Electromagnetic Shocks in the Quantum Vacuum

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The interaction of two counter-propagating electromagnetic waves in a vacuum is analyzed within the framework of the Heisenberg-Euler formalism in quantum electrodynamics. The nonlinear electromagnetic wave in the quantum vacuum is characterized by wave steepening, subsequent generation of high order harmonics and electromagnetic shock wave formation with electron-positron pair generation at the shock wave front.

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I. INTRODUCTION

In contrast to classical electrodynamics where electromagnetic waves do not interact in a vacuum, in quantum electrodynamics (QED), the photon–photon scattering in a vacuum occurs via the generation of virtual electron–positron pairs resulting in vacuum polarization, Lamb shift, vacuum birefringence, Coulomb field modification, etc. [1]. Off–shell photon-photon scattering was indirectly observed in collisions of heavy ions accelerated by standard charged particles accelerators (see review article [2] and in results of experiments obtained with the ATLAS detector at the Large Hadron Collider [3]). Further study of the process will allow extensions of the Standard Model to be tested, in which new particles can participate in loop diagrams [4, 5].

The increasing availability of high power lasers raises interest in experimental observation and motivates theoretical studies of such processes in laser-laser scattering [6–13], scattering of the XFEL emitted photons [5], and the interaction of relatively long wavelength high intensity laser light with short wavelength X-ray photons [14–17].

In the relatively low photon energy limit, for photon energy below the electron rest-mass energy, $\mathcal{E}_{\gamma} = \hbar \omega < m_e c^2$, the total photon–photon scattering cross section for non-polarized photons is proportional to the sixth power of the photon energy,

$$\sigma_{\gamma-\gamma} = \left(\frac{973}{10125\pi}\right) \alpha^2 r_e^2 \left(\frac{\hbar\omega}{m_e c^2}\right)^6,\tag{1}$$

reaching its maximum at $\hbar\omega \approx 1.5 m_e c^2$ and decreases proportionally to the inverse of the second power of the photon energy for $\hbar\omega > m_e c^2$ (see Ref. [1]), where $\alpha = e^2/\hbar c \approx 1/137$ is the fine structure constant, $r_e = e^2/m_e c^2 = 2.82 \times 10^{-13} \, \mathrm{cm}$ is the classical electron radius, e and m_e are electron electric charge and mass, c is speed of light in a vacuum and \hbar is the reduced Planck constant.

From Eq. (1), it seems that by using the maximal frequency of the electromagnetic wave, we can reach a

higher number of scattering events. This would be so if we assume the same number of photons in the colliding beams having the same transverse size for the beams with different frequencies. However, if we think of highest field amplitude and highest luminosity of the colliding photon beams, we must consider the smallest transverse size of the beams, i.e., they should be focused on a spot of one-lambda size, which is different for beams of different frequencies. In general, this approach corresponds to the Gerard Mourou's lambda-cube concept [18].

To find the number of photon–photon scattering events in the low frequency limit, we estimate the number of photons in the electromagnetic pulse with the amplitude E in the λ^3 volume, where $\lambda=2\pi c/\omega$ is the electromagnetic wave wavelength. The number of photons in a laser pulse is then equal to

$$N_{\gamma} = \frac{E^2 \lambda^3}{4\pi\hbar\omega}.$$
 (2)

Using these relationships it is easy to find the number of scattering events per 4-volume $2\pi\lambda^3/\omega$. It is proportional to the scattering cross section given by Eq. (1), to the product of the photon numbers in colliding photon bunches, and it is inverse proportional to the square of the wavelength. Assuming the equal photon numbers in colliding bunches and the equal photon frequencies we obtain

$$N_{\gamma-\gamma} = \sigma_{\gamma-\gamma} \frac{N_{\gamma}^2}{\lambda^2} = \frac{973}{10125\pi} \alpha^2 \left(\frac{E}{E_S}\right)^4, \quad (3)$$

where $E_S=m_e^2c^3/e\hbar$ is the critical field of quantum electrodynamics. It is also known as the Sauter-Schwinger electric field. The corresponding to this field electromagnetic radiation intensity is $I_S=cE_S^2/4\pi\approx 10^{29} {\rm W/cm^2}$. Finally and importantly, we observe that the number of scatterings does not depend on the electromagnetic wave frequency. It is determined by the radiation intensity $I=cE^2/4\pi$ as

$$N_{\gamma-\gamma} \propto \alpha^2 (I/I_S)^2,$$
 (4)

i.e., the frequency independent dimensionless parameter characterizing photon-photon scattering is $\alpha(I/I_S)$,

as in the case when photon–photon scattering is described within the framework of the approach based on the Heisenberg-Euler lagrangian used below. We note that a similar (but not the same) analysis can be found in the papers [14, 15].

At the limit of extremely high amplitude of an electromagnetic field with a strength approaching the QED critical field E_S , nonlinear modification of the vacuum refraction index via the polarization of virtual electron–positron pairs supports electromagnetic wave self–interaction.

The nonlinear properties of the QED vacuum have been extensively addressed in a number of publications. The theoretical problem of non-linear effects of light propagation is considered in Ref. [19], where they study photon splitting in an external field in the full Heisenberg-Euler theory. Another extensive studies can be found in Refs. [20–22]. Other results on nontrivial vacua and on curved spacetimes can be found in Refs. [23–25]. The photon splitting in crossed electric and magnetic fields is considered, for example, in [26]. Nonlinear wave mixing in cavities is analyzed in [27]. Nonlinear interaction between an electromagnetic pulse and a radiation background is investigated in [28]. In the monograph [29], the vacuum birefringence phenomena is described within the framework of the geometrical optics approximation by using unified formalism. In the work [31], they incorporate weakest dispersion into Heisenberg-Euler theory, and in [32] the approach used in Ref. [29] is generalized allowing one to obtain the dispersion equation for the electromagnetic wave frequency and wavenumber. This process, in particular, results in decreasing the velocity of counter-propagating electromagnetic waves. As well known, the co-propagating waves do not change their propagation velocity because the copropagating photons do not interact, e.g., see Ref. [30].

The finite amplitude wave interaction in the QED vacuum results in the high order harmonics generation [31, 33–36]. High frequency harmonics generation can be a powerful tool to explore the physics of nonlinear QED vacuum. The highest harmonics can be used to probe the high energy region because they are naturally co-propagating and allow one to measure of QED effects in the coherent harmonic focus. High–order harmonics generation in vacuum is studied in detail in [33, 34].

Nonlinear properties of the QED vacuum in the long wavelength and low frequency limit are described by the Heisenberg-Euler Lagrangian [37], describing electromagnetic fields in dispersionless media whose refraction index depends on the electromagnetic field. In the media where the refraction index dependense on the field amplitude leads to the nonlinear response, the electromagnetic wave can evolve into a configuration with singularities [38].

The appearance of singularities in the Heisenberg-Euler electrodynamics is noticed in Ref. [39] where a singular particular solution of equations derived from the Heisenberg-Euler Lagrangian is obtained. In Ref. [36], the wave steepening is demonstrated by numerical inte-

gration of nonlinear QED vacuum electrodynamics equations.

In this paper we address the problem of nonlinear wave evolution in quantum vacuum using the low frequency and long wavelength approximation aiming at finding theoretical description of the electromagnetic shock wave formation in nonlinear QED vacuum. We present and analyze an analytical solution of the Heisenberg–Euler electrodynamics equations for the finite amplitude electromagnetic wave counter-propagating to the crossed electromagnetic field. This configuration may correspond to the collision of the low-frequency very high intensity laser pulse with high frequency X-ray pulse generated by XFEL. The first, long-wavelength, electromagnetic wave is approximated by a constant crossed field. We derive the corresponding nonlinear field equations containing expressions for the relatively short wavelength pulse. The solution of the nonlinear field equations is found in a form of the simple wave or the Riemann wave. This solution describes high order harmonic generation, wave steepening and formation of the electromagnetic shock wave in a vacuum. We investigate these characteristics in more detail together with discussion of the shock wave front formation process.

II. ON THE HEISENBERG-EULER LAGRANGIAN

The Heisenberg-Euler Lagrangian is given by

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}',\tag{5}$$

where

$$\mathcal{L}_0 = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} \tag{6}$$

is the Lagrangian in classical electrodynamics, $F_{\mu\nu}$ is the electromagnetic field tensor $(F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu})$, with A_{μ} being the 4-vector of the electromagnetic field and $\mu = 0, 1, 2, 3$. In the Heisenberg–Euler theory, the radiation corrections are described by \mathcal{L}' on the right hand side of Eq. (5), which in the weak field approximation is given by [40],

$$\mathcal{L}' = \frac{\kappa}{4} \left\{ (F_{\mu\nu} F^{\mu\nu})^2 + \frac{7}{4} \left(F_{\mu\nu} \tilde{F}^{\mu\nu} \right)^2 + \frac{90}{315} \left(F_{\mu\nu} F^{\mu\nu} \right) \left[(F_{\mu\nu} F^{\mu\nu})^2 + \frac{13}{16} \left(F_{\mu\nu} \tilde{F}^{\mu\nu} \right)^2 \right] \right\}$$
(7)

where $\kappa=e^4/360\pi^2m^4$, $F_{\mu\nu}=\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu}$ is and $\tilde{F}^{\mu\nu}=\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ is a dual tensor to a tensor of electromagnetic field, $F_{\mu\nu}$ with $\varepsilon^{\mu\nu\rho\sigma}$ being the Levi-Civita symbol in four dimensions.

In the following text, we use the units $c = \hbar = 1$, and the electromagnetic field is normalized on the QED critical field E_S .

To describe the singular solutions, we should keep the terms within the weak field approximation to the sixth order in the field amplitude, because the terms in the contributions of the fourth order cancel each other in calculation of dispersive properties of the QED vacuum. The remaining contribution is of the same order as from the Heisenberg–Euler Lagrangian expansion to the sixth order in the fields. We note that in Ref. [39], the Heisenberg–Euler Lagrangian expansion to the fourth order was used.

In the Lagrangian (7) the first two terms on the right hand side describe four interacting photons and the last two terms correspond to six photon interaction.

III. COUNTER-PROPAGATING ELECTROMAGNETIC WAVES

For the sake of brevity, we consider counterpropagating electromagnetic waves of the same polarization. They are given by the vector potential having one component, $\mathbf{A} = A\mathbf{e}_z$, with \mathbf{e}_z being a unit vector along the z axis.

We assume that the electromagnetic field 4-potential written in the light cone coordinates,

$$x_{+} = (x+t)/\sqrt{2}, \quad x_{-} = (x-t)/\sqrt{2},$$
 (8)

equals

$$A = Wx_{+} + a(x_{+}, x_{-}). \tag{9}$$

The term Wx_+ describes the crossed electric and magnetic fields $(E_0 = B_0 = -W/\sqrt{2})$, whose Poynting vector is antiparallel to the x axis:

$$\mathbf{P} = \frac{1}{4\pi} \mathbf{E} \times \mathbf{B} = -\frac{W^2}{8\pi} \mathbf{e}_x. \tag{10}$$

Here \mathbf{e}_x is a unit vector along the x-axis. In this case, the Lagrangian (5) with \mathcal{L}_0 and \mathcal{L}' given by Eqs. (6) and (7) takes the form

$$\mathcal{L} = -\frac{1}{4\pi} \left[(W+w)u - \epsilon_2 (W+w)^2 u^2 - \epsilon_3 (W+w)^3 u^3 \right]$$
(11)

It depends on the functions $u = \partial_{x_{-}} a$ and $w = \partial_{x_{+}} a$. The dimensionless parameters ϵ_2 and ϵ_3 in Eq. (11) are equal to

$$\epsilon_2 = 2e^2/45\pi = (2/45\pi)\alpha,$$
 (12)

$$\epsilon_3 = 32e^2/315\pi = (32/315\pi)\alpha,$$
 (13)

i.e., $\epsilon_2 \approx 10^{-4}$ and $\epsilon_3 \approx 2 \times 10^{-4}$.

The field equations can be found by varying the Lagrangian. It yields

$$\partial_{x_{-}}(\partial \mathcal{L}/\partial u) + \partial_{x_{+}}(\partial \mathcal{L}/\partial w) = 0.$$
 (14)

As a result, we obtain the system of equations

$$\partial_{x_{-}}w - \partial_{x_{+}}u = 0, \tag{15}$$

$$[1 - 4\epsilon_2(W + w)u - 9\epsilon_3 u^2(W + w)^2]\partial_{x_+}u$$

$$-[\epsilon_2(W + w)^2 + 3\epsilon_3 u(W + w)^3]\partial_{x_-}u \qquad (16)$$

$$-[\epsilon_2 u^2 + 3\epsilon_3 u^3(W + w)]\partial_{x_+}w = 0,$$

where the first equation comes from the equality of mixed partial derivatives: $\partial_{x_-x_+}a = \partial_{x_+x_-}a$.

The equations (15, 16) have a solution for which u=0 and $\partial_{x_-}w=0$, i.e., w is an arbitrary function depending on the variable x_+ . This is a finite amplitude electromagnetic wave propagating from the right to the left with a propagation velocity equal to speed of light in a vacuum. Its form does not change in time. The electric and magnetic field components are equal to each other $(E=B=-w/\sqrt{2})$, i.e., its superposition with the crossed electromagnetic field gives $E=B=-1/\sqrt{2}(W+w)$.

Linearizing Eqs. (15, 16), it is easy to find expressions describing the small amplitude wave for which we have

$$u(x_{+}, x_{-}) = u_{0} \left(x_{-} + \epsilon_{2} W^{2} x_{+} \right) \tag{17}$$

and

$$w(x_+, x_-) = \epsilon_2 W^2 u_0 \left(x_- + \epsilon_2 W^2 x_+ \right) + w_0(x_+). \tag{18}$$

In Eqs. (17, 18) the functions u_0 and w_0 are determined by the initial conditions. The function $u(x_+, x_-)$ depends on the light-cone coordinates (x_+, x_-) in combination

$$\psi(x_+, x_-) = x_- + \epsilon_2 W^2 x_+. \tag{19}$$

The wave phase ψ can be rewritten as

$$\psi(x,t) = \frac{1}{\sqrt{2}} \left[x \left(1 + \epsilon_2 W^2 \right) - t \left(1 - \epsilon_2 W^2 \right) \right]. \tag{20}$$

The constant phase condition shows that the wave propagates from the left to the right with the speed

$$v_W = \frac{1 - \epsilon_2 W^2}{1 + \epsilon_2 W^2} \approx 1 - 2\epsilon_2 W^2 + 2\epsilon_2^2 W^4.$$
 (21)

It is less than unity, i.e., the wave phase (group) velocity is below the speed of light in a vacuum (see also Refs. [7, 19, 29] and the literature cited therein).

Measuring the phase difference between the phase of the electromagnetic pulse colliding with the counterpropagating wave and the phase of the pulse which does not interact with high intensity wave, it is equal to

$$\delta\psi = 4\pi \frac{d}{\lambda} \epsilon_2 W^2, \tag{22}$$

where λ is the wavelength of high frequency pulse and d is the interaction length, plays a central role in discussion of experimental verification of the QED vacuum birefringence [10, 16]. For 10 petawatt laser the radiation intensity can reach 10^{24}W/cm^2 , for which $W^2 \approx 10^{-5}$. Taking the ratio equal to $d/\lambda \approx 10^4$, i.e. equal to the ratio between the optical and x-ray radiation wavelength, and using for ϵ_2 the expression (12), we find that $\psi \approx 10^{-4}$.

IV. NONLINEAR WAVE EVOLUTION

To analyze the nonlinear wave evolution, we seek a self-similar solution to Eqs. (15, 16) of a simple wave (e.g.,

see [41–43]), in which w is considered as a function of u: w(u). The simple wave (or Riemann wave) represents an exact solution of self–similar type of the nonlinear wave equations describing the finite amplitude wave propagating in a continuous media. With this assumption, we obtain from Eqs. (15, 16) the system of equations

$$\partial_{x_{+}} u = J \partial_{x_{-}} u,$$

$$\partial_{x_{+}} u =$$

$$(23)$$

$$\frac{(W+w)^{2}(\epsilon_{2}+3\epsilon_{3}(W+w)u)\partial_{x_{-}}u}{1-(W+w)u[4\epsilon_{2}+9\epsilon_{3}u(W+w)+3\epsilon_{3}u^{2}J]-\epsilon_{2}u^{2}J},$$
(24)

where we express

$$\partial_{x_{\perp}} w = J \partial_{x_{\perp}} u \tag{25}$$

and

$$\partial_{x_{-}}w = J\partial_{x_{-}}u \tag{26}$$

with the Jakobian J = dw/du. Equations (23) and (24) are consistent provided that the coefficients in front of $\partial_{x_-}u$ on the right hand sides are equal to each other. This condition yields an equation for J:

$$[\epsilon_2 u^2 + 3\epsilon_3 (W+w)u^3] J^2 - [1 - 4\epsilon_2 (W+w)u - 9\epsilon_3 u^2 (W+w)^2] J + (W+w)^2 [\epsilon_2 + 3\epsilon_3 u (W+w)] = 0.$$
(27)

Using smallness of the parameters ϵ_2 and ϵ_3 and the relationship $w \approx \epsilon_2 W^2$, which follows from Eq. (18), we obtain an expression for the function J(u) in the form of the power series:

$$J(u) = \epsilon_2 W^2 + 4\epsilon_2^2 W^3 u + 3\epsilon_3 W^3 u + \dots$$
 (28)

Taking into account that J = dw/du and integrating the r.h.s. of Eq. (28) with respect to the variable u we obtain for the function w(u) the following expression

$$w(u) = \epsilon_2 W^2 u + 2\epsilon_2 W^3 u^2 + \frac{3}{2} \epsilon_3 W^3 u^2 + \dots$$
 (29)

As a result, we find the electric and magnetic field components in the electromagnetic wave propagating from the left to the right

$$E = (w - u)/\sqrt{2} \approx -\sqrt{2}u(1 - \epsilon_2 W^2)$$
 (30)

and

$$B = (u+w)/\sqrt{2} \approx \sqrt{2}u(1+\epsilon_2 W^2)$$
 (31)

respectively.

Substitution of this expression to the right hand side of Eq. (23) results in

$$\partial_{x_{+}}u - \left[\epsilon_{2}W^{2} + (4\epsilon_{2}^{2} + 3\epsilon_{3})W^{3}u\right]\partial_{x_{-}}u = 0.$$
 (32)

For the variables x, t the equation for the function

$$\bar{u} = -2(4\epsilon_2^2 + 3\epsilon_3)W^3u \tag{33}$$

can be written as

$$\partial_t \bar{u} + (v_W + \bar{u}) \, \partial_x \bar{u} = 0. \tag{34}$$

with the velocity of linear wave, v_W , given by Eq. (21).

A solution to this equation can be obtained in a standard manner (see Refs. [41, 43]). According to this solution, the function $\bar{u}(x,t)$ transfers along the characteristic x_0 without distortion:

$$\bar{u} = \bar{u}_0(x_0) \tag{35}$$

The characteristic equation for Eq. (34) is

$$\frac{dx}{dt} = v_W + \bar{u} \tag{36}$$

with the solution

$$x = x_0 + (v_W + \bar{u}_0(x_0))t. \tag{37}$$

Combining these relationships, we obtain the solution to Eq. (34) in the implicit form, where the function u(x,t) should be found from equation

$$\bar{u} = \bar{u}_0 \left(x - (v_W + \bar{u})t \right).$$
 (38)

In particular, this expression describes high order harmonics generation and wave steepening in a vacuum.

Various mechanisms for generating high order harmonics in the QED vacuum are analyzed in Refs. [31, 33, 34, 36]. In particular, the parametric wave interaction process was considered in [31] and the "relativistic oscillating mirror" concept (for details of this concept, see [44–46]) was applied in [36]. Here, we formulate perhaps one of the simplest mechanisms. To obtain the scaling of high order harmonics generation within the framework of this mechanism, we choose the initial electromagnetic wave as

$$\bar{u}_0 = \bar{a}_1 \cos(k(x - v_W t)),\tag{39}$$

where \bar{a}_1 and $k=\omega/c$ are the wave amplitude and wave number (ω is the wave frequency), respectively. Using the weakness of nonlinearity ($\bar{a}_1 \ll 1$), we find from Eqs. (38, 39) that

$$\bar{u}(x,t) = \bar{a}_1 \cos(k(x - v_W t)) - \frac{\bar{a}_1^2}{2} k v_W t \sin(2k(x - v_W t))...$$
(40)

Taking into account normalization of the wave amplitude given by Eq. (33), we find that the ratio of the second harmonic amplitude to the amplitude of the wave with fundamental frequency scales as $(2\epsilon_2^2W^3+3\epsilon_3W^3/2)\bar{a}_1kv_Wt$. It is proportional to the duration of the electromagnetic wave interaction. Assuming $kv_Wt=2\pi d/\lambda$ as in the case corresponding to Eq. (22), and the intensity of the x-ray

pulse of 10^{21} W/cm² we obtain that the ratio is approximately equal to 10^{-11} .

From expression (38), it follows that the electromagnetic field gradient increases with time, i.e., wave steepening occurs. Differentiating u(x,t) with respect to the coordinate x, we find

$$\partial_x u = \frac{\partial_{x_0} u_0(x_0)}{1 - 2(4\epsilon_2^2 + 3\epsilon_3) W^3 \partial_{x_0} u_0(x_0) t},$$
 (41)

where the dependence of the Lagrange coordinate x_0 on time and the Euler coordinate x is given by Eq. (37). As shown, the gradient $\partial_x u$ becomes infinite at time

$$t_{br} = \frac{1}{2(4\epsilon_2^2 + 3\epsilon_3)W^3|\partial_{x_0}u_0(x_0)|}$$
 (42)

and at the coordinate x_0 where the derivative $\partial_{x_0}u_0(x_0)$ has its maximum. This singularity is called the "gradient catastrophe" or "the wave breaking".

The formation of singularity during the evolution of a finite amplitude electromagnetic wave in the quantum vacuum is illustrated in Figures 1 and 2. The electromagnetic pulse at t=0 takes the form

$$u_0(x_0) = a_0 \exp(-x_0^2/2L^2)\cos(kx_0),$$
 (43)

where $L = 4\pi$ and k = 2. The parameter $4\epsilon_2^2 W^3 + 3\epsilon_3 W^3$ is assumed to be equal to 0.125 and $a_0 = 1$. As clearly shown in Fig. 1, wave steepening evolves with time. Wave breaking occurs due to the characteristic intersection as shown in Fig. 2.

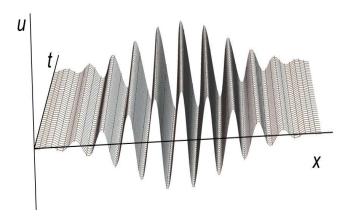


FIG. 1: The function u(x,t) given by Eq. (43) for $u_0(x_0)$ given by the expression (43).

We note that the singularity formed at the electromagnetic wave breaking corresponds to the rarefaction shock wave formation (the wave steepens and breaks in the backwards direction, as shown in Fig. 1), because the wave crest propagates with a speed lower than the propagation speed of the part of the pulse with lower amplitude.

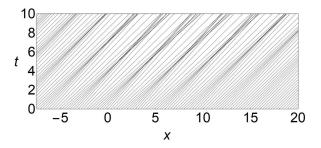


FIG. 2: Characteristics of Eq. (38) plotted in the x, t plane for the same parameters as in Fig. (1).

When a wave approaches the wave breaking point, wave steepening is equivalent to harmonics generation with higher numbers. Because of this fact the long-wavelength approximation used above becomes inapplicable and the Heisenberg-Euler Lagrangian cannot be used to describe the wave evolution in the vicinity of the gradient catastrophe, i.e., at the shock wave front. According to the shock wave paradigm [42], the basic properties of the shock wave (the relationships between the shock wave velocity and the average parameters of the medium before and after the shock wave front) can be found within the framework in the long-wavelength approximation if we consider the shock wave front region as a discontinuity.

V. ELECTROMAGNETIC SHOCK WAVE IN VACUUM

The long-wavelength approximation breaks when the frequencies of the interacting waves, ω_{γ} and Ω become high enough, i.e., when their product becomes of the order or higher than

$$\omega_{\alpha}\Omega > m_{\alpha}^2 c^4/\hbar^2. \tag{44}$$

At this photon energy level the photon–photon interaction can result in the creation of real electron–positron pairs during the Breit-Wheeler process [47], in the saturation of wave steepening, and in electromagnetic shock wave formation. Here, ω_{γ} and Ω are the frequencies of high energy photons and low frequency counterpropagating electromagnetic waves, respectively.

The electromagnetic shock wave separates two regions, I and II, where the function u(x,t) takes the values u_I and u_{II} , respectively. The shock wave front (it is an interface between regions I and II) moves with a velocity equal to v_{sw} , in other words, the shock wave front is localized at the position $x_{sw} = v_{sw}t$. Integrating Eq. (34) over an infinitely small interval $(-\delta + x_{sw}, x_{sw} + \delta)$, where

 $\delta \to 0$, we obtain

$$\{-v_{sw}u + v_Wu - (4\epsilon_2^2 + 3\epsilon_3)W^3u^2\}_{x=x_{sw}} = 0.$$
 (45)

Here, $\{f\}_x = f(x+\delta) - f(x-\delta)$ at $\delta \to 0$ denotes the discontinuity of the function f(x) at the point x. From Eq. (45), it follows that

$$v_{sw} = v_W - (4\epsilon_2^2 + 3\epsilon_3)W^3(u_I + u_{II}).$$
 (46)

Near the threshold $\omega\Omega \geq m_e^2 c^4/\hbar^2$, the electron–positron creation cross section equals [1, 48]

$$\sigma_{e-p} = \pi r_e^2 \sqrt{\frac{\hbar^2 \omega \Omega}{m_e^2 c^4} - 1}.$$
 (47)

The width of the shock wave front can be estimated to be of the order of the length $l_{sw} = 1/n_{\gamma}\sigma_{e-p}$, over which the photon with energy of the order of $m_e c^2/\hbar$ creates the electron-positron pair. The photon density n_{γ} is related to the electromagnetic pulse energy \mathcal{E}_{em} as $n_{\gamma} \approx \mathcal{E}_{em}/\hbar\omega\lambda^{3}N_{em}$, where $N_{em} = l_{em}/\lambda$ is the electromagnetic pulse length divided by the wavelength. It yields $l_{sw} \approx \hbar \omega \lambda^3 N_{em} / \pi r_e^2 \mathcal{E}_{em}$. Since it is assumed that the photon energy is at the threshold of the electronpositron pair creation, the shock wave front width should be of the order of the Compton wavelength, $\lambda_C = \hbar/m_e c$. This condition imposes a constraint from below on the electromagnetic pulse energy of $\mathcal{E}_{em} \geq m_e c^2 (\lambda/r_e)^2$. For a $1\mu m$ wavelength laser with $N_{em}=10$, it requires $\mathcal{E}_{em} \geq 100 \text{kJ}$. If $\lambda = 10^{-8} \text{cm}^2$, which corresponds to an X-ray pulse of 10 KeV, we have $\mathcal{E}_{em} \geq 10^{-4} \text{J}$.

The electron-positron pairs created at the electromagnetic shock wave front being accelerated by the electromagnetic wave emit gamma-ray photons which lead to the electron-positron avalanche via the multi-photon Breit-Wheeler mechanism [49] as discussed in Refs. [50, 51] (see also review article [8] and the literature cited therein). This process requires the dimensionless parameters $\chi_e \approx (E/E_S)\gamma_e$ and $\chi_{\gamma} \approx (E/E_S)(\hbar\omega_{\gamma}/m_ec^2)$ to be greater than one. At $\chi_{\gamma} > 1$, the QED vacuum becomes a dispersive and dissipative media [55, 56]. The effects of the electromagnetic wave dispersion originate formally from the high order derivatives in the corrections to the Heisenberg-Euler Lagrangian found in Refs. [57–59]. Discussions of higher-order derivatives when describing the QED vacuum beyond the Heisenberg-Euler Lagrangian model can be found in Refs. [57] and [58, 59]. As it has been noted in Refs. [62, 63] the implementation of the high derivatives into the description of nonlinear wave interaction in QED vacuum can result in the soliton formation. In [63] it is noted that the counterplay of nonlinearity and dispersion in nonlinear QED vacuum can lead to the dark soliton formation, which can be interpreted as the shock wave of the electromagnetic pulse envelope. Details on the dark soliton properties can be found in Refs. [64–66] and the literature cited therein.

In our case, the dispersion can result in modulations of the electromagnetic field in the vicinity of the shock wave front. Whether the dispersive properties come into play well below the pair-production threshold (44) depends not only on the wave amplitudes but also on the frequency of the interacting waves. For example, in the case of 10 KeV X-ray radiation, the dispersive effects prevail at an intensity above $10^{26}\ W/cm^2$. In any case, we do not expect this would change the main result of our paper because dissipation/dispersion determines the shock front structure.

With additional terms with derivatives in the Heisenberg–Euler Lagrangian (5), the theory predicts also the existence of bright spatial solitons [62]. With the presence of dispersion, the resulting wave break is prevented and it results in the process when the first soliton is formed. This process continues until the initial pulse is completely splitted into the chain of the solitons (see for more details Refs. [60, 61]). Such scenario needs huge peak intensity $10^{33}W/cm^2$ which can be decreased for experimental observation by making the size of the soliton large compared to the carier wavelength. For $\lambda = 10 \, \mathrm{nm}$ the peak intensity is $10^{25} \, W/cm^2$ [62].

VI. CONCLUSION

In conclusion, we presented and analyzed an analytical solution of the Heisenberg–Euler electrodynamics equations describing the finite amplitude electromagnetic wave counter-propagating to the crossed electromagnetic field. The solution belongs to the family of self-similar solutions corresponding to the Rieman wave. It describes the wave steepening and formation of the electromagnetic shock wave in the vacuum and the high order harmonic generation.

The singularity formed at the electromagnetic wave breaking has rarefaction shock wave character (the wave steepens and breaks in the backwards direction, as illustrated in Fig. 1), because the wave crest propagates with a speed lower than the propagation speed of the part of the pulse with lower amplitude.

In general, photon-photon scattering in a vacuum is governed by the dimensionless parameter $\alpha(I_{em}/I_S)$, as it concerns shock-like configuration formation, high order harmonics generation and the electron-positron and gamma ray flash at the electromagnetic shock wave front. Observation of these phenomena in a high power laser or x-ray interaction with matter implies high precision measurements as in experiments [2, 3] or achieving an electromagnetic field amplitude approaching the critical QED field E_S . One of the ways of reaching these regimes is to increase the laser power. For example, observation of one scattered photon per day with a 1 Hz laser requires an intensity of the order of $8 \times 10^{27} \text{W/cm}^2$, i.e., several hundred kJ laser energy. Another way of approaching the critical QED field limit is associated with the relativistic flying mirror concept [12] (for relativistic flying mirror theory and experiments see Refs. [46, 52–54]), where light intensity can be increased during the nonlinear laser-plasma interaction.

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