Stability of stealth magnetic field in de Sitter spacetime

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A detailed stability analysis is presented for the de Sitter solution with a homogeneous magnetic field that was recently found in the context of a $U(1)$ gauge theory nonminimally coupled to scalar-tensor gravity. The magnetic field is "stealth" in the sense that the corresponding stress-energy tensor is of the form of an effective cosmological constant and thus is isotropic despite the fact that the magnetic field has a preferred spatial direction. We study the stability of the solution against linear perturbations in the subhorizon and superhorizon limits. We then present some explicit examples that satisfy all stability conditions. The stable de Sitter solution with a homogeneous magnetic field opens up a new possibility for inflationary magnetogenesis, in which magnetic fields in the Universe at all scales may originate from a classical, homogeneous magnetic field sustained during inflation.

I. INTRODUCTION

The origin of magnetic fields in the Universe at various scales is one of the mysteries in modern cosmology. There thus have been a large number of attempts to find a theoretical framework in which magnetic fields in the Universe are generated in the course of the cosmic history (see [1–4] for reviews). The author recently proposed a new scenario of inflationary magnetogenesis in which magnetic fields in the Universe at all scales may originate from a classical, homogeneous magnetic field sustained during inflation [5]. It was found that a $U(1)$ gauge theory nonminimally coupled to scalar-tensor gravity admits a cosmological attractor solution that represents a de Sitter universe with a homogeneous magnetic field, fully taking into account the backreaction of the magnetic field on the geometry.

In the standard Einstein-Maxwell theory, a homogeneous magnetic field would introduce anisotropies in the geometry through the stress-energy tensor, making it impossible to admit a de Sitter solution. One thus needs to modify the standard Einstein-Maxwell theory one way or another to make a homogeneous magnetic field and an isotropic expansion of the universe consistent with each other. In the solution found in [5], a homogeneous magnetic field has a preferred spatial direction but its stress-energy tensor is isotropic and is of the form of an effective cosmological constant. In this sense the magnetic field in the solution is "stealth".

The basic idea behind the model of [5] is as follows. For simplicity let us first consider an action for a $U(1)$ gauge field A_μ of the form

$$I_w = \int d^4x \sqrt{-g}\, f(w)\,, \quad w = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\,, \quad (1.1)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength and $g_{\mu\nu}$ is the spacetime metric. By taking the variation of the action with respect to $g_{\mu\nu}$, one obtains the corresponding stress-energy tensor as

$$T_w^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta I_w}{\delta g_{\mu\nu}} = f'(w)F^\mu_{\ \rho}F^{\nu\rho} + f(w)g^{\mu\nu}\,. \quad (1.2)$$

If $f'(w) = 0$ then the stress-energy tensor is proportional to $g^{\mu\nu}$ and thus admits an exact de Sitter solution. For example, this would be the case if $f(w) = f_0 + f_2(w - w_0)^2$ with f_0, f_2 and w_0 constant and if $w = w_0$. However, in Friedmann-Lemaître-Robertson-Walker (FLRW) backgrounds (including the de Sitter spacetime), w for a homogeneous magnetic field decays as $1/a^4$, where a is the scale factor, and thus the only constant value of w that is consistent with the expansion of the universe is zero. For this reason, this simple action does not work. Let us next consider an action of the form

$$I_W = \int d^4x \sqrt{-g}\, f(W)\,, \quad W = -\frac{1}{4}e^{2\phi}F_{\mu\nu}F^{\mu\nu}\,, \quad (1.3)$$

where ϕ is a scalar field and it is understood that some kinetic terms for ϕ are also added to the action. In this case the corresponding stress-energy tensor is again proportional to $g^{\mu\nu}$ if $f'(W) = 0$. The main difference from the previous case is that W for a homogeneous magnetic field is proportional to $e^{2\phi}/a^4$ in FLRW backgrounds and thus is constant if $e^\phi \propto a^2$. This is the basic idea behind the model of [5].

The model action of [5] however includes not only a nonlinear function of W but also other terms such as the Horndeski's nonminimal vector coupling and the Horndeski scalar terms. This is because, while the simple action (1.3) supplemented by kinetic terms for ϕ in principle admits a de Sitter solution with a homogeneous magnetic field, the stability of such a solution requires inclusion of additional terms in the action.

The purpose of the present paper is to show a detailed stability analysis for the solution representing a de Sitter spacetime with a stealth homogeneous magnetic field. In particular we study the stability of the solution against

linear perturbations in the subhorizon and superhorizon limits. We show that the stability of the solution requires inclusion of the Horndeski's nonminimal vector coupling as well as the Horndeski scalar terms. We then present some explicit examples that satisfy all stability conditions.

The rest of the present paper is organized as follows. In section II we briefly review the model and the solutions studied in [5], focusing on those without the electric field. In section III we formulate the linear perturbation analysis and study the stability of the de Sitter solution with the stealth magnetic field against subhorizon perturbations. In section IV we then study the stability of the solution against superhorizon perturbations. In section V, after showing a couple of classes of models that are unstable, we present explicit models that satisfy all stability conditions. Section VI is then devoted to a summary of the paper and some discussions. Appendices A-D show explicit expressions of some matrices and coefficients.

II. REVIEW OF THE MODEL

In this section we briefly review the model and the de Sitter solution with a homogeneous magnetic field studied in [5].

A. Action

The model consists of a metric $g_{\mu\nu}$, a $U(1)$ gauge field A_μ and a scalar field ϕ, described by the action

$$I = \int d^4x \sqrt{-g} \left[L + L_3 + L_4 + L_5 + L_H \right], \quad (2.1)$$

where $L = L(X, W, Y, Z)$ is an arbitrary function of

$$X \equiv -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi, \quad W \equiv -\frac{1}{4} \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu},$$
$$Y \equiv \mathcal{F}_{\mu\nu} \tilde{\mathcal{F}}^{\mu\nu}, \quad Z \equiv \mathcal{F}^{\rho\mu} \mathcal{F}_\rho{}^\nu \partial_\mu \phi \partial_\nu \phi; \quad (2.2)$$

$\mathcal{F}_{\mu\nu}$ and $\tilde{\mathcal{F}}^{\mu\nu}$ are defined by

$$\mathcal{F}_{\mu\nu} \equiv e^\phi F_{\mu\nu}, \quad \tilde{\mathcal{F}}^{\mu\nu} \equiv e^\phi \tilde{F}^{\mu\nu},$$
$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu, \quad \tilde{F}^{\mu\nu} \equiv \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}, \quad (2.3)$$

and $\epsilon^{0123} = -1/\sqrt{-g}$;

$$L_3 = -G_3(X) \Box \phi,$$
$$L_4 = G_4(X) R + G_{4X}(X) \left[(\Box \phi)^2 - (\nabla^\mu \nabla_\nu \phi)(\nabla^\nu \nabla_\mu \phi) \right],$$
$$L_5 = G_5(X) G^{\mu\nu} \nabla_\mu \nabla_\nu \phi - \frac{1}{6} G_{5X}(X) \left[(\Box \phi)^3 \right.$$
$$- 3(\Box \phi)(\nabla^\mu \nabla_\nu \phi)(\nabla^\nu \nabla_\mu \phi)$$
$$\left. + 2(\nabla^\mu \nabla_\nu \phi)(\nabla^\nu \nabla_\rho \phi)(\nabla^\rho \nabla_\mu \phi) \right], \quad (2.4)$$

are shift-symmetric Horndeski scalar terms [6, 7]; and

$$L_H = \xi \tilde{\mathcal{F}}^{\mu\nu} \tilde{\mathcal{F}}^{\rho\sigma} R_{\mu\nu\rho\sigma}, \quad (2.5)$$

is a simple modification ($\tilde{F}^{\mu\nu} \to \tilde{\mathcal{F}}^{\mu\nu}$) of the Horndeski's nonminimal coupling of the $U(1)$ gauge field to the Riemann tensor $R^\mu{}_{\nu\rho\sigma}$ of the metric $g_{\mu\nu}$ [8]. Here, the scalar field ϕ is normalized so that its mass dimension is zero, $G_{3,4,5}(X)$ are arbitrary functions of X, the subscript X denotes derivative with respect to X, and ξ is an arbitrary constant.

The action is diffeomorphism invariant and enjoys the $U(1)$ gauge symmetry

$$A_\mu \to A_\mu + \partial_\mu \lambda, \quad (2.6)$$

where λ is an arbitrary function. Furthermore, the action respects the global symmetry

$$\phi \to \phi + \phi_0, \quad A_\mu \to e^{-\phi_0} A_\mu, \quad (2.7)$$

where ϕ_0 is an arbitrary constant. We assume that the function $L(X, W, Y, Z)$ is even with respect to Y

$$L(X, W, Y, Z) = L(X, W, -Y, Z). \quad (2.8)$$

This assumption ensures that the equations of motion admit a solution without the electric field [5].

B. Bianchi I solution with magnetic field

We first consider a Bianchi I spacetime

$$g_{\mu\nu} dx^\mu dx^\nu = -N(t)^2 dt^2 + a(t)^2 \left[e^{4\sigma(t)} dx^2 \right.$$
$$\left. + e^{-2\sigma(t)} (dy^2 + dz^2) \right], \quad (2.9)$$

with

$$\frac{\dot{a}}{Na} = const. \equiv H_0, \quad \frac{\dot{\sigma}}{N} = const. \equiv \Sigma_0, \quad (2.10)$$

where an over-dot represents derivative with respect to t. We further assume that the scalar field is homogeneous, $\phi = \phi(t)$, and has a constant "velocity" as

$$\frac{\dot{\phi}}{N} = const., \quad (2.11)$$

and that the $U(1)$ gauge field represents a homogeneous magnetic field as

$$A_t = A_x = 0, \quad A_y = \frac{1}{2} Bz, \quad A_z = -\frac{1}{2} By, \quad (2.12)$$

where B is a constant.

Assuming that not only H_0, Σ_0, B and $\dot{\phi}/N$ but also $\chi \equiv e^\phi/a^2$ are constant (see the sentences just after (1.3) for the motivation for this extra assumption) and rescaling the spatial coordinates so that $\chi = 1$, the equations of motion are reduced to

$$0 = bL_Y, \quad (2.13)$$

and

$$0 =16(1-s)^6 H_0^6 G_{5X} - 6G_4(1-s)^2 H_0^2 - L$$
$$+ 4\left[4\xi b^2 + 6(1-s)^2 G_{4X}\right](1-s)^2 H_0^4\,,$$
$$0 =8G_{5XX}(1+2s)(1-s)^6 H_0^6$$
$$+ 6(1-s)^4\left[4(1+s)G_{4XX} + (1+4s)G_{5X}\right]H_0^4$$
$$+ 2\left\{-(1-4s)\xi b^2 + 3(1-s)^2[G_{3X}\right.$$
$$\left.+(1+3s)G_{4X}]\right\}H_0^2 - 3sG_4 + (1-s)L_X\,,$$
$$0 =4\left[18s(1-s)^2 G_{4X} - \xi(5-4s+8s^2)b^2\right]H_0^2$$
$$+ 72G_{5X}(1-s)^4 s H_0^4 - b^2 L_W - 18sG_4\,, \quad (2.14)$$

where $b \equiv B/H_0$ and $s \equiv \Sigma_0/H_0$. Hereafter, subscripts W, Y and Z represent derivatives with respect to them. Since $Y = 0$ for the ansatz (2.12) and we have assumed (2.8), the first equation (2.13) is automatically satisfied. Therefore we have three algebraic equations (2.14) to be solved with respect to the three parameters (H_0, b, s) of the ansatz. Generically they admit a solution (or a set of solutions).

C. de Sitter solution with magnetic field

By fine-tuning one of parameters in the action, one can then take the limit $s \to 0$ so that the spacetime becomes a de Sitter,

$$g_{\mu\nu}dx^\mu dx^\nu = -N(t)^2 dt^2 + a(t)^2(dx^2+dy^2+dz^2)\,. \quad (2.15)$$

Then, under the assumption (2.8), the independent equations of motion are

$$0 =16G_{5X}H_0^6 + 8(2\xi b^2 + 3G_{4X})H_0^4 - 6G_4 H_0^2 - L\,,$$
$$0 =8G_{5XX}H_0^6 + 6(4G_{4XX} + G_{5X})H_0^4$$
$$+ 2\left[-\xi b^2 + 3(G_{3X} + G_{4X})\right]H_0^2 + L_X\,,$$
$$0 =20\xi H_0^2 + L_W\,. \quad (2.16)$$

These are three algebraic equations. One of the three equations (or one combination of them) simply represents the fine-tuning of one of parameters in the action and the remaining two equations can generically be solved with respect to the two parameters (H_0, b) of the ansatz.

If we relax the fine-tuning without abandoning the discrete symmetry (2.8) then the solution goes back to the Bianchi I solution with the three parameters (H_0, b, s) described in the previous subsection.

D. Attractor condition

Ref. [5] found the necessary and sufficient condition under which the de Sitter solution with a homogeneous magnetic field introduced in subsection II C is stable against homogeneous linear perturbations, under the assumption (2.8). The stability condition, or the attractor condition, is $\mathcal{A}/\mathcal{N} > 0$, where

$$\mathcal{N} =2\zeta_3 g_h(\zeta_3 - 8\zeta_1)b^2 + \zeta_1(\zeta_1\zeta_2 + 3\zeta_3^2)\,,$$
$$\mathcal{A} =56b^6 g_h^3 - 4(9\zeta_1 + \zeta_2 + 15\zeta_3)g_h^2 b^4 - 2g_h\zeta_4(\zeta_1 - \zeta_3)b^3$$
$$+ \left[6(-\zeta_1^2 + \zeta_1\zeta_2 + 2\zeta_1\zeta_3 + 2\zeta_3^2)g_h\right.$$
$$\left.+\zeta_5(\zeta_1 - \zeta_3)^2\right]b^2 + \frac{3}{2}\zeta_1\zeta_4(\zeta_1 - \zeta_3)b\,, \quad (2.17)$$

g_h and ζ_i ($i = 1, \cdots 5$) are constants defined by

$$g_h =\xi H_0^2/M_{\rm Pl}^2\,, \quad \zeta_1 = 2b^2 g_h + g_4 - 4g_{4x} - 4g_{5x}\,,$$
$$\zeta_2 =2b^2 g_h + 6g_{3x} + 24g_{3xx} + 72g_{4xx} + 96g_{4xxx}$$
$$+ 6g_{5x} + 48g_{5xx} + 32g_{5xxx} + 4l_{xx}\,,$$
$$\zeta_3 =4b^2 g_h + 2g_{3x} + 4g_{4x} + 16g_{4xx} + 6g_{5x} + 8g_{5xx}\,,$$
$$\zeta_4 =- 4(g_h + l_{xw})b\,, \quad \zeta_5 = -12g_h - b^2 l_{ww}\,, \quad (2.18)$$

and

$$L_{XX} = l_{xx}\frac{M_{\rm Pl}^2}{H_0^2}\,, \ L_{XW} = l_{xw}\frac{M_{\rm Pl}^2}{H_0^2}\,, \ L_{WW} = l_{ww}\frac{M_{\rm Pl}^2}{H_0^2}\,,$$
$$G_{3X} = g_{3x}\frac{M_{\rm Pl}^2}{H_0^2}\,, \ G_{3XX} = g_{3xx}\frac{M_{\rm Pl}^2}{H_0^4}\,,$$
$$G_4 = g_4 M_{\rm Pl}^2\,, \ G_{4X} = g_{4x}\frac{M_{\rm Pl}^2}{H_0^2}\,, \ G_{4XX} = g_{4xx}\frac{M_{\rm Pl}^2}{H_0^4}\,,$$
$$G_{4XXX} = g_{4xxx}\frac{M_{\rm Pl}^2}{H_0^6}\,, \ G_{5X} = g_{5x}\frac{M_{\rm Pl}^2}{H_0^4}\,,$$
$$G_{5XX} = g_{5xx}\frac{M_{\rm Pl}^2}{H_0^6}\,, \ G_{5XXX} = g_{5xxx}\frac{M_{\rm Pl}^2}{H_0^8}\,. \quad (2.19)$$

It is understood that the left hand sides of (2.19) are evaluated at the de Sitter solution with a homogeneous magnetic field and thus are constant.

We shall later see that the absence of ghosts implies that $\mathcal{N} > 0$. Therefore the attractor condition is

$$\mathcal{A} > 0\,. \quad (2.20)$$

III. LINEAR PERTURBATION

In this section we consider inhomogeneous linear perturbations around the de Sitter attractor solution with a homogeneous magnetic field ($H_0 > 0$, $s = 0$, $b \neq 0$) and seek the no-ghost conditions and the sound speeds in the subhorizon limit.

A. Decomposition of linear perturbation

Around the general background introduced in subsection II B we introduce the components Φ, V_i and h_{ij} ($i, j = 1, 2, 3$) of the metric perturbation $\delta g_{\mu\nu}$, the components δA_a ($a = 0, \cdots, 3$) of the perturbation of the

vector field δA_μ and the perturbation π of the scalar field ϕ as

$$\delta g_{\mu\nu}dx^\mu dx^\nu = -2\Phi\, e^0 e^0 + V_i\, e^0 e^i + V_i\, e^i e^0 + h_{ij}\, e^i e^j\,,$$
$$\delta A_\mu dx^\mu = \delta A_a e^a\,, \quad \delta\phi = \pi\,, \tag{3.1}$$

where

$$e^0 = N(t)dt\,, \quad e^1 = a(t)e^{2\sigma(t)}dx\,,$$
$$e^2 = a(t)e^{-\sigma(t)}dy\,, \quad e^3 = a(t)e^{-\sigma(t)}dz\,, \tag{3.2}$$

and the Einstein's summation rule is employed. We then decompose each component of the perturbation into spatial Fourier modes. For this purpose, we first introduce scalar harmonics Y_n ($n = cc, cs, sc, ss$) as

$$Y_{cc} = \cos(k_x x)\cos(k_y y + k_z z)\,,$$
$$Y_{cs} = \cos(k_x x)\sin(k_y y + k_z z)\,,$$
$$Y_{sc} = \sin(k_x x)\cos(k_y y + k_z z)\,,$$
$$Y_{ss} = \sin(k_x x)\sin(k_y y + k_z z)\,, \tag{3.3}$$

and then define odd and even vector harmonics $Y_{n,p}^{\rm odd/even}$ ($n = cc, cs, sc, ss; \ p, q = 2, 3$) as

$$Y_{n,p}^{\rm odd} = \epsilon_{pq}\frac{\partial}{\partial x^q}Y_n\,, \quad Y_{n,p}^{\rm even} = \frac{\partial}{\partial x^p}Y_n\,, \tag{3.4}$$

where $x^2 = y$, $x^3 = z$, and ϵ_{pq} is the two-dimensional Levi-Civita symbol ($\epsilon_{23} = -\epsilon_{32} = 1$ and $\epsilon_{22} = \epsilon_{33} = 0$), and odd and even tensor harmonics $Y_{n,pq}^{\rm odd/even}$ ($n = cc, cs, sc, ss; \ p, q = 2, 3$) as

$$Y_{n,pq}^{\rm odd/even} = \frac{1}{2}\left(\frac{\partial}{\partial x^p}Y_{n,q}^{\rm odd/even} + \frac{\partial}{\partial x^q}Y_{n,p}^{\rm odd/even}\right)\,. \tag{3.5}$$

The components of the perturbations are then decomposed as

$$\pi = \int\frac{d^3k}{(2\pi)^3}\pi_n Y_n\,, \quad \Phi = \int\frac{d^3k}{(2\pi)^3}\Phi_n Y_n\,,$$
$$V_1 = \int\frac{d^3k}{(2\pi)^3}\chi_n Y_n\,, \quad h_{11} = \int\frac{d^3k}{(2\pi)^3}\psi_n Y_n\,,$$
$$\delta A_0 = \int\frac{d^3k}{(2\pi)^3}A_{0,n}Y_n\,, \quad \delta A_1 = \int\frac{d^3k}{(2\pi)^3}A_{1,n}Y_n\,,$$
$$V_p = \int\frac{d^3k}{(2\pi)^3}(V_n^{\rm odd}Y_{n,p}^{\rm odd} + V_n^{\rm even}Y_{n,p}^{\rm even})\,,$$
$$h_{1p} = h_{p1} = \int\frac{d^3k}{(2\pi)^3}(h_{1,n}^{\rm odd}Y_{n,p}^{\rm odd} + \beta_n Y_{n,p}^{\rm even})\,,$$
$$\delta A_p = \int\frac{d^3k}{(2\pi)^3}(A_n^{\rm odd}Y_{n,p}^{\rm odd} + A_n^{\rm even}Y_{n,p}^{\rm even})\,,$$
$$h_{pq} = \int\frac{d^3k}{(2\pi)^3}(h_n^{\rm odd}Y_{n,pq}^{\rm odd} + E_n Y_{n,pq}^{\rm even} + \tau_n Y_n \delta_{pq})\,, \tag{3.6}$$

where $p, q = 2, 3$ and the Einstein's summation rule for $n = cc, cs, sc, ss$ is understood.

We fix gauge freedom associated with the spacetime coordinate transformation and the $U(1)$ gauge transformation as

$$\beta_n = h_n^{\rm odd} = E_n = \tau_n = 0\,, \quad A_n^{\rm even} = 0\,. \tag{3.7}$$

B. General quadratic action

Substituting the decomposition of perturbations to the action, imposing the gauge condition and expanding the action up to the quadratic order in perturbations, we obtain the quadratic action for the linear perturbations around a general background introduced in subsection II B as

$$I^{(2)} = \frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\int Na^3 dt\left[\frac{1}{N^2}\dot{\mathbf{Y}}_n^\intercal\mathbf{K}\dot{\mathbf{Y}}_n + \frac{1}{N}\left(\dot{\mathbf{Y}}_n^\intercal\mathbf{M}\mathbf{Y}_n\right.\right.$$
$$\left.+\mathbf{Y}_n^\intercal\mathbf{M}^\intercal\dot{\mathbf{Y}}_n\right) - \mathbf{Y}_n^\intercal\mathbf{V}\mathbf{Y}_n + (\mathbf{Z}_n^\intercal\mathbf{A}\mathbf{Y}_n + \mathbf{Y}_n^\intercal\mathbf{A}^\intercal\mathbf{Z}_n)$$
$$\left.+\frac{1}{N}\left(\mathbf{Z}_n^\intercal\mathbf{B}\dot{\mathbf{Y}}_n + \dot{\mathbf{Y}}_n^\intercal\mathbf{B}^\intercal\mathbf{Z}_n\right) + \mathbf{Z}_n^\intercal\mathbf{C}\mathbf{Z}_n\right]\,, \tag{3.8}$$

where the superscript $\intercal$ represents the transpose operation,

$$\mathbf{Y}_n = \begin{pmatrix} k_\perp a^2 A_n^{\rm odd} \\ k_\perp h_{1,n}^{\rm odd} \\ a^2 A_{1,n} \\ \pi_n \\ \psi_n \end{pmatrix}\,, \tag{3.9}$$

represent dynamical degrees of freedom,

$$\mathbf{Z}_{cc} = \begin{pmatrix} k_\perp V_{sc}^{\rm odd} \\ k_\perp V_{cc}^{\rm even} \\ a^2 A_{0,sc} \\ \chi_{sc} \\ \frac{a}{k_\perp}H_0\Phi_{cc} \end{pmatrix}\,, \quad \mathbf{Z}_{cs} = \begin{pmatrix} k_\perp V_{ss}^{\rm odd} \\ k_\perp V_{cs}^{\rm even} \\ a^2 A_{0,ss} \\ \chi_{ss} \\ \frac{a}{k_\perp}H_0\Phi_{cs} \end{pmatrix}\,,$$

$$\mathbf{Z}_{sc} = \begin{pmatrix} -k_\perp V_{cc}^{\rm odd} \\ k_\perp V_{sc}^{\rm even} \\ -a^2 A_{0,cc} \\ -\chi_{cc} \\ \frac{a}{k_\perp}H_0\Phi_{sc} \end{pmatrix}\,, \quad \mathbf{Z}_{ss} = \begin{pmatrix} -k_\perp V_{cs}^{\rm odd} \\ k_\perp V_{ss}^{\rm even} \\ -a^2 A_{0,cs} \\ -\chi_{cs} \\ \frac{a}{k_\perp}H_0\Phi_{ss} \end{pmatrix}\,, \tag{3.10}$$

represent non-dynamical degrees of freedom, $k_\perp = \sqrt{(k_y)^2 + (k_z)^2}$ and the Einstein's summation rule for $n = cc, cs, sc, ss$ is again understood. Important points are that the matrices $\mathbf{K}$, $\mathbf{M}$, $\mathbf{V}$, $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are common for all $n = cc, cs, sc, ss$ and that there are no coupling between $(\mathbf{Y}_n, \mathbf{Z}_n)$ and $(\mathbf{Y}_{n'}, \mathbf{Z}_{n'})$ with $n \neq n'$. In general, however, even and odd sectors within $(\mathbf{Y}_n, \mathbf{Z}_n)$ couple with each other.

Integrating out the non-dynamical variables $\mathbf{Z}_n$, one obtains the quadratic action for the dynamical variables $\mathbf{Y}_n$ as

$$\tilde{I}^{(2)} = \frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\int Na^3 dt\left[\frac{1}{N^2}\dot{\mathbf{Y}}_n^\intercal\bar{\mathbf{K}}\dot{\mathbf{Y}}_n + \frac{1}{N}\left(\dot{\mathbf{Y}}_n^\intercal\bar{\mathbf{M}}\mathbf{Y}_n\right.\right.$$
$$\left.\left.+\mathbf{Y}_n^\intercal\bar{\mathbf{M}}^\intercal\dot{\mathbf{Y}}_n\right) - \mathbf{Y}_n^\intercal\bar{\mathbf{V}}\mathbf{Y}_n\right]\,, \tag{3.11}$$

where

$$\bar{\mathbf{K}} = \mathbf{K} - \mathbf{B}^\intercal\mathbf{C}^{-1}\mathbf{B}\,, \quad \bar{\mathbf{M}} = \frac{1}{2}\left(\bar{\mathbf{m}} - \bar{\mathbf{m}}^\intercal\right)\,,$$
$$\bar{\mathbf{V}} = \mathbf{V} + \mathbf{A}^\intercal\mathbf{C}^{-1}\mathbf{A} + \frac{1}{2Na^3}\partial_t\left[a^3\left(\bar{\mathbf{m}} + \bar{\mathbf{m}}^\intercal\right)\right]\,, \tag{3.12}$$

and

$$\bar{\mathbf{m}} = \mathbf{M} - \mathbf{B}^{\mathsf{T}}\mathbf{C}^{-1}\mathbf{A}\,, \qquad (3.13)$$

It is then straightforward to diagonalize the kinetic matrix $\bar{\mathbf{K}}$ as

$$\bar{\mathbf{K}} \to \tilde{\mathbf{K}} = \mathbf{U}^{\mathsf{T}}\bar{\mathbf{K}}\mathbf{U}, \quad \mathbf{U} = \mathbf{U}_1\mathbf{U}_2\mathbf{U}_3\mathbf{U}_4\mathbf{U}_5\,, \qquad (3.14)$$

where

$$\mathbf{U}_1 = \begin{pmatrix} 1 & -\frac{\bar{K}_{12}}{\bar{K}_{11}} & -\frac{\bar{K}_{13}}{\bar{K}_{11}} & -\frac{\bar{K}_{14}}{\bar{K}_{11}} & -\frac{\bar{K}_{15}}{\bar{K}_{11}} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{U}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -\frac{\bar{K}_{23}}{\bar{K}_{22}} & -\frac{\bar{K}_{24}}{\bar{K}_{22}} & -\frac{\bar{K}_{25}}{\bar{K}_{22}} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{U}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{\bar{K}_{34}}{\bar{K}_{33}} & -\frac{\bar{K}_{35}}{\bar{K}_{33}} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mathbf{U}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{\bar{K}_{45}}{\bar{K}_{44}} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad (3.15)$$

and $\mathbf{U}_5 = \mathrm{diag}(1, k/k_\perp, k/k_\perp, 1, k^2/k_\perp^2)$. Here $k = \sqrt{(k_x)^2 + (k_\perp)^2}$. Correspondingly, the friction matrix $\bar{\mathbf{M}}$ and the mass matrix $\bar{\mathbf{V}}$ transform as

$$\bar{\mathbf{M}} \to \tilde{\mathbf{M}} = \frac{1}{2}\left(\tilde{\mathbf{m}} - \tilde{\mathbf{m}}^{\mathsf{T}}\right)\,,$$

$$\bar{\mathbf{V}} \to \tilde{\mathbf{V}} = \mathbf{U}^{\mathsf{T}}\bar{\mathbf{V}}\mathbf{U} - \frac{1}{N}\left(\dot{\mathbf{U}}^{\mathsf{T}}\bar{\mathbf{M}}\mathbf{U} + \mathbf{U}^{\mathsf{T}}\bar{\mathbf{M}}^{\mathsf{T}}\dot{\mathbf{U}}\right)$$

$$- \frac{1}{N^2}\dot{\mathbf{U}}^{\mathsf{T}}\bar{\mathbf{K}}\dot{\mathbf{U}} + \frac{1}{2Na^3}\partial_t\left[a^3\left(\tilde{\mathbf{m}} + \tilde{\mathbf{m}}^{\mathsf{T}}\right)\right]\,, \quad (3.16)$$

where

$$\tilde{\mathbf{m}} = \mathbf{U}^{\mathsf{T}}\bar{\mathbf{M}}\mathbf{U} + \frac{1}{N}\mathbf{U}^{\mathsf{T}}\bar{\mathbf{K}}\dot{\mathbf{U}}\,. \qquad (3.17)$$

The quadratic action is now written as

$$\tilde{I}^{(2)} = \frac{1}{2}\int\frac{d^3k}{(2\pi)^3}\int Na^3dt\left[\frac{1}{N^2}\dot{\tilde{\mathbf{Y}}}_{\mathbf{n}}^{\mathsf{T}}\tilde{\mathbf{K}}\dot{\tilde{\mathbf{Y}}}_{\mathbf{n}}\right.$$

$$\left.+ \frac{1}{N}\left(\dot{\tilde{\mathbf{Y}}}_{\mathbf{n}}^{\mathsf{T}}\tilde{\mathbf{M}}\tilde{\mathbf{Y}}_{\mathbf{n}} + \tilde{\mathbf{Y}}_{\mathbf{n}}^{\mathsf{T}}\tilde{\mathbf{M}}^{\mathsf{T}}\dot{\tilde{\mathbf{Y}}}_{\mathbf{n}}\right) - \tilde{\mathbf{Y}}_n^{\mathsf{T}}\tilde{\mathbf{V}}\tilde{\mathbf{Y}}_n\right]\,, \qquad (3.18)$$

where the new variables $\tilde{\mathbf{Y}}_n$ are related to the original variables $\mathbf{Y}_n$ via $\mathbf{Y}_n = \mathbf{U}\tilde{\mathbf{Y}}_n$.

C. Quadratic action around de Sitter background with magnetic field

In the de Sitter background with a homogeneous magnetic field, the matrices in (3.8) are greatly simplified and all their components are written in terms of $p_\perp \equiv k_\perp/a$, $p_x \equiv k_x/a$, b, g_h, g_4 and ζ_α ($\alpha = 1, \cdots, 7$), where g_h and ζ_α are defined in (2.18) and

$$\zeta_6 = 4(-3g_h + 2l_z)\,, \quad \zeta_7 = 4(-3g_h + 2l_z + 4b^2l_{yy})\,. \quad (3.19)$$

Here, l_z and l_{yy} are defined as

$$L_Z = l_z\frac{M_{\mathrm{Pl}}^2}{H_0^2}\,, \quad L_{YY} = l_{yy}\frac{M_{\mathrm{Pl}}^2}{H_0^2}\,, \qquad (3.20)$$

where it is understood that the left hand sides of (3.20) are evaluated at the de Sitter solution with a homogeneous magnetic field. It is convenient to decompose the matrices into sub-matrices as

$$\mathbf{K} = \begin{pmatrix} \mathbf{K}_{\mathrm{odd}} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{\mathrm{even}} \end{pmatrix}, \mathbf{M} = \begin{pmatrix} \mathbf{0} & \mathbf{M}_{\mathrm{mix}} \\ -\mathbf{M}_{\mathrm{mix}}^{\mathsf{T}} & \mathbf{M}_{\mathrm{even}} \end{pmatrix},$$

$$\mathbf{V} = \begin{pmatrix} \mathbf{V}_{\mathrm{odd}} & \mathbf{V}_{\mathrm{mix}} \\ \mathbf{V}_{\mathrm{mix}}^{\mathsf{T}} & \mathbf{V}_{\mathrm{even}} \end{pmatrix}, \mathbf{A} = \begin{pmatrix} \mathbf{A}_{\mathrm{odd}} & \mathbf{A}_{\mathrm{mixI}} \\ \mathbf{A}_{\mathrm{mixII}} & \mathbf{A}_{\mathrm{even}} \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} \mathbf{B}_{\mathrm{odd}} & \mathbf{0} \\ \mathbf{B}_{\mathrm{mix}} & \mathbf{B}_{\mathrm{even}} \end{pmatrix}, \mathbf{C} = \begin{pmatrix} \mathbf{C}_{\mathrm{odd}} & \mathbf{C}_{\mathrm{mix}} \\ \mathbf{C}_{\mathrm{mix}}^{\mathsf{T}} & \mathbf{C}_{\mathrm{even}} \end{pmatrix}, \quad (3.21)$$

where explicit expressions for the sub-matrices are given in Appendix A.

If $b = 0$ (we have already set $s = 0$) then, since L is assumed to be even with respect to Y in (2.8), the theory and the background respect the parity invariance and, as a result, the mixing matrices $\mathbf{M}_{\mathrm{mix}}$, $\mathbf{V}_{\mathrm{mix}}$, $\mathbf{A}_{\mathrm{mixI}}$, $\mathbf{A}_{\mathrm{mixII}}$, $\mathbf{B}_{\mathrm{mix}}$ and $\mathbf{C}_{\mathrm{mix}}$ vanish. In general they do not vanish and thus the even and odd perturbations do mix.

The quadratic action for the dynamical variables $\mathbf{Y}_n$ is then given by the formulas (3.11)-(3.13). After diagonalization of the kinetic matrix, one obtains (3.18) with (3.14)-(3.17).

D. Subhorizon limit

For the theoretical consistency, one needs to analyze the stability of the background in the UV, i.e. in the subhorizon limit $p^2 \gg H_0^2$, where $p = \sqrt{p_x^2 + p_\perp^2} = k/a$. This ensures the absence of instability whose timescale is parametrically shorter than the cosmological timescale $\sim 1/H_0$. The stability of the solution against perturbations with $p^2 \ll H_0^2$ shall be studied in the next section.

In the subhorizon limit $p^2 \gg H_0^2$, introducing a small bookkeeping parameter ϵ so that $H_0^2/p^2 = \mathcal{O}(\epsilon^2)$, components of the matrices in the quadratic action (3.18) are

simplified as

$$\tilde{\mathbf{K}} = \begin{pmatrix} \tilde{K}_1 & 0 & 0 & 0 & 0 \\ 0 & \tilde{K}_2 & 0 & 0 & 0 \\ 0 & 0 & \tilde{K}_3 & 0 & 0 \\ 0 & 0 & 0 & \tilde{K}_4 & 0 \\ 0 & 0 & 0 & 0 & \tilde{K}_5 \end{pmatrix} + \mathcal{O}(\epsilon^2) \,,$$

$$\tilde{\mathbf{M}} = \begin{pmatrix} 0 & 0 & 0 & \tilde{M}_1 & \tilde{M}_2 \\ 0 & 0 & \tilde{M}_3 & 0 & 0 \\ 0 & -\tilde{M}_3 & 0 & 0 & 0 \\ -\tilde{M}_1 & 0 & 0 & 0 & 0 \\ -\tilde{M}_2 & 0 & 0 & 0 & 0 \end{pmatrix} + p\mathcal{O}(\epsilon^2) \,,$$

$$\tilde{\mathbf{V}} = \begin{pmatrix} \tilde{V}_1 & 0 & 0 & 0 & 0 \\ 0 & \tilde{V}_2 & 0 & 0 & 0 \\ 0 & 0 & \tilde{V}_3 & 0 & 0 \\ 0 & 0 & 0 & \tilde{V}_4 & \tilde{V}_6 \\ 0 & 0 & 0 & \tilde{V}_6 & \tilde{V}_5 \end{pmatrix} + p^2\mathcal{O}(\epsilon^2) \,, \qquad (3.22)$$

where

$$\tilde{K}_1 = \zeta_6 \,,$$

$$\tilde{K}_2 = \left(\frac{p_\perp^2}{\zeta_1} + \frac{p_x^2}{\zeta_1 - 2b^2 g_h} \right)^{-1} p^2 \,,$$

$$\tilde{K}_3 = \left(\frac{p_\perp^2}{\zeta_7} + \frac{p_x^2}{\zeta_6} \right)^{-1} p^2 \,,$$

$$\tilde{K}_4 = \frac{c_1 p_\perp^4 + 2c_2 p_\perp^2 p_x^2 + \mathcal{N} p_x^4}{[(\zeta_1 - \zeta_3)p^2 + 3b^2 g_h p_\perp^2]^2} \,,$$

$$\tilde{K}_5 = \frac{(\zeta_1 - 2b^2 g_h)\mathcal{N} p^4}{4(c_1 p_\perp^4 + 2c_2 p_\perp^2 p_x^2 + \mathcal{N} p_x^4)} \,,$$

$$\tilde{M}_1 = \frac{p_\perp(c_3 p_\perp^2 + c_4 p_x^2)}{(\zeta_1 - \zeta_3)p^2 + 3b^2 g_h p_\perp^2} \,,$$

$$\tilde{M}_2 = \frac{b g_h p_\perp p^2 (\zeta_1 - 2b^2 g_h)(c_5 p_\perp^2 + c_6 p_x^2)}{\zeta_1 (c_1 p_\perp^4 + 2c_2 p_\perp^2 p_x^2 + \mathcal{N} p_x^4)} \,,$$

$$\tilde{M}_3 = -\frac{2b g_h p_\perp}{\zeta_1} \tilde{K}_2 \,,$$

$$\tilde{V}_1 = \zeta_5 p_\perp^2 + 4g_h \left(\frac{4b^2 g_h}{\zeta_1} - 3 \right) p_x^2 \,,$$

$$\tilde{V}_2 = g_4 p^2 \,,$$

$$\tilde{V}_3 = \frac{4g_h p^2 [3(2b^2 g_h - \zeta_1)p_\perp^2 + (4b^2 g_h - 3\zeta_1)p_x^2]}{(\zeta_1 - 2b^2 g_h)p_\perp^2 + \zeta_1 p_x^2} \,,$$

$$\tilde{V}_4 = \frac{c_7 p_\perp^6 + c_8 p_\perp^4 p_x^2 + c_9 p_\perp^2 p_x^4 + c_{10} p_x^6}{[(\zeta_1 - \zeta_3)p^2 + 3b^2 g_h p_\perp^2]^2} \,,$$

$$\tilde{V}_5 = \frac{p^4(c_{11}p_\perp^6 + c_{12}p_\perp^4 p_x^2 + c_{13}p_\perp^2 p_x^4 + c_{14}p_x^6)}{[c_1 p_\perp^4 + 2c_2 p_\perp^2 p_x^2 + \mathcal{N} p_x^4]^2} \,,$$

$$\tilde{V}_6 = \frac{b^2 g_h p_\perp^2 p^2 (c_{15}p_\perp^4 + 2c_{16}p_\perp^2 p_x^2 + c_{17}p_x^4)}{[(\zeta_1 - \zeta_3)p^2 + 3b^2 g_h p_\perp^2][c_1 p_\perp^4 + 2c_2 p_\perp^2 p_x^2 + \mathcal{N} p_x^4]} \,. \qquad (3.23)$$

Here, $\mathcal{N}$ is defined in (2.17) and $c_1, \cdots, c_{17}$ are shown in Appendix B.

E. No-ghost conditions

All components $\tilde{K}_{1,2,3,4,5}$ of the diagonalized kinetic matrix in the subhorizon limit are positive for $^\forall p_\perp$ and $^\forall p_x$ such that $p^2 \neq 0$, if and only if $\mathcal{N}_a > 0$ $(a = 1, \cdots, 7)$, where

$$\mathcal{N}_1 = \zeta_1 \,, \quad \mathcal{N}_2 = \zeta_1 - 2b^2 g_h \,,$$
$$\mathcal{N}_3 = \zeta_6 \,, \quad \mathcal{N}_4 = \zeta_7 \,, \quad \mathcal{N}_5 = \mathcal{N} \,, \qquad (3.24)$$

and

$$\mathcal{N}_6 = c_1 \,, \quad \mathcal{N}_7 = c_2 + \sqrt{c_1 \mathcal{N}} \,. \qquad (3.25)$$

Here, the positivity of the last three $\mathcal{N}_{5,6,7}$ is necessary and sufficient for $c_1 p_\perp^4 + 2c_2 p_\perp^2 p_x^2 + \mathcal{N} p_x^4$ to be positive for $^\forall p_\perp$ and $^\forall p_x$ such that $p^2 \neq 0$.

One can actually show that

$$\mathcal{N}_6 = \frac{\mathcal{N}_2}{4\mathcal{N}_1^2}(\mathcal{N}_1 - \mathcal{N}_2)^2(4\mathcal{N}_1 - \zeta_3)^2 + \frac{\mathcal{N}_5}{4\mathcal{N}_1^2}(\mathcal{N}_1 + \mathcal{N}_2)^2 \,,$$

$$\mathcal{N}_7 = \frac{\mathcal{N}_5}{2\mathcal{N}_1}(\mathcal{N}_1 + \mathcal{N}_2) + \sqrt{\mathcal{N}_5 \mathcal{N}_6} \,. \qquad (3.26)$$

Therefore the positivity of $\mathcal{N}_6$ and $\mathcal{N}_7$ follows from other conditions and the no-ghost condition is simply

$$\mathcal{N}_a > 0 \,, \quad (a = 1, \cdots, 5) \,. \qquad (3.27)$$

F. Sound speeds

The squared sound speeds c_s^2 of the five modes are determined by

$$0 = \lim_{\epsilon \to 0} \det \left[c_s^2 \tilde{\mathbf{K}} - 2i\sqrt{c_s^2}\tilde{\mathbf{M}}/p - \tilde{\mathbf{V}}/p^2 \right]$$
$$= \tilde{K}_1 \tilde{K}_2 \tilde{K}_3 \tilde{K}_4 \tilde{K}_5 \times \left[(c_s^2)^2 - 2\alpha_1 c_s^2 + \alpha_2 \right]$$
$$\times \left[(c_s^2)^3 - 3\alpha_3 (c_s^2)^2 + 3\alpha_4 c_s^2 - \alpha_5 \right] \,, \qquad (3.28)$$

where

$$\alpha_1 = \frac{b^2 \mathcal{N}_1 \mathcal{N}_3 \mathcal{S}_1 + 2\mathcal{N}_2(\mathcal{N}_1 + 2\mathcal{N}_2)\mathcal{S}_2}{2b^2 \mathcal{N}_1 \mathcal{N}_2 \mathcal{N}_3} \frac{p_x^2}{p^2}$$
$$+ \frac{b^2 \mathcal{N}_4 \mathcal{S}_1 + 2(\mathcal{N}_1 + 2\mathcal{N}_2)\mathcal{S}_2}{2b^2 \mathcal{N}_1 \mathcal{N}_4} \frac{p_\perp^2}{p^2} \,,$$

$$\alpha_2 = \frac{2[(\mathcal{N}_1 + 2\mathcal{N}_2)p_x^2 + 3\mathcal{N}_2 p_\perp^2](\mathcal{N}_3 p_\perp^2 + \mathcal{N}_4 p_x^2)}{b^2 \mathcal{N}_1 \mathcal{N}_2 \mathcal{N}_3 \mathcal{N}_4 p^4} \mathcal{S}_1 \mathcal{S}_2 \,,$$

$$\alpha_3 = \frac{\mathcal{N}_1 \mathcal{S}_3 p_x^2 + \mathcal{N}_2 \mathcal{S}_4 p_\perp^2}{3b^2 \mathcal{N}_1^2 \mathcal{N}_2 \mathcal{N}_3 \mathcal{N}_5 p^2} \,,$$

$$\alpha_4 = \frac{\mathcal{S}_5 p_x^4 + 2(\mathcal{S}_7 - \sqrt{\mathcal{S}_5 \mathcal{S}_6})p_x^2 p_\perp^2 + \mathcal{S}_6 p_\perp^4}{3b^2 \mathcal{N}_1 \mathcal{N}_2 \mathcal{N}_3 \mathcal{N}_5 p^4} \,,$$

$$\alpha_5 = \frac{\mathcal{S}_9 p_x^4 + 2(\mathcal{S}_{11} - \sqrt{\mathcal{S}_9 \mathcal{S}_{10}})p_x^2 p_\perp^2 + \mathcal{S}_{10} p_\perp^4}{b^2 \mathcal{N}_1 \mathcal{N}_2 \mathcal{N}_3 \mathcal{N}_5 p^6}$$
$$\times [2(\mathcal{N}_1 + \mathcal{N}_2)\mathcal{S}_2 p_x^2 + b^2 \mathcal{N}_1 \mathcal{S}_8 p_\perp^2] \,, \qquad (3.29)$$

and $\mathcal{S}_{1,\cdots,11}$ are given in Appendix C.

For the absence of gradient instabilities, it is necessary and sufficient to impose

$$\alpha_1 > 0\,,\ \alpha_2 > 0\,,\ \alpha_3 > 0\,,\ \alpha_4 > 0\,,\ \alpha_5 > 0\,,$$
$$\text{for } {}^\forall p_\perp \text{ and } {}^\forall p_x \text{ such that } p^2 \neq 0\,, \qquad (3.30)$$

and

$$\alpha_1^2 - \alpha_2 > 0\,, \quad \alpha_3^2 - \alpha_4 > 0\,,$$
$$-\alpha_5^2 + 2(3\alpha_4 - 2\alpha_3^2)\alpha_3\alpha_5 + (3\alpha_3^2 - 4\alpha_4)\alpha_4^2 > 0\,,$$
$$\text{for } {}^\forall p_\perp \text{ and } {}^\forall p_x \text{ such that } p^2 \neq 0\,. \qquad (3.31)$$

Here, we supposed that the no-ghost condition (3.27) is satisfied. The condition (3.30) is equivalent to

$$\mathcal{S}_a > 0\,, \quad (a = 1, \cdots, 11)\,. \qquad (3.32)$$

IV. LONG WAVELENGTH PERTURBATIONS

In the previous section we studied the stability of the de Sitter solution with a homogeneous magnetic field in the UV limit, i.e. the limit where the wavelengths of perturbations are sufficiently shorter than the size of the de Sitter horizon. In this section we study the stability of the solution in the opposite limit, namely the limit where the wavelengths of perturbations are sufficiently longer than the size of the horizon.

As argued in [9], instabilities for modes with $p^2 = \mathcal{O}(H_0^2)$ can be as harmless as the standard Jeans instability. It is nonetheless necessary to impose the stability for those modes with $p^2 \ll H_0^2$ if one wants the background under investigation to realize dynamically as an attractor of the system. In general relativity, the standard Jeans instability prevents the matter-dominated FLRW background from being an attractor of the system, but this is not a problem since it is not the matter-dominated epoch but the inflationary epoch that sets the initial condition of our local patch of the universe. On the other hand, for the system under investigation in the present paper, we would like the homogeneous magnetic field to realize dynamically and to be sustained during a de Sitter phase representing the inflationary epoch. For this reason, we require the solution to be an attractor of the system. Namely, we require the solution to be stable against perturbations with $p^2 \ll H_0^2$.

For this purpose we send $p_\perp/H_0$ and p_x/H_0 to zero after obtaining equations of motion for properly normalized dynamical variables. Since the vector harmonics $Y_{n,p}^{\text{odd/even}}$ ($n = cc, cs, sc, ss$; $p, q = 2, 3$) vanish in the limit and $Y_{n,p}^{\text{odd/even}}/k_\perp$ remain finite, we introduce $\tilde{h}_{1,n}^{\text{odd}} \equiv k_\perp h_{1,n}^{\text{odd}}$, $\tilde{A}_n^{\text{odd}} \equiv k_\perp A_n^{\text{odd}}$ and $\tilde{V}_n^{\text{odd/even}} \equiv k_\perp V_n^{\text{odd/even}}$. The stability of the background solution against anisotropic perturbations with long wavelengths can then be analyzed by studying the equations of motion for the dynamical variables ($\tilde{h}_{1,n}^{\text{odd}}$, $A_{1,n}$, π_n, ψ_n, $\tilde{A}_n^{\text{odd}}$) in the limit where $p_\perp/H_0$ and p_x/H_0 are sent to zero, after

eliminating non-dynamical variables. For simplicity we set $N(t) = 1$ in the rest of this section.

The equation of motion for $\tilde{h}_{1,n}^{\text{odd}}$ in the long wavelength limit does not involve other variables and is

$$\ddot{\tilde{h}}_{1,n}^{\text{odd}} + 3H_0\dot{\tilde{h}}_{1,n}^{\text{odd}} = 0\,, \qquad (4.1)$$

where an over-dot represents derivative with respect to the time variable t. This gives the solution

$$\tilde{h}_{1,n}^{\text{odd}} = C_1 + C_2 e^{-3H_0 t}\,, \qquad (4.2)$$

where C_1 and C_2 are constants. The first solution, which is constant in space and time, is actually a pure gauge. Therefore, $\tilde{h}_{1,n}^{\text{odd}}$ is stable.

The equation of motion for $A_{1,n}$ in the long wavelength limit also does not involve other variables and is

$$\ddot{A}_{1,n} + 7H_0\dot{A}_{1,n} + 6H_0^2 A_{1,n} = 0\,, \qquad (4.3)$$

which gives the solution

$$A_{1,n} = C_3 e^{-H_0 t} + C_4 e^{-6H_0 t}\,, \qquad (4.4)$$

where C_3 and C_4 are constants. Therefore, $A_{1,n}$ is stable.

The equations of motion for π_n and ψ_n in the long wavelength limit are coupled but can be solved easily by setting

$$\pi_n = \pi_n^0 e^{\Gamma H_0 t}\,, \quad \psi_n = \psi_n^0 e^{\Gamma H_0 t}\,, \qquad (4.5)$$

where π_n^0 and ψ_n^0 are constants. The general solution is of course a linear combination of solutions of this form for all allowed values of Γ. By substituting (4.5) to the equations of motion for π_n and ψ_n, one obtains

$$\Gamma(\Gamma + 3)\left(\Gamma^2 + 3\Gamma + \frac{\mathcal{A}}{\mathcal{N}}\right) = 0\,, \qquad (4.6)$$

where $\mathcal{A}$ and $\mathcal{N}$ are the same as those defined in (2.17) and thus in particular $\mathcal{N} = \mathcal{N}_5$ (see (3.24)). For $\Gamma = 0$, $\pi_n^0 = 0$ and ψ_n is a pure gauge. Therefore, π_n and ψ_n are also stable, provided that the attractor condition (2.20) and the no-ghost condition (3.27) are satisfied.

The equation of motion for $\tilde{A}_n^{\text{odd}}$ in the long wavelength limit does not involve other variables and is

$$\ddot{\tilde{A}}_n^{\text{odd}} + 5H_0\dot{\tilde{A}}_n^{\text{odd}} + \mathcal{B}H_0^2\tilde{A}_n^{\text{odd}} = 0\,,$$
$$\mathcal{B} = \frac{\mathcal{A}_1 k_x^4 + 2(\mathcal{A}_3 - \sqrt{\mathcal{A}_1\mathcal{A}_2})k_x^2 k_\perp^2 + \mathcal{A}_2 k_\perp^4}{[(\zeta_1 - \zeta_3)k_x^2 + (3g_h b^2 + \zeta_1 - \zeta_3)k_\perp^2]^2}\,, \qquad (4.7)$$

where $\mathcal{A}_{1,2,3}$ are given in Appendix D. Therefore, $\tilde{A}_n^{\text{odd}}$ is stable if and only if

$$\mathcal{A}_a > 0\,, \quad (a = 1, 2, 3)\,. \qquad (4.8)$$

V. EXAMPLES

In this section we first show a couple of classes of models that always violate either (3.27) or (3.32). After that, we show some concrete examples of stable models with specific choices of parameters, which satisfy not only (3.27) and (3.32) but also (3.31), (2.20) and (4.8).

A. Unstable classes of models

Let us first consider the case without the Horndeski's non-minimal vector coupling ($\xi = 0$). Setting $g_h = 0$, we obtain $\mathcal{S}_2 = 0$. This implies that $\alpha_2 = 0$ and thus one of the five modes has vanishing sound speed.

Let us next consider the case with the Horndeski's non-minimal vector coupling ($\xi \neq 0$) but without Horndeski scalar terms. Setting $g_{3x} = g_{3xx} = g_{4x} = g_{4xx} = g_{5x} = g_{5xx} = 0$, we obtain $\mathcal{S}_9/\mathcal{S}_2 = -(2\mathcal{N}_2 - \mathcal{N}_1)^2\mathcal{N}_2$, which is non-positive if we impose the no-ghost condition. Therefore, either $\mathcal{S}_9$ or $\mathcal{S}_2$ is non-positive if the no-ghost condition is imposed.

These two unstable examples imply that the stability requires both the Horndeski's nonminimal vector coupling and Horndeski scalar terms.

B. Stable models

Our strategy here is to find a set of parameters that satisfies (3.27) and (3.32), and then to check if (3.31), (2.20) and (4.8) are also satisfied.

We have already assumed (2.8). For simplicity, in all the examples considered in this subsection we further set

$$0 = g_{3xx} = g_{4xx} = g_{4xxx} = g_{5x}$$
$$= g_{5xx} = g_{5xxx} = l_{yy} = l_{xw} = l_z. \qquad (5.1)$$

In each of the following examples we specify (g_h, g_4, g_{4x}, g_{3x}, b, l_{xx}, l_{ww}). We then solve the background equations of motion (2.16) with respect to (l, l_x, l_w), where

$$L = l\, M_{\rm Pl}^2 H_0^2\,, \;\; L_X = l_x\, M_{\rm Pl}^2\,, \;\; L_W = l_w\, M_{\rm Pl}^2\,, \qquad (5.2)$$

and it is understood that the left hand sides of (5.2) are evaluated at the de Sitter solution with a homogeneous magnetic field. After that, we confirm that (3.27) and (3.32) as well as (3.31), (2.20) and (4.8) are satisfied.

For the set of parameters (g_h, g_4, g_{4x}, g_{3x}, b, l_{xx}, l_{ww}; l, l_x, l_w) and the assumptions (2.8) and (5.1), one can easily reconstruct the Lagrangian assuming a simple ansatz and noting that $X = 2H_0^2$ and $W = -b^2 H_0^2/2$ on the de Sitter solution parameterized by (H_0, b). For example, if we assume that L is quadratic in X and W and independent of Y and Z then

$$\frac{L}{M_{\rm Pl}^2} = \left(l - 2l_x + 2l_{xx} + \frac{1}{2}b^2 l_w + \frac{1}{8}b^4 l_{ww}\right)H_0^2$$

$$+ (l_x - 2l_{xx})X + \left(l_w + \frac{1}{2}b^2 l_{ww}\right)W$$

$$+ \frac{1}{2H_0^2}(l_{xx}X^2 + l_{ww}W^2)\,. \qquad (5.3)$$

The simplest choice of $G_{3,4,5}$ is

$$G_3 = g_{3x}\frac{M_{\rm Pl}^2}{H_0^2}X\,, \;\; G_4 = g_4 M_{\rm Pl}^2 + g_{4x}\frac{M_{\rm Pl}^2}{H_0^2}X\,, \;\; G_5 = 0\,. \qquad (5.4)$$

The value of ξ is

$$\xi = g_h\frac{H_0^2}{M_{\rm Pl}^2}\,. \qquad (5.5)$$

1. Example 1

Let us consider the following choice of parameters.

$$g_h = -1\,, \;\; g_4 = \frac{1}{2}\,, \;\; g_{4x} = 0\,, \;\; g_{3x} = \frac{1}{10}\,,$$
$$b = \frac{1}{10}\,, \;\; l_{xx} = -\frac{1}{10}\,, \;\; l_{ww} = -1\,, \qquad (5.6)$$

for which the background equations of motion (2.16) give

$$l = -\frac{79}{25}\,, \;\; l_x = -\frac{31}{50}\,, \;\; l_w = 20\,. \qquad (5.7)$$

In this case we have

$$\mathcal{N}_1 = \frac{12}{25}\,, \;\; \mathcal{N}_2 = \frac{1}{2}\,, \;\; \mathcal{N}_3 = 12\,,$$
$$\mathcal{N}_4 = 12\,, \;\; \mathcal{N}_5 = \frac{1408}{15625}\,, \qquad (5.8)$$

and

$$\mathcal{S}_1 = \frac{1}{2}\,, \;\; \mathcal{S}_2 = \frac{1}{50}\,, \;\; \mathcal{S}_3 = \frac{59456}{9765625}\,,$$
$$\mathcal{S}_4 = \frac{35469972}{6103515625}\,, \;\; \mathcal{S}_5 = \frac{1696}{390625}\,, \;\; \mathcal{S}_6 = \frac{7940769}{1953125000}\,,$$
$$\mathcal{S}_7 = \frac{2060321}{488281250} + \sqrt{\mathcal{S}_5\mathcal{S}_6}\,, \;\; \mathcal{S}_8 = \frac{1201}{100}\,, \;\; \mathcal{S}_9 = \frac{224}{15625}\,,$$
$$\mathcal{S}_{10} = \frac{29}{2500}\,, \;\; \mathcal{S}_{11} = \frac{823}{62500} + \sqrt{\mathcal{S}_9\mathcal{S}_{10}}\,. \qquad (5.9)$$

Thus (3.27) and (3.32) are satisfied.

The conditions (2.20) and (4.8) are also satisfied as

$$\mathcal{A} = \frac{5979}{390625}\,, \;\; \mathcal{A}_1 = \frac{256}{625}\,, \;\; \mathcal{A}_2 = \frac{87451}{250000}\,,$$
$$\mathcal{A}_3 = \frac{47493}{125000} + \sqrt{\mathcal{A}_1\mathcal{A}_2}\,. \qquad (5.10)$$

After substituting (5.9) into (3.29) to obtain explicit expressions of $\alpha_{1,\cdots,5}$, it is straightforward to compute the three expressions on the left hand sides of the inequalities in (3.31) and to show that they all have the form A/B, where A and B are polynomials of $p_\perp^2$ and p_x^2 with positive coefficients. Therefore, (3.31) is also satisfied.

2. Example 2

Let us consider the following choice of parameters.

$$g_h = -1\,, \;\; g_4 = \frac{1}{2}\,, \;\; g_{4x} = 0\,, \;\; g_{3x} = \frac{1}{10}\,,$$
$$b = \frac{1}{10}\,, \;\; l_{xx} = -\frac{1}{10}\,, \;\; l_{ww} = 0\,, \qquad (5.11)$$

for which the background equations of motion (2.16) give the same values of l, l_x and l_w as shown in (5.7).

In this case we have

$$\mathcal{N}_1 = \frac{12}{25}\,, \ \mathcal{N}_2 = \frac{1}{2}\,, \ \mathcal{N}_3 = 12\,,$$
$$\mathcal{N}_4 = 12\,, \ \mathcal{N}_5 = \frac{1408}{15625}\,, \tag{5.12}$$

and

$$\mathcal{S}_1 = \frac{1}{2}\,, \ \mathcal{S}_2 = \frac{1}{50}\,, \ \mathcal{S}_3 = \frac{59456}{9765625}\,,$$
$$\mathcal{S}_4 = \frac{1418292}{244140625}\,, \ \mathcal{S}_5 = \frac{1696}{390625}\,, \ \mathcal{S}_6 = \frac{79353}{19531250}\,,$$
$$\mathcal{S}_7 = \frac{16477}{3906250} + \sqrt{\mathcal{S}_5\mathcal{S}_6}\,, \ \mathcal{S}_8 = 12\,, \ \mathcal{S}_9 = \frac{224}{15625}\,,$$
$$\mathcal{S}_{10} = \frac{29}{2500}\,, \ \mathcal{S}_{11} = \frac{823}{62500} + \sqrt{\mathcal{S}_9\mathcal{S}_{10}}\,. \tag{5.13}$$

Thus (3.27) and (3.32) are satisfied.

The conditions (2.20) and (4.8) are also satisfied as

$$\mathcal{A} = \frac{239}{15625}\,, \ \mathcal{A}_1 = \frac{256}{625}\,, \ \mathcal{A}_2 = \frac{1749}{5000}\,,$$
$$\mathcal{A}_3 = \frac{5699}{15000} + \sqrt{\mathcal{A}_1\mathcal{A}_2}\,. \tag{5.14}$$

After substituting (5.18) into (3.29) to obtain explicit expressions of $\alpha_{1,\cdots,5}$, it is straightforward to compute the three expressions on the left hand sides of the inequalities in (3.31) and to show that they all have the form A/B, where A and B are polynomials of $p_\perp^2$ and p_x^2 with positive coefficients. Therefore, (3.31) is also satisfied.

3. Example 3

Let us consider the following choice of parameters.

$$g_h = -\frac{1}{20}\,, \ g_4 = \frac{1}{2}\,, \ g_{4x} = 0\,, \ g_{3x} = \frac{1}{10}\,,$$
$$b = \frac{1}{10}\,, \ l_{xx} = -\frac{1}{10}\,, \ l_{ww} = 0\,, \tag{5.15}$$

for which the background equations of motion (2.16) give

$$l = -\frac{376}{125}\,, \ l_x = -\frac{601}{1000}\,, \ l_w = 1\,. \tag{5.16}$$

In this case we have

$$\mathcal{N}_1 = \frac{499}{1000}\,, \ \mathcal{N}_2 = \frac{1}{2}\,, \ \mathcal{N}_3 = \frac{3}{5}\,,$$
$$\mathcal{N}_4 = \frac{3}{5}\,, \ \mathcal{N}_5 = \frac{108990799}{1000000000}\,, \tag{5.17}$$

and

$$\mathcal{S}_1 = \frac{1}{2}\,, \ \mathcal{S}_2 = \frac{1}{1000}\,, \ \mathcal{S}_3 = \frac{371370384707}{1000000000000000}\,,$$
$$\mathcal{S}_4 = \frac{92615485451247}{250000000000000000}\,, \ \mathcal{S}_5 = \frac{50621001581}{200000000000000}\,,$$
$$\mathcal{S}_6 = \frac{252272242903}{1000000000000000}\,,$$
$$\mathcal{S}_7 = \frac{50538124821}{200000000000000} + \sqrt{\mathcal{S}_5\mathcal{S}_6}\,, \ \mathcal{S}_8 = \frac{3}{5}\,,$$
$$\mathcal{S}_9 = \frac{29949199}{2000000000}\,, \ \mathcal{S}_{10} = \frac{59301}{4000000}\,,$$
$$\mathcal{S}_{11} = \frac{59601699}{4000000000} + \sqrt{\mathcal{S}_9\mathcal{S}_{10}}\,. \tag{5.18}$$

Thus (3.27) and (3.32) are satisfied.

The conditions (2.20) and (4.8) are also satisfied as

$$\mathcal{A} = \frac{304101}{500000000}\,, \ \mathcal{A}_1 = \frac{90601}{250000}\,, \ \mathcal{A}_2 = \frac{718809}{2000000}\,,$$
$$\mathcal{A}_3 = \frac{90045653}{249500000} + \sqrt{\mathcal{A}_1\mathcal{A}_2}\,. \tag{5.19}$$

After substituting (5.13) into (3.29) to obtain explicit expressions of $\alpha_{1,\cdots,5}$, it is straightforward to compute the three expressions on the left hand sides of the inequalities in (3.31) and to show that they all have the form A/B, where A and B are polynomials of $p_\perp^2$ and p_x^2 with positive coefficients. Therefore, (3.31) is also satisfied.

4. Example 4

Let us consider the following choice of parameters.

$$g_h = -\frac{1}{20}\,, \ g_4 = \frac{1}{2}\,, \ g_{4x} = -\frac{1}{10}\,, \ g_{3x} = 0\,,$$
$$b = \frac{1}{10}\,, \ l_{xx} = -\frac{1}{10}\,, \ l_{ww} = 0\,, \tag{5.20}$$

for which the background equations of motion (2.16) give

$$l = -\frac{676}{125}\,, \ l_x = \frac{599}{1000}\,, \ l_w = 1\,. \tag{5.21}$$

In this case we have

$$\mathcal{N}_1 = \frac{899}{1000}\,, \ \mathcal{N}_2 = \frac{9}{10}\,, \ \mathcal{N}_3 = \frac{3}{5}\,,$$
$$\mathcal{N}_4 = \frac{3}{5}\,, \ \mathcal{N}_5 = \frac{108704599}{1000000000}\,, \tag{5.22}$$

and

$$\mathcal{S}_1 = \frac{1}{2}\,,\; \mathcal{S}_2 = \frac{1}{1000}\,,\; \mathcal{S}_3 = \frac{9143271446151}{5000000000000000}\,,$$
$$\mathcal{S}_4 = \frac{456009833453647}{250000000000000000}\,,\; \mathcal{S}_5 = \frac{9305798288929}{5000000000000000}\,,$$
$$\mathcal{S}_6 = \frac{9284212649127}{5000000000000000}\,,$$
$$\mathcal{S}_7 = \frac{9295017384133}{5000000000000000} + \sqrt{\mathcal{S}_5\mathcal{S}_6}\,,\; \mathcal{S}_8 = \frac{3}{5}\,,$$
$$\mathcal{S}_9 = \frac{207508199}{2000000000}\,,\; \mathcal{S}_{10} = \frac{10351541}{100000000}\,,$$
$$\mathcal{S}_{11} = \frac{414541899}{4000000000} + \sqrt{\mathcal{S}_9\mathcal{S}_{10}}\,. \tag{5.23}$$

Thus (3.27) and (3.32) are satisfied.

The conditions (2.20) and (4.8) are also satisfied as

$$\mathcal{A} = \frac{9185101}{500000000}\,,\; \mathcal{A}_1 = \frac{1692601}{250000}\,,\; \mathcal{A}_2 = \frac{13530009}{2000000}\,,$$
$$\mathcal{A}_3 = \frac{3042086553}{449500000} + \sqrt{\mathcal{A}_1\mathcal{A}_2}\,. \tag{5.24}$$

After substituting (5.23) into (3.29) to obtain explicit expressions of $\alpha_{1,\cdots,5}$, it is straightforward to compute the three expressions on the left hand sides of the inequalities in (3.31) and to show that they all have the form A/B, where A and B are polynomials of $p_\perp^2$ and p_x^2 with positive coefficients. Therefore, (3.31) is also satisfied.

VI. SUMMARY AND DISCUSSION

We have presented a detailed stability analysis for the de Sitter solution with a homogeneous magnetic field that was recently found in [5] in the context of a $U(1)$ gauge theory nonminimally coupled to scalar-tensor gravity. The magnetic field is "stealth" in the sense that the corresponding stress-energy tensor is of the form of an effective cosmological constant and thus is isotropic despite the fact that the magnetic field has a preferred spatial direction. We have studied the stability of the solution against linear perturbations in the subhorizon and superhorizon limits and have shown some explicit examples that satisfy all stability conditions. Stable models include both Horndeski's nonminimal vector coupling and Horndeski scalar terms.

The stable de Sitter solution with a homogeneous magnetic field opens up a new possibility for inflationary magnetogenesis, in which magnetic fields in the Universe at all scales may originate from a classical, homogeneous magnetic field sustained during inflation. Towards such a new scenario of inflationary magnetogenesis, an important step forward is to show a graceful exit from the de Sitter solution with constant values of (X, W, Y, Z, χ). In particular, the scalar field ϕ should be stabilized around a local minimum of its potential with a sufficiently large mass at late time after inflation in order to recover the standard Einstein-Maxwell theory. (To be more precise, what is recovered is the Einstein-Maxwell theory with the Horndeski's nonminimal vector coupling.) Also, as already pointed out in [5], a field other than ϕ needs to be introduced as the main source of curvature perturbation, i.e. either an inflaton or a curvaton, in order to avoid too large statistical anisotropies and non-Gaussianities. One eventually needs to establish the stability of the whole system all the way from the inflationary epoch to the standard expansion history at late time through the graceful exit. The present paper has established the stability of the inflationary epoch and thus can be considered as the first step towards the new inflationary magnetogenesis scenario based on a classical, homogeneous magnetic field sustained during inflation.

The recent multi-messenger detection of binary neutron stars put a strong constraint on the deviation of the propagation speed of gravitational waves from that of light at the present [10, 11]. If the Horndeski scalar-tensor theory is used as a model of the late-time acceleration of the universe then the functions G_4 and G_5 are strongly constrained. On the other hand, the propagation speed of gravitational waves in the early universe such as the inflationary epoch is not constrained at all by such observations. Therefore, the model considered in [5] and in the present paper is consistent with the multi-messenger detection of binary neutron stars as far as the scalar field is stabilized around a local minimum of its potential with a sufficiently large mass at late time after inflation, as required anyway for the recovery of the Einstein-Maxwell theory (with the Horndeski's nonminimal vector coupling).

After inflation and the stabilization of ϕ around a local minimum of its potential, the homogeneous magnetic field is no longer stealth and thus the background spacetime is then described by a Bianchi I geometry (instead of FLRW one). It is therefore important to investigate phenomenology of the Bianchi I universe with a homogeneous magnetic field and the standard content of the universe (radiation, matter and the cosmological constant) in the context of the Einstein-Maxwell theory with the Horndeski's nonminimal vector coupling [12].

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Appendix A: Explicit expressions of sub-matrices

The sub-matrices in (3.21) are

$$\mathbf{K}_{\text{odd}} = \begin{pmatrix} \zeta_6 & 0 \\ 0 & \zeta_1 \end{pmatrix}, \tag{A1}$$

$$\mathbf{K}_{\text{even}} = \begin{pmatrix} \zeta_7 & 0 & 0 \\ 0 & \zeta_2 & \zeta_3 \\ 0 & \zeta_3 & 0 \end{pmatrix}, \tag{A2}$$

$$\mathbf{M}_{\text{even}} = H_0 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -b^2 g_h \\ 0 & b^2 g_h & 0 \end{pmatrix}, \tag{A3}$$

$$\mathbf{M}_{\text{mix}} = p_\perp \begin{pmatrix} 0 & w_1 & 2bg_h \\ -2bg_h & 0 & 0 \end{pmatrix}, \tag{A4}$$

$$\mathbf{V}_{\text{odd}} = p_\perp^2 \begin{pmatrix} \zeta_5 & 0 \\ 0 & g_4 \end{pmatrix} + p_x^2 \begin{pmatrix} -12g_h & 0 \\ 0 & 0 \end{pmatrix}$$
$$+ H_0^2 \begin{pmatrix} -4\zeta_6 & 0 \\ 0 & 0 \end{pmatrix}, \tag{A5}$$

$$\mathbf{V}_{\text{even}} = p_\perp^2 \begin{pmatrix} -12g_h & 0 & 0 \\ 0 & w_2 & w_3 \\ 0 & w_3 & 0 \end{pmatrix} + p_x^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & w_4 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$+ H_0^2 \begin{pmatrix} -4\zeta_7 & 0 & 0 \\ 0 & w_5 & 3b^2 g_h \\ 0 & 3b^2 g_h & 0 \end{pmatrix}, \tag{A6}$$

$$\mathbf{V}_{\text{mix}} = p_\perp H_0 \begin{pmatrix} 0 & w_6 & 2bg_h \\ 8bg_h & 0 & 0 \end{pmatrix}, \tag{A7}$$

$$\mathbf{A}_{\text{odd}} = p_x H_0 \begin{pmatrix} 0 & -4b^2 g_h \end{pmatrix}, \tag{A8}$$

$$\mathbf{A}_{\text{even}} = \frac{p_\perp^3}{H_0} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 8b^2 g_h - 2\zeta_3 & \zeta_1 \end{pmatrix}$$
$$+ \frac{p_\perp p_x^2}{H_0} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -2\zeta_3 & 0 \end{pmatrix}$$
$$+ p_\perp H_0 \begin{pmatrix} 0 & b^2 \left(2g_h + \frac{1}{2}\zeta_6\right) & -2b^2 g_h \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -b\zeta_4 & 0 \end{pmatrix}$$
$$+ p_x H_0 \begin{pmatrix} 0 & 0 & 0 \\ -\zeta_7 & 0 & 0 \\ 0 & -4b^2 g_h & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{A9}$$

$$\mathbf{A}_{\text{mixI}} = p_\perp p_x \begin{pmatrix} 4bg_h & 0 & 0 \end{pmatrix}, \tag{A10}$$

$$\mathbf{A}_{\text{mixII}} = p_\perp^2 \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \zeta_4 & 0 \end{pmatrix} + p_\perp p_x \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 4bg_h & 0 \\ 0 & 0 \end{pmatrix}$$

$$+ p_x^2 \begin{pmatrix} 4bg_h & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + H_0^2 \begin{pmatrix} b\zeta_6 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{A11}$$

$$\mathbf{B}_{\text{odd}} = p_x \begin{pmatrix} 0 & -\zeta_1 \end{pmatrix}, \tag{A12}$$

$$\mathbf{B}_{\text{even}} = p_\perp \begin{pmatrix} 0 & -8b^2 g_h + 2\zeta_3 & -\zeta_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & w_7 & 2(\zeta_1 - \zeta_3) \end{pmatrix}$$
$$+ p_x \begin{pmatrix} 0 & 0 & 0 \\ \zeta_7 & 0 & 0 \\ 0 & -2\zeta_3 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{A13}$$

$$\mathbf{B}_{\text{mix}} = H_0 \begin{pmatrix} -b\zeta_6 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{A14}$$

$$\mathbf{C}_{\text{odd}} = p_\perp^2 \begin{pmatrix} -2b^2 g_h + \zeta_1 \end{pmatrix} + p_x^2 \begin{pmatrix} \zeta_1 \end{pmatrix} + H_0^2 \begin{pmatrix} b^2\zeta_6 \end{pmatrix}, \tag{A15}$$

$$\mathbf{C}_{\text{even}} = p_\perp^2 \begin{pmatrix} 0 & 0 & 0 & w_8 \\ 0 & \zeta_6 & 0 & 0 \\ 0 & 0 & \zeta_1 & 0 \\ w_8 & 0 & 0 & w_9 \end{pmatrix}$$
$$+ p_\perp p_x \begin{pmatrix} 0 & 0 & \zeta_1 & 0 \\ 0 & 0 & 0 & 0 \\ \zeta_1 & 0 & 0 & 4(\zeta_3 - \zeta_1) \\ 0 & 0 & 4(\zeta_3 - \zeta_1) & 0 \end{pmatrix}$$
$$+ p_x^2 \begin{pmatrix} \zeta_1 & 0 & 0 & 0 \\ 0 & \zeta_7 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + H_0^2 \begin{pmatrix} b^2\zeta_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{A16}$$

$$\mathbf{C}_{\text{mix}} = p_\perp H_0 \begin{pmatrix} 0 & b\zeta_6 & 0 & 0 \end{pmatrix}, \tag{A17}$$

where

$$w_1 = -b\left(2g_h + \frac{1}{4}\zeta_6\right) + \frac{1}{4}\zeta_4,$$

$$w_2 = -b^2\left(7g_h + \frac{1}{4}\zeta_6\right) + \zeta_1 + \zeta_3 - g_4,$$

$$w_3 = b^2 g_h - \frac{1}{2}\zeta_1 + \frac{1}{2}g_4,$$

$$w_4 = -4b^2 g_h + \zeta_1 + \zeta_3 - g_4,$$

$$w_5 = (-6g_h + \zeta_5)b^2 + \frac{3}{2}b\zeta_4,$$

$$w_6 = -b(2g_h + \zeta_5 + \zeta_6) - \frac{1}{2}\zeta_4,$$

$$w_7 = 16b^2 g_h - 2(\zeta_2 + 3\zeta_3),$$

$$w_8 = 12b^2 g_h + 4(\zeta_1 - \zeta_3),$$

$$w_9 = -56b^2 g_h + 4(-3\zeta_1 + \zeta_2 + 6\zeta_3). \tag{A18}$$

Appendix B: Coefficients of matrix components

The coefficients $c_1, \cdots, c_{17}$ in (3.23) are as follows.

$$c_1 = \zeta_1(\zeta_1\zeta_2 + 3\zeta_3^2) - 2b^2 g_h(\zeta_1\zeta_2 + 8\zeta_1\zeta_3 + 2\zeta_3^2)$$
$$+ b^4 g_h^2(16\zeta_1 + \zeta_2 + 24\zeta_3) - 32b^6 g_h^3, \quad \text{(B1)}$$

$$c_2 = \zeta_1(\zeta_1\zeta_2 + 3\zeta_3^2) - b^2 g_h(\zeta_1\zeta_2 + 16\zeta_1\zeta_3 + \zeta_3^2)$$
$$+ 2b^4 g_h^2 \zeta_3 \left(8 - \frac{\zeta_3}{\zeta_1}\right), \quad \text{(B2)}$$

$$c_3 = \frac{1}{4}\zeta_1(\zeta_4 - b\zeta_6) + \frac{1}{4}bg_h[-8(\zeta_1 - \zeta_3) - b\zeta_4 + b^2\zeta_6]$$
$$- 6b^3 g_h^2, \quad \text{(B3)}$$

$$c_4 = \frac{1}{4}\zeta_1(\zeta_4 - b\zeta_6) - 2bg_h(\zeta_1 - \zeta_3) - 4b^3 g_h^2 \left(4 - \frac{\zeta_3}{\zeta_1}\right), \quad \text{(B4)}$$

$$c_5 = \zeta_1 \left[(\zeta_1\zeta_2 + 3\zeta_3^2) - \frac{b}{8}(\zeta_4 - b\zeta_6)(4\zeta_1 - \zeta_3)\right]$$
$$+ b^2 g_h \zeta_1(4\zeta_1 - \zeta_2 - 21\zeta_3) + 32b^4 g_h^2 \zeta_1, \quad \text{(B5)}$$

$$c_6 = \zeta_1(\zeta_1\zeta_2 + 3\zeta_3^2) - 2b^2 g_h \zeta_3(8\zeta_1 - \zeta_3), \quad \text{(B6)}$$

$$c_7 = -\frac{1}{4}\zeta_1^2\zeta_6 b^2 - (\zeta_1 - \zeta_3)^2 g_4 + (\zeta_1 - \zeta_3)\zeta_1^2$$
$$+ \left[\frac{b^2}{2}\zeta_1\zeta_6 + 6(\zeta_3 - \zeta_1)g_4 - \zeta_1(\zeta_1 - 4\zeta_3)\right]b^2 g_h$$
$$- \left[\frac{1}{4}\zeta_6 b^2 + 3(3\zeta_1 + 3g_4 + \zeta_3)\right]b^4 g_h^2 + 9b^6 g_h^3, \quad \text{(B7)}$$

$$c_8 = -\frac{b^2}{2}\zeta_1^2\zeta_6 - 3(\zeta_1 - \zeta_3)^2 g_4 + 3(\zeta_1 - \zeta_3)\zeta_1^2$$
$$+ \left[\frac{b^2}{2}\zeta_1\zeta_6 + 12(\zeta_3 - \zeta_1)g_4 + 6\zeta_1(2\zeta_3 - \zeta_1)\right]b^2 g_h$$
$$- (17\zeta_1 + 7\zeta_3 + 9g_4)b^4 g_h^2 + 12b^6 g_h^3, \quad \text{(B8)}$$

$$c_9 = -\frac{b^2}{4}\zeta_1^2\zeta_6 - 3(\zeta_1 - \zeta_3)^2 g_4 + 3(\zeta_1 - \zeta_3)\zeta_1^2$$
$$+ 3\left[2(\zeta_3 - \zeta_1)g_4 + \zeta_1(4\zeta_3 - 3\zeta_1)\right]b^2 g_h$$
$$- 4(2\zeta_1 + \zeta_3)b^4 g_h^2, \quad \text{(B9)}$$

$$c_{10} = -(\zeta_1 - \zeta_3)^2 g_4 + \zeta_1(\zeta_1 - \zeta_3)(\zeta_1 - 4b^2 g_h), \quad \text{(B10)}$$

$$c_{11} = \frac{1}{4}\zeta_1^2 g_4(\zeta_1\zeta_2 + 3\zeta_3^2)^2$$
$$- \zeta_1(\zeta_1\zeta_2 + 3\zeta_3^2)(\zeta_1\zeta_2 + 8\zeta_1\zeta_3 + 2\zeta_3^2)g_4 b^2 g_h$$
$$+ \left[-\frac{1}{16}\zeta_1^2\zeta_6(4\zeta_1 - \zeta_3)^2 b^2 + (-4\zeta_1^4 + 10\zeta_1^3\zeta_2\right.$$
$$+ 10\zeta_1^3\zeta_3 + \frac{5}{4}\zeta_1^2\zeta_2^2 + \frac{51}{2}\zeta_1^2\zeta_3\zeta_2 + \frac{343}{4}\zeta_1^2\zeta_3^2$$
$$+ \frac{9}{2}\zeta_1\zeta_3^2\zeta_2 + 63\zeta_1\zeta_3^3 + 3\zeta_3^4)g_4 + \frac{1}{4}\zeta_1^2(4\zeta_1 - \zeta_3)$$
$$\left.\times (4\zeta_1^2 - 2\zeta_1\zeta_2 - 5\zeta_1\zeta_3 - 5\zeta_3^2)\right]b^4 g_h^2$$
$$+ \left[\frac{1}{4}\zeta_1\zeta_6(4\zeta_1 - \zeta_3)^2 b^2 + (-16\zeta_1^3 - 30\zeta_1^2\zeta_2\right.$$

$$-144\zeta_1^2\zeta_3 - \frac{41}{2}\zeta_1\zeta_3\zeta_2 - 219\zeta_1\zeta_3^2 - \frac{1}{2}\zeta_1\zeta_2^2$$
$$-\zeta_3^2\zeta_2 - 30\zeta_3^3)g_4 - \frac{1}{4}\zeta_1(4\zeta_1 - \zeta_3)(12\zeta_1^2$$
$$\left.-10\zeta_1\zeta_2 - 59\zeta_1\zeta_3 - 20\zeta_3^2)\right]b^6 g_h^3$$
$$+ \left[-\frac{1}{4}\zeta_6(4\zeta_1 - \zeta_3)^2 b^2 + (112\zeta_1^2 + 20\zeta_1\zeta_2\right.$$
$$+296\zeta_1\zeta_3 + 3\zeta_3\zeta_2 + 95\zeta_3^2)g_4 - (4\zeta_1 - \zeta_3)$$
$$\left.\times(16\zeta_1^2 + 4\zeta_1\zeta_2 + 44\zeta_1\zeta_3 + 5\zeta_3^2)\right]b^8 g_h^4$$
$$+ [-32(4\zeta_1 + 3\zeta_3)g_4$$
$$+(4\zeta_1 - \zeta_3)(68\zeta_1 + 2\zeta_2 + 39\zeta_3)]b^{10}g_h^5$$
$$+ 64(\zeta_3 - 4\zeta_1)b^{12}g_h^6, \quad \text{(B11)}$$

$$c_{12} = \frac{3}{4}\zeta_1^2 g_4(\zeta_1\zeta_2 + 3\zeta_3^2)^2$$
$$- \zeta_1(\zeta_1\zeta_2 + 3\zeta_3^2)(2\zeta_1\zeta_2 + 24\zeta_1\zeta_3 + 3\zeta_3^2)g_4 b^2 g_h$$
$$+ \left[\left(-4\zeta_1^4 + 10\zeta_1^3\zeta_2 + 10\zeta_1^3\zeta_3 + \frac{5}{4}\zeta_1^2\zeta_2^2 + \frac{1}{2}\zeta_1\zeta_3^2\zeta_2\right.\right.$$
$$\left.+\frac{115}{2}\zeta_1^2\zeta_3\zeta_2 + \frac{855}{4}\zeta_1^2\zeta_3^2 + 127\zeta_1\zeta_3^3 - 7\zeta_3^4\right)g_4$$
$$\left.+\frac{1}{4}\zeta_1^2(4\zeta_1 - \zeta_3)(4\zeta_1^2 - 2\zeta_1\zeta_2 - 5\zeta_1\zeta_3 - 5\zeta_3^2)\right]b^4 g_h^2$$
$$+ \left[(16\zeta_1^3 - 20\zeta_1^2\zeta_2 - 200\zeta_1^2\zeta_3 - 27\zeta_1\zeta_3\zeta_2\right.$$
$$\left.-415\zeta_1\zeta_3^2 + 3\zeta_3^2\zeta_2 + 8\zeta_3^3 + 4\frac{\zeta_3^4}{\zeta_1}\right)g_4$$
$$-\zeta_1(4\zeta_1 - \zeta_3)(8\zeta_1^2 - 2\zeta_1\zeta_2 - 18\zeta_1\zeta_3 - 3\zeta_3^2)]b^6 g_h^3$$
$$+ \left[\left(-16\zeta_1^2 + 360\zeta_1\zeta_3 + 39\zeta_3^2 - 12\frac{\zeta_3^3}{\zeta_1}\right)g_4\right.$$
$$\left.+(4\zeta_1 - \zeta_3)(20\zeta_1^2 - 2\zeta_1\zeta_2 - 57\zeta_1\zeta_3 + 3\zeta_3^2)\right]b^8 g_h^4$$
$$- 4\left(4 - \frac{\zeta_3}{\zeta_1}\right)(4\zeta_1^2 - 13\zeta_1\zeta_3 + 2\zeta_3^2)b^{10}g_h^5, \quad \text{(B12)}$$

$$c_{13} = \frac{3}{4}\zeta_1^2 g_4(\zeta_1\zeta_2 + 3\zeta_3^2)^2$$
$$- \zeta_1^2(\zeta_1\zeta_2 + 3\zeta_3^2)(\zeta_2 + 24\zeta_3)g_4 b^2 g_h$$
$$+ \zeta_3(8\zeta_1 - \zeta_3)(4\zeta_1\zeta_2 + 24\zeta_1\zeta_3 + 9\zeta_3^2)g_4 b^4 g_h^2$$
$$- 4\frac{\zeta_3^2}{\zeta_1}(8\zeta_1 - \zeta_3)^2 g_4 b^6 g_h^3, \quad \text{(B13)}$$

$$c_{14} = \frac{1}{4}\zeta_1^2 g_4(\zeta_1\zeta_2 + 3\zeta_3^2)^2$$
$$- \zeta_1\zeta_3(\zeta_1\zeta_2 + 3\zeta_3^2)(8\zeta_1 - \zeta_3)g_4 b^2 g_h$$
$$+ \zeta_3^2(8\zeta_1 - \zeta_3)^2 g_4 b^4 g_h^2, \quad \text{(B14)}$$

$$c_{15} = \frac{1}{8}\zeta_1^2\zeta_6(4\zeta_1 - \zeta_3)b^2$$
$$+ \frac{1}{2}(4\zeta_1^2 - \zeta_1\zeta_2 - 5\zeta_1\zeta_3 - 2\zeta_3^2)\left[(\zeta_1 - \zeta_3)g_4 - \zeta_1^2\right]$$
$$+ \left[-\frac{3}{8}\zeta_1\zeta_6(4\zeta_1 - \zeta_3)b^2\right.$$
$$+ \left(10\zeta_1^2 - \zeta_1\zeta_2 - \frac{5}{2}\zeta_1\zeta_3 - \frac{1}{2}\zeta_3\zeta_2 - 12\zeta_3^2\right)g_4$$

$$+\zeta_1\left(4\zeta_1^2 - 2\zeta_1\zeta_2 - 19\zeta_1\zeta_3 - \frac{5}{2}\zeta_3^2\right)\Big]b^2 g_h$$

$$+\left[\frac{1}{4}\zeta_6(4\zeta_1 - \zeta_3)b^2 + \left(-4\zeta_1 + \frac{3}{2}\zeta_2 + 43\zeta_3\right)g_4\right.$$

$$\left.+26\zeta_1^2 + \frac{5}{2}\zeta_1\zeta_2 + \frac{71}{2}\zeta_1\zeta_3 + \zeta_3^2\right]b^4 g_h^2$$

$$+ (-68\zeta_1 - 48g_4 - \zeta_2 - 15\zeta_3)b^6 g_h^3 + 32b^8 g_h^4\,, \tag{B15}$$

$$c_{16} = \frac{1}{16}\zeta_1^2\zeta_6(4\zeta_1 - \zeta_3)b^2 + \frac{1}{2}(4\zeta_1^2 - \zeta_1\zeta_2 - 5\zeta_1\zeta_3 - 2\zeta_3^2)$$

$$\times (\zeta_1 - \zeta_3)g_4 - \frac{1}{2}(4\zeta_1^2 - \zeta_1\zeta_2 - 5\zeta_1\zeta_3 - 2\zeta_3^2)\zeta_1^2$$

$$+\left[-\frac{1}{8}\zeta_1\zeta_6(4\zeta_1 - \zeta_3)b^2 + \frac{1}{4}\left(12\zeta_1^2\right.\right.$$

$$\left.-2\zeta_1\zeta_2 + 29\zeta_1\zeta_3 - \zeta_3\zeta_2 - 54\zeta_3^2 + 4\frac{\zeta_3^3}{\zeta_1}\right)g_4$$

$$\left.+\frac{1}{4}\zeta_1(32\zeta_1^2 - 6\zeta_1\zeta_2 - 84\zeta_1\zeta_3 - 3\zeta_3^2)\right]b^2 g_h$$

$$+\left[-\frac{1}{2\zeta_1}(28\zeta_1^2 - 55\zeta_1\zeta_3 + 6\zeta_3^2)g_4\right.$$

$$\left.+4\zeta_1^2 + \zeta_1\zeta_2 + 37\zeta_1\zeta_3 - \frac{7}{2}\zeta_3^2\right]b^4 g_h^2$$

$$- 2\left(12\zeta_1 + 5\zeta_3 - \frac{\zeta_3^2}{\zeta_1}\right)b^6 g_h^3\,, \tag{B16}$$

$$c_{17} = \frac{1}{2}(4\zeta_1^2 - \zeta_1\zeta_2 - 5\zeta_1\zeta_3 - 2\zeta_3^2)(\zeta_1 - \zeta_3)g_4$$

$$-\frac{1}{2}(4\zeta_1^2 - \zeta_1\zeta_2 - 5\zeta_1\zeta_3 - 2\zeta_3^2)\zeta_1^2$$

$$+\left[-\left(1 - \frac{\zeta_3}{\zeta_1}\right)(4\zeta_1^2 - 13\zeta_1\zeta_3 + 2\zeta_3^2)g_4\right.$$

$$\left.+\zeta_1(12\zeta_1^2 - \zeta_1\zeta_2 - 23\zeta_1\zeta_3 + \zeta_3^2)\right]b^2 g_h$$

$$+ 2(-8\zeta_1^2 + 18\zeta_1\zeta_3 - 3\zeta_3^2)b^4 g_h^2\,. \tag{B17}$$

Appendix C: Coefficients of dispersion relation

In this appendix we show the coefficients $\mathcal{S}_{1,\cdots,11}$ of the dispersion relation (3.28)-(3.29).

$$\mathcal{S}_1 = g_4\,, \tag{C1}$$

$$\mathcal{S}_2 = \mathcal{N}_2 - \mathcal{N}_1\,, \tag{C2}$$

$$\mathcal{S}_3 = -b^2 g_4 \mathcal{N}_1 \mathcal{N}_2 \mathcal{N}_3 \zeta_3^2 - \mathcal{N}_1 \mathcal{N}_3(\mathcal{N}_1^2 \mathcal{N}_2 - \mathcal{N}_5)b^2 g_4$$

$$+ [2\mathcal{N}_3 \mathcal{N}_2 \mathcal{N}_1^2 b^2 g_4 + \mathcal{N}_3 \mathcal{N}_2 \mathcal{N}_1^2(\mathcal{N}_1 - 2\mathcal{N}_2)b^2]\zeta_3$$

$$- \mathcal{N}_1^3 \mathcal{N}_2 \mathcal{N}_3(\mathcal{N}_1 - 2\mathcal{N}_2)b^2$$

$$- 2\mathcal{N}_2 \mathcal{N}_5(\mathcal{N}_1 + 2\mathcal{N}_2)(\mathcal{N}_1 - \mathcal{N}_2)\,, \tag{C3}$$

$$\mathcal{S}_4 = [-\mathcal{N}_1 \mathcal{N}_3(2\mathcal{N}_1 - \mathcal{N}_2)b^2 g_4 + 4(2\mathcal{N}_1 - \mathcal{N}_2)^2(\mathcal{N}_1 - \mathcal{N}_2)^2]\zeta_3^2$$

$$+ [2\mathcal{N}_1^2(2\mathcal{N}_1 - \mathcal{N}_2)(\mathcal{N}_1 - \mathcal{N}_2)b\zeta_4$$

$$+ \mathcal{N}_3 \mathcal{N}_1^2(7\mathcal{N}_1 - 5\mathcal{N}_2)b^2 g_4$$

$$- \mathcal{N}_3 \mathcal{N}_1^2(3\mathcal{N}_1^2 - 3\mathcal{N}_1 \mathcal{N}_2 + \mathcal{N}_2^2)b^2$$

$$- 8\mathcal{N}_1(2\mathcal{N}_1 - \mathcal{N}_2)(5\mathcal{N}_1 - 4\mathcal{N}_2)(\mathcal{N}_1 - \mathcal{N}_2)^2]\zeta_3$$

$$+ b^2 \mathcal{N}_1^4 \zeta_4^2/4$$

$$+ [-b^3 \mathcal{N}_1^4 \mathcal{N}_3/2 - 2\mathcal{N}_1^3(5\mathcal{N}_1 - 4\mathcal{N}_2)(\mathcal{N}_1 - \mathcal{N}_2)b]\zeta_4$$

$$+ b^2 \mathcal{N}_1^2 \mathcal{N}_5 \zeta_5 - \mathcal{N}_1 \mathcal{N}_3(5\mathcal{N}_1^3 - 4\mathcal{N}_1^2 \mathcal{N}_2 - \mathcal{N}_5)b^2 g_4$$

$$+ \mathcal{N}_1^3 \mathcal{N}_3(15\mathcal{N}_1^2 - 21\mathcal{N}_1 \mathcal{N}_2 + 8\mathcal{N}_2^2)b^2/2$$

$$+ 4(\mathcal{N}_1 - \mathcal{N}_2)^2(25\mathcal{N}_1^4 - 40\mathcal{N}_1^3 \mathcal{N}_2 + 16\mathcal{N}_1^2 \mathcal{N}_2^2 + \mathcal{N}_2 \mathcal{N}_5)\,, \tag{C4}$$

$$\mathcal{S}_5 = [-b^2 \mathcal{N}_1 \mathcal{N}_3 g_4^2 + 2\mathcal{N}_2(\mathcal{N}_1 + 2\mathcal{N}_2)(\mathcal{N}_1 - \mathcal{N}_2)g_4]\zeta_3^2$$

$$+ \{2\mathcal{N}_3 \mathcal{N}_1^2 b^2 g_4^2 + [\mathcal{N}_3 \mathcal{N}_1^2(\mathcal{N}_1 - 2\mathcal{N}_2)b^2$$

$$- 4\mathcal{N}_1 \mathcal{N}_2(\mathcal{N}_1 + 2\mathcal{N}_2)(\mathcal{N}_1 - \mathcal{N}_2)]g_4$$

$$- 2\mathcal{N}_1 \mathcal{N}_2(\mathcal{N}_1 + 2\mathcal{N}_2)(\mathcal{N}_1 - \mathcal{N}_2)(\mathcal{N}_1 - 2\mathcal{N}_2)\}\zeta_3$$

$$- b^2 g_4^2 \mathcal{N}_1^3 \mathcal{N}_3 + [-\mathcal{N}_1^3 \mathcal{N}_3(\mathcal{N}_1 - 2\mathcal{N}_2)b^2$$

$$+ 2(\mathcal{N}_1 + 2\mathcal{N}_2)(\mathcal{N}_1 - \mathcal{N}_2)(\mathcal{N}_1^2 \mathcal{N}_2 - \mathcal{N}_5)]g_4$$

$$+ 2\mathcal{N}_1^2 \mathcal{N}_2(\mathcal{N}_1 + 2\mathcal{N}_2)(\mathcal{N}_1 - \mathcal{N}_2)(\mathcal{N}_1 - 2\mathcal{N}_2)\,, \tag{C5}$$

$$\mathcal{S}_6 = [-\mathcal{N}_2(2\mathcal{N}_1 - \mathcal{N}_2)b^2 g_4 \zeta_5 - b^2 \mathcal{N}_1 \mathcal{N}_3 g_4^2]\zeta_3^2$$

$$+ \{2\mathcal{N}_1 \mathcal{N}_2(\mathcal{N}_1 - \mathcal{N}_2)b g_4 \zeta_4$$

$$+ [\mathcal{N}_1 \mathcal{N}_2(7\mathcal{N}_1 - 5\mathcal{N}_2)b^2 g_4$$

$$+ \mathcal{N}_1 \mathcal{N}_2(\mathcal{N}_1^2 - 3\mathcal{N}_1 \mathcal{N}_2 + \mathcal{N}_2^2)b^2]\zeta_5$$

$$+ \mathcal{N}_1 \mathcal{N}_3(5\mathcal{N}_1 - 3\mathcal{N}_2)b^2 g_4^2 - \mathcal{N}_1 \mathcal{N}_2 \mathcal{N}_3(2\mathcal{N}_1 - \mathcal{N}_2)b^2 g_4$$

$$- 4\mathcal{N}_1 \mathcal{N}_2^2(\mathcal{N}_1 - \mathcal{N}_2)^2\}\zeta_3 + b^2 g_4 \mathcal{N}_1^2 \mathcal{N}_2 \zeta_4^2/4$$

$$+ \{[-b^3 \mathcal{N}_1^2 \mathcal{N}_2 \mathcal{N}_3/2$$

$$- \mathcal{N}_1 \mathcal{N}_2(5\mathcal{N}_1 - 3\mathcal{N}_2)(\mathcal{N}_1 - \mathcal{N}_2)b]g_4$$

$$- \mathcal{N}_1 \mathcal{N}_2^2(\mathcal{N}_1 - \mathcal{N}_2)^2 b\}\zeta_4$$

$$+ [-\mathcal{N}_2(5\mathcal{N}_1^3 - 4\mathcal{N}_1^2 \mathcal{N}_2 - \mathcal{N}_5)b^2 g_4 - b^4 \mathcal{N}_1^3 \mathcal{N}_2 \mathcal{N}_3/4$$

$$- \mathcal{N}_1^2 \mathcal{N}_2(5\mathcal{N}_1^2 - 15\mathcal{N}_1 \mathcal{N}_2 + 8\mathcal{N}_2^2)b^2/2]\zeta_5$$

$$- \mathcal{N}_1 \mathcal{N}_3(5\mathcal{N}_1 - 3\mathcal{N}_2)^2 b^2 g_4^2/4$$

$$+ \mathcal{N}_1 \mathcal{N}_2 \mathcal{N}_3(2\mathcal{N}_1 - \mathcal{N}_2)(5\mathcal{N}_1 - 3\mathcal{N}_2)b^2 g_4/2$$

$$- \mathcal{N}_1 \mathcal{N}_2^2 \mathcal{N}_3(\mathcal{N}_1 - \mathcal{N}_2)^2 b^2/4$$

$$+ 2\mathcal{N}_1 \mathcal{N}_2^2(5\mathcal{N}_1 - 3\mathcal{N}_2)(\mathcal{N}_1 - \mathcal{N}_2)^2\,, \tag{C6}$$

$$\mathcal{S}_7 = [-b^2 g_4 \mathcal{N}_1 \mathcal{N}_2 \zeta_5/2 - b^2 \mathcal{N}_1 \mathcal{N}_3 g_4^2$$

$$+ (\mathcal{N}_1 - \mathcal{N}_2)(-\mathcal{N}_2 + 4\mathcal{N}_1)(2\mathcal{N}_1^2 - 3\mathcal{N}_1 \mathcal{N}_2$$

$$+ 2\mathcal{N}_2^2)g_4/\mathcal{N}_1]\zeta_3^2 + \{\mathcal{N}_1(2\mathcal{N}_1 - \mathcal{N}_2)(\mathcal{N}_1 - \mathcal{N}_2)b g_4 \zeta_4$$

$$+ [b^2 g_4 \mathcal{N}_1^2 \mathcal{N}_2 + \mathcal{N}_1^2 \mathcal{N}_2(\mathcal{N}_1 - 2\mathcal{N}_2)b^2/2]\zeta_5$$

$$+ \mathcal{N}_3 \mathcal{N}_1(7\mathcal{N}_1 - 3\mathcal{N}_2)b^2 g_4^2/2$$

$$+ [-\mathcal{N}_3 \mathcal{N}_1(3\mathcal{N}_1^2 - 4\mathcal{N}_1 \mathcal{N}_2 + 3\mathcal{N}_2^2)b^2/2$$

$$- (\mathcal{N}_1 - \mathcal{N}_2)(40\mathcal{N}_1^3 - 85\mathcal{N}_1^2 \mathcal{N}_2 + 73\mathcal{N}_1 \mathcal{N}_2^2 - 22\mathcal{N}_2^3)]g_4$$

$$- \mathcal{N}_2(\mathcal{N}_1 - \mathcal{N}_2)(\mathcal{N}_1^3 - 3\mathcal{N}_1^2 \mathcal{N}_2 + \mathcal{N}_1 \mathcal{N}_2^2 - 2\mathcal{N}_2^3)\}\zeta_3$$

$$+ b^2 g_4 \mathcal{N}_1^3 \zeta_4^2/8 + [-\mathcal{N}_1^3 \mathcal{N}_3 b^3/4$$

$$- \mathcal{N}_1^2(5\mathcal{N}_1 - 4\mathcal{N}_2)(\mathcal{N}_1 - \mathcal{N}_2)b]g_4 \zeta_4$$

$$+ [-\mathcal{N}_1(\mathcal{N}_1^2 \mathcal{N}_2 - \mathcal{N}_5)b^2 g_4$$

$$- \mathcal{N}_1^3 \mathcal{N}_2(\mathcal{N}_1 - 2\mathcal{N}_2)b^2]\zeta_5/2$$

$$- \mathcal{N}_1^2 \mathcal{N}_3(5\mathcal{N}_1 - 3\mathcal{N}_2)b^2 g_4^2/2$$

$$+ [\mathcal{N}_1^2\mathcal{N}_3(15\mathcal{N}_1^2 - 23\mathcal{N}_1\mathcal{N}_2 + 12\mathcal{N}_2^2)b^2/4$$
$$+ (\mathcal{N}_1 - \mathcal{N}_2)(50\mathcal{N}_1^5 - 125\mathcal{N}_1^4\mathcal{N}_2 + 116\mathcal{N}_1^3\mathcal{N}_2^2$$
$$- 38\mathcal{N}_1^2\mathcal{N}_2^3 - \mathcal{N}_1\mathcal{N}_2\mathcal{N}_5 - 2\mathcal{N}_2^2\mathcal{N}_5)/\mathcal{N}_1]g_4$$
$$+ \mathcal{N}_1^2\mathcal{N}_2\mathcal{N}_3(\mathcal{N}_1 + 2\mathcal{N}_2)(\mathcal{N}_1 - \mathcal{N}_2)b^2/4$$
$$+ \mathcal{N}_1\mathcal{N}_2(\mathcal{N}_1 - \mathcal{N}_2)(5\mathcal{N}_1^3 - 9\mathcal{N}_1^2\mathcal{N}_2$$
$$- 10\mathcal{N}_1\mathcal{N}_2^2 + 8\mathcal{N}_2^3)/2 + \sqrt{\mathcal{S}_5\mathcal{S}_6}\,, \tag{C7}$$

$$\mathcal{S}_8 = \zeta_5\,, \tag{C8}$$

$$\mathcal{S}_9 = -g_4(\mathcal{N}_1 - \zeta_3)(-\zeta_3 g_4 + g_4\mathcal{N}_1 + \mathcal{N}_1^2 - 2\mathcal{N}_1\mathcal{N}_2)\,, \tag{C9}$$

$$\mathcal{S}_{10} = -\zeta_3^2 g_4^2 + [(5\mathcal{N}_1 - 3\mathcal{N}_2)g_4^2 - g_4\mathcal{N}_2^2]\zeta_3$$
$$- (5\mathcal{N}_1 - 3\mathcal{N}_2)^2 g_4^2/4 + [-b^2\mathcal{N}_1\mathcal{N}_2\mathcal{N}_3/4$$
$$+ \mathcal{N}_2^2(5\mathcal{N}_1 - 3\mathcal{N}_2)/2]g_4 - \mathcal{N}_2^2(\mathcal{N}_1 - \mathcal{N}_2)^2/4\,, \tag{C10}$$

$$\mathcal{S}_{11} = \{-\zeta_3^2 g_4 + [(7\mathcal{N}_1 - 3\mathcal{N}_2)g_4/2 + \mathcal{N}_1^2/2$$
$$- \mathcal{N}_1\mathcal{N}_2 - \mathcal{N}_2^2/2]\zeta_3 - \mathcal{N}_1(5\mathcal{N}_1 - 3\mathcal{N}_2)g_4/2$$
$$- \mathcal{N}_3\mathcal{N}_1^2 b^2/8 - \mathcal{N}_1(5\mathcal{N}_1^2 - 13\mathcal{N}_1\mathcal{N}_2 + 4\mathcal{N}_2^2)/4\}g_4$$
$$+ \sqrt{\mathcal{S}_9\mathcal{S}_{10}}\,. \tag{C11}$$

Appendix D: Coefficients of the equation of motion in the long wavelength limit

The coefficients $\mathcal{A}_{1,2,3}$ in (4.7) are as follows.

$$\mathcal{A}_1 = 4(-\zeta_3 + \zeta_1)^2\,, \tag{D1}$$

$$\mathcal{A}_2 = (36g_h + 7\zeta_5/2)g_h b^4 + 9b^3\zeta_4 g_h/4$$
$$+ [24(\zeta_1 - \zeta_3)g_h + (3\zeta_1 - 6\zeta_3 - \zeta_2)\zeta_5/4 - \zeta_4^2/16]b^2$$
$$+ 3\zeta_4(-\zeta_3 + \zeta_1)b/4 + 4(-\zeta_3 + \zeta_1)^2\,, \tag{D2}$$

$$\mathcal{A}_3 = 28g_h^3 b^6/\zeta_1 + (3\zeta_1 - 2\zeta_2 - 30\zeta_3)g_h^2 b^4/\zeta_1$$
$$- g_h\zeta_4(-\zeta_3 + \zeta_1)b^3/\zeta_1$$
$$+ [(27\zeta_1^2 + (3\zeta_2 - 30\zeta_3)\zeta_1 + 12\zeta_3^2)g_h/(2\zeta_1)$$
$$+ (-\zeta_3 + \zeta_1)^2\zeta_5/(2\zeta_1)]b^2 + 3\zeta_4(-\zeta_3 + \zeta_1)b/8$$
$$+ 4(-\zeta_3 + \zeta_1)^2 + \sqrt{\mathcal{A}_1\mathcal{A}_2}\,. \tag{D3}$$

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