

Thermodynamic phase transitions from dynamical compact dimensions

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ABSTRACT: We perform a double quotient of global AdS₄ and study its thermal properties. We find that a double quotient yields a spacetime with an expanding compact dimension. Studying the entanglement of the dual CFT we find that, at early times, the spacetime has thermal properties which disappear after a critical time. For slow expansion, this critical time depends on the expansion rate as expected, but becomes much more sensitive with more rapid expansion rates.

Contents

1	Introduction	1
2	Single Quotient of Global AdS₄	2
3	Particle Modes in the Boundary Theory	4
4	Second Quotient of AdS₄	9
5	Action of Quotient on D_f and D_p	11
6	Entanglement of Boundary CFT	12
7	Conclusions	14

1 Introduction

Over two decades ago, a new solution to Einstein's equations in $2 + 1$ dimensions with a negative cosmological constant was discovered by Bañados *et al* [5]. This solution, known as the BTZ black hole, is an eternal black hole spacetime which is locally identical to Anti-de Sitter (AdS) space. It was also shown that this spacetime could be obtained from a quotient of global AdS [5, 12].

The BTZ black hole has proven to be an invaluable tool for studying quantum gravity, especially in regards to the AdS/CFT correspondence due its exhibiting interesting global properties, while remaining locally trivial [2, 3, 4]. While the BTZ black hole has been generalized to higher dimensions, calculations in these higher-dimensional generalizations are made more complicated due to time-dependences in the spacetimes.

Of interest to this paper is the case of a double quotient of global AdS in $3 + 1$ dimensions. In $2 + 1$ dimensions, it has been shown that, depending on the identification chosen, the second quotient either produces a third exterior region and creates a multi-boundary wormhole, or identifies the two asymptotic boundaries of the BTZ black hole to create a single-exterior black hole [10, 16, 17]. The implications of this in terms of multipartite entanglement of $(1 + 1)$ -dimensional CFTs has also been discussed [11]. Where a single quotient of $(3 + 1)$ -dimensional global AdS yields an exotic single-exterior black hole, we find that a double quotient results in a four-exterior black hole spacetime reminiscent of a big bang. This is not a typical big bang, however, in the sense that the entire universe spawns from a single point. Instead, we see that only a single, compact dimension is created which then expands to become macroscopic.

We begin by reviewing the single quotient of global AdS in $3 + 1$ dimensions. We go on to study the thermodynamics of this spacetime using holographic methods. Then we

perform the second quotient which yields a new spacetime whose thermodynamic behavior we study. We assign a Hartle-Hawking state to show that the dual CFTs are initially in an entangled thermofield double state. We then apply Ryu and Takayanagi's Holographic Entanglement Entropy (HEE) to the double-quotient spacetime to find the time-dependent behavior of the entanglement and discuss its implications on the thermodynamics of the spacetime. We end with our conclusions about the work.

2 Single Quotient of Global AdS₄

This section serves as an overview of the process used to obtain the (3+1)-dimensional topological black hole in [7] as well introduce different coordinate systems which will be useful later on. To perform the first quotient, we begin with global (3 + 1)-dimensional AdS (AdS₄), defined as the surface

$$-T_1^2 - T_2^2 + X_1^2 + X_2^2 + X_3^2 = -l^2 \quad (2.1)$$

embedded in $\mathbb{R}^{2,3}$ with metric

$$ds^2 = -dT_1^2 - dT_2^2 + dX_1^2 + dX_2^2 + dX_3^2 \quad (2.2)$$

where l is the AdS radius. We will consider the boost-like isometry given by

$$\xi = -X_1\partial_{T_1} - T_1\partial_{X_1} \quad , \quad \xi^2 = -X_1^2 + T_1^2 \quad (2.3)$$

to generate the quotient. To avoid closed timelike curves, we remove regions of the spacetime where $\xi^2 < 0$ after the quotient [9, 10]. This creates a singularity in the causal structure where timelike geodesics can end at

$$-X_1^2 + T_1^2 = 0 \quad (2.4)$$

or equivalently, by using (2.1),

$$-T_2^2 + X_2^2 + X_3^2 = -l^2. \quad (2.5)$$

This singularity asymptotes to the null cone given by

$$T_2^2 = X_2^2 + X_3^2 \quad \Rightarrow \quad X_1^2 - T_1^2 = -l^2 \quad (2.6)$$

which we identify as the event horizon of the black hole. These surfaces are plotted in Figure 1.

We can define coordinates on the surface (2.1), which are related to the coordinates of the embedding space by

$$T_1 = l \frac{1 - t^2 + y_1^2 + y_2^2}{1 + t^2 - y_1^2 - y_2^2} \cosh\left(\frac{r_+}{l}\phi\right) \quad , \quad T_2 = \frac{2lt}{1 + t^2 - y_1^2 - y_2^2}$$

$$X_1 = l \frac{1 - t^2 + y_1^2 + y_2^2}{1 + t^2 - y_1^2 - y_2^2} \sinh\left(\frac{r_+}{l}\phi\right) \quad , \quad X_2 = \frac{2ly_1}{1 + t^2 - y_1^2 - y_2^2} \quad , \quad X_3 = \frac{2ly_2}{1 + t^2 - y_1^2 - y_2^2} \quad (2.7)$$

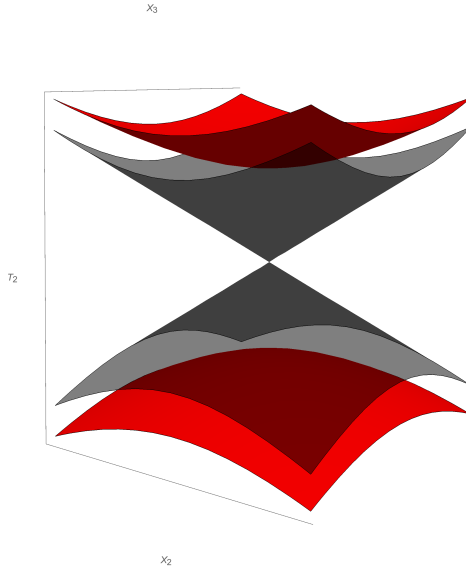


Figure 1. Singularity (red) and event horizon (black) created from a quotient by ξ . Here, the T_1 and X_1 coordinates are suppressed.

where r_+ is a constant. In terms of these coordinates the induced metric on the surface becomes

$$ds^2 = \frac{4l^2}{(1+t^2-y_1^2-y_2^2)^2}(-dt^2 + dy_1^2 + dy_2^2) + r_+^2 \left(\frac{1-t^2+y_1^2+y_2^2}{1+t^2-y_1^2-y_2^2} \right)^2 d\phi^2 \quad (2.8)$$

and the Killing vector which generates the quotient becomes $\xi = \partial_\phi$ [7]. The quotient makes the identification $\phi = \phi + 2\pi$, and the coordinate ranges after the quotient are given by $t, y_i \in (-\infty, \infty)$ and $\phi \in [0, 2\pi)$. In these coordinates the singularity is given by the surface

$$\xi^2 = l^2 \left(\frac{1-t^2+y_1^2+y_2^2}{1+t^2-y_1^2-y_2^2} \right)^2 = 0 \quad \rightarrow \quad -t^2 + y_1^2 + y_2^2 = -1 \quad (2.9)$$

and the event horizon is given by

$$T_1^2 - X_1^2 = l^2 \left(\frac{1-t^2+y_1^2+y_2^2}{1+t^2-y_1^2-y_2^2} \right)^2 = l^2 \quad \rightarrow \quad -t^2 + y_1^2 + y_2^2 = 0. \quad (2.10)$$

We can also see that the embedding coordinates diverge at the surface

$$1 + t^2 - y_1^2 - y_2^2 = 0 \quad (2.11)$$

which we associate with the boundary of the spacetime. Here we note that, aside from the singularity, the spacetime in these coordinates is geodesically complete, and therefore no further maximal extension is necessary [7]. In addition we see that this spacetime has only a single, connected asymptotic boundary, as opposed to the $(2+1)$ -dimensional case, which has two asymptotically distinct boundaries [5, 7, 12].

For future reference, it is also helpful to define “Schwarzschild coordinates” which cover the full exterior of the black hole. These can be defined by

$$t = \chi \sinh t' \quad , \quad y_1 = \chi \cos \psi \cosh t' \quad , \quad y_2 = \chi \sin \psi \cosh t' \quad (2.12)$$

giving the metric [7]

$$ds^2 = \frac{4l^2}{(1-\chi^2)^2} (-\chi^2 dt'^2 + d\chi^2 + \chi^2 \cosh^2 t' d\psi^2) + r_+^2 \left(\frac{1+\chi^2}{1-\chi^2} \right)^2 d\phi^2. \quad (2.13)$$

The first quotient of AdS₄ is an interesting case to study due to its single asymptotic boundary. Once we perform the second quotient of the spacetime, this single boundary will be split into four distinct asymptotic boundaries, and will also exhibit non-trivial thermal behavior.

3 Particle Modes in the Boundary Theory

It has been shown [1, 2, 3, 4] that the thermodynamics exhibited by asymptotically AdS spacetimes are mirrored in CFTs defined on the conformal boundary of the spacetime. We will use this to our advantage to track the thermal behavior of the double-quotient spacetime in the next section. First, we will define global coordinates on AdS₄ in terms of the embedding coordinates by

$$T_1 = l \frac{1+r^2}{1-r^2} \cos t \quad , \quad T_2 = l \frac{1+r^2}{1-r^2} \sin t$$

$$X_1 = \frac{2lr}{1-r^2} \cos \lambda \quad , \quad X_2 = \frac{2lr}{1-r^2} \cos \psi \sin \lambda \quad , \quad X_3 = \frac{2lr}{1-r^2} \sin \psi \sin \lambda \quad (3.1)$$

with ranges $r \in [0, 1)$, $\psi \in [0, 2\pi)$, $\lambda \in [0, \pi]$, and $t \in (-\infty, \infty)$ after enforcing a universal covering to avoid closed timelike curves [6]. The metric becomes

$$ds^2 = \frac{4l^2}{(1-r^2)^2} \left[-\frac{(1+r^2)^2}{4} dt^2 + dr^2 + r^2 d\lambda^2 + r^2 \cos^2 \lambda d\theta^2 \right] \quad (3.2)$$

where the boundary is located at $r = 1$. The Killing vector that generates the quotient to form the black hole is given by (2.3). There is another Killing vector orthogonal to ξ given by $\eta = X_2 \partial_{T_2} + T_2 \partial_{X_2}$, where the motivation for introducing η will become clear. To get the metric on the boundary, we multiply (3.2) by the conformal factor

$$\Omega^2 = \frac{(1-r^2)^2}{4l^2} \quad (3.3)$$

and take $r \rightarrow 1$ to obtain

$$d\sigma^2 = -dt^2 + d\lambda^2 + \cos^2 \lambda d\psi^2. \quad (3.4)$$

The conformal Killing vectors on the boundary corresponding to ξ and η are given by

$$\begin{aligned} \xi_b &= \cos \lambda \sin t \partial_t + \cos t \sin \lambda \partial_\lambda \\ \eta_b &= \cos t \sin \lambda \cos \psi \partial_t + \sin t \cos \lambda \cos \psi \partial_\lambda - \sin t \csc \lambda \sin \psi \partial_\psi \end{aligned} \quad (3.5)$$

respectively [15]. To find the region of the boundary that will survive the quotient, we must find the region where $\xi_b^2 > 0$. This region corresponds to the single diamond

$$0 < \lambda < \pi, \quad |t| < \frac{\pi}{2} - \left| \lambda - \frac{\pi}{2} \right| \quad (3.6)$$

in contrast to the BTZ black hole, which gives two separate diamonds [16, 17]. This is a reflection of the fact that the $(3+1)$ -dimensional topological black hole has a single, connected boundary.

To better understand the action of ξ_b on the boundary, we define the coordinates (inspired by those used in [16])

$$\begin{aligned} \alpha &= -\log \left[\tan \left(\frac{\lambda - t}{2} \right) \right] \\ \beta &= \log \left[\tan \left(\frac{\lambda + t}{2} \right) \right] \end{aligned} \quad (3.7)$$

giving the metric

$$d\sigma^2 = \frac{d\alpha d\beta + \cosh^2 [(\alpha + \beta)/2] d\psi^2}{\cosh \alpha \cosh \beta} \quad (3.8)$$

where the conformal Killing vectors are now given by

$$\begin{aligned} \xi_b &= -\partial_\alpha + \partial_\beta \\ \eta_b &= \cos \psi (\partial_\alpha + \partial_\beta) - \sin \psi \tanh \left(\frac{\alpha + \beta}{2} \right) \partial_\psi. \end{aligned} \quad (3.9)$$

The vector ξ_b maps the point

$$(\alpha, \beta, \psi) \rightarrow (\alpha - c, \beta + c, \psi) \quad (3.10)$$

where c is a constant. Unfortunately, the metric (3.8) is not invariant under an action generated by ξ_b , but the conformally related metric

$$d\sigma^2 = -d\alpha d\beta + \cosh^2 \left(\frac{\alpha + \beta}{2} \right) d\psi^2 \quad (3.11)$$

is. Finally, defining $\alpha = t' - r_+ \phi$ and $\beta = t' + r_+ \phi$ where $\tau, \phi \in (-\infty, \infty)$, we obtain the metric

$$d\sigma^2 = -dt'^2 + \cosh^2 t' d\psi^2 + r_+^2 d\phi^2 \quad (3.12)$$

and the conformal Killing vectors take the form

$$\begin{aligned} \xi_b &= \partial_\phi \\ \eta_b &= \cos \psi \partial_{t'} - \sin \psi \tanh t' \partial_\psi. \end{aligned} \quad (3.13)$$

A quotient by ξ_b makes the identification $\phi = \phi + 2\pi$ and we see that the metric (3.12) is the conformal boundary of the topological black hole in the coordinates (2.13).

Near $t' = 0$, we find $\eta_b = \cos \psi \partial_{t'}$ is purely timelike, except for the points $\psi = \pm\pi/2$. Moreover, for the region

$$D_f \doteq \left\{ \psi \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right) \right\} \quad (3.14)$$

η_b is future-directed, while for the region

$$D_p \doteq \left\{ \psi \in \left(\frac{\pi}{2}, \frac{3\pi}{2} \right) \right\} \quad (3.15)$$

η_b is past-directed. Therefore, we naturally associate η_b with particle modes in the boundary theory, where positive energy modes are associated with η_b in D_f and negative energy modes are associated with η_b in D_p .

As the system evolves away from $t' = 0$, we find that η_b is no longer timelike over the full boundary and thus η_b will only generate particles on the region of the spacetime corresponding to

$$\cosh^2 t' \cos^2 \psi - \sinh^2 t' > 0. \quad (3.16)$$

Looking at the form of (2.12), we can see that this is just the region of the boundary $y_1^2 - t^2 > 0$.

We would now like to show that the regions D_f and D_p are in pure states. Since these regions have a global timelike Killing vector associated with them, we can use the Ryu-Takayanagi minimal-area holographic entanglement entropy proposal to find the entanglement entropy of these regions [3, 14].

The holographic entanglement entropy conjecture of Ryu and Takayanagi states that the entanglement entropy, S_A of a spatial region A of a CFT can be calculated from the area of the minimal area surface, γ , in the dual spacetime which terminates on the boundary of A . This relationship is given by

$$S_A = \frac{\text{Area}(\gamma)}{4G_N^{(d)}} \quad (3.17)$$

where $G_N^{(d)}$ is the d -dimensional gravitational constant [3, 14].

We can now find the entanglement entropy of the region of the boundary where η_b is timelike by simply relating coordinates (2.8) to Poincaré coordinates [3, 4, 14]. The metric

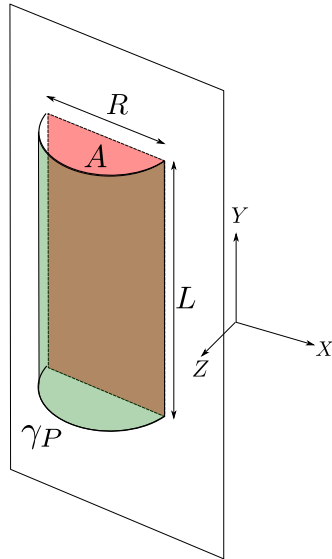


Figure 2. Minimal-area surface, γ_P , (green) corresponding to the strip, A , (red) at a constant-time slice of global AdS_4 .

of global AdS_4 in Poincaré coordinates is given by

$$ds^2 = \frac{l^2}{Z^2} (dW_+ dW_- + dY^2 + dZ^2) \quad (3.18)$$

where $W_{\pm}, Y \in (-\infty, \infty)$ and $Z \in (0, \infty)$ [8]. Note that this is the null coordinate form of the Poincaré metric where $W_{\pm} = X \pm T$ with T and X being timelike and spacelike coordinates, respectively. The boundary of the spacetime is at $Z = 0$. If we consider a strip on the boundary at fixed time given by

$$W_+ - W_- = \text{const} \Rightarrow \Delta W_+ - \Delta W_- = 0, \quad (3.19)$$

the length of the strip will be given by $\Delta Y = L$ and the width will be given by

$$R^2 = \Delta W_+ \Delta W_-. \quad (3.20)$$

If we choose a minimal-area surface, γ_P , which terminates only on the boundary in the $X = 1/2(W_+ + W_-)$ direction as shown in Figure 2, the area of γ_P will be given by

$$\text{Area}(\gamma_P) = 2l^2 \left(\frac{L}{\epsilon} \right) - \kappa l^2 \left(\frac{L}{R} \right) \quad (3.21)$$

where ϵ is a cutoff introduced to prevent the expression from diverging and

$$\kappa = 4\pi \left(\frac{\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} \right)^2 \quad (3.22)$$

is a positive constant [4, 14]. For the expression (3.21) to be consistent, we can see that the area of the surface γ_P must vanish when the region, A , on the boundary also vanishes, i.e. $\text{Area}(\gamma_P) \rightarrow 0$ as $\epsilon, R \rightarrow 0$.

Now, we can relate the Poincaré coordinates to those used in (2.8), but to ease the calculations, we will introduce a polar form of the coordinates with $y_1 = r \cos \psi$ and $y_2 = r \sin \psi$. These coordinates are related to the null form of Poincaré coordinates by

$$\begin{aligned} W_{\pm} &= \frac{2}{1-t^2+r^2} (r \cos \psi \pm t) e^{\phi} \\ Y &= \frac{2}{1-t^2+r^2} r \sin \psi e^{\phi} \\ Z &= \frac{1+t^2-r^2}{1-t^2+r^2} e^{\phi}. \end{aligned} \quad (3.23)$$

In these coordinates, the boundary is the surface where $Z = 0$, which corresponds to $r = \sqrt{1+t^2}$. If we anchor the minimal-area surface, γ , to the boundary at fixed time $t = t_0$ such that $\psi \in [\psi_1, \psi_2]$ and $\phi \in [\phi_1, \phi_2]$, the cutoff at $Z = \epsilon \ll 1$ corresponds to

$$\epsilon_{1,2} = a e^{\phi_{1,2}} \quad \text{where} \quad 0 < a \ll 1. \quad (3.24)$$

For simplicity, we will center the region A on the boundary such that $\psi_1 = -\Psi$ and $\psi_2 = \Psi$. Thus, the length, L , of A is given by

$$L = \Delta Y = \sqrt{1+t_0^2} (e^{\phi_2} + e^{\phi_1}) \sin \Psi \quad (3.25)$$

and the width, R , is given by (3.20)

$$R^2 = (e^{\phi_2} - e^{\phi_1})^2 [(1+t_0^2) \cos^2 \Psi - t_0^2]. \quad (3.26)$$

Using (3.24), (3.25), and (3.26) in (3.21), we find an expression for the area of γ using $\epsilon^2 = \epsilon_1 \epsilon_2$

$$\text{Area}(\gamma) = 4l^2 \left(\frac{\cosh \Delta}{a} \right) \sqrt{1+t_0^2} \sin \Psi - \kappa l^2 \left(\frac{\sqrt{1+t_0^2} \sin \Psi}{\sqrt{(1+t_0^2) \cos^2 \Psi - t_0^2}} \right) \coth \Delta \quad (3.27)$$

where $\Delta = (\phi_2 - \phi_1)/2$. Here, we recall that the area must vanish when both $\epsilon, R \rightarrow 0$, corresponding to

$$a = 0 \quad , \quad (1+t_0^2) \cos^2 \Psi - t_0^2 = 0. \quad (3.28)$$

We notice that the latter of these requirements is equivalent to $y_1^2 - t^2 = 0$, which is exactly the boundary of D_f . Since the area of γ vanishes on the boundary of D_f , we can conclude that the CFT on D_f is in a pure state. This also holds true for D_p .

The regions D_f and D_p will serve as useful tools for analyzing the thermodynamics of the double-quotient spacetime. This is mainly due to the fact that any CFTs defined on these regions will be “complete” theories i.e. we do not need any information from the rest of the boundary to describe the state of these theories.

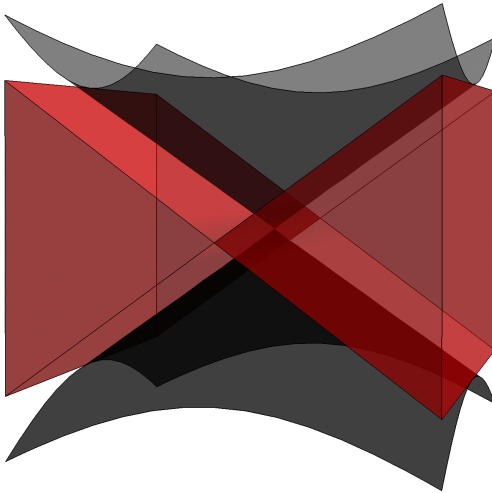


Figure 3. The regions excised during second quotient (red) end on null surfaces, running along the event horizon of the black hole (black). Since these surfaces are null, they are not bounded by event horizons and therefore, the resulting singularity is naked.

4 Second Quotient of AdS_4

To perform a second quotient of AdS_4 , we first recognize that the metric (2.8) has a boost-like isometry with Killing vector

$$\xi' = y_2 \partial_t + t \partial_{y_2}. \quad (4.1)$$

As we will be performing a quotient defined by this Killing vector, we will again need to excise regions of the spacetime where $\xi'^2 < 0$ to avoid closed timelike curves. These regions correspond to

$$y_2^2 - t^2 > 0. \quad (4.2)$$

Since the boundary of these regions are null lines $t = \pm y_2$, they are not bounded by event horizons and therefore are naked. Figure 3 shows these naked singularities. It is also worthy to note that the introduction of these new singularities separates the spacetime into four asymptotically distinct regions: two future regions and two past regions.

We now define new coordinates

$$t = |\tau| \cosh(\alpha \theta) \quad , \quad y_2 = |\tau| \sinh(\alpha \theta) \quad (4.3)$$

where $\theta \in (-\infty, \infty)$ and $\tau \in (-\infty, \infty)$. This transforms the metric (2.8) to

$$ds^2 = \frac{4l^2}{(1 + \tau^2 - y_1^2)^2} (-d\tau^2 + dy_1^2 + \alpha^2 \tau^2 d\theta^2) + r_+^2 \left(\frac{1 - \tau^2 + y_1^2}{1 + \tau^2 - y_1^2} \right)^2 d\phi^2. \quad (4.4)$$

and the Killing vector (4.1) to

$$\xi' = \frac{1}{\alpha} \partial_\theta. \quad (4.5)$$

This Killing vector maps the points $\theta \rightarrow \theta + a/\alpha$ where a is an arbitrary parameter. Defining $\alpha = a/2\pi$, we can see that a quotient by ξ' will make the θ coordinate periodic, with range $\theta \in [0, 2\pi)$. The black hole singularity is now located at the surface

$$-\tau^2 + y_1^2 = -1, \quad (4.6)$$

the black hole event horizon is given by the surface

$$-\tau^2 + y_1^2 = 0, \quad (4.7)$$

and the boundary of the spacetime is at

$$-\tau^2 + y_1^2 = 1. \quad (4.8)$$

In these coordinates, the naked singularity is located at $\xi'^2 = 0$ or $\tau = 0$.

Since the θ coordinate is scaled by τ , for very early times, the θ dimension will be microscopic and we may effectively neglect the $d\theta$ term in our metric. We are then left with the metric of the BTZ black hole [2]

$$ds^2 = \frac{4l^2}{1 - y_1^2} (-d\tau^2 + dy_1^2) + r_+^2 \left(\frac{1 + y_1^2}{1 - y_1^2} \right) d\phi^2. \quad (4.9)$$

Therefore, for times near $\tau = 0$, we should expect the black hole to radiate at inverse temperature [12]

$$\beta = \frac{r_+}{2\pi}. \quad (4.10)$$

However, away from $\tau = 0$, the black hole cools to zero temperature. This can be seen by defining the coordinates

$$\tau = \rho \sinh(\alpha\zeta) \quad , \quad y_2 = \rho \cosh(\alpha\zeta) \quad (4.11)$$

which transform the metric (4.4) to

$$ds^2 = \frac{4l^2}{(1 - \rho^2)^2} [-\alpha^2 \rho^2 d\zeta^2 + d\rho^2 + \alpha^2 \rho^2 \sinh^2(\alpha\zeta) d\theta^2] + r_+^2 \left(\frac{1 + \rho^2}{1 - \rho^2} \right)^2 d\phi^2. \quad (4.12)$$

Here, $\zeta \in (-\infty, \infty)$ and $\rho \in [0, 1)$. These coordinates correspond to an observer living in one exterior of the black hole, whose event horizon is located at $\rho = 0$. It is worth noting that, in these coordinates, the event horizon is 1-dimensional. We can perform a Wick rotation by taking $\zeta \rightarrow -i\zeta_E$ and the metric becomes

$$ds^2 = \frac{4l^2}{(1 - \rho^2)^2} [\alpha^2 \rho^2 d\zeta_E^2 + d\rho^2 + \alpha^2 \rho^2 \sin^2(\alpha\zeta_E) d\theta^2] + r_+^2 \left(\frac{1 + \rho^2}{1 - \rho^2} \right)^2 d\phi^2. \quad (4.13)$$

The term in square brackets is simply the metric on \mathbb{R}^3 in spherical coordinates where ζ_E is the polar coordinate and is therefore non-periodic. Since imaginary time is non-periodic after a Wick rotation, this implies that the black hole is not radiating far from $\tau = 0$. We will see more evidence for the freezing out of radiation in Section 6.

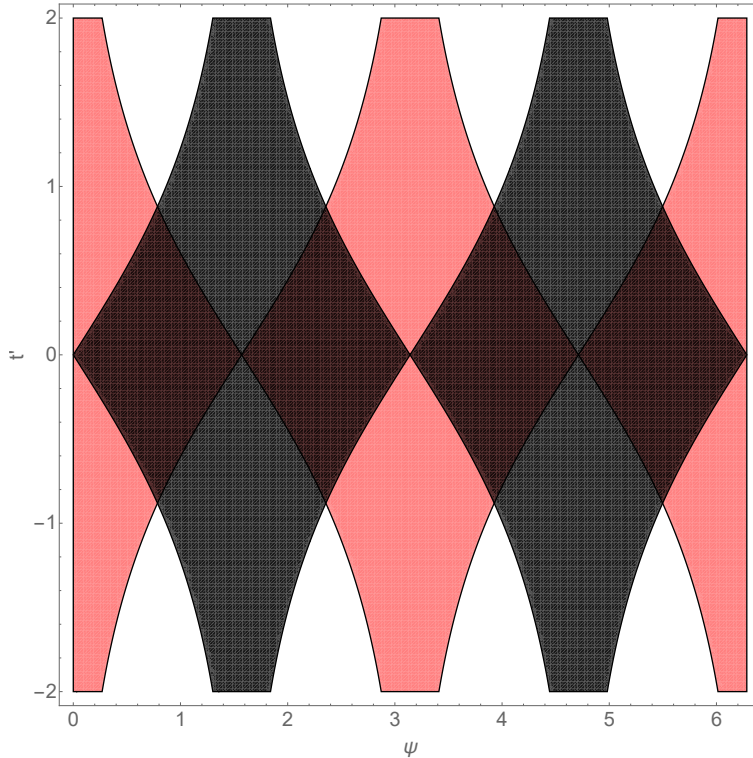


Figure 4. Regions of the boundary removed during second quotient (black) overlap the regions of the boundary in a pure state (red) at early times. However, at later times, this overlap ceases, reflecting that, at early times, the boundary CFT should be in a mixed state, but should purify at later times.

5 Action of Quotient on D_f and D_p

Now that we have shown how the quotient acts on the spacetime, we would next like to consider how the boundary theory behaves under this quotient. Specifically, we wish to look at what regions of the boundary are excised by the quotient. As we have seen, the quotient removes the portions of the spacetime corresponding to $y_2^2 - t^2 > 0$. We have already found that D_f and D_p are the regions of the boundary $y_1^2 - t^2 > 0$. Since $y_1 \rightarrow y_2$ is the same as taking $\psi \rightarrow \psi + \pi/2$, by (2.12), it is clear that the region of the boundary removed by the quotient is given by

$$\cosh^2 t' \sin^2 \psi - \sin^2 t' > 0 \quad (5.1)$$

in coordinates (2.13). These removed regions overlap D_f and D_p at early times, as shown in Figure 4.

The fact that we have removed regions of the boundary which can house CFTs in pure states tells us that, at early times, the CFT should, in general, be in a mixed state. However, at some later time, this overlap ceases and the states should purify.

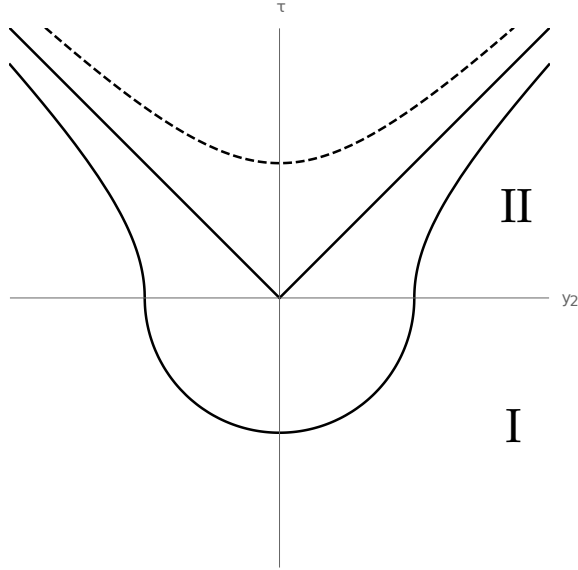


Figure 5. Hartle-Hawking initial state of the big bang spacetime. Region I is a 3-dimensional Euclidean space, where Region II is a (3 + 1)-dimensional Lorentzian spacetime. In Region I, the ϕ dimension is suppressed, and in Region II, the ϕ and θ dimensions are suppressed. Despite the spaces having different dimensionality, we can smoothly glue them along $\tau = 0$ since the θ dimension collapses as $\tau \rightarrow 0$.

6 Entanglement of Boundary CFT

We have shown that the boundary CFT begins in a mixed state, but purifies after some time. We now consider what state the CFT is initially in and explicitly show this purification through the use of holographic entanglement.

Since the metric (4.4) exhibits a symmetry under $\tau \rightarrow -\tau$, we should be able to assign a Hartle-Hawking initial condition to the spacetime [18]. However, we cannot obtain a Euclidean-signature metric from a single Wick rotation, $\tau \rightarrow -i\tau$. Instead, we will utilize the fact that as $\tau \rightarrow 0$, the θ coordinate collapses and the metric becomes that of BTZ. We can therefore smoothly glue the 3-dimensional Euclidean metric of the BTZ spacetime, given by

$$ds_E^2 = \frac{4l^2}{(1 - \tau_E^2 - y_2^2)^2} (d\tau_E^2 + dy_2^2) + r_+^2 \left(\frac{1 + \tau_E^2 + y_2^2}{1 - \tau_E^2 - y_2^2} \right) d\phi^2, \quad (6.1)$$

to the metric (4.4) along $\tau = 0$ as shown in Figure 5.

Since the initial condition of the spacetime is identical to that of BTZ, the state of the CFTs will also be identical very near $\tau = 0$. Namely they will be in a thermofield double state, given by

$$|\Psi\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{-\beta E_n/2} |E_n\rangle_{D_f} \otimes |E_n\rangle_{D_p} \quad (6.2)$$

where D_f and D_p refer to the two CFTs on each region of the boundary, respectively [2]. We again see that at early times, the CFTs are in a mixed state, and furthermore,

are entangled with each other. More importantly, this mixed state can be interpreted as thermal, a direct reflection of the fact that the spacetime is thermal at early times.

To see how the entanglement between the CFTs on D_f and D_p evolves, we will use the same procedure as in Section 3.

The coordinates in (4.4) are related to Poincaré coordinates by

$$W_{\pm} = \frac{\pm 2l|\tau|}{1 - \tau^2 + y_2^2} e^{\frac{r_{\pm}}{l}\phi \pm \alpha\theta}, \quad Y = \frac{2ly_1}{1 - \tau^2 + y_2^2} e^{\frac{r_{\pm}}{l}\phi}, \quad Z = l \left(\frac{1 + \tau^2 - y_1^2}{1 - \tau^2 + y_1^2} \right) e^{\frac{r_{\pm}}{l}\phi}. \quad (6.3)$$

The boundary of the spacetime will be located at $Z = 0$ or $y_1 = \sqrt{1 + \tau^2}$ ¹. We can now choose to anchor the surface, γ , to the boundary at fixed time $\tau = \tau_0$, such that $\theta \in [\theta_1, \theta_2]$ and $\phi \in [\phi_1, \phi_2]$, the cutoff $Z = \epsilon \ll 1$ will be given by

$$\epsilon_{1,2} = la e^{\frac{r_{\pm}}{l}\phi_{1,2}} \quad (6.4)$$

where $0 < la \ll 1$ and $\epsilon_{1,2} \rightarrow 0$ corresponds to $a \rightarrow 0$. Similar to above, we will center the strip such that $\theta_1 = -\Theta$ and $\theta_2 = \Theta$. Then, the width of the strip is given by

$$R^2 = l \left(e^{\frac{r_{\pm}}{l}\phi_2} - e^{\frac{r_{\pm}}{l}\phi_1} \right)^2 [1 - \tau_0^2 \sinh^2(\alpha\Theta)] \quad (6.5)$$

and the length is given by

$$L = l|\tau_0| \left(e^{\frac{r_{\pm}}{l}\phi_2} + e^{\frac{r_{\pm}}{l}\phi_1} \right) \sinh(\alpha\Theta). \quad (6.6)$$

The width, R , vanishes either when $\phi_1 = \phi_2$ or $\tau_0 \sinh(\alpha\Theta) = 1$.

Using these expressions in (3.21), we obtain an expression for the area of the minimal area surface in coordinates (4.4)

$$\text{Area}(\gamma) = 4l^2 \frac{|\tau_0|}{a} \sinh(\alpha\Theta) \cosh \Delta - \kappa l^2 \left(\frac{|\tau_0| \sinh(\alpha\Theta)}{\sqrt{1 - \tau_0^2 \sinh^2(\alpha\Theta)}} \right) \coth \Delta. \quad (6.7)$$

We can see that, as $a \rightarrow 0$ and $\tau_0 \sinh(\alpha\Theta) \rightarrow 1$, the area goes to zero and the region on the boundary is pure. This implies that there is a minimum time for a region of the boundary to reach a pure state

$$\tau_{min} = \frac{1}{\sinh(\alpha\Theta)}. \quad (6.8)$$

When $\Theta = \pi$, the region, A , will be one full future asymptotic boundary. Therefore, the minimum time, τ_{crit} , for the CFT on this copy of the future boundary of the spacetime to purify will be

$$\tau_{crit} = \frac{1}{\sinh(\alpha\pi)}. \quad (6.9)$$

For $\tau < \tau_{crit}$, the boundary of either future exterior is in a mixed state, which we know from the Hartle-Hawking state to be a thermofield double. For $\tau > \tau_{crit}$, the boundaries

¹Since the asymptotic boundaries are distinct, the region A can only exist on one of these boundaries, so we choose the one corresponding to $y_1, \tau > 0$.

of both future exteriors are in pure states. Again, this is a reflection of the fact that, for early times, the spacetime is thermal but cools after a certain time.

We can also gain information from this about how the thermodynamic phase transition of the spacetime depends on the scaling parameter α . Naïvely, by the form of the metric (4.4), we may expect this phase transition to occur when the θ dimension grows from microscopic to macroscopic at times

$$\tau \sim \frac{1}{\alpha}. \tag{6.10}$$

For small values of α , (6.9) appears to confirm this scaling relation. However, for $\alpha \gtrsim 1$, the boundary purifies much faster than $1/\alpha$, suggesting that the thermodynamics of the spacetime are much more sensitive to this scaling parameter than the metric would lead us to believe.

7 Conclusions

We have arrived at a black hole spacetime from a double quotient of global AdS_4 which has a time-dependent, expanding compact dimension. At early times, when this spatial dimension is small, the black hole emits thermal radiation. During this period, we also see entanglement between two of the boundary CFTs. After a critical time, the radiation stops and the entanglement is broken between the CFTs. For slow expansion, the critical time is inversely proportional to the expansion rate of the compact dimension, a result we might expect from the form of the metric. However, for rapid expansion, we have found that the purification time is much shorter than this naïve inverse relation.

This work further exemplifies the power of the AdS/CFT correspondence. Using relativity or quantum mechanics alone, it is only possible to study the early and late time limits of the spacetime, but not the transition period. However, this is the epoch in which the spacetime exhibits particularly interesting characteristics. Through the application of the holographic entanglement entropy calculation, we were able to uncover specific behaviors of this transition period. We also see from the holographic calculation that the intuition we may have from just the spacetime metric alone is not necessarily correct. Moreover, this work further confirms the holographic entanglement entropy conjecture, as the holographic calculation agreed with both the early and late time limits in the relativistic and quantum calculations.

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