

A Brief Summary Of Global Anomaly Cancellation In Six-Dimensional Supergravity

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1. Introduction

This paper is a brief summary of a talk at Strings 2018 in Okinawa [38]. It is also an informal and less-technical summary of our paper [34].

The motivation for the present work is the general question of the relation of apparently consistent low energy theories of quantum gravity to string theory: We would like to find new consistency conditions to restrict the possible low energy effective supergravity theories and at the same time find new string constructions. The interesting question is whether every consistent low energy supergravity theory can be obtained from a string-theory construction. The state of the art in this general subject is summarized in [6, 46]. One way to impose consistency conditions on low energy supergravity theories is via anomaly cancellation. In the case of six-dimensional supergravity, while the Green-Schwarz mechanism was described some time ago [23, 42, 43], only recently have the anomaly cancellation conditions been systematically investigated. See [46] for a review. Moreover, while previous investigations have included some considerations of global anomaly cancellation, no completely systematic account of global anomalies in this context has yet been given. It is the purpose of [33, 34] to begin to fill this gap.

The papers [33, 34] contain three main results: First, a necessary condition for anomaly cancellation is stated which summarizes and extends all previous results.¹ Second, a necessary and sufficient condition is given for global anomaly cancellation in terms of

¹More accurately, [33] uses both global anomaly cancellation and tadpole cancellation of string charge. It is possible the tadpole cancellation conditions can be rederived from global anomaly cancellation on the worldsheets of the six-dimensional strings, (somewhat in the spirit of the work of Polchinski and Cai [39]) but that is beyond the scope of the present work.

triviality of a certain 7-dimensional spin topological field theory (defined below). Third, the necessary conditions of [33] were verified (in that same paper) to hold in the case of F-theory compactifications. A novel aspect of this third result is that one must know the global structure of the connected component of the gauge group for the vectormultiplets. In this summary we will focus on just the first two of these three results.

2. Data For Six-Dimensional Supergravity

The field content of six-dimensional $(1, 0)$ supergravity is determined by giving three pieces of data: First, a compact Lie group G for the vectormultiplets, second, a quaternionic representation of G , denoted \mathcal{R} , for the half-hypermultiplets, and third an integral lattice Λ of Lorentzian signature $(1, T)$ defining the tensormultiplets. The lattice may be regarded as a lattice of string charges for strings charged under the 2-form gauge potentials in the tensormultiplets. Of course, in addition to this, we add the supergravity multiplet. Given this data the equations of motion of the classical supergravity theory are uniquely determined [41].

In the quantum theory the chiral fermions and the (anti-) self-dual tensor fields have gauge and gravitational anomalies [1]. The Green-Schwarz mechanism for the cancellation of the perturbative anomalies is well known [23, 24, 40, 42, 43]: We compute the anomaly 8-form from the data $(G, \mathcal{R}, \Lambda)$:

$$I_8 = \frac{1}{2} \left[\text{Tr}_{\mathcal{R}} e^{\frac{F}{2\pi i} \hat{A}} + \dots \right]_8 \quad (2.1)$$

$$\sim (\dim_{\mathbb{H}} \mathcal{R} - \dim_{\mathbb{R}} G + 29T - 273) \text{Tr}(R^4) + (9 - T)(\text{Tr} R^2)^2 + \dots$$

(some complicated numerical coefficients are suppressed in the second line). Then we seek to factorize it

$$I_8 = \frac{1}{2} Y^2 \quad (2.2)$$

where $Y \in \Omega^4(\mathcal{W}; \Lambda_{\mathbb{R}})$ is a 4-form on some (as yet unspecified) 8-manifold \mathcal{W} with values in the vector space $\Lambda_{\mathbb{R}} := \Lambda \otimes \mathbb{R}$. Note that this vector space inherits a Lorentzian signature quadratic form from Λ and in (2.2) we implicitly use that form to contract the two factors of Y . This convention will be used henceforth without comment. Next, we interpret Y as a background magnetic current for the tensormultiplets and modify the Bianchi identity for the fieldstrength $H \in \Omega^3(\mathcal{M}_6; \Lambda_{\mathbb{R}})$ to $dH = Y$. Here \mathcal{M}_6 is the (compact, Euclidean) spacetime of the six-dimensional supergravity. As is well-known, this forces the 2-form gauge potentials to shift under diffeomorphisms and to transform under vectormultiplet gauge transformations. We now merely add the counterterm to the exponentiated action e^{-S} in the supergravity path integral:

$$e^{-S} \rightarrow e^{-S} e^{-2\pi i \frac{1}{2} \int_{\mathcal{M}_6} BY} . \quad (2.3)$$

It is worth noting that there are a few difficulties with this textbook story: First, in topologically nontrivial situations B is not globally defined, so the six-form $BY \in \Omega^6(\mathcal{M}_6)$ is not globally defined. A standard way to deal with this is to extend B and Y to forms

on a seven-dimensional manifold \mathcal{U}_7 so that $\partial\mathcal{U}_7 = \mathcal{M}_6$. Then, since dB is better-defined than B we can attempt to substitute

$$\frac{1}{2} \int_{\mathcal{M}_6} BY \rightarrow \frac{1}{2} \int_{\mathcal{U}_7} dBY \quad (2.4)$$

Of course this only makes sense if the result is independent of extension. In fact, it is not. Even the difference of Green-Schwarz terms $\frac{1}{2} \int_{\mathcal{U}_7} (H_1 - H_2)Y$ for two tensormultiplet fields in the same gauge and gravitational background is ill-defined if we use the standard quantization condition on the periods: $[H_1 - H_2] \in H_{dR}^3(\mathcal{M}_6; \Lambda)$. The problem here is that the factor of $1/2$ leads to a sign ambiguity. These considerations motivate our more precise mathematical discussion of the Green-Schwarz term given below.

In order to proceed it is useful to decompose the Lie algebra \mathfrak{g} of G as a sum over simple Lie algebras \mathfrak{g}_i and a basis of $\mathfrak{u}(1)$ factors:

$$\mathfrak{g} = \oplus_i \mathfrak{g}_i \oplus_I \mathfrak{u}(1)_I. \quad (2.5)$$

Then the general form for Y is

$$Y = \frac{a}{4} p_1 - \sum_i b_i c_2^i + \frac{1}{2} \sum_{I,J} b_{IJ} c_1^I c_1^J \quad (2.6)$$

where p_1 , c_2^i , and c_1^I are the Chern-Weil representatives of the indicated cohomology classes provided by the metric and vectormultiplet gauge fields:

$$p_1 := \frac{1}{8\pi^2} \text{Tr}_{\text{vec}} R^2 \quad c_2^i := \frac{1}{16\pi^2 h_i^\vee} \text{Tr}_{\text{adj}} F_i^2 \quad c_1^I := \frac{1}{2\pi} F_I \quad (2.7)$$

Here h_i^\vee is the dual Coxeter number of \mathfrak{g}_i . The vectors $a, b_i, b_{IJ} \in \Lambda_{\mathbb{R}}$ appearing in (2.6) are, by definition, the *anomaly coefficients*. (We will see below that there are, in addition, new anomaly coefficients associated with torsion classes which have not been discussed before.)

We note that just the very existence of a factorization such as (2.2) puts nontrivial constraints on the data $(G, \mathcal{R}, \Lambda)$. For example, we have the famous constraint

$$\dim_{\mathbb{H}} \mathcal{R} - \dim_{\mathbb{R}} G + 29T - 273 = 0. \quad (2.8)$$

Moreover, the existence of a factorization strongly constrains the anomaly coefficients. For example, we must have $a^2 = 9 - T$. The full set of constraints for factorization is expressed in terms of group-theoretical factors associated with G and \mathcal{R} . See, for example, equations (2.19) and (2.20) in [33] for the full list of conditions. These constraints have been extensively analyzed in the literature.

Even when a factorization (2.2) exists it is not unique. Therefore, the full data needed to define a perturbatively consistent supergravity theory is the set of data $(G, \mathcal{R}, \Lambda, a, b_i, b_{IJ})$. The global anomaly conditions found in [33] are best stated as follows: First, we must have $a \in \Lambda$. Next, the data (b_i, b_{IJ}) should be regarded as defining a quadratic form on \mathfrak{g} valued

in $\Lambda_{\mathbb{R}}$. After all, we use a symmetric invariant bilinear form to define the Chern-Weil representatives in (2.7). We denote the corresponding quadratic form by \bar{b} . It turns out that the space of such bilinear forms has an interpretation in topology. It is naturally isomorphic to the cohomology group $\mathcal{Q} := H^4(BG_1; \Lambda_{\mathbb{R}})$, where G_1 is the connected component of the identity in G . Inside the vector space \mathcal{Q} there is a lattice $H^4(BG_1; \Lambda) \subset \mathcal{Q}$. The strongest necessary condition in [33] states that

$$\bar{b} \in 2H^4(BG_1; \Lambda). \quad (2.9)$$

The derivation in [33] proceeds by requiring that the supergravity theory can be put on an arbitrary spin 6-manifold with arbitrary G -bundle. Then, cancellation of string charge in the compact Euclidean spacetime implies that for all $\Sigma \in H_4(\mathcal{M}_6; \mathbb{Z})$ we must have $\int_{\Sigma} Y \in \Lambda$ in order to cancel the background string charge with strings.² The conditions on (a, \bar{b}) mentioned thus far are necessary conditions for cancellation of some global anomalies but are not, in general, necessary and sufficient. In order to derive necessary and sufficient conditions we must view anomaly cancellation from a more geometric perspective.

3. Geometrical Anomaly Cancellation

The space of all fields in a six-dimensional supergravity is fibered over the space of nonanomalous fields

$$\mathfrak{B} = \text{Met}(\mathcal{M}_6) \times \text{Conn}(\mathcal{P}) \times \mathcal{S} \quad (3.1)$$

where $\text{Conn}(\mathcal{P})$ is the set of connections on a principal G bundle $\mathcal{P} \rightarrow \mathcal{M}_6$ and \mathcal{S} is the set of scalar fields coming from the (half-)hypermultiplets and tensor multiplets. Path integrals in the theory, such as the partition function, can thus be written as an integral first over the anomalous fields and second over the fields \mathfrak{B} :

$$Z_{\text{Sugra}} = \int_{\mathfrak{B}/\mathfrak{G}} \int_{\text{Fermi+B}} e^{-S_{\mathfrak{B}} - S_{\text{Fermi+B}}} \quad (3.2)$$

where we have separated the action as a sum of an action $S_{\mathfrak{B}}$ involving only the nonanomalous fields and a term $S_{\text{Fermi+B}}$ giving the coupling of the anomalous fields to \mathfrak{B} . Also, \mathfrak{G} is the group of diffeomorphisms and vectormultiplet gauge transformations. The problem is that the path integral over the anomalous fields involves determinants of chiral Dirac operators and is hence, *a priori*, a section of a line bundle $\mathcal{L}_{\text{Anomaly}} \rightarrow \mathfrak{B}/\mathfrak{G}$:

$$\Psi_{\text{Anomaly}}(\mathfrak{b}) := \int_{\text{Fermi+B}} e^{-S_{\text{Fermi+B}}} \quad (3.3)$$

²By contrast, arguments in [33], similar to those in [28] and other references that are based on manipulations of Casimirs in group theory, together with considerations of certain global anomalies lead to weaker conditions, namely conditions 1-8 in Section 2.3 of [33]. The condition (2.9) is rederived in [34] from consistency of the construction of the Green-Schwarz term. We note that the weaker condition $\bar{b} \in 2H^4(B\widetilde{G}_1; \Lambda)$ where \widetilde{G}_1 takes the universal cover of the semisimple part of G_1 can be stated more concretely as the condition that $b_i, \frac{1}{2}b_{IJ}, b_{IJ} \in \Lambda$. Another way of stating $\bar{b} \in 2H^4(B\widetilde{G}_1; \Lambda)$ is that \bar{b} is an even Λ -valued quadratic form on the coroot lattice of G , while (2.9) requires it to be an even Λ -valued form on the cocharacter lattice of G .

where $\mathfrak{b} \in \mathfrak{B}$. Indeed, this line bundle is a determinant (or Pfaffian) line bundle constructed from the relevant Dirac operators. Even in the most formal sense it does not make sense to integrate a section of a line bundle against a well-defined measure. Rather one needs to provide a *geometrical trivialization* of the line bundle $\mathcal{L}_{\text{Anomaly}}$. A geometrical trivialization is a (natural) choice of a flat connection with trivial holonomy. Cancellation of local anomalies is the flatness condition. Cancellation of global anomalies is the trivial holonomy condition. Given a geometrical trivialization one can identify Ψ_{Anomaly} with a well-defined function on $\mathfrak{B}/\mathfrak{G}$, which can then be integrated against the measure $[d\mathfrak{b}]e^{-S_{\mathfrak{B}}}$ to produce a well-defined path integral. (Of course, this path integral only makes sense in a rather formal sense since one then needs to deal with the problems of functional integration of a six-dimensional field theory including gravity.) This is the geometrical formulation of anomalies that was quite popular in the 1980's. See [1, 2, 4, 12, 35, 36] for accounts.

A useful modern perspective on geometrical anomaly cancellation makes use of the notion of invertible field theory [14]. This was explained nicely in [19] and we adopt that viewpoint here: Attached to our six-dimensional field theory will be a seven-dimensional invertible field theory $\mathcal{F}_{\text{Anomaly}}$ called the *anomaly field theory* of the supergravity theory. Here and henceforth we use the term “field theory” in the mathematical sense of [3, 17, 20, 29, 44] wherein a d -dimensional field theory is identified with a functor \mathcal{F} from a bordism category of manifolds, of dimension $\leq d$, equipped with some geometrical and/or topological structure, to some tensor category, often the category of vector spaces. Thus, in a d -dimensional field theory the partition function on a closed d -manifold is a complex number, the partition function on a closed $(d-1)$ -manifold is the vector space of states, and so on. In an invertible field theory the partition function is always a nonzero complex number, while the space of states is a one-dimensional vector space. One can put unitary structures on invertible field theories and the deformation classes of such theories have been classified [20].

To make contact between $\mathcal{F}_{\text{Anomaly}}$ and the older geometrical formulation of anomalies we note that the anomaly field theory of a d -dimensional theory is a $(d+1)$ -dimensional field theory defined on a bordism category of spin manifolds endowed with the non-anomalous fields \mathfrak{B} in the d -dimensional theory (together with their $(d+1)$ -dimensional counterparts). The evaluation of an invertible $(d+1)$ -dimensional field theory on a closed d -manifold \mathcal{M}_d equipped with a point $\mathfrak{b} \in \mathfrak{B}$ is a one-dimensional vector space. In particular, for an anomaly field theory, $\mathcal{F}_{\text{Anomaly}}(\mathcal{M}_d, \mathfrak{b})$ is a one-dimensional vector space, but it does not have a canonical choice of basis vector. So, varying the point \mathfrak{b} in \mathfrak{B} we get a line bundle over \mathfrak{B} . This is the anomaly line bundle $\mathcal{L}_{\text{Anomaly}}$ of the older geometrical formulation of anomalies.³ In the modern language of anomaly field theories, geometrical anomaly cancellation is then a two-step procedure:

1. Construct a “counterterm” invertible field theory \mathcal{F}_{CT} which produces a line bundle over $\mathfrak{B}/\mathfrak{G}$ complex conjugate to $\mathcal{L}_{\text{Anomaly}}$, or, equivalently, a family of one-

³A useful perspective on the anomaly field theory is that it is an example of a “field theory valued in a field theory in one higher dimension,” a notion discussed in [37]. This notion is closely related to “relative field theory,” discussed in [18].

dimensional vector spaces $\mathcal{F}_{\text{CT}}(\mathcal{M}_d, \mathfrak{b})$ with equivariance properties under the gauge group \mathfrak{G} on \mathfrak{B} conjugate to those of the family $\mathcal{F}_{\text{Anomaly}}(\mathcal{M}_d, \mathfrak{b})$.

2. Use the data of the d -dimensional fields to give a local construction of a section $\Psi_{\text{CT}} \in \mathcal{F}_{\text{Anomaly}}(\mathcal{M}_d)$ so that

$$\int_{\text{Fermi+B}} e^{-S_{\text{Fermi+B}}} \Psi_{\text{CT}} \quad (3.4)$$

descends to an honest function on $\mathfrak{B}/\mathfrak{G}$.

One might well ask: Why not simply take $\mathcal{F}_{\text{CT}} = \mathcal{F}_{\text{Anomaly}}^*$? This would certainly trivially cancel the anomalies! The point is that one must then also construct a vector in $\mathcal{F}_{\text{Anomaly}}^*(\mathcal{M}_d, \mathfrak{b})$ using just the fields of the d -dimensional field theory in a way which is local. In older language, one must construct a local counterterm. That is the real challenge.

The anomaly field theory $\mathcal{F}_{\text{Anomaly}}$ of a six-dimensional supergravity based on

$$(G, \mathcal{R}, \Lambda, a, \bar{b}) \quad (3.5)$$

is a product of 7-dimensional Dai-Freed field theories [8] associated with the chiral fermions and (anti-) self-dual fields. (This is essentially a restatement of the formulae for global anomalies derived by Witten in [47, 49].) To define the Dai-Freed theory recall that if D is a Dirac operator on odd-dimensional manifolds then the APS eta invariant:

$$\xi(D) := \frac{1}{2}(\eta(D) + \dim \ker(D)) \quad (3.6)$$

defines an invertible field theory on the category of Riemannian spin manifolds equipped with suitable gauge bundle and connection. Indeed $\mathcal{F}_{\text{Dai-Freed}}(\mathcal{M}_{2n+1}) = e^{2\pi i \xi(D)}$ is the nonzero partition function while the same expression $e^{2\pi i \xi(D)}$ on $2n$ -dimensional manifolds defines a section of a nontrivial line bundle over the space of equivalence classes of metrics and connections.⁴ In fact, that line bundle is again a determinant line bundle of a six-dimensional chiral Dirac operator. See [50] for an account in the physics literature.

It is shown in [34] how to construct a Dirac operator D from the data $(G, \mathcal{R}, \Lambda)$ so that the anomaly field theory for the six-dimensional supergravity is the Dai-Freed field theory for D .⁵ It is useful to examine the partition function of this seven-dimensional invertible field theory on closed seven-dimensional spin manifolds \mathcal{U}_7 equipped with $\mathfrak{b} \in \mathfrak{B}$. In general, η invariants are impossible to compute in simpler terms. However, if the principal G -bundle $\mathcal{P} \rightarrow \mathcal{U}_7$ extends to a principal G -bundle over an eight-dimensional spin manifold \mathcal{W}_8 , with

⁴The η invariant is closely related to the Chern-Simons invariant. It is well-known that the Chern-Simons invariant on a $(2n+1)$ -manifold with $2n$ -dimensional boundary is not gauge invariant. This failure of gauge invariance can be interpreted as the statement that the exponentiated Chern-Simons invariant is a section of a line bundle over the space of gauge equivalence classes of connections. In a similar way the exponentiated η invariant should be interpreted as a section of a line bundle.

⁵The contribution of the (anti-)self-dual fields involves some tricky factors of two. It is really a “quarter Dai-Freed theory.”

$\partial\mathcal{W}_8 = \mathcal{U}_7$, and if we choose anomaly coefficients (a, \bar{b}) so that $I_8 = \frac{1}{2}Y^2$ then a rather simple expression emerges [34]:

$$\mathcal{F}_{\text{Anomaly}}(\mathcal{U}_7, \mathbf{b}) = \exp \left\{ 2\pi i \left(\int_{\mathcal{W}_8} \frac{1}{2}Y^2 - \frac{\sigma}{8} \right) \right\} \quad (3.7)$$

where σ denotes the signature of the free quotient $H^{4,free}(\mathcal{W}_8, \partial\mathcal{W}_8; \Lambda)$ of the relative cohomology group. Since the signature is multiplicative this is the product of the signature $\sigma(\Lambda) = (1 - T)$ of Λ and the signature of $H^{4,free}(\mathcal{W}_8, \partial\mathcal{W}_8; \mathbb{Z})$. As we will see momentarily, equation (3.7) provides the key to the construction of the counterterm invertible field theory.

The simple expression (3.7) raises the question of whether there can be topological obstructions to extending $\mathcal{P} \rightarrow \mathcal{U}_7$ to a bounding spin manifold. Indeed such obstructions can exist. The bordism group $\Omega_7^{\text{spin}}(pt) = 0$, meaning that every closed spin seven-dimensional manifold is a boundary of some spin eight-dimensional manifold. However, if we wish to extend a principal G bundle to 8 dimensions then the appropriate bordism group is $\Omega_7^{\text{spin}}(BG)$, and this group can be nonzero. For example, if $G = O(n)$ then the integral $\int_{\mathcal{U}_7} w_1(\mathcal{P})^7$ is a bordism invariant. It would vanish if the principal $O(n)$ bundle \mathcal{P} extended to \mathcal{W}_8 . But it is easy to construct examples where this integral is nonzero. Nevertheless, it is shown in [34] that $\Omega_7^{\text{spin}}(BG) = 0$ for many groups of interest, including $G = U(n), SU(n), USp(2n)$ and products thereof. (It can also be shown to vanish for some special cases, such as $G = E_8$.) On the other hand, $\Omega_7^{\text{spin}}(BG)$ is nonzero and computable for all nontrivial cyclic groups.

We now make an elementary algebraic manipulation of uncompleting the square. We define $X := Y - \frac{1}{2}\lambda'$ where $\lambda' := a \otimes \lambda''$ and λ'' is, roughly speaking, $\lambda'' \sim \frac{1}{2}p_1$. Then X is a closed 4-form with periods in Λ (thanks to (2.9)) and (3.7) takes the form

$$\mathcal{F}_{\text{Anomaly}}(\mathcal{U}_7, \mathbf{b}) = \exp \left\{ 2\pi i \left(\int_{\mathcal{W}_8} \frac{1}{2}X(X + \lambda') + \frac{\lambda'^2 - \sigma}{8} \right) \right\}. \quad (3.8)$$

Here λ'^2 is the scalar given by the natural pairing obtained by multiplying, contracting with the metric on Λ , and integrating over \mathcal{W}_8 .

(The reason for the qualifier ‘‘roughly speaking’’ in the previous paragraph is that there is a technical, but quite important subtlety. It leads to much of the hard work in [34] and we only briefly describe it here. Some readers will wish to skip this paragraph. When \mathcal{W}_8 is a compact spin 8-manifold without boundary there is a canonical cohomology class $\lambda \in H^4(\mathcal{W}_8; \mathbb{Z})$ such that $p_1(\mathcal{W}_8) = 2\lambda$. The class λ descends to a characteristic vector for the unimodular lattice $H^{4,free}(\mathcal{W}_8; \mathbb{Z})$. Happily, such a characteristic vector satisfies $\int_{\mathcal{W}_8} \lambda^2 = \sigma(H^{4,free}(\mathcal{W}_8; \mathbb{Z})) \text{ mod } 8$. Finally, λ is an integral lift of the fourth Wu class of \mathcal{W}_8 . However, in our application \mathcal{W}_8 is a manifold with nontrivial boundary \mathcal{U}_7 , so $H^{4,free}(\mathcal{W}_8; \mathbb{Z})$, and more to the point, the relative cohomology group $H^{4,free}(\mathcal{W}_8, \partial\mathcal{W}_8; \mathbb{Z})$ are not unimodular and λ is no longer a characteristic vector. What we must do is choose a smooth relative cocycle λ'' which, when reduced modulo two represents the fourth Wu class of \mathcal{W}_8 and yet vanishes on the boundary. This is possible since the fourth Wu class vanishes on closed manifolds of dimension lower than 8. The relative cocycle λ'' is of the

form $d\eta + \hat{\nu}$, where $\hat{\nu}$ is an integral lift of the Wu class and $d\eta$ trivializes the Wu class on the boundary \mathcal{U}_7 . We then define $\lambda' := a \otimes \lambda''$. If \mathcal{W}_8 were closed then, since $a^2 = \sigma(\Lambda) \bmod 8$ by the factorization conditions, the “extra” eighth root of unity in the second term on the right hand side of (3.8) could be dropped. But since $\partial\mathcal{W}_8 = \mathcal{U}_7$ is nonempty the extra phase is nontrivial and cannot be dropped. It will lead to a related technicality in our construction of the counterterm invertible field theory.)

4. A Construction Of The Counterterm Invertible Field Theory From Wu Chern-Simons Theory

The importance of (3.8) is that it is also closely related to a formula for both the exponentiated action and the partition function of a 7-dimensional analog of Abelian spin-Chern-Simons theory known as Wu Chern-Simons (WCS) theory. We will use a modified version of the WCS partition function to construct the counterterm invertible field theory \mathcal{F}_{CT} .

A general theory of Wu Chern-Simons theories has been presented in [32] and the detailed computations of [34] make extensive use of the results of [32]. In the case of six-dimensional supergravity we need the seven-dimensional WCS theory of a 3-form gauge potential with 4-form fieldstrength $X \in \Omega^4(\mathcal{U}_7; \Lambda)$. The analogy with 3-dimensional spin Chern-Simons theory where the gauge group is a torus $\Lambda_{\mathbb{R}}/\Lambda$ and Λ is an integral lattice is quite illuminating. (See, for example [5] for a detailed discussion of these theories.) In the latter theories, if we normalize the fieldstrength $F \in \Omega^2(\mathcal{U}_3; \Lambda_{\mathbb{R}})$ so that its periods are in Λ then the action is

$$\exp \left\{ 2\pi i \left(\int_{\mathcal{W}_4} \frac{1}{2} F(F + \lambda') \right) \right\}. \quad (4.1)$$

Here $\lambda' = W \otimes \hat{w}_2$, while W is a characteristic vector of Λ , ⁶ \hat{w}_2 is an integral lift of the second Stiefel-Whitney class, and the fields have been extended to a bounding four-manifold \mathcal{W}_4 (again so that F has periods in Λ). Because (4.1) involves an integral over a manifold with boundary we need to take into account the boundary behavior of the integrand. In particular, it is necessary to choose a trivialization of \hat{w}_2 on the boundary. Such a trivialization amounts to a choice of spin structure on \mathcal{U}_3 , and the action (4.1) depends on the choice of spin structure.

Just as the Abelian spin Chern-Simons theory in three dimensions depends on a choice of spin structure, the construction of the 7-dimensional Wu Chern-Simons theory requires a choice of *Wu structure* ω . Wu structures are higher-form generalizations of spin structures. In our case a Wu structure is a trivialization of the fourth Wu class. (Since our manifolds are orientable and spin the fourth Wu class coincides with the fourth Stiefel-Whitney class.) The isomorphism classes of Wu structures on \mathcal{U}_7 or \mathcal{M}_6 form a torsor for $H^3(\mathcal{U}_7; \mathbb{Z}_2)$ or $H^3(\mathcal{M}_6; \mathbb{Z}_2)$, respectively. In addition, in complete analogy to the three-dimensional case, the formulation of the Wu Chern-Simons theory requires a choice of a characteristic vector

⁶In general, if Λ is an integral lattice then $v^2 \bmod 2$ is a linear function on vectors $v \in \Lambda$. A characteristic vector $W \in \Lambda^\vee$ is a vector such that $v^2 = v \cdot W \bmod 2$, thus making the obvious manifest.

$\tilde{a} \in \Lambda^\vee$ [32]. With these choices the action of the WCS theory on \mathcal{U}_7 can be written as:

$$\mathcal{F}_{\text{WCS}}^{\text{PQ}}(\mathcal{U}_7; \omega; X) = \exp \left\{ 2\pi i \int_{\mathcal{W}_8} \frac{1}{2} X(X + \tilde{\lambda}) \right\} \quad (4.2)$$

where the superscript PQ (for “pre-quantum”) indicates that we are defining an invertible field theory from the classical action of the WCS theory. On the RHS of (4.2) we have $\tilde{\lambda} = \tilde{a} \otimes \lambda''$. Again, roughly speaking, $\lambda'' \sim \frac{1}{2}p_1$. More accurately, it is described in the final paragraph of Section 3, and it depends on a choice of Wu structure. There is no topological obstruction to extending the field X , initially defined on \mathcal{U}_7 , to \mathcal{W}_8 . However, it is critical that \tilde{a} be a characteristic vector of Λ in order for the action to be independent of extension to eight dimensions.

Comparing equations (3.8) and (4.2) it is evident that we should try to use $\mathcal{F}_{\text{WCS}}^{\text{PQ}}$ to define the counterterm invertible field theory. This is not exactly the invertible field theory we need because we must take into account the “extra” eighth root of unity appearing in (3.8). Rather, what we do instead is regard X as a background, nondynamical field coupled to a dynamical flat 4-form $Z \in \Omega^4(\mathcal{U}_7; \Lambda_{\mathbb{R}})$ (or, to be more accurate, a flat differential 4-cocycle with coefficients in Λ - see below). The partition function of the Z -theory, where we integrate over all flat 4-forms, depends on X (and Wu structure) and differs from the action $\mathcal{F}_{\text{WCS}}^{\text{PQ}}$ by the Arf invariant of a certain quadratic refinement q of the link pairing of torsion classes in $H^4(\mathcal{U}_7; \Lambda)$.⁷ This partition function satisfies gluing rules so it is part of a 7-dimensional field theory which we call the *Wu Chern-Simons theory*. To pursue the analogy with Abelian spin Chern-Simons theory, X plays the role of the external Maxwell electromagnetic field and the Wu Chern-Simons theory is analogous to the effective action for the Maxwell field after integrating out the statistical Chern-Simons fields used when modeling the FQHE. Returning to our Wu Chern-Simons field theory, its value on a 7-manifold \mathcal{U}_7 endowed with a Wu structure and with a background field X is

$$\mathcal{F}_{\text{WCS}}(\mathcal{U}_7; \omega; X) = \exp \left\{ 2\pi i \int_{\mathcal{W}_8} \frac{1}{2} X(X + \tilde{\lambda}) \right\} \text{Arf}(q). \quad (4.3)$$

It turns out that the phase $\text{Arf}(q)$ is

$$\exp \left\{ 2\pi i \left(\frac{\tilde{\lambda}^2 - \sigma}{8} \right) \right\} \quad (4.4)$$

and is thus exactly of the same form as the “extra” phase in (3.8). Moreover, when we construct \mathcal{F}_{WCS} as a field theory then it is only an invertible field theory when Λ is unimodular.

We are now ready to construct the counterterm theory. We take $\tilde{a} = a$. This makes sense when Λ is unimodular and identifies $\tilde{\lambda} = \lambda'$. Next, we take the background field to be $X = Y(\mathfrak{b}) - \frac{1}{2}\lambda'$, where we write $Y(\mathfrak{b})$ rather than Y to emphasize the dependence on the nonanomalous fields $\mathfrak{b} \in \mathfrak{B}$, as in equation (2.6). Finally, since we want to cancel

⁷The quadratic refinement q is itself defined in terms of a Wu Chern-Simons action $q(z) = S(Z)$ where Z is a field with torsion topological class z .

anomalies, we take a complex conjugate. As we have stressed, the quantities in (4.2) and (4.3) depend on a choice of Wu structure ω . We do not wish to add a choice of Wu structure to the defining data of a supergravity theory! (And, more rationally, the bordism category on which $\mathcal{F}_{\text{Anomaly}}$ is defined does not include a choice of Wu structure.) Recall that λ' is ω -dependent. In [34] we show that the dependence can be chosen so that the choice of Wu structure actually cancels out when we evaluate

$$\mathcal{F}_{\text{WCS}}(\mathcal{U}_7; \omega; X = Y(\mathfrak{b}) - \frac{1}{2}\lambda') \quad (4.5)$$

(and similarly if we substitute $\mathcal{U}_7 \rightarrow \mathcal{M}_6$). Since $Y(\mathfrak{b})$ (or rather, its lift to a differential cocycle - see below) depends on the metric and vectormultiplet gauge connection the counterterm field theory

$$\mathcal{F}_{\text{CT}}(\mathcal{M}_6; \mathfrak{b}) := \left(\mathcal{F}_{\text{WCS}}(\mathcal{M}_6; X = Y(\mathfrak{b}) - \frac{1}{2}\lambda') \right)^* \quad (4.6)$$

is defined on the same geometrical category as $\mathcal{F}_{\text{Anomaly}}$. This is our counterterm invertible field theory.

We now consider the product theory:

$$\mathcal{F}_{\text{Top}} := \mathcal{F}_{\text{Anomaly}} \times \mathcal{F}_{\text{CT}}. \quad (4.7)$$

It follows from equation (3.8) and equation (4.2) that the dependence on metric and gauge connection cancel out in this invertible field theory and we are therefore left with a spin topological field theory. This theory is determined by its seven-dimensional partition function. By construction, the partition function is trivial when the principal bundle $\mathcal{P} \rightarrow \mathcal{U}_7$ extends to 8 dimensions, and therefore the partition function is a homomorphism $\Omega_7^{\text{spin}}(BG) \rightarrow U(1)$. If this homomorphism is nontrivial we can be confident that the Green-Schwarz mechanism, at least insofar as we define it here, cannot cancel the global gravitational anomalies.

While it is in general impossible to evaluate eta invariants in simpler terms, one can show, using results from [21], that for certain G -bundles over Lens spaces \mathcal{F}_{Top} will in general be nontrivial when G is a cyclic group. On the other hand, if the theory \mathcal{F}_{Top} is the trivial theory we can proceed to the next step in anomaly cancellation, namely a local construction of a vector in the statespace $\mathcal{F}_{\text{CT}}(\mathcal{M}_6; \mathfrak{b})$ where $\mathfrak{b} \in \mathfrak{B}$. Note that if $\Omega_7^{\text{spin}}(BG) = 0$ then \mathcal{F}_{Top} is necessarily trivial. Moreover, when $\Omega_6^{\text{spin}}(BG) \neq 0$ there can be different “settings” [14] of the integrand of the supergravity path integral. This leads to new θ -like angles, which do not seem to have been discussed in the literature on 6d anomaly cancellation. (See also pp. 19-20 of [48] where a similar phenomenon was foreseen in a different context.)

We remark that it is significant that our construction requires Λ to be unimodular. This condition has been previously derived in [45] by consideration of global anomalies. We can consider the present discussion as an independent derivation that Λ is self-dual. Moreover we have found that $a = \tilde{a}$ must be a characteristic vector of Λ . While it has been found that in all F -theory constructions a must be a characteristic vector, no derivation

of this condition within the context of low energy supergravity has previously been given. We can regard the present discussion as a low-energy demonstration that a must be a characteristic vector.

5. Differential Cohomology And The Green-Schwarz Term

In order to make precise sense of several of the claims made above, and, especially, to give an explicit local construction of the Green-Schwarz term in topologically nontrivial situations one must make use of the formalism of differential cohomology. Differential cohomology is a mathematical framework for describing higher form abelian gauge fields (and global symmetries). See [7, 9, 13, 15, 16, 25] for expositions.

Briefly, associated to a p -form Abelian gauge field A are three distinct gauge invariant quantities. There is, of course, the fieldstrength, (or curvature), $F = dA$. Moreover, there are “Wilson lines” (or holonomies), $\exp(2\pi i \int_{\Sigma} A)$ where Σ is a p -cycle and A is normalized so that F has integer periods. Finally, there is the topological class, or characteristic class, which is valued in E^* , where E^* is some generalized cohomology theory. The gauge equivalence class of a p -form gauge field is an element of a differential cohomology group associated to E^* . The differential cohomology group is an infinite dimensional Abelian group and the relation of this group to the above three gauge invariant quantities is summarized by some standard exact sequences.

In our case, we have modeled the B field using the generalized cohomology theory $E^*(\cdot) = H^*(\cdot; \Lambda)$. In order to write a precise form of the Green-Schwarz term Ψ_{CT} generalizing (2.4) to topologically nontrivial situations we need to lift $Y \in \Omega^4(\mathcal{M}_6; \Lambda)$ to a differential cocycle. Given a metric, spin structure, and vectormultiplet gauge connection there is a way of lifting Y to a differential cocycle \check{Y} .⁸ The lift is not quite canonical, and while all lifts \check{Y} have the same fieldstrength (by definition) they can differ by exact differential cocycles and, more importantly, they can differ in their torsion components. While these torsion components are not visible in the fieldstrength Y specified in (2.6), they do affect global anomaly cancellation. The anomaly coefficient of \check{Y} associated to the vectormultiplet gauge symmetry is really an element $b \in H^4(BG; \Lambda)$. The projection to the free quotient is the quantity $\frac{1}{2}\bar{b} \in H^4(BG_1; \Lambda)$ discussed above. Any lift of $\frac{1}{2}\bar{b}$ to $H^4(BG; \Lambda)$ yields an acceptable set of anomaly coefficients. Any lift that cancels global anomalies (if it exists) is physically acceptable. Given any reference lift, we can encode the torsion component of the anomaly coefficients by an element

$$b_T \in \text{Tors}(H^4(BG; \Lambda)). \quad (5.1)$$

b_T is a new piece of data that must be added to the defining data of a low energy six-dimensional supergravity theory. Hence we arrive at a new necessary condition for global anomaly cancellation: A supergravity theory based on $(G, \mathcal{R}, \Lambda, a, b)$ can only be nonanomalous if the topological field theory defined in (4.7) is trivial.

⁸The procedure is described in detail in Appendix A of [34] and involves making some universal choices on classifying spaces.

In section 6 of [34] we use the formalism of differential cohomology to construct an explicit Green-Schwarz counterterm Ψ_{CT} which can be seen as a section of the counterterm line bundle over \mathfrak{B} . The construction is independent of any choice of Wu structure, is completely local in the six-dimensional fields (yet makes sense in nontrivial topology), and of course it reduces to the standard Green-Schwarz counterterm in topologically trivial situations. When the topological field theory (4.7) is trivial this counterterm cancels all local and global anomalies. Thus, the triviality of (4.7) is both necessary and sufficient.

6. The Main Result, And A Corollary

In summary, the main result of [34] is the following theorem: If a supergravity theory is defined by the data $(G, \mathcal{R}, \Lambda, a, b)$ such that

1. $I_8 = \frac{1}{2}Y^2$;
2. Λ is unimodular and $a \in \Lambda$ is a characteristic vector;
3. $\bar{b} \in 2H^4(BG_1; \Lambda)$;

then a necessary and sufficient condition for the cancellation of all global anomalies is that the topological field theory (4.7) is trivial. This reduces to the condition that the partition function on 7-manifolds, which is a homomorphism $\Omega_7^{\text{Spin}}(BG) \rightarrow U(1)$, must be trivial.

An important corollary is that if the above three criteria are met and G is such that $\Omega_7^{\text{Spin}}(BG) = 0$ then all anomalies, both local and global, are cancelled by the Green-Schwarz mechanism.

We must close with a word of caution. In our work we have made what we consider to be the most reasonable choice of the generalized cohomology theory used to model the (anti-) self-dual fields of six-dimensional supergravity. We would note, however, that other generalized cohomology theories are used for anomaly cancellation in type II strings and orientifolds [10, 11, 13, 30]. It would be very interesting if the adoption of a different generalized cohomology theory for the B -fields of six-dimensional supergravity led to different conditions for a nonanomalous field content $(G, \mathcal{R}, \Lambda)$ of six-dimensional supergravity.

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References

- [1] L. Alvarez-Gaume and E. Witten, “Gravitational Anomalies,” Nucl. Phys. B **234**, 269 (1984). doi:10.1016/0550-3213(84)90066-X

- [2] L. Alvarez-Gaume and P. H. Ginsparg, “The Structure of Gauge and Gravitational Anomalies,” *Annals Phys.* **161**, 423 (1985) Erratum: [*Annals Phys.* **171**, 233 (1986)]. doi:10.1016/0003-4916(85)90087-9
- [3] M. Atiyah, “Topological quantum field theories,” *Inst. Hautes Etudes Sci. Publ. Math.* **68**, 175 (1989). doi:10.1007/BF02698547
- [4] M. F. Atiyah and I. M. Singer, “Dirac Operators Coupled to Vector Potentials,” *Proc. Nat. Acad. Sci.* **81**, 2597 (1984). doi:10.1073/pnas.81.8.2597
- [5] D. Belov and G. W. Moore, *Classification of abelian spin Chern-Simons theories*, hep-th/0505235.
- [6] T. D. Brennan, F. Carta and C. Vafa, “The String Landscape, the Swampland, and the Missing Corner,” arXiv:1711.00864 [hep-th].
- [7] J. Cheeger and J. Simons, *Differential characters and geometric invariants*, in *Geometry and Topology*, vol. 1167 of *Lecture Notes in Mathematics*, pp. 50–80. Springer Berlin / Heidelberg, 1985. 10.1007/BFb0075216.
- [8] X.-Z. Dai and D. S. Freed, *Eta invariants and determinant lines*, *J. Math. Phys.* **35** (1994) 5155–5194, [hep-th/9405012].
- [9] R. Dijkgraaf and E. Witten, *Topological gauge theories and group cohomology*, *Commun. Math. Phys.* **129** (1990) 393.
- [10] J. Distler, D. S. Freed and G. W. Moore, “Orientifold Precipice,” In *Sati, Hisham (ed.) et al.: Mathematical Foundations of Quantum Field theory and Perturbative String Theory** 159–171 [arXiv:0906.0795 [hep-th]].
- [11] J. Distler, D. S. Freed and G. W. Moore, “Spin structures and superstrings,” arXiv:1007.4581 [hep-th].
- [12] D. S. Freed, “Determinants, Torsion, and Strings,” *Commun. Math. Phys.* **107**, 483 (1986). doi:10.1007/BF01221001
- [13] D. S. Freed, *Dirac charge quantization and generalized differential cohomology*, hep-th/0011220.
- [14] D. S. Freed and G. W. Moore, *Setting the quantum integrand of M-theory*, *Commun. Math. Phys.* **263** (2006) 89–132, [hep-th/0409135].
- [15] D. S. Freed, G. W. Moore and G. Segal, “The Uncertainty of Fluxes,” *Commun. Math. Phys.* **271**, 247 (2007) doi:10.1007/s00220-006-0181-3 [hep-th/0605198].
- [16] D. S. Freed, G. W. Moore, and G. Segal, *Heisenberg groups and noncommutative fluxes*, *Annals Phys.* **322** (2007) 236–285, [hep-th/0605200].
- [17] D. S. Freed, “The cobordism hypothesis,” arXiv:1210.5100 [math.AT].
- [18] D. S. Freed and C. Teleman, “Relative quantum field theory,” *Commun. Math. Phys.* **326**, 459 (2014) doi:10.1007/s00220-013-1880-1 [arXiv:1212.1692 [hep-th]].
- [19] D. S. Freed, *Anomalies and invertible field theories*, arXiv:1404.7224.
- [20] D. S. Freed and M. J. Hopkins, *Reflection positivity and invertible topological phases*, arXiv:1604.0652.

- [21] Peter B. Gilkey, John V Leahy, JeongHyeong Park, *Spectral Geometry, Riemannian Submersions, and the Gromov-Lawson Conjecture*
- [22] M. B. Green and J. H. Schwarz, *Anomaly cancellation in supersymmetric D=10 gauge theory and superstring theory*, *Phys. Lett.* **B149** (1984) 117–122.
- [23] M. B. Green, J. H. Schwarz, and P. C. West, *Anomaly Free Chiral Theories in Six-Dimensions*, *Nucl. Phys.* **B254** (1985) 327–348.
- [24] M. B. Green, J. H. Schwarz and E. Witten, “Superstring Theory. Vol. 1: Introduction,” Cambridge, Uk: Univ. Pr. (1987) 469 P. (Cambridge Monographs On Mathematical Physics)
- [25] M. J. Hopkins and I. M. Singer, *Quadratic functions in geometry, topology, and M-theory*, *J. Diff. Geom.* **70** (2005) 329, [[math/0211216](#)].
- [26] J. A. Jenquin, *Classical Chern-Simons on manifolds with spin structure*, [math/0504524](#).
- [27] J. A. Jenquin, *Spin Chern-Simons and spin TQFTs*, [math/0605239](#).
- [28] V. Kumar, D. R. Morrison, and W. Taylor, *Global aspects of the space of 6D N = 1 supergravities*, *JHEP* **11** (2010) 118, [[arXiv:1008.1062](#)].
- [29] J. Lurie, “On the Classification of Topological Field Theories,” [arXiv:0905.0465](#) [[math.CT](#)].
- [30] R. Minasian and G. W. Moore, *K-theory and Ramond-Ramond charge*, *JHEP* **11** (1997) 002, [[hep-th/9710230](#)].
- [31] S. Monnier, *Hamiltonian anomalies from extended field theories*, *Commun. Math. Phys.* **338** (2015) 1327–1361, [[arXiv:1410.7442](#)].
- [32] S. Monnier, *Topological field theories on manifolds with Wu structures*, *Rev. Math. Phys.* **29** (2017), no. 1750015 [[arXiv:1607.0139](#)].
- [33] S. Monnier, G. W. Moore and D. S. Park, “Quantization of anomaly coefficients in 6D $\mathcal{N} = (1, 0)$ supergravity,” *JHEP* **1802**, 020 (2018) doi:10.1007/JHEP02(2018)020 [[arXiv:1711.04777](#) [[hep-th](#)]].
- [34] S. Monnier and G. W. Moore, “Remarks on the Green-Schwarz terms of six-dimensional supergravity theories,” *Commun. Math. Phys.*, [[arXiv:1808.01334](#)]
- [35] G. W. Moore and P. C. Nelson, “Anomalies in Nonlinear σ Models,” *Phys. Rev. Lett.* **53**, 1519 (1984). doi:10.1103/PhysRevLett.53.1519
- [36] G. W. Moore and P. C. Nelson, “The Etiology of σ Model Anomalies,” *Commun. Math. Phys.* **100**, 83 (1985). doi:10.1007/BF01212688
- [37] G. Moore, “A minicourse on generalized abelian gauge theory, self-dual theories and differential cohomology.” Available at <http://www.physics.rutgers.edu/~gmoore/SCGP-Minicourse.pdf>, 2011; “Lecture notes for Felix Klein lectures on N=2 gauge theories.” Available at <http://www.physics.rutgers.edu/~gmoore/FelixKleinLectureNotes.pdf>, 2012.
- [38] G. Moore, talk at <https://indico.oist.jp/indico/event/5/> . Or see talk # 83 at <http://www.physics.rutgers.edu/~gmoore/> .
- [39] J. Polchinski and Y. Cai, “Consistency of Open Superstring Theories,” *Nucl. Phys. B* **296**, 91 (1988). doi:10.1016/0550-3213(88)90382-3

- [40] J. Polchinski, “String theory. Vol. 2: Superstring theory and beyond,” doi:10.1017/CBO9780511618123
- [41] F. Riccioni, *All couplings of minimal six-dimensional supergravity*, *Nucl. Phys.* **B605** (2001) 245–265, [[hep-th/0101074](#)].
- [42] V. Sadov, *Generalized Green-Schwarz mechanism in F theory*, *Phys. Lett.* **B388** (1996) 45–50, [[hep-th/9606008](#)].
- [43] A. Sagnotti, *A Note on the Green-Schwarz mechanism in open string theories*, *Phys. Lett.* **B294** (1992) 196–203, [[hep-th/9210127](#)].
- [44] G. Segal, “The definition of conformal field theory,” in *Topology, Geometry, and Quantum Field Theory*, U. Tillmann ed., London Mathematical Society Lecture Note Series 308
- [45] N. Seiberg and W. Taylor, *Charge Lattices and Consistency of 6D Supergravity*, *JHEP* **06** (2011) 001, [[arXiv:1103.0019](#)].
- [46] W. Taylor, *TASI Lectures on Supergravity and String Vacua in Various Dimensions*, [arXiv:1104.2051](#).
- [47] E. Witten, *Global gravitational anomalies*, *Commun. Math. Phys.* **100** (1985) 197–229.
- [48] E. Witten, “Five-brane effective action in M theory,” *J. Geom. Phys.* **22**, 103 (1997) doi:10.1016/S0393-0440(97)80160-X [[hep-th/9610234](#)].
- [49] E. Witten, “World sheet corrections via D instantons,” *JHEP* **0002**, 030 (2000) doi:10.1088/1126-6708/2000/02/030 [[hep-th/9907041](#)].
- [50] E. Witten, “Fermion Path Integrals And Topological Phases,” *Rev. Mod. Phys.* **88**, no. 3, 035001 (2016) doi:10.1103/RevModPhys.88.035001, 10.1103/RevModPhys.88.35001 [[arXiv:1508.04715 \[cond-mat.mes-hall\]](#)].