Complex symmetric Hamiltonians and exceptional points of order four and five

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Abstract

A systematic elementary linear-algebraic construction of non-Hermitian Hamiltonians $H = H(\gamma)$ possessing exceptional points $\gamma = \gamma^{(EP)}$ of higher orders is proposed. The implementation of the method leading to the EPs of orders K = 4 and K = 5 is described in detail. Two distinct areas of applicability of our user-friendly benchmark models are conjectured (1) in *quantum* mechanics of non-Hermitian systems, or (2) in their experimental simulations via *classical* systems (e.g., coupled waveguides).

Keywords

non-Hermitian quantum dynamics; multilevel degeneracies; exceptional points of higher orders; non-quantum simulations; coupled waveguides;

1 Introduction

The recent increase of popularity of Schrödinger equations

$$i \partial_t |\psi\rangle = H |\psi\rangle \tag{1}$$

with non-Hermitian (and, say, parameter-dependent) Hamiltonians

$$H = H(\gamma) \neq H^{\dagger}(\gamma) \tag{2}$$

opened (or re-opened) a number of mathematical questions [1]. It also evoked many challenges in theoretical quantum physics [2, 3] as well as in experimental classical physics and optics [4].

Inside the community of quantum physicists many new, promising phenomenological models have been conjectured and studied, ranging from their ordinary differential versions (to which the attention has been attracted, in 1998, by the pioneering letter by Bender and Boettcher [5]) up to their truly sophisticated quantum-field descendants (cf., e.g., an exhaustive review [6] of this most ambitious project).

The Bender's and Boettcher's toy-model Hamiltonians $H_{(BB)}(\gamma) = -d^2/dx^2 + V_{(BB)}(x,\gamma)$ were chosen non-selfadjoint but, for technical reasons, \mathcal{PT} -symmetric, $H_{(BB)}\mathcal{PT} = \mathcal{PT}H_{(BB)}$, with parity \mathcal{P} and antilinear time-reversal \mathcal{T} . Several unexpected features of these unconventional quantum Hamiltonians (e.g., the often-occurring reality of spectra) inspired mathematicians who enhanced their interest in a systematic study of similar models (cf., *pars pro toto*, a truly nice monograph [7]).

The idea proved inspiring also beyond quantum physics. In classical optics, for example, the Bender's and Boettcher's \mathcal{PT} -symmetric Schrödinger equation was found equivalent to the classical Maxwell equation in paraxial approximation [8]. Complex function V(x) acquired a new physical meaning of the complex refraction index admitting both the gain and loss of the intensity of the beam. The quick growth of popularity of the Maxwell-equation-related models followed. One of the reasons behind their successful tests in the laboratory may be seen in the current progress in nanotechnologies. This helped people to simulate various *ad hoc* features and forms of the non-Hermiticity (e.g., \mathcal{PT} -symmetry) *experimentally*, say, in the context of physics of photonic molecules [9] or for the devices composed of coupled waveguides [10].

In our present paper we felt inspired by the mutual enrichment between the quantum and nonquantum considerations, especially when related to the concept of exceptional point (EP, cf. p. 64 in [11]). In section 2 we will recall the particular role played by the EPs in quantum physics. As an illustration we will recall the \mathcal{PT} -symmetric version of the Bose-Hubbard manybody model in its non-Hermitian version which has been proposed and studied, in 2008, by Graefe et al [12] (cf. subsection 2.1). In subsequent subsection 2.2 an explanation will be added of the less widely known possibility of the full theoretical compatibility between the non-Hermiticity of a quantum Hamiltonian in an auxiliary, "false but friendly" [13] Hilbert space (endowed with an unphysical but more easily calculated inner product) and the unitarity of the evolution it generates in another, amended Hilbert space using a non-trivial Hilbert-space metric $\Theta \neq I$ in the definition of the necessary correct and physical inner product.

In the alternative, non-quantum applications the theoretical as well as phenomenological role of the EPs is different. In section 3 we shall explain why the change of the perspective enhances the appeal of non-Hermiticity in experimental physics as well as in the mathematics of elementary algebraic construction methods. In section 4, in particular, our main message will be then based on the turn of attention from the Bose-Hubbard model to a more general family. Complex and symmetric tridiagonal-matrix non-Hermitian Hamiltonians will be considered. Using the straightforward linear-algebraic methods, a systematic search for the EP singularities will be performed. Their exhaustive classification will be shown fully non-numerical at N = 4 and N = 5.

An extensive review and analysis of the various physical aspects of these mathematical results will finally be given in section 5, with all of this material summarized in section 6.

2 Quantum systems with exceptional points

Drawing attention to the occurrence and unfolding of exceptional points (EPs) we intend to consider the class of finite-dimensional matrix Hamiltonians (2) with dimension $N < \infty$ and with the special complex-symmetric structure,

$$H_{mn}(\gamma) = H_{nm}^*(\gamma) \,. \tag{3}$$

In the context of pure mathematics such a choice could have been found promoted by review paper [14] and talk [15]. From *loc. cit.* one deduces, i.a., an intimate relationship between the complex symmetry of matrices and the \mathcal{PT} -symmetry of operators.

An equally strong encouragement of our study was provided by physicists, especially via papers [12] - [16] in which the class of complex symmetric Hamiltonians has further been narrowed to their tridiagonal-matrix subfamily such that

$$H_{mn}(\gamma) = 0$$
 whenever $|m - n| \ge 2$. (4)

What attracted our attention to the finite-dimensional Hilbert spaces were also certain older quantum models and results of the physics of atomic nuclei (a recommended compact review may be found in Ref. [17]). Last but not least, we should mention that during our study we found a deeper physical inspiration in Refs. [18, 19] and in several other recent studies of optical waveguides with gain and loss.

2.1 \mathcal{PT} -symmetric Bose-Hubbard model of Graefe et al [12]

In phenomenological context the proposal and study of the \mathcal{PT} -symmetric Bose-Hubbard model of Ref. [12] was motivated by the search for an elementary simulation of the process of the Bose-Einstein condensation. Up to the purely numerically tractable interaction term the non-Hermitian Hamiltonian of the model was chosen in the form

$$H_{(BH)}(\gamma) = 2\left(-i\gamma L_z + L_x\right) \tag{5}$$

of the complex linear combination of the two angular-momentum generators $L_{z,x}$ of the real Lie algebra su(2). The underlying representation theory enables one to treat operator (5) as decomposed into an infinite family of finite-dimensional N by N matrices

$$H_{(BH)}^{(2)}(\gamma) = \begin{bmatrix} -i\gamma & 1\\ 1 & i\gamma \end{bmatrix},$$
(6)

$$H_{(BH)}^{(3)}(\gamma) = \begin{bmatrix} -2i\gamma & \sqrt{2} & 0\\ \sqrt{2} & 0 & \sqrt{2}\\ 0 & \sqrt{2} & 2i\gamma \end{bmatrix},$$
(7)

etc. An inessential change of parameter $\gamma \to \sqrt{z}$ can be recommended in the subsequent items, i.e., in

$$H_{(BH)}^{(4)}(z) = \begin{bmatrix} -3i\sqrt{z} & \sqrt{B} & 0 & 0\\ \sqrt{B} & -i\sqrt{z} & \sqrt{A} & 0\\ 0 & \sqrt{A} & i\sqrt{z} & \sqrt{B}\\ 0 & 0 & \sqrt{B} & 3i\sqrt{z} \end{bmatrix}$$
(8)

with B = 3 and A = 4 as well as in

$$H_{(BH)}^{(5)}(z) = \begin{bmatrix} -4i\sqrt{z} & \sqrt{B} & 0 & 0 & 0\\ \sqrt{B} & -2i\sqrt{z} & \sqrt{A} & 0 & 0\\ 0 & \sqrt{A} & 0 & \sqrt{A} & 0\\ 0 & 0 & \sqrt{A} & 2i\sqrt{z} & \sqrt{B}\\ 0 & 0 & 0 & \sqrt{B} & 4i\sqrt{z} \end{bmatrix}$$
(9)

with B = 4 and A = 6, etc. Such a change of notation is motivated by the simplification of the respective secular equations (see below). Still, the possibility of the alternative choice of sign of $\gamma \rightarrow -\sqrt{z}$ should be kept in mind as a helpful symmetry of the Bose-Hubbard model (see also

picture Nr. 1 in [12]). Due to this symmetry and due to the reality of the spectra, all of the matrices (6) - (9) (etc) may be declared eligible, together with their $\gamma \rightarrow -\gamma$ counterparts, as non-numerical toy-model generators of quantum evolution.

2.2 The procedure of Hermitization

In the conventional studies of quantum evolution one usually requires that it is unitary. For our sample generators $H_{(BH)}^{(N)}(\gamma)$ (i.e., for the diagonalizable, hiddenly Hermitian Bose-Hubbard Hamiltonians) this means that the spectrum must be real. The authors of Ref. [12] emphasized that the latter condition is satisfied if and only if $-1 < \gamma < 1$. For the conventional non-negative $\gamma \geq 0$ and real $z \geq 0$ they concluded that every element $H_{(BH)}^{(N)}(z)$ of the sequence is exactly solvable and, moreover, that each toy-model Hamiltonian $H_{(BH)}^{(N)}(z)$ in the family possesses a unique and real *exceptional point* $z^{(EP)} = 1$ of order N.

For the admissible parameters $\gamma \in (-1, 1)$ one can find a (non-unique) Hermitian and positive matrix of Hilbert-space metric Θ such that

$$H^{\dagger}\Theta = \Theta H \tag{10}$$

i.e., such that $H = H_{(BH)}^{(N)}(\gamma)$ may be declared Hermitian with respect to an amended inner product $\langle \cdot | \Theta \cdot \rangle$. The details of the theory using nontrivial metrics $\Theta \neq I$ may be found, say, in reviews [2, 17]. For our present purposes it is only necessary to emphasize that the first principles of the theory remain unchanged. Thus, any observable phenomenon must still be represented by a hiddenly Hermitian operator Λ such that $\Lambda^{\dagger} \Theta = \Theta \Lambda$. In other words, only the initial knowledge of the Hamiltonian-dependent metric $\Theta = \Theta(H)$ makes the model theoretically complete.

Needless to add, the necessary guarantee of validity of Eq. (10) is far from trivial in practice. The difficulty of the construction of $\Theta = \Theta(H)$ is, incidentally, one of the reasons why the matrix phenomenological models with small dimensions $N \ll \infty$ are so important, both in the theory and in its applications.

2.2.1 The construction of metric at N = 2

At N = 2 we may insert quantum Hamiltonian $H_{(BH)}^{(2)}(\gamma)$ of Eq. (6) in Eq. (10). The routine solution of this linear algebraic problem yields *all* of the eligible (i.e., positive definite) metrics. Up to an overall inessential multiplicative factor they are all defined by the following one-parametric formula

$$\Theta^{(2)}(\beta) = I^{(2)} + \begin{bmatrix} 0 & \beta + i\gamma \\ \beta - i\gamma & 0 \end{bmatrix}, \quad -\sqrt{1 - \gamma^2} < \beta < \sqrt{1 - \gamma^2}.$$
(11)

Obviously, in the arbitrarily small vicinities of $\gamma = \gamma^{(EP)}$ such a metric only exists, in the spirit of Ref. [20], after the minimally anisotropic metric constant choice of $\beta = 0$.

2.2.2 The construction of metric at N = 3

The fully analogous treatment of the next quantum system with Hamiltonian (7) (in which we abbreviate $g = \sqrt{2\gamma}$) will lead to the two-parametric family of metrics

$$\Theta^{(3)}(\beta,\delta) = I^{(3)} + \begin{bmatrix} 0 & \beta + ig & \delta + ig \beta \\ \beta - ig & \delta + g^2 & \beta + ig \\ \delta - ig \beta & \beta - ig & 0 \end{bmatrix}.$$
 (12)

In the spirit of Ref. [20] one may again prefer a "minimally anisotropic" choice of $\beta = 0$ and $\delta = 0$ which yields the metric with eigenvalues

$$\theta_0 = 1, \quad \theta_{\pm} = 1 + \frac{1}{2}g^2 \pm \frac{1}{2}\sqrt{8g^2 + g^4}.$$
 (13)

Obviously, the resulting special metric $\Theta^{(3)}(0,0)$ still exists up to the maximal admissible (i.e., real-energy-admitting) non-Hermiticity limit of $g \to g^{(EP)} = 1$.

An entirely analogous procedure will also work at any higher matrix dimension N.

3 Exceptional points in *non-quantum* optics

Schrödinger-like equations and their solutions appear in many non-quantum branches of physics. Thus, in the physical context of classical optics the quantum effects may be simulated via microwave devices [21] or coupled waveguides [22]. Once these devices prove characterized by a symmetry between the gain and loss in the medium, the mathematics of solutions becomes shared with the quantum theories exhibiting the combined parity times time reversal symmetry alias \mathcal{PT} -symmetry [6]. Thus, after one leaves the unitary quantum mechanics and after one turns attention to classical optics, the related Schrödinger-to-Maxwell change of the meaning of the Schrödinger-like evolution Eq. (1) gets accompanied by the slightly easier, formally less restrictive mathematics. For example, the "wave function" solutions $|\psi\rangle$ need not be normalizable anymore.

In the new context it is crucial that the evolution generated by the Hamiltonian-resembling operators $H = H^{(N)}(\gamma)$ need not be required unitary. The spectra of "energies" may be complex while the components of the "wave functions" themselves may become directly measurable [4]. Last but not least, the simplification of mathematics may be accompanied by the feasibility of experimental setups in which the generators are allowed time-dependent, H = H(t) [23].

3.1 Two coupled waveguides

The theoretical studies of the non-stationary systems as well as the realizations of the experiments remain highly nontrivial. Fortunately, this direction of research also leads to multiple surprising results reconnecting the quantum and classical physics. For illustration one may recall the really surprising recent discovery of the failure of applicability of the intuitive adiabatic hypothesis in the non-Hermitian setting [18] which was first reported during the 15th International Workshop on Pseudo-Hermitian Hamiltonians in Quantum Physics in Palermo [24] in 2015.

For explanation, just a version of the two-by-two toy model of Eq. (6) proved sufficient. In the recent compact review [19] the authors described an interesting application of the new phenomenon to the description of scattering of the classical (say, electromagnetic) waves through a two-mode waveguide device. A comparatively satisfactory theoretical explanation of the phenomenon of the breakdown of adiabatic approximation has been achieved via the numerical solution of the manifestly time-dependent N = 2 evolution rule (1).

On this background a number of "elusive effects" has been predicted, resulting from the fact that the Hamiltonian matrix itself is symmetric but not real. Indeed, the requirement of its reality would make it Hermitian so that all of the "elusive effects" would disappear. Theoretically the device was described by a slightly modified version

$$H(\delta, g, \gamma_1, \gamma_2) = \begin{bmatrix} \delta - i\gamma_1/2 & g \\ g & -i\gamma_2/2 \end{bmatrix}$$
(14)

of the most elementary model (6). All of the four parameters (viz., δ responsible for the so called detuning, g denoting the mutual symmetric coupling of the modes, and γ_1 and γ_2 gauging the losses in the medium of the two respective waveguides) were chosen real.

The model yields a pair of instantaneous eigenenergies $E_{\pm}(z) = E_{\pm}[\delta(z), g(z), \gamma_1(z), \gamma_2(z)]$ which are distinct and complex in general. The secular equation is trivial yielding the two values $E_{\pm}(z)$ in closed form. These values may be perceived as evaluations of a two-sheeted analytic function $\mathbb{F}(z)$. According to Kato [11] such an explicit representation of the spectrum also enables us to localize the so called "exceptional points" $z^{(EP)}$ at which the two levels of the energy happen to merge, $E_{\pm}(z^{(EP)}) = E_{-}(z^{(EP)})$.

In [19], the authors performed the search. In their two-by-two matrix model (14) the analyticfunction representation of the energies appeared to have the square-root form near $z^{(EP)}$, with $\mathbb{F}(z) \sim \sqrt{z - z^{(EP)}}$. In an experimental setup the "no-detuning" choice of $\delta = 0$ and the z-independent choice of the losses $\gamma_{1,2}$ enabled the authors to determine the degeneracy-responsible EP couplings in closed form,

$$g = g[z^{(EP)}] = \frac{1}{4} (\gamma_1 - \gamma_2) .$$
(15)

The existence of the non-diagonalizable EP limit of the Hamiltonian

$$\lim_{z \to z^{(EP)}} H[\delta(z), g(z), \gamma_1(z), \gamma_2(z)] = \frac{1}{4} \begin{bmatrix} -2i\gamma_1 & \gamma_1 - \gamma_2 \\ \gamma_1 - \gamma_2 & -2i\gamma_2 \end{bmatrix}$$
(16)

has been also found detectable, experimentally, due to the mathematical property of the eigenvalues which form, in the EP vicinity, the so called cycle of period two (cf. p. 65 in [11]). In the context of physics the latter mathematical peculiarity of the model enabled the authors to reveal the limits of the applicability of the conventional adiabatic hypothesis in the non-Hermitian and non-stationary dynamical setting [19].

Multiple analogous searches for the signatures of the two-sheeted nature of the energy Riemann surfaces were performed, recently, by many independent experimental groups [25]. In the language of mathematics the existence of the EP2 singularity means that one might perform such a variation of the parameters that the system would be forced to circumscribe its EP2 branch point and to move to the second sheet of the Riemann surface. In order to force the wave function to return to its initial value, one has to circumscribe the EP2 singularity twice.

In an alternative experimental setup one could try to force the system to pass strictly through the EP2 singularity. This would lead to the coalescence $|\psi_0(z^{(EP)})\rangle = |\psi_1(z^{(EP)})\rangle$ of wave functions, tractable as a simulation of a quantum phase transition [26]. Beyond quantum world, the effect is equally interesting. In the context of waveguides, for example, one could even force the classical photons to stop at EP2 [27].

An analogous theoretical as well as experimental analysis would become much less easily accessible in the general dynamical scenarios characterized by the Kato's EPs of order K > 2. Any experimental study of the dynamical K > 2 mode switching would require a rather sophisticated equipments working, say, with complicated but still tractable waveguide systems as sampled, say, by Fig. Nr. 6 in Ref. [28].

3.2 Three coupled waveguides

In paper [29] the authors contemplated an experimentally feasible arrangement of a coupled *triplet* of semiconductor waveguides. Their system (exhibiting, in addition, the so called \mathcal{PT} -symmetric distribution of the gain and loss in the medium of the waveguides) found a formal theoretical description in another elementary schematic Hamiltonian, viz., matrix (7). This is a complex and symmetric matrix, not too dissimilar to its two-dimensional predecessor (6). In Ref. [29] we may read that "In principle the approach of extending the system with additional waveguides ... can be continued [but] ... all further extensions should first be studied in [simplified] approaches."

These sentences may be re-read as the most concise formulation of the aims of our present paper. One still has to expect that any extension of the EP2-related results to the case of the K-sheeted Riemann-surface scenarios with K > 2 will be complicated. Even in the case of K = 3the proper design of the experimental setup required a careful fine-tuning of parameters [16, 30]. On positive side, a strong encouragement comes from the observation that a decisive theoretical progress has been achieved after the scope of the experiment-oriented searches had been restricted to the tridiagonal complex symmetric models as sampled by Eq. (7) above. In this sense, a decisive theoretical step forward has been made by the authors of Ref. [16]. Subsequently, the practical experimental appeal of the tridiagonal matrix model has been emphasized in [10] and [29].

All of the latter K = 3 projects were aimed at a slow motion along a path over the threesheeted Riemann surface while circumscribing, three times, the carefully localized exceptional point of order three (EP3). The details may be found in Figure Nr. 3 of Ref. [29]. What the latter studies revealed was, first of all, the phenomenon of the characteristic permutation of the components of $|\psi\rangle$. This offered a signature of the coalescence of all of the three eigenfunctions at $z = z^{(EP3)}$. It also appeared to make sense to change the parameters in

$$H = H^{(3)}(z) = \begin{bmatrix} -i\sqrt{z} & \sqrt{A(z)} & 0\\ \sqrt{A(z)} & 0 & \sqrt{A(z)}\\ 0 & \sqrt{A(z)} & i\sqrt{z} \end{bmatrix}$$

This simplified the secular polynomial as well as its three energy roots $E_{\pm}(z) = \pm \sqrt{2 A(z) - z}$ and $E_0 = 0 \neq E_0(z)$. One could spot the EP3 singularity at $z = z^{(EP3)} = 1$ and A(1) = 1/2,

$$\lim_{z \to z^{(EP3)}} H^{(3)}(z) = H^{(3)}_{(EP3)} = \begin{bmatrix} -i & 1/\sqrt{2} & 0\\ 1/\sqrt{2} & 0 & 1/\sqrt{2}\\ 0 & 1/\sqrt{2} & i \end{bmatrix}.$$
 (17)

An extensive analysis of consequences may be found in Ref. [16]. It made us to conclude that the models using K > 3 deserve to be studied along similar lines.

4 Tridiagonal complex symmetric Hamiltonians revisited

Theoretical papers [12, 31] paid attention to the systems which are assigned a multi-sheeted Riemann surface \mathbb{F} admitting the existence and, perhaps, experimental detection of EPs of the K-th order. At any integer K one can work, locally, with the K-sheeted analytic function $\mathbb{F}(z) \sim (z - z^{(EP)})^{1/K}$ representing the energy eigenvalues. In what follows we will try to support this point of view constructively.

4.1 Four by four Hamiltonian matrices

Let us pick up K = 4 and contemplate the eligible Hamiltonian matrices in the tridiagonal and complex symmetric form of Eq. (8) above. The key merit of this choice is that it yields energies $E = \pm \sqrt{x}$ determined in terms of roots of the exactly solvable secular equation

$$x^{2} + 10 xz - 2 Bx - Ax + 9 z^{2} + 6 Bz - 9 zA + B^{2} = 0.$$
 (18)

The availability of closed formula

$$x_{\pm} = B - 5z + \frac{1}{2}A \pm \frac{1}{2}\sqrt{-64Bz + 4BA + 64z^2 + 16zA + A^2}$$
(19)

reduces the search for the EP4 confluence of the roots $x_{\pm}(z^{(EP4)}) = 0$ to the analysis of Eq. (19), yielding the following two algebraic equations for three unknowns,

$$B - 5z + 1/2A = 0, \quad -64Bz + 4BA + 64z^{2} + 16zA + A^{2} = 0.$$
⁽²⁰⁾

They have the two independent EP4 solutions, viz., the well known Bose-Hubbard solution of Ref. [12],

$$B^{(EP4)} = 3 z , \qquad A^{(EP4)} = 4 z \tag{21}$$

and the new solution

$$B^{(EP4)} = -27 z , \qquad A^{(EP4)} = 64 z .$$
(22)

As long as both of these solutions are non-numerical, they may easily be analyzed in detail.

4.1.1 Bose-Hubbard model revisited

Once we pick up the first solution and set, tentatively, B = 3 and A = 4 we arrive at the one-parametric family of Hamiltonians

$$H = H^{(4)}(z) = \begin{bmatrix} -3i\sqrt{z} & \sqrt{3} & 0 & 0\\ \sqrt{3} & -i\sqrt{z} & 2 & 0\\ 0 & 2 & i\sqrt{z} & \sqrt{3}\\ 0 & 0 & \sqrt{3} & 3i\sqrt{z} \end{bmatrix}$$
(23)

with energies

$$E_{\pm,\pm}(z) = \pm (2\pm 1)\sqrt{1-z}.$$

All four of them remain real for $z \in (0, 1)$, and all of them vanish at $z = z^{(EP4)} = 1$. Beyond this point, all of the levels become purely imaginary so that the model is to be assigned the physical interpretation of an open quantum system or of a device in classical optics.

Inside the interval of $z \in (0, 1)$, on the contrary, the guaranteed reality of the spectrum leads to the possibility of the construction of the metric which would render possible the consistent unitary-evolution interpretation of the system in quantum setting. The authors of study [12] left, unfortunately, this construction of $\Theta(H)$ as well as the discussion of its properties to the readers as an elementary exercise. Here, we shall return to this point in paragraph 5.3.2 below.

4.1.2 Jordan blocks

By definition, the Bose-Hubbard Hamiltonian (23) ceases to be diagonalizable at the EP4 singularity. It can only be assigned there the canonical four-dimensional non-diagonal Jordan-block representation. In general, at any EP energy degeneracy of order K we may postulate

$$H^{(K)}(z^{(EPK)})Q^{(K)} = Q^{(K)}J^{(K)}(E).$$
(24)

The symbol $J^{(K)}(E)$ stands here for the K by K Jordan block

$$J^{(K)}(E) = \begin{bmatrix} E & 1 & 0 & \dots & 0 \\ 0 & E & 1 & \ddots & \vdots \\ 0 & 0 & E & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & 0 & E \end{bmatrix}.$$
 (25)

The other symbol $Q = Q^{(K)}$ is Hamiltonian-dependent. It denotes the object called transition matrix. For example, in our present Bose-Hubbard illustrative example with $E = E^{(EP4)} = 0$ and $z^{(EP4)} = 1$ it is entirely routine to evaluate

$$Q^{(4)} = \begin{bmatrix} 6 i z^{3/2} & -6 z & -3 i \sqrt{z} & 1 \\ -6 z^{3/2} \sqrt{3} & -4 i z \sqrt{3} & \sqrt{3} \sqrt{z} & 0 \\ -3 i z^{3/2} \sqrt{3} \sqrt{4} & \sqrt{3} z \sqrt{4} & 0 & 0 \\ 3 z^{3/2} \sqrt{4} & 0 & 0 & 0 \end{bmatrix}$$

Such a transition matrix plays a key role in the perturbation-expansion analysis of Schrödinger equations (1) in the vicinity of EPs of any order $K \ge 2$. The physical consequences as well as the mathematical details may be found explained in Ref. [12] and, beyond the Bose-Hubbard illustrative example, in Ref. [31].

4.1.3 Generalized Bose-Hubbard model

Our first new, non-Bose-Hubbard result is that we may insert B = -27 and A = 64 in Eq.(8). This yields the EP4-supporting complex symmetric toy model

$$H^{(4)}(z) = \begin{bmatrix} -3i\sqrt{z} & 3i\sqrt{3} & 0 & 0\\ 3i\sqrt{3} & -i\sqrt{z} & 8 & 0\\ 0 & 8 & i\sqrt{z} & 3i\sqrt{3}\\ 0 & 0 & 3i\sqrt{3} & 3i\sqrt{z} \end{bmatrix}$$
(26)

with energies

$$E_{\pm,\pm}(z) = \pm \sqrt{5 - 5 \, z \pm 4 \, \sqrt{-44 + 43 \, z + z^2}} \,. \tag{27}$$

The analysis is again straightforward. Graphically, the z-dependence of the real and imaginary parts of the eigenvalues of matrix (26) is displayed in respective Figs. 1 and 2. The picture shows that all the four energies vanish at z = 1 (EP4). The inner square root expression remains purely imaginary at smaller positive $z \in (0, 1)$ so that all of the four related energies are complex. Otherwise, the inner square root expression is real so that one only has to distinguish between the interval of $z \in (1, 81)$ (in which we have two real and two purely imaginary energies) and the interval of $z \in (81, \infty)$.



Figure 1: Real parts of eigenvalues $E_n(z)$ of the non-Hermitian N = 4 matrix $H^{(N)}(z)$ of Eq. (26).

In the latter interval all of the four energy roots are purely imaginary so that an *ad hoc* premultiplication of matrix (26) by imaginary unit would make the spectrum real, rendering the resulting complex symmetric matrix $\tilde{H}^{(4)}(z) = iH^{(4)}(z)$ eligible as a correct and physical hiddenly Hermitian generator of unitary evolution in a new, slightly exotic quantum model with just two imaginary matrix elements.

One can summarize that at positive $z \ge 0$ the model exhibits the EP4 degeneracy at z = 1and the EP2 degeneracy at z = 81. It is worth adding that at z = 1 the model satisfies Eq. (24)



Figure 2: Imaginary parts of eigenvalues $E_n(z)$ of the non-Hermitian N = 4 matrix $H^{(N)}(z)$ of Eq. (26).

with the following transition matrix

$$Q^{(4)} = \begin{bmatrix} 216 i z^{3/2} & -36 z & -3 i \sqrt{z} & 1 \\ 72 z \sqrt{-3 z} & -12 i \sqrt{z} \sqrt{-3 z} & 3 \sqrt{-3 z} & 0 \\ -9 i z \sqrt{-3 z} \sqrt{64} & 3 \sqrt{-3 z} \sqrt{64} \sqrt{z} & 0 & 0 \\ -27 z^{3/2} \sqrt{64} & 0 & 0 & 0 \end{bmatrix}$$

In the case of the hiddenly Hermitian Hamiltonian $\tilde{H}^{(4)}(z)$ only two of the energies vanish in the limit $z \to z^{(EP2)} = 81$ so that the perturbation study of its vicinity would only be based on the use of the transition matrix of rank two.

4.2 Five by five Hamiltonian matrices

With N = 5 and with the Bose-Hubbard choice of B = 4, A = 6 and $z \ge 0$ in (9) we have to deal with the tridiagonal complex symmetric Hamiltonian possessing, up to a constant level $E_0 = 0$, the following four z-dependent energy eigenvalues

$$E_{\pm,\pm}(z) = \pm (3\pm 1)\sqrt{(1-z)}.$$

They stay real for $z \in (0, 1)$, and all of them vanish at z = 1. The EP5 presence is readily verified. The z-dependent quadruplet becomes purely imaginary at any z > 1 [12]. At $z = z^{(EP5)} = 1$ the Hamiltonian ceases to be diagonalizable. With transition matrix

$$Q^{(5)} = \begin{bmatrix} 24 & 24i & -12 & -4i & 1 \\ 48i & -36 & -12i & 2 & 0 \\ -24\sqrt{6} & -12i\sqrt{6} & 2\sqrt{6} & 0 & 0 \\ -48i & 12 & 0 & 0 & 0 \\ 24 & 0 & 0 & 0 & 0 \end{bmatrix}$$

it may still be shown to satisfy Eq. (24).

4.2.1 Bose-Hubbard model revisited

Even if we do not specify parameters A and B, the five-by-five complex symmetric tridiagonal matrix Hamiltonian (9) preserves the symmetry of the energies $E = \pm \sqrt{x}$. Keeping in mind the z-independent root $x_0 = 0$, i.e., $E_0 = 0$ we have still an utterly elementary secular equation

$$x^{2} + 20 xz - 2 xB - 2 Ax + 2 AB - 32 zA + 64 z^{2} + 16 zB + B^{2} = 0.$$
 (28)

It is straightforward to deduce that

$$x_{\pm} = B - 10 \, z + A \pm \sqrt{-36 \, zB + 36 \, z^2 + 12 \, zA + A^2} \,.$$

The search for the EP5 confluence proceeds again via the pair of relations

$$-36 zB + 36 z^{2} + 12 zA + A^{2} = 0, \qquad B - 10 z + A = 0$$
⁽²⁹⁾

with solutions

$$B^{(EP5)} = 4z, \quad A^{(EP5)} = 6z \tag{30}$$

and

$$A^{(EP5)} = -54 z, \quad B^{(EP5)} = 64 z.$$
(31)

This means that the conclusions may be expected to remain qualitatively the same as at N = 4 above.

4.2.2 Generalized Bose-Hubbard model

The non-Bose-Hubbard choice of B = 64 and A = -54 yields the "anomalous" Hamiltonian

$$H^{(5)}(z) = \begin{bmatrix} -4i\sqrt{z} & 8 & 0 & 0 & 0 \\ 8 & -2i\sqrt{z} & 3i\sqrt{6} & 0 & 0 \\ 0 & 3i\sqrt{6} & 0 & 3i\sqrt{6} & 0 \\ 0 & 0 & 3i\sqrt{6} & 2i\sqrt{z} & 8 \\ 0 & 0 & 0 & 8 & 4i\sqrt{z} \end{bmatrix}.$$

Up to the constant "observer energy" E = 0, the quadruplet of its nontrivial energies reads

$$E_{\pm,\pm}(z) = \pm \sqrt{-10 \, z + 10 \pm 6 \, \sqrt{z^2 - 82 \, z + 81}}$$

They vanish at z = 1 (EP5). One can detect the occurrence of two coupled EP2 singularities at z = 81, with the two non-vanishing, purely imaginary energies $E_{\pm}^{(EP2)} = \pm \sqrt{-800}$ in Eq. (24).

The classification of the reality/imaginarity of the energies proceeds as before. One finds four complex energy roots in the interval of $z \in (1, 81)$, the quadruplet of the purely imaginary roots at the larger $z \in (81, \infty)$ and, finally, two real and two imaginary energies at the smallest eligible $z \in (0, 1)$. The evaluation of the EP5 transition matrix

$$Q^{(5)} = \begin{bmatrix} -3456 & -576i & 48 & -4i & 1 \\ -1728i & -144 & -48i & 8 & 0 \\ -1728i\sqrt{6} & 144\sqrt{6} & 24i\sqrt{6} & 0 & 0 \\ 1728i & -432 & 0 & 0 & 0 \\ -3456 & 0 & 0 & 0 & 0 \end{bmatrix}$$

makes the whole construction completed.

5 Discussion

Once a Hamiltonian admits complex eigenvalues, we lose the possibility of its Hermitization, i.e., of the unitarity of quantum evolution based on the introduction of a suitable *ad hoc* metric $\Theta = \Theta(H)$ amending the Hilbert space. Two consistent physical interpretations of the model remain available. In one we can treat the model as an incomplete, effective description of the so called open quantum system (cf. [3, 32]). In an alternative interpretation one returns to classical physics and one accepts the loss of unitarity as a characteristic feature of the evolution process in question.

In our present paper we considered both the real- and complex-spectrum scenarios. We pointed out the similarities as well as differences. A number of technical challenges was considered, indicating that the most difficult part of the build-up of an acceptable *quantum* non-Hermitian unitary model may be seen in its necessary Hermitization, i.e., in the construction of the Hamiltoniandependent metric $\Theta(H)$. In comparison, the most difficult part of the proposals of the analogous *classical-physics* models relates to their realization in the laboratory.

Let us now add a few remarks. We shall stress a few most important differences, mathematical as well as phenomenological. We also intend to emphasize the existence of the shared features including (1) the phenomenological relevance of the EP singularities and (2) the practical feasibility of predictions.

5.1 Physics represented by classical non-Hermitian Hamiltonians

Inside the non-quantum setup the most difficult obstacles are currently well known to emerge in a design of the experimental realizations. The absence of the necessity of the construction of the physical inner products simplifies mathematics. The majority of the existing experiments seems connected with the most elementary picture of non-Hermitian eigenvalue degeneracies *alias* phase transitions (we restricted our consideration to the finite-dimensional models mainly due to this reason).

The mathematical core of our message may be seen in the construction of the models in which just a few eigenvalues merge at an elementary exceptional point of order K (EPK) with a not too large K. We just returned to our constructive studies of the quantum models possessing higher-order exceptional points [33], and we just omitted the requirement of the reality of the spectrum. Such a change of perspective was inspired by the recent growth of interest in sophisticated experiments localizing the exceptional points of higher order in classical systems. We felt mostly inspired by the recent conference report [29] in which the exceptional points of the third order (EP3) were studied in a system of coupled waveguides. We also noticed that in Ref. [34], optical microcavities were identified as prime candidates for the sensing applications of the EPs. The third-order exceptional points were identified, in Ref. [35], for a system of the three coupled micro-rings made from a semiconductor material. Using the same, rather restricted and more or less purely numerical mathematics people also considered the waves coupled in acoustic cavities with asymmetric losses in which the realization of the fourth-order EP4 proved feasible [36].

We decided to concentrate on the not yet fully clarified theoretical aspects of the similar experiments. In contrast to the difficulties of the practical fine-tuning of parameters in experiments, the construction of the non-quantum theoretical models supporting the K-th-order exceptional points (EPK) may already be declared well advanced at present. These developments proceeded along several independent lines. People tested and, subsequently, widely accepted that, first of all, one can hardly move beyond the N = 3 matrix models without the heuristically helpful requirement of \mathcal{PT} -symmetry [37].

The acceptance of \mathcal{PT} -symmetry opened the way towards the not entirely expected nonnumerical (albeit computer-assisted and symbolic-manipulation-based [38]) constructions of suitable Hs and, in particular, to the localization of their EPs via closed formulae [33, 39, 40]. Another efficient EP-construction tool has been found in the standard self-adjoint toy models known and used in condensed-matter physics. Via a straightforward replacement of some of their real parameters by the purely imaginary or complex quantities it was possible to preserve the underlying algebraic solvability features. Interesting generalizations were obtained for the \mathcal{PT} -symmetrized Su-Schrieffer-Heeger model [41], for the Aubry-André and Harper models [42], etc.

In the \mathcal{PT} -symmetrized Bose-Hubbard model of Graefe et al. [12] the authors were particularly

successful when they recalled the representation theory of angular momentum algebra. This facilitated their study of the unfolding of the EP degeneracy under perturbations. In our present paper, another feature of the model (viz, the matrix tridiagonality and complex symmetry of the Hamiltonian) were found almost equally useful for generalizations, especially at the lower dimensions N. Along these lines we discovered the existence of a family of new, in general non-unitary and non-Bose-Hubbard models possessing higher-order exceptional points.

5.2 Mathematics behind the exceptional-point unfolding

The turn of our attention to classical optics enabled us to avoid various mathematical challenges connected with the proper probabilistic interpretation of the evolving state vectors $|\psi\rangle$ in the consequent quantum setting. We did not need to consider the Hermitian-conjugate form of Eq. (1) which must be solved in the full-fledged non-Hermitian quantum mechanics [43]. We also did not need to insist on the reality of the eigenvalues of H itself which may be necessary for the very observability of quantum systems. At the same time, our interest in the systems near EPs may be perceived as shared by both the classical and quantum physicists, in both cases being separated into their experimental and theoretical subcategories.

One of the most interesting questions emerging in the vicinity of EPs concerns the role of perturbations in a removal of the EP-related spectral degeneracy. A key mathematical subtlety of such an "EP-unfolding" reflects the fact that one has to distinguish between the unfolding of a single EP of order K and the unfolding of a family of the lower-order EPs of the respective orders K_1, \ldots, K_n such that $K_1 + \ldots + K_n = K$. In mathematical language, one has to speak here about the so called "cycles" [11]. For our present purposes, an optimal clarification of physics behind such a scenario may be mediated by examples.

5.2.1 The cycles and their degeneracy at N = 3

In the constructive analysis of the non-Hermitian three-by-three-matrix toy model Hamiltonian of Ref. [16] the authors emphasized that the existence of the standard, EP3-related Jordan-block canonical form

$$J^{(3)}(E) = \begin{bmatrix} E & 1 & 0 \\ 0 & E & 1 \\ 0 & 0 & E \end{bmatrix}$$
(32)

of their three-dimensional tridiagonal toy model Hamiltonian implies that in the vicinity of the EP3 singularity the eigenenergies may be perceived as represented by the analytic function which lives on a three-sheeted Riemann surface (see, e.g., pp. 63 - 65 in [11]) for details). This is a purely theoretical feature of the model which is in a one-to-one correspondence with the possibility of the experimental confirmation that whenever one succeeds in circumscribing the singularity, three

circles are needed for the system to return to its initial state, i.e., in the language of quantum mechanics, to the initial wave function. In parallel, the authors of Ref. [16] also added a remark that after some other choice of the parameters in their Hamiltonian, one can encounter an alternative scenario in which the system returns to its initial state after the mere *two* circles.

An explanation of the apparent paradox is easy: the K = 2 nature of the new situation will merely reflect the partial survival of the diagonalizability of the Hamiltonian. The canonical form of H will be mediated by Eq. (24) in which one obtains the K = 2 Jordan-block result of the following form

$$J^{(1+2)}(E',E) = \begin{bmatrix} E' & 0 & 0\\ 0 & E & 1\\ 0 & 0 & E \end{bmatrix}.$$
(33)

Thus, a cycle of lower order appears here due to the accidental degeneracy $E' \to E$ between a non-EP and an EP energy level.

5.2.2 Degenerate cycles at N = 4

Similar effects may be encountered also at the higher matrix dimensions N of course. The explanation is given by the relationship between the number of circles and the respective dimensions K_j of the canonical Jordan blocks given by the so called periods of the cycles of the eigenvalues in the vicinity of the degenerate EPs (cf., e.g., p. 65 and equation Nr. (1.6) in [11])).

The canonical form of our present N = 4 complex symmetric Hamiltonian (8) can mimic the reduction phenomenon at the EP2 parameters B = -27, A = 64 and z = 81, yielding, via Eq. (24), the mere K = 2 Jordan-block canonical representation

$$J^{(1+1+2)}(E',E'',E) = \begin{bmatrix} E' & 0 & 0 & 0\\ 0 & E'' & 0 & 0\\ 0 & 0 & E & 1\\ 0 & 0 & 0 & E \end{bmatrix}, \quad E = 0.$$
(34)

This means that, hypothetically, the system returns to its original state after the mere two circles circumscribing the EP singularity.

Along the same lines one could even get an exhaustive classification of the alternative scenarios. For the four-dimensional Hamiltonians, for example, one could complement the above-mentioned EP4 and single-EP2 scenarios by the single-EP3 possibility

$$J^{(1+3)}(E',E) = \begin{bmatrix} E' & 0 & 0 & 0\\ 0 & E & 1 & 0\\ 0 & 0 & E & 1\\ 0 & 0 & 0 & E \end{bmatrix}$$
(35)

or, finally, by its double-EP2 alternative

$$J^{(2+2)}(E',E) = \begin{bmatrix} E' & 1 & 0 & 0 \\ 0 & E' & 0 & 0 \\ 0 & 0 & E & 1 \\ 0 & 0 & 0 & E \end{bmatrix}.$$
(36)

An exhaustive classification of the five-dimensional (or higher) non-equivalent scenarios would be slightly more complicated but equally straightforward.

5.3 The physics behind the *quantum* non-Hermitian models

In quantum systems one has to deal with several paradoxical new aspects of the old question of the stability or instability of systems exposed to small perturbations. In the quantum unitaryevolution setting the emergence of instabilities *alias* quantum catastrophes is not always sufficiently well understood and explained in the literature, in spite of being one of the most characteristic consequences of the occurrence of EPs. Along these lines, a complementary inspiration of our research was provided also by the conventional Hermitian treatment of quantum information. The \mathcal{PT} -symmetrization recipe opened the way, e.g., towards the perfect-transfer-of-states protocol which has been based on the choice and treatment of the Hamiltonian as an angular momentum in an external magnetic field [44].

There exist two remarkable byproducts of the latter choices of H. One of them lies in the exact, non-numerical description of the critical behavior at the EP2 singularities even after a fairly nontrivial hypercube-graph generalization of the model. This might prove inspiring in the future. Another source of inspiration may be found in Ref. [28] where the authors developed a recursive bosonic quantization technique which is able to generate the generalized classes of the \mathcal{PT} -symmetric networks and other classical photonic structures exhibiting numerous interesting topological and symmetry features (cf., e.g., Figure Nr. 2 in *loc. cit.*).

Once one turns attention to non-Hermitian quantum systems, one reveals the existence of a number of paradoxes created, often, by the lack of the necessary mathematical insight. The physics of quantum systems represented by non-Hermitian phenomenological observables $H^{(N)}$, $\Lambda^{(N)}$, ... (with real spectra and with any matrix dimension N, finite or infinite) becomes particulary interesting in the vicinity of their EP-singularity boundaries. For illustration, a few quantitative studies of the emergence of related quantum phase transitions may be found in [33]. The authors of some other studies did not always keep in mind the fundamental requirements of the consistent probabilistic interpretation of their quantum models. This may lead to misunderstandings, indeed.

5.3.1 The strength-of-perturbation paradox

The physics represented by non-Hermitian operators is still under intensive development. In most cases, one just has to avoid certain more or less elementary misunderstandings. Many of them were already clarified in review [2]. One encounters their subtler forms also in the more recent literature. For example, in Ref. [45] the authors claimed to have detected "unexpected wild properties of operators familiar from PT-symmetric quantum mechanics" and, as a consequence, they "propose giving the mathematical concept of the pseudospectrum a central role in quantum mechanics with non-Hermitian operators." The phenomenologically disturbing "immanent instability" claims are also illustrated, by the authors of Ref. [45], via a number of $N = \infty$ ordinary differential models.

Despite the use of a high-quality functional analysis the authors' correct mathematical results are accompanied by their misleading physical interpretation. We cannot endorse their claims. The point lies very close to our above discussion: One must distinguish between the classical, Maxwellequation systems and the unitary quantum models. Exclusively in such a case the picture of the "wild" physics is realistic (cf. also a number of further examples of the non-Hermiticity-caused instabilities in [7]). In the quantum-physics approach, in contrast, it would be necessary to amend the inner-product metric $I \to \Theta$ making the quantum version of the theory consistent by means of the necessary transition from the auxiliary, "user-friendly" but "false" Hilbert space $\mathcal{H}^{(F)}$ to its correct, physical, "standard" alternative $\mathcal{H}^{(S)}$.

The latter link is missing in [45]. In more detail, the purely mathematical explanation of the misunderstanding lies in the use of the "false" pseudospectrum

$$\Lambda_{\epsilon}^{(F)}(A) = \{\lambda \in \mathbb{C} \mid \exists \psi \in \mathcal{H}^{(F)} \setminus \{0\}, \exists V : (A+V)\psi = \lambda\psi, ||V||_{\mathcal{H}^{(F)}} \le \epsilon\}$$

This definition is inappropriate, based on the use of the norm of the perturbation V in the manifestly unphysical Hilbert space $\mathcal{H}^{(F)}$. The analysis of the latter, ill-defined (or, better, unphysical) pseudospectrum is, in the unitary quantum-theory setting, irrelevant. The calculation does not take into account the realistic metric, i.e., the different geometry of the unique physical Hilbert space $\mathcal{H}^{(S)}$.

In quantum world the influence of the perturbations must necessarily be characterized by another, correctly defined, *metric-dependent* pseudospectrum

$$\Lambda_{\epsilon}^{(S)}(A) = \{\lambda \in \mathbb{C} \mid \exists \psi \in \mathcal{H}^{(S)} \setminus \{0\}, \exists V : (A+V)\psi = \lambda \psi, ||V||_{\mathcal{H}^{(S)}} \leq \epsilon \}.$$

In spite of the technically much more complicated nature of the latter, amended pseudospectrum, one cannot expect that its evaluation would lead to the "unexpected wild properties" of any admissible \mathcal{PT} -symmetric operators [31]. Indeed, whenever one satisfies the (necessary) hidden-Hermiticity constraint (10), and whenever the perturbations remain small in the correct physical geometry of Hilbert space $\mathcal{H}^{(S)}$, one *does not* encounter any instabilities.

5.3.2 The Bose-Hubbard unfolding paradox

In the non-Hermitian but \mathcal{PT} -symmetric Bose-Hubbard quantum model the authors of Ref. [12] studied the phenomenon of the unfolding (i.e., of the removal of degeneracy) of the higher-order exceptional-point energy under a well-defined perturbation V. Unfortunately, they also did not formulate the task in a consistent manner, i.e., in the correct physical Hilbert space $\mathcal{H}^{(S)}$ with metric $\Theta^{(correct)}(H_0 + V)$. For this reason, the description of the unfolding of the multiply degenerate (i.e., higher-order EP) Bose-Hubbard bound-state energies $E_n(\gamma^{(EP)})$ after perturbation {cf. sections # 4 (numerical results) and # 5 (perturbation results) in Ref. [12]} must be characterized as incomplete. The main reason is that the self-consistent nature of perturbation theory in non-Hermitian quantum picture makes the calculations, in effect, non-linear. We have to keep in mind that

$$\Theta_0 = \Theta(H_0) \neq \Theta^{(correct)}(H_0 + V) = \Theta_0 + K(\Theta_0, H_0, V) \,.$$

For each separate perturbation V we have to repeat the solution of Eq. (10). For the "sufficiently small" perturbations (whatever it means) we can simplify the process and neglect the higher-order terms. Equation (10) gets replaced by

$$H_0^{\dagger} K - K H_0 = \Theta_0 V - V^{\dagger} \Theta_0 \tag{37}$$

i.e., by the implicit definition of the leading-order form of K. Whenever one skips this apparently merely technical step, the results cannot be declared physical.

In the \mathcal{PT} -symmetric Bose-Hubbard case the criticism of subsection 5.3.1 re-applies. The proper probabilistic interpretation cannot be provided without the knowledge of $\Theta^{(correct)}(H_0+V)$. Without this knowledge, the perturbed system is merely tractable as classical. Moreover, in [12], the perturbed eigenvalues cease to be real and form the rings in complex plane. The perturbed Bose-Hubbard quantum system ceases to be observable so that it must be declared non-unitary and unstable.

Beyond the concrete Bose-Hubbard model the generic change of non-Hermitian perturbations will always imply a nontrivial change of the metric Θ . The mutual relationship between these changes can be best studied in quantum systems with small $N \ll \infty$. The correlations become particularly relevant near the dynamical EP-singularity. In the EP limit the eligible Hamiltonian-Hermitization metrics $\Theta(H)$ will cease to exist because all of the candidates for the metric (i.e., *all* of the solutions of Eq. (10)) will cease to be invertible. The geometry of the physical Hilbert space will become, in the EP vicinity, strongly anisotropic [20].

6 Summary

Although the notion of exceptional point (EP) emerged, in the context of perturbation theory, in mathematics [11], it very quickly acquired applications in several branches of physics [46]. The conference "The Physics of Exceptional Points" in 2010 [47] covered, for example, the domains of physics as different as the study of Bose-Einstein condensates and of the light-matter interactions, or the behavior of molecules during photo-dissociation and the phase transitions related to the spontaneous breakdown of \mathcal{PT} -symmetry, or the questions of stability in many-body quantum systems as well as in classical magnetohydrodynamics. Still, the most immediate motivation of the meeting seems to have been provided by the series of speculations and experiments [25] which demonstrated the presence of an EP singularity in the eigenvalue and eigenvector spectra of various classical devices. In [48], for example, two modes of a certain classical electromagnetic microwave billiard (i.e., eigenvectors $\psi_1(z)$ and $\psi_2(z)$ of its "Hamiltonian" H(z)) were shown to coalesce, at the EP singularity (i.e., at a parameter $z = z^{(EP)}$), with a phase difference of $\pi/2$.

The EP singularity in question was shown to be of the square-root type (abbreviated as EP2). From the point of view of elementary linear algebra this means that the two-by-two matrix $H(z^{(EP)})$ ceased to be diagonalizable and that it acquired a canonical form of two-by-two Jordan block. In the language of functional analysis the related two eigenvalues $E_1(z)$ and $E_2(z)$ of H(z) with $z \neq z^{(EP2)}$ could be called "cyclic", of period two (cf. p. 64 in [11]). The importance of this feature in physics can be deduced from the very logo of conference [47]. In an illustration of the consequences of the cyclicity this logo samples the intensity of the electromagnetic field in the microwave during a step-by-step variation of parameter z = z(t). The parameter is made to circumscribe its critical value $z^{(EP2)}$ so that one can see that the billiard can only return to its initial state after *two* circles.

In our present paper we emphasized that under the assumption of complex symmetry and tridiagonality of Hamiltonians, the transition to the more general N by N matrices $H^{(N)}(z^{(EP)})$ can been found, both theoretically and experimentally, feasible. A decisive encouragement can be sought in the purely empirical fact that in the dedicated literature, virtually all of the successful benchmark Hamiltonians $H^{(N)}(z)$ are being chosen, even at the lowest dimensions N = 2 and N = 3, in the very specific complex symmetric and tridiagonal matrix forms exhibiting a number of features shared with the angular-momentum representation methods, say, of Refs. [12, 39] or [28]. In this spirit we performed here an extended search using a straightforward linear algebraic method. Considering just a few-parametric class of the eligible candidates $H^{(N)}(z)$ we filled the gap, on the side of theory, up to N = 4 and N = 5. The resulting models appeared, unexpectedly, exactly solvable. We believe that the experimental simulations based on these matrices will prove equally user-friendly in the future, especially in the vicinity of the various EP-limiting cases.

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