infinite oscillator model [13][14][15][16] of Brownian motion. As the model involves an infinite set of unobservable degrees of freedom the statistical description is unavoidable. We have an environment of an infinite set of scalar χ fields which have an arbitrary initial distributions. We begin their evolution from an equilibrium thermal state. In such a case we obtain a random physical system driven by thermal fluctuations of the environment. We could also take into account quantum fluctuations extracting them as high momentum modes of the inflaton as is done in the Starobinsky stochastic inflation [17]. In such a framework a description of a quantum random evolution is reduced to a classical stochastic process. We can apply the thermodynamic formalism to the study of evolution of stochastic systems. In such a framework we can calculate the probability of a transition from one state to another. In particular, a vacuum decay can be treated as a stochastic process leading to a production of radiation [18].

The plan of this paper is the following. In sec.2 we derive the stochastic wave equation for an inflaton interacting with an infinite set of scalar fields in a homogeneous expanding metric. In sec.3 we briefly discuss a generalization to inhomogeneous perturbations of the metric satisfying Einstein-Klein-Gordon equations. In sec.4 we approximate the non-linear system by a linear inhomogeneous stochastic wave equation with a space-time dependent mass. The Fokker-Planck equation for the probablity distribution of the inflaton is derived. A solution of the Foker-Planck equation in the form of a Gibbs state with a time-dependent temperature is obtained. In sec.5 a general linear system is discussed. In sec.6 we obtain partial differential equations for the correlation functions of this system. In sec.7 Gaussian solutions of the Fokker-Planck equation and their relation to the equations for correlation functions are studied. In sec.8 we discuss thermodynamics of time-dependent (non-equilibrium) diffusive systems based on the notion of the relative entropy (free energy). In the Appendix we treat a simple system of stochastic oscillators in order to show that the formalism works well for this system.

2 Scalar fields interacting linearly with an environment

The CMB observations show that the universe was once in an equilibrium state. The Hamiltonian dynamics of scalar fields usually discussed in the model of inflation do not equilibrate. We can achieve an equilibration if the scalar field interacts with an environment. We suggest a field theoretic model which is an extension of the well-known oscillator model discussed in [13][14][15][16]. We consider the Lagrangian

$$\mathcal{L} = R\sqrt{g} + \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) + \sum_b(\tfrac{1}{2}\partial_\mu\chi^b\partial^\mu\chi^b - \tfrac{1}{2}m_b^2\chi^b\chi^b - \lambda_b\phi\chi^b - U_b(\chi^b)). \tag{1}$$

Equations of motion for the scalar fields read

$$g^{-\frac{1}{2}}\partial_\mu(g^{\frac{1}{2}}\partial^\mu)\phi = -V' - \sum_b \lambda_b\chi^b, \tag{2}$$

$$g^{-\frac{1}{2}}\partial_\mu(g^{\frac{1}{2}}\partial^\mu)\chi^b + m_b^2\chi^b = -\lambda_b\phi - \frac{\partial U_b}{\partial\chi^b}, \tag{3}$$

where $g_{\mu\nu}$ is the metric tensor and $g = |\det[g_{\mu\nu}]|$.

We consider the flat expanding metric

$$ds^2 = dt^2 - a^2 d\mathbf{x}^2, \tag{4}$$

In the metric (4) eq.(3) reads

$$\partial_t^2\chi^b + 3H\partial_t\chi^b - a^{-2}\triangle\chi^b + m_b^2\chi^b = -\lambda_b\phi - \frac{\partial U_b}{\partial\chi^b} \tag{5}$$

We may choose

$$U_b(\chi^b) = \kappa_b(\chi_b^2 - v_b^2)^2 \tag{6}$$

We write

$$\chi = v + a^{-\frac{3}{2}}\tilde{\chi} \tag{7}$$

Then

$$\partial_t^2\tilde{\chi}^b - a^{-2}\triangle\tilde{\chi}^b + \omega_b^2\tilde{\chi}^b = -\lambda_b a^{\frac{3}{2}}\phi + a^{-\frac{3}{2}}o(\tilde{\chi}^2) \tag{8}$$

where

$$\omega_b^2 = m_b^2 + 8\kappa_b v_b^2 - \frac{3}{2}\partial_t H - \frac{9}{4}H^2 \tag{9}$$

We consider large $a \to \infty$ so that for a large time we may neglect a^{-2} term. Moreover, we assume that $\omega_b^2 > 0$ and that ω_b is approximately constant (this is exactly so for the de Sitter space and approximate for power-law expansion when the H dependent term decays as t^{-2}). Then, the solution of eq.(8) is

$$\tilde{\chi}^b = \cos(\omega_b t)\tilde{\chi}_0^b + \sin(\omega_b t)\omega_b^{-1}\tilde{\pi}_0^b - \lambda_b\int_{t_0}^t \sin(\omega_b(t-s))\omega_b^{-1}a(s)^{\frac{3}{2}}\phi_s ds \tag{10}$$

Inserting the solution of eq.(3) in eq.(2) we obtain an equation of the form

$$g^{-\frac{1}{2}}\partial_\mu(g^{\frac{1}{2}}\partial^\mu)\phi + V'(\phi) = \int_{t_0}^{t}\mathcal{K}(t,t')\phi(t')dt' + a^{-\frac{3}{2}}\eta(\chi(0),\tilde{\pi}), \tag{11}$$

where

$$\mathcal{K}(t,s) = a(t)^{-\frac{3}{2}}\sum_b \lambda_b^2 \sin(\omega_b(t-s))\omega_b^{-1}a(s)^{\frac{3}{2}} \tag{12}$$

and the noise η depends linearly on the initial conditions $(\tilde{\chi}(0),\tilde{\pi}(0))$

$$\eta_t = -\sum_b \lambda_b \cos(\omega_b t)\tilde{\chi}_0^b + \lambda_b \sin(\omega_b t)\omega_b^{-1}\tilde{\pi}_0^b \tag{13}$$

If the correlation function of the noise is to be stationary (depend on time difference) then we need

$$\langle\tilde{\chi}_0^b\tilde{\chi}_0^b\rangle = \langle\omega_b^{-2}\tilde{\pi}_0^b\tilde{\pi}_0^b\rangle \tag{14}$$

Then, (assuming that $\langle\tilde{\chi}\tilde{\pi}\rangle = 0$, true in the classical Gibbs state) we get

$$\langle\eta_t(\mathbf{x})\eta_s(\mathbf{y})\rangle = a^{-3}\sum_b \lambda_b^2\langle\Big(\omega_b^{-2}\cos(\omega_b(t-s))\tilde{\pi}_0^b(\mathbf{x}),\tilde{\pi}_0^b(\mathbf{y})\Big)\rangle \tag{15}$$

We assume a certain probability distribution for initial values. The relation (14) is satisfied for classical as well as quantum Gibbs distribution with the Hamiltonian $\tilde{H}_b = \frac{1}{2}(\tilde{\pi}^2 + \omega_b^2\tilde{\chi}_b^2)$. In the classical field theory in the Gibbs state the covariance of the fields in eq.(14) is $(-a^{-2}\triangle + \omega_b^2)^{-1}$. If $a^{-2}\triangle = 0$ then this covariance is approximated by [19] $\beta^{-1}\omega_b^{-2}\delta(\mathbf{x}-\mathbf{y})$. We choose

$$\lambda_b \simeq \sqrt{\beta}\gamma\pi^{-\frac{1}{2}}\omega_b \tag{16}$$

Under the assumption (16) and a continuous spectrum of ω_b in eq.(11) we shall have

$$\begin{aligned}&\int_{t_0}^{t} ds\mathcal{K}(t,s)\phi(s) = -\gamma^2\partial_t\phi(t) - \tfrac{3}{2}\gamma^2 H(t)\phi(t)\\ &+\gamma^2\delta(0)\phi(t) - \gamma^2\delta(t-t_0)\phi(t_0)a(t)^{-\frac{3}{2}}a(t_0)^{\frac{3}{2}}\end{aligned}$$

Here, $\delta(t)$ comes from $\beta^{-1}\sum_b\lambda_b^2\omega_b^{-2}\cos(\omega_b t)$; the $\delta(0)$ term is (an infinite) mass renormalization which appears already in the Caldeira-Leggett model [15], it could be included in m^2.The last term can be neglected when t_0 tends to $-\infty$. We shall omit these terms in further discussion.

In an expanding metric eq.(11) takes the form

$$\partial_t^2\phi - a^{-2}\triangle\phi + (3H+\gamma^2)\partial_t\phi + \frac{3}{2}\gamma^2 H\phi + V'(\phi) = \gamma a^{-\frac{3}{2}}\eta. \tag{17}$$

where

$$\langle\eta(t,\mathbf{x})\eta(t',\mathbf{x}')\rangle = \delta(t-t')K_t(\mathbf{x},\mathbf{x}'). \tag{18}$$

Here, $K(\mathbf{x},\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}')$ comes from the expectation value of the initial values $\tilde{\chi}$ and $\tilde{\pi}$ with the neglect of $a^{-2}\triangle$. If we do not neglect $a^{-2}\triangle$ in eq.(8) then the form of eq.(18) would be much more complicated (we would obtain a non-local equation). We make the approximation (18) which preserves the Markov property and provides stochastic fields which are regular functions of $\mathbf{x}$ (it can be considered as a cutoff ignoring high momenta components of ϕ). Eq.(17) has been derived earlier in [19]. It is applied in the model of warm inflation [20].

3 Stochastic equations in a perturbed inhomogeneous metric

We did not write yet equations for the metric which result from Lagrangean (1). The power spectrum of inflaton perturbations depends on the metric [3][4][21][22][23] . In general, the equations for the metric are difficult to solve. They are solved in perturbation theory. We consider only scalar perturbations of the homogenous metric using the scalar fields A, B, E, ψ. Then, the metric is expressed in the form [3][23]

$$ds^2 == -(1+2A)dt^2 + 2a\partial_j B dx^j dt + a^2\Big((1-2\psi)\delta_{ij} + 2\partial_i\partial_j E\Big)dx^i dx^j \quad (19)$$

Inserting the metric in eqs.(2)-(3) we obtain

$$\partial_t^2\phi + (3H+\Gamma)\partial_t\phi - a^{-2}\triangle\phi + V'(\phi)(1+2A) = -\sum_b \lambda_b\chi_b(1+2A) \quad (20)$$

and

$$\partial_t^2\chi_b + (3H+\Gamma)\partial_t\chi_b - a^{-2}\triangle\chi_b + m_b^2\chi_b(1+2A) = -\lambda_b\phi(1+2A) - \frac{\partial U_b}{\partial\chi_b}(1+2A) \quad (21)$$

where

$$-\Gamma = \partial_t A + 3\partial_t\psi - a^{-2}\triangle(a^2\partial_t E - aB) \quad (22)$$

Let

$$\chi_b = \exp(\sigma)\tilde{\chi}_b + v_b \quad (23)$$

with

$$\partial_t\sigma = -\frac{1}{2}(3H+\Gamma) \quad (24)$$

Then, in the approximation neglecting higher powers of $\tilde{\chi}$ in U

$$\partial_t^2\tilde{\chi}_b - a^{-2}\triangle\tilde{\chi}_b + \Omega_b^2\chi_b = -\lambda_b\exp(-\sigma)(1+2A)\phi \quad (25)$$

where

$$\Omega_b^2 = (m_b^2 + 8v_b^2\kappa_b)(1+2A) - \frac{1}{4}(3H+\Gamma)^2 - \frac{1}{2}\partial_t(3H+\Gamma) \quad (26)$$

We again assume that $\Omega^2 \simeq m_b^2 + 8v_b^2\kappa_b$. Then, in the derivation of eq.(17) for ϕ the only change comes from the differentiation of $\exp(\sigma)$ inside the integral (11) and the factor $(1+2A)$ multiplying the fields. so,

$$\frac{3}{2}\gamma^2 H\phi \rightarrow \frac{1}{2}\gamma^2(3H+\Gamma)\phi(1+2A)^2 \tag{27}$$

and

$$\gamma^2\partial_t\phi \rightarrow \gamma^2(1+2A)\partial_t((1+2A)\phi) \tag{28}$$

Hence, our final equation is (for a general theory of a stochastic wave equation on a Riemannian manifold see [26] and references cited there)

$$\begin{array}{l}\partial_t^2\phi + (3H+\gamma^2+\Gamma)\partial_t\phi - a^{-2}\triangle\phi + \frac{1}{2}\gamma^2(3H+\Gamma)\phi \\ +6\gamma^2 AH\phi + 2\gamma^2\partial_t A\phi + 4\gamma^2 A\partial_t\phi + V'(\phi)(1+2A) = \gamma(1+2A)a^{-\frac{3}{2}}\eta\end{array} \tag{29}$$

4 A simplified system of a decaying inflaton

The metric (A, B, ψ, E) can be expressed by ϕ from Einstein equations resulting from the Lagragean (1). We expand inflaton equation with a potential V around the classical solution of Klein-Gordon-Einstein equation. The linearized version of the equation for fluctuations takes the form of the Klein-Gordon equation with a space-time dependent mass [3][4][24][23][27]

$$\partial_t\phi = \Pi$$

$$d\Pi + (3H+\gamma^2)\Pi dt + \frac{3}{2}\gamma^2 H\phi dt + \nu\phi dt + a^{-2}\triangle\phi dt = \gamma a^{-\frac{3}{2}}dB \tag{30}$$

where we write $\eta = \frac{dB}{dt}$ and treat (30) as Ito stochastic differential equation [28]. The function ν depends on the potential V in eq.(17) and on the choice of coordinates (t, x) (the choice of gauge [29]). We do not discuss ν in this paper. We consider in this section the simplified version of eq.(30) without the ϕ terms

$$d\Pi + (3H+\gamma^2)\Pi dt = \gamma a^{-\frac{3}{2}}dB \tag{31}$$

We define the energy density

$$\rho = \frac{1}{2}\Pi^2 \tag{32}$$

Then, from eq.(31) applying the stochastic calculus [28][31]and eq.(31) we obtain

$$d\Pi^2 = 2\Pi d\Pi + d\Pi d\Pi = -2(3H+\gamma^2)\Pi^2 dt + 2\gamma a^{-\frac{-3}{2}}\Pi dB + \gamma^2 a^{-3}K(x,x)dt$$

We may first integrate this equation and use $\langle\int_0^t f dB\rangle = 0$ for the Ito integral. Differentiating the expectation value over t we obtain

$$d\langle\rho\rangle + 6H\langle\rho\rangle dt = -2\gamma^2\langle\rho\rangle dt + \frac{1}{2}\gamma^2 K(x,x)a^{-3}dt \tag{33}$$

Eq.(33) describes the inflaton density with $w = \frac{\rho}{p} = 1$ and a cosmological term varying with the speed a^{-3}. The term $-2\gamma^2\langle\rho\rangle$ violates the energy conservation of the inflaton . It describes a decay of the inflaton into the χ fields. If we couple the χ fields to radiation then if χ fields are invisible the observable effect will be detected as a production of radiation from the decay of the inflaton [18][30].

For the stochastic system (31) the Fokker-Planck equation reads

$$\partial_t P = \frac{\gamma^2}{2}\int d\mathbf{x}d\mathbf{x}'\mathcal{G}_t(\mathbf{x},\mathbf{x}')\frac{\delta^2}{\delta\Pi(\mathbf{x})\delta\Pi(\mathbf{x}')}P + \int d\mathbf{x}\frac{\delta}{\delta\Pi(\mathbf{x})}(3H+\gamma^2)\Pi P. \tag{34}$$

Then, the Gaussian solution is

$$P = L\exp(-\frac{1}{2}\int \sqrt{g}\Pi\beta\Pi) \tag{35}$$

where $\sqrt{g} = a^3$. It can be checked that

$$\beta = \exp(2\gamma^2 t)a^3\Big(R + \gamma^2\int_0^t dsa(s)^3\exp(2\gamma^2 s)\Big)^{-1} \tag{36}$$

and

$$L^{-1}\partial_t L = -\frac{1}{2}\gamma^2 a^{-3}Tr(K\beta) + (3H+\gamma^2)\delta(0)\int d\mathbf{x}. \tag{37}$$

This normalization factor is infinite (needs renormalization) but the value of L does not appear in the expectation values $\langle F\rangle = (\int P)^{-1}\int PF$.

β has the meaning of the inverse temperature. The dependence (36) of the temperature of the diffusing system on the scale factor a has been derived (for $\gamma = 0$ and arbitrary w) in [32][33][34]for any system with $w = 1$ and the a^{-3} correction (33) to the cosmological term (in eq.(36) $w = 1$)(for time dependent cosmological term see [35][36][37]) .

5 The linearized wave equation

We can rewrite eqs.(4)-(5) in a way that they do not contain first order time derivatives of fields. Let

$$\phi = a^{-\frac{3}{2}}\exp(-\frac{1}{2}\gamma^2 t)\Phi$$

Then, the linearized version of the inflaton equation expanded around the classical solution (with an account of metric perturbations) reads

$$\partial_t\Phi = \Pi$$

$$\begin{aligned}&\partial_t\Pi+K^2\Phi-\tfrac{3}{2}\partial_t H\Phi-\tfrac{9}{4}H^2\Phi-\tfrac{1}{4}\gamma^2\Phi+\nu\Phi\\&=\partial_t\Pi+K^2\Phi+\tilde{\nu}\Phi=\gamma a^{\frac{3}{2}}\exp(\tfrac{1}{2}\gamma^2 t)\eta\end{aligned}\tag{38}$$

where

$$K^2=-a^{-2}\triangle+m^2\tag{39}$$

For the stochastic system (38) the Fokker-Planck equation reads

$$\begin{aligned}&\partial_t P=\tfrac{\gamma^2}{2}\int d\mathbf{x}d\mathbf{x}'\mathcal{G}_t(\mathbf{x},\mathbf{x}')\tfrac{\delta^2}{\delta\Pi(\mathbf{x})\delta\Pi(\mathbf{x}')}P\\&+\int d\mathbf{x}(K^2\Phi+\tilde{\nu}\Phi)\tfrac{\delta}{\delta\Pi(\mathbf{x})}P-\int d\mathbf{x}\Pi(\mathbf{x})\tfrac{\delta}{\delta\Phi(\mathbf{x})}P\equiv\mathcal{A}P.\end{aligned}\tag{40}$$

where

$$\mathcal{G}_t(\mathbf{x},\mathbf{x}')=a^3\exp(\gamma^2 t)K_t(\mathbf{x},\mathbf{x}')\tag{41}$$

and ν depends on the classical solution in the potential V [24][23].

We may write the noise in the Fourier momentum space

$$\langle\eta(t,\mathbf{k})\eta(t',\mathbf{k}')\rangle=\mathcal{G}_t(\mathbf{k})\delta(\mathbf{k}+\mathbf{k}')\delta(t-t')\tag{42}$$

Then, eq.(38) is rewritten as an ordinary (instead of partial) differential equation.

6 A differential equation for correlations

Let us consider eq.(38) expressed in the form

$$\partial_t\Phi=\Pi\tag{43}$$

$$\partial_t\Pi=A\Phi+\gamma a^{\frac{3}{2}}\exp(\frac{1}{2}\gamma^2 t)\eta\tag{44}$$

where

$$-A=K^2+\tilde{\nu}=-a^{-2}\triangle+m^2+\tilde{\nu}\tag{45}$$

Let us denote

$$\langle\Phi_t(\mathbf{x})\Phi_t(\mathbf{y})\rangle=C_t(\mathbf{x},\mathbf{y})\tag{46}$$

$$\langle\Phi_t(\mathbf{x})\Pi_t(\mathbf{y})\rangle=E_t(\mathbf{x},\mathbf{y})\tag{47}$$

$$\langle\Pi_t(\mathbf{x})\Pi_t(\mathbf{y})\rangle=D_t(\mathbf{x},\mathbf{y})\tag{48}$$

Using the stochastic calculus [28][31] and taking the expectation value we get a system of differential equations for the correlation functions

$$\partial_t C_t(\mathbf{x},\mathbf{y})=E_t(\mathbf{x},\mathbf{y})+E_t(\mathbf{y},\mathbf{x})\tag{49}$$

$$\partial_t D_t(\mathbf{x},\mathbf{y})=A_x E_t(\mathbf{x},\mathbf{y})+A_y E_t(\mathbf{y},\mathbf{x})+\gamma^2\mathcal{G}_t(\mathbf{x},\mathbf{y})\tag{50}$$

$$\partial_t E_t(\mathbf{y},\mathbf{x})=A_x C_t(\mathbf{x},\mathbf{y})+D_t(\mathbf{x},\mathbf{y})\tag{51}$$

If the system is translation invariant then we can Fourier transform these equations obtaining a system of ordinary differential equations for Fourier transforms

$$\partial_t C_t(k) = E_t(k) + E_t(-k) \tag{52}$$

$$\partial_t D_t(k) = -(a^{-2}k^2 + m^2 + \tilde{\nu})(E_t(k) + E_t(-k)) + \gamma^2 \mathcal{G}_t(k) \tag{53}$$

$$\partial_t E_t(k) = -(a^{-2}k^2 + m^2 + \tilde{\nu})C_t(k) + D_t(k) \tag{54}$$

where $\mathcal{G}_t(k)$ is defined in eq.(42).

7 Gaussian solutions of the Fokker-Planck equation

We look for a solution of the Fokker-Planck equation (40) in the form

$$P_t^I = L(t)\exp\Big(-\gamma^{-2}\int d\mathbf{x}d\mathbf{x}'\Big(\frac{1}{2}\Pi\beta_1(t,\mathbf{x},\mathbf{x}')\Pi + \Pi\beta_2(t,\mathbf{x},\mathbf{x}')\Phi + \frac{1}{2}\Phi\beta_3(t,\mathbf{x},\mathbf{x}')\Phi\Big) + \int d\mathbf{x}M\Phi + \int d\mathbf{x}N\Pi\Big). \tag{55}$$

or in the momentum space

$$P_t^I = L(t)\exp\Big(-\gamma^{-2}\int d\mathbf{k}\Big(\frac{1}{2}\Pi\beta_1(t,\mathbf{k})\Pi + \Pi\beta_2(t,\mathbf{k})\Phi + \frac{1}{2}\Phi\beta_3(t,\mathbf{k})\Phi\Big) + \int d\mathbf{k}M(\mathbf{k})\Phi(-\mathbf{k}) + \int d\mathbf{k}N(\mathbf{k})\Pi(-\mathbf{k})\Big). \tag{56}$$

In the configuration space β is an operator and in the momentum space a function of $\mathbf{k}$. $L(t)$ is determined by normalization or directly from the Fokker-Planck equation

$$L^{-1}\partial_t L = -\frac{1}{2}\int d\mathbf{x}d\mathbf{x}'\mathcal{G}_t(\mathbf{x},\mathbf{x}')\beta_1(\mathbf{x},\mathbf{x}') \tag{57}$$

The equations for β read

$$\partial_t\beta_1 = -\beta_1\mathcal{G}_t\beta_1 - 2\beta_2 \tag{58}$$

$$\partial_t\beta_2 = -\beta_2\mathcal{G}_t\beta_1 + \omega^2\beta_1 - \beta_3 \tag{59}$$

$$\partial_t\beta_3 = -\beta_2\mathcal{G}_t\beta_2 + 2\omega^2\beta_2 \tag{60}$$

where

$$\omega^2 = a^{-2}k^2 + m^2 + \tilde{\nu} \tag{61}$$

We skip the equations for M and N.

It is useful to introduce instead of β_j the variables (defined by Fourier transforms of β)

$$X = \beta_3 - \frac{\beta_2^2}{\beta_1} \tag{62}$$

$$Y = \frac{\beta_2}{\beta_1} \tag{63}$$

$$Z = \frac{\beta_3}{\beta_1} \tag{64}$$

We can invert these relations

$$\beta_1 = \frac{X}{Z - Y^2} \tag{65}$$

$$\beta_2 = \frac{XY}{Z - Y^2} \tag{66}$$

$$\beta_3 = \frac{ZX}{Z - Y^2} \tag{67}$$

P_t^I can be expressed as

$$P_t^I = L(t) \exp\Big(- \int d\mathbf{x} d\mathbf{x}' \frac{1}{2} \gamma^{-2} \Big((\Pi + Y\Phi)\beta_1(\Pi + Y\Phi) + \Phi X \Phi \Big) \Big). \tag{68}$$

From eq.(68) it can be seen that the probability distribution is diagonal in the variables Φ and $\Pi + Y\Phi$. So, we obtain the expectation value $\langle (\Pi + Y\Phi)(\mathbf{x})(\Pi + Y\Phi)(\mathbf{x}') \rangle = \gamma^2 \beta_1^{-1}(\mathbf{x}, \mathbf{x}')$.

Assume that we calculate the expectation values at time t

$$D = \langle \Pi_t^2 \rangle = \gamma^2 \beta_3 (\beta_1 \beta_3 - \beta_2^2)^{-1} = \gamma^2 Z X^{-1} \tag{69}$$

$$C = \langle \Phi_t^2 \rangle = \gamma^2 \beta_1 (\beta_1 \beta_3 - \beta_2^2)^{-1} = \gamma^2 X^{-1} \tag{70}$$

$$E = \langle \Phi_t \Pi_t \rangle = -\gamma^2 \beta_2 (\beta_1 \beta_3 - \beta_2^2)^{-1} = -\gamma^2 Y X^{-1} \tag{71}$$

X, Y, Z can be expressed by D, C, E as

$$Z = DC^{-1} \tag{72}$$

$$Y = -EC^{-1} \tag{73}$$

$$X = \gamma^2 C^{-1} \tag{74}$$

If we know D, E, C then we can express

$$\beta_1 = C(DC - E^2)^{-1} \tag{75}$$

$$\beta_2 = -E(DC - E^2)^{-1} \tag{76}$$

$$\beta_3 = D(DC - E^2)^{-1} \tag{77}$$

Note that $\beta_2(t = 0) = 0$ means $E(t = 0) = 0$.

The relations (63)-(77) allow to relate the solutions of the stochastic equation (38) with the solutions of the differential equations (58)-(60) and the solution of the Fokker-Planck equation (40). In fact, the solution P^I can be expressed by the Fokker-Planck transition function P_t (which is defined by the solution of the stochastic equation (38)[28]) as follows

$$P_t^I(\phi,\Pi)=\int d\phi' d\Pi' P_0^I(\phi',\Pi')P_t(\phi',\Pi';\phi,\Pi) \tag{78}$$

The expectation values of the solution of the stochastic equation with the initial condition (ϕ',Π') is

$$\Big\langle F\Big(\phi_t(\phi',\Pi'),\Pi_t(\phi',\Pi')\Big)\Big\rangle=\int d\phi d\Pi P_t(\phi',\Pi';\phi,\Pi)F(\phi,\Pi)$$

Hence,

$$\begin{aligned}&\int d\phi' d\Pi' P_0^I(\phi',\Pi')\langle F(\phi_t(\phi',\Pi'),\Pi_t(\phi',\Pi'))\rangle\\ &=\int d\phi' d\Pi' P_0^I(\phi',\Pi')\int d\phi d\Pi P_t(\phi',\Pi';\phi,\Pi)F(\phi,\Pi)=\int d\phi d\Pi P_t^I(\phi,\Pi)F(\phi,\Pi)\end{aligned} \tag{79}$$

Note that the initial value P_0^I in eq.(79) according to eq.(55) is determined by the initial values of β_j. A possible choice for the initial value is the thermal Gibbs distribution

$$P_0^I=\exp\Big(-\frac{1}{2T}\int d\mathbf{x}(\Pi^2+(\nabla\phi)^2+m^2\phi^2)\Big)$$

which corresponds to the initial condition $\beta_2(t=0)=0$, $\beta_1(t=0,\mathbf{x},\mathbf{x}')=\frac{1}{T}\delta(\mathbf{x}-\mathbf{x}')$ and $\beta_3(t=0,\mathbf{x},\mathbf{x}')=\frac{1}{T}\triangle\delta(\mathbf{x}-\mathbf{x}')$. We could also consider the initial probability distribution

$$P_0^I=\exp\Big(-\frac{1}{2\sigma^2}\int d\mathbf{x}(\phi(\mathbf{x})-v)^2\Big)$$

describing the field concentrated at v. Then, in eq.(55) $M(t=0,\mathbf{x}))=\sigma^{-2}v$. In such a case, the probability distribution (55) describes the probability of the transition from v to ϕ (see [38] for such calculations in quantum mechanics).

8 The relative entropy

Assume we have a functional equation of the form (like eq.(40))

$$\begin{aligned}\partial_t P&=\frac{1}{2}\int d\mathbf{x}d\mathbf{x}'\mathcal{D}_t(\mathbf{x},\mathbf{x}')\frac{\delta^2}{\delta\Pi(\mathbf{x})\delta\Pi(\mathbf{x}')}P\\ &+\int d\mathbf{x}\frac{\delta}{\delta\Pi(\mathbf{x})}D_1(\phi(\mathbf{x}),\Pi(\mathbf{x}))P+\int d\mathbf{x}\frac{\delta}{\delta\Phi(\mathbf{x})}D_2(\Phi(\mathbf{x}),\Pi(\mathbf{x}))P\equiv\mathcal{A}P\end{aligned} \tag{80}$$

Let us assume that we have two solutions P_1 and P_2 of this equation. Define the relative entropy

$$F=\int d\Phi d\Pi Z_1^{-1}P_1\ln\Big(Z_2Z_1^{-1}P_1P_2^{-1}\Big) \tag{81}$$

where

$$Z_1 = \int d\Phi d\Pi P_1 \tag{82}$$

$$Z_2 = \int d\Phi d\Pi P_2 \tag{83}$$

From the definition of F it follows that [39]

$$F \geq 0 \tag{84}$$

Calculation of the time derivative of F gives

$$\partial_t F = -\frac{1}{2}\int d\phi d\Pi P_1 \Big(P_2 P_1^{-1}\Big)^2 d\mathbf{x}d\mathbf{x}'\mathcal{D}_t(\mathbf{x},\mathbf{x}')\frac{\delta}{\delta\Pi(\mathbf{x}))}(P_1 P_2^{-1})\frac{\delta}{\delta\Pi(\mathbf{x}')}(P_1 P_2^{-1}) \leq 0 \tag{85}$$

We choose $P_2 = P^I$ (then $Z_2 = 1$ because $L(t)$ is the normalization factor). Then, we define the entropy (the entropy of inflaton and gravitational perturbations has been discussed earlier in [40]-[41])

$$S = -Z^{-1}\int d\Phi d\Pi P \ln(Z^{-1}P) \tag{86}$$

Using eq.(80) we calculate the time derivative

$$\partial_t S = -Z^{-1}\int d\Phi d\Pi \mathcal{A}P \ln P - Z^{-1}\int d\Phi d\Pi \mathcal{A}P \tag{87}$$

The second term is zero, whereas the first term is equal to

$$\begin{aligned}&\partial_t S = \frac{1}{2}\int d\Phi d\Pi P^{-1} d\mathbf{x}d\mathbf{x}'\mathcal{D}_t(\mathbf{x},\mathbf{x}')\frac{\delta}{\delta\Pi(\mathbf{x})}P\frac{\delta}{\delta\Pi(\mathbf{x}')}P\\&+\int d\phi d\Pi\int d\mathbf{x}\Big(D_1\frac{\delta P}{\delta\Pi(\mathbf{x})} + D_2\frac{\delta P}{\delta\Phi(\mathbf{x})}\Big)\end{aligned} \tag{88}$$

In eq.(88) the first term is positive whereas the second term depends on the dynamics (it is vanishing for Hamiltonian dynamics). Using the formula (55) for $P_2 = P^I$ we obtain

$$F + S = Z^{-1}\int d\Phi d\Pi P\Big(\frac{1}{2}\gamma^{-2}\Pi\beta_1\Pi + \gamma^{-2}\Phi\beta_2\Pi + \frac{1}{2}\gamma^{-2}\Phi\beta_3\Phi\Big) - \ln L(t) \tag{89}$$

The formula (89) has a thermodynamic meaning relating the sum of free energy F and entropy S to the internal energy expressed by the rhs of eq.(89). At the initial time (with the initial conditions discussed at the end of sec.7) the rhs of eq.(89) is the mean value of the energy

$$U_0 = \frac{1}{2}\int d\mathbf{x}(\Pi^2 + (\nabla\Phi)^2 + m^2\Phi^2)$$

In the static universe we would have an equilibrium distribution as P_2. In such a case the thermodynamic relation (88) would describe the standard version of the second law of thermodynamics of diffusing systems. F with $\partial_t F \leq 0$ in eq.(85) would show the approach to equilibrium. In the expanding universe the relation (89) can serve for a comparison of various probability measures starting from different initial conditions.

9 Summary

The main source of observational data [1]-[2] comes from measurements of the cosmic microwave background (CMB) and observations of galaxies evolution (including galaxies distribution). The CMB spectrum and its fluctuations are the test ground for models involving quantum and thermal fluctuations. A simplified description of an interaction of a relativistic system with an environment leads to a stochastic wave equation for an inflaton generating the expansion (inflation) of the universe. We considered a linearization of the wave equation. We discussed the Fokker-Planck equation for the probability distribution of the inflaton. Gaussian solutions of the Fokker-Planck equation for linearized systems can be treated as Gibbs states with a time-dependent temperature. The model leads to a formula for density and temperature evolution. We have derived the density evolution law in eq.(33) and the temperature evolution in eq.(36) in a simplified model. In order to obtain the results in the complete model we would have to solve (numerically) equations of sec.7. The comparison of density evolution (36) with observations is discussed in [33] and in similar models with the decaying vacuum (see [35][37] and references cited there). The model allows to calculate (and compare with observations) the power spectrum resulting from thermal fluctuations which may go beyond the approximations applied in the warm inflation of ref.[20]. We have introduced a thermodynamic description of the expanding diffusive systems in terms of the relative entropy (free energy) and entropy. The state of a stochastic system can be identified with its probability distribution. The relative entropy allows to compare the evolution of the probability distributions with different initial conditions. In this sense relative entropy can be treated as a quantitative measure of a decay of one state into another state (as an alternative to a quantum description of vacuum decay in cosmology [42][43]).

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10 Appendix:Statistical physics of a static finite dimensional model

A finite dimensional analog of the wave equation is ($\mathbf{x} \in R^n$)

$$\frac{dx^k}{dt} = p^k$$

$$\frac{dp^k}{dt} = -\Gamma p^k - \omega^2 x^k + \gamma\eta^k \tag{90}$$

The Fokker-Planck equation reads

$$\partial_t P = -p^k \frac{\partial}{\partial x^k} + \frac{\partial}{\partial p^k}(\Gamma p^k + \omega^2 x^k)P + \frac{\gamma^2}{2}\frac{\partial^2 P}{\partial p^k \partial p^k} \tag{91}$$

The stationary solution is

$$P_\infty = \exp\Big(-\frac{\Gamma}{\gamma^2}(\mathbf{p}^2 + \omega^2\mathbf{x}^2)\Big) = \exp(-\frac{\mathcal{E}}{T}) \tag{92}$$

where $\mathcal{E}$ is the energy of the oscillator. It describes a Gibbs state with the temperature

$$T = \frac{\gamma^2}{2\Gamma} \tag{93}$$

We look for a solution of eq.(91) in the form

$$P_t = L(t)\exp\Big(-\frac{1}{2}\alpha_1\mathbf{p}^2 - \alpha_2\mathbf{x}\mathbf{p} - \frac{1}{2}\alpha_3\mathbf{x}^2\Big) \tag{94}$$

Then

$$L^{-1}\partial_t L = -\frac{n}{2}\gamma^2\alpha_1 + \Gamma n \tag{95}$$

$$\partial_t\alpha_1 = -2\alpha_2 - \gamma^2\alpha_1^2 + 2\Gamma\alpha_1 \tag{96}$$

$$\partial_t\alpha_2 = -\alpha_3 - \gamma^2\alpha_1\alpha_2 + \Gamma\alpha_2 + \omega^2\alpha_1 \tag{97}$$

$$\partial_t\alpha_3 = -\gamma^2\alpha_2^2 + 2\omega^2\alpha_2 \tag{98}$$

Let us write

$$x^k = \exp(-\frac{\Gamma}{2}t)y^k \tag{99}$$

Then, the stochastic equation for y reads

$$\frac{d^2y^k}{dt^2} = -\Omega^2 y^k + \gamma\exp(\frac{\Gamma}{2}t)\eta^k \tag{100}$$

where

$$\Omega^2 = \omega^2 - \frac{\Gamma^2}{4} \tag{101}$$

The solution of eq.(100) is

$$y^k(t) = \cos(\Omega t)y^k(0) + \sin(\Omega t)\Omega^{-1}\partial_t y^k(0) + \gamma\int_0^t \sin(\Omega(t-s))\Omega^{-1}\exp(\frac{\Gamma}{2}s)w(s)ds \tag{102}$$

We can easily calculate the correlation functions of $x^k(s)$ and $p^k(s)$ in two ways: either from the stochastic equations or using the probability distribution P_t resulting from the solution of the differential equations (96)-(98).

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