Bion non-perturbative contributions versus infrared renormalons in two-dimensional $\mathbb{C}P^{N-1}$ models

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We derive the semiclassical contributions from the real and complex bions in the two-dimensional $\mathbb{C}P^{N-1}$ sigma model on $\mathbb{R} \times S^1$ with a twisted boundary condition. The bion configurations are saddle points of the complexified Euclidean action, which can be viewed as bound states of a pair of fractional instantons with opposite topological charges. We first derive the bion solutions by solving the equation of motion in the model with a potential which simulates an interaction induced by fermions in the $\mathbb{C}P^{N-1}$ quantum mechanics. The bion solutions have quasi-moduli parameters corresponding to the relative distance and phase between the constituent fractional instantons. By summing over the Kaluza-Klein modes of the quantum fluctuations around the bion backgrounds, we find that the effective action for the quasi-moduli parameters is renormalized and becomes a function of the dynamical scale (or the renormalized coupling constant). Based on the renormalized effective action, we obtain the semiclassical bion contribution in a weak coupling limit by making use of the Lefschetz thimble method. We find that the non-perturbative contribution vanishes in the supersymmetric case and it has an imaginary ambiguity which is consistent with the expected infrared renormalon ambiguity in non-supersymmetric cases. This is the first explicit result indicating the relation between the complex bion and the infrared renormalon.

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I. INTRODUCTION

Understanding the non-perturbative aspects of quantum field theory (QFT) is one of the most long-standing problems in theoretical physics. Although lattice simulations may uncover non-perturbative phenomena such as confinement and dynamical mass generation in asymptotically free field theories including quantum chromodynamics (QCD), a systematic and analytical method to study non-perturbative aspects of QFT has not yet been established in general.

The "infrared renormalon" observed in the perturbative expansion in QFT [1, 2] is believed to be related to non-perturbative phenomena. In QCD, a specific set of Feynman diagrams with an internal chain of loops gives a factorial divergence of perturbation series with respect to the coupling constant $\alpha_s(\mu)$ renormalized at the energy scale μ. The Borel transform of such a divergent series has singularities on the positive real axis of the Borel plane, leading to imaginary ambiguities of the Borel resummation. The first Borel singularity, which gives the leading imaginary ambiguity for small $\alpha_s(\mu)$, is located at $t = -2S_I/\beta_0$, and hence the corresponding imaginary ambiguity is proportional to $e^{2S_I/\beta_0} \approx |\Lambda_{\mathrm{QCD}}/\mu|^4$, where β_0 (< 0) is the beta function coefficient, S_I is the instanton action and Λ_{QCD} is the dynamical QCD scale. This indicates that the ambiguity arising in the perturbation series is associated with the low-energy non-perturbative physics. It is notable that the location of the Borel singularity is not twice of the instanton action $2S_I$, but $-2S_I/\beta_0$. Thus this singularity cannot be identified with an instanton–antiinstanton contribution unlike the quantum mechanical systems such as the double-well and sine-Gordon models [3–38]. The ambiguities of perturbation series associated with this type of Borel singularity in asymptotic-free QFTs are called "infrared renormalon ambiguities".

The resurgence theory [39–58], which was originally discussed in the study of ordinary differential equations, has been investigated in various contexts including matrix models and supersymmetric gauge theories [59–92]. From the viewpoint of the resurgence theory, it has been conjectured in four-dimensional (4D) QCD(adj.) and two-dimensional (2D) $\mathbb{C}P^{N-1}$ models compactified on S^1 with a small compactification radius [93–96] that the renormalon could be identified as an object called the bion, which is composed of a pair of fractional instanton and anti-instanton. In these models, fractional instantons with fractional topological charges ($Q = 1/N$) emerge [97–102] due to the $\mathbb{Z}_N$-symmetric Polyakov-loop holonomy in the compactified dimension (equivalent to the $\mathbb{Z}_N$-symmetric twisted boundary condition). The conjecture states that the Borel singularity corresponding to the bion could become the renormalon singularity at $-2S_I/\beta_0$ due to renormalization or decompactification. In spite of the recent progress on compactified $\mathbb{C}P^{N-1}$ models with twisted boundary conditions [103–107] and the intensive studies on the resurgence in the 2D models [108–122], the conjecture has not yet been verified even in the 2D sigma models.

In the $\mathbb{C}P^{N-1}$ quantum mechanics corresponding to the small compactification radius limit of the $\mathbb{C}P^{N-1}$ model on $\mathbb{R} \times S^1$ with the twisted boundary condition, it was shown that the semiclassical contributions from the bion saddle points of the complexified action cancel the imaginary ambiguity in the Borel resummation of the perturbation series. Furthermore, it was confirmed that the full resurgent trans-series composed of the contributions from an infinite tower of the complex saddle points correctly gives the exact result [25, 31, 35]. It is notable that the single complex bion solution is a composite of a kink and an anti-kink corresponding to the fractional instanton and fractional anti-instanton in 2D, respectively. This fact indicates that in the 2D $\mathbb{C}P^{N-1}$ sigma model, bion solutions composed of fractional instantons with opposite topological charges are the saddle points corresponding to the perturbative Borel singularity. The question is whether the 2D complex bion can be identified as the infrared renormalon.

In this work, we discuss bions in the $\mathbb{C}P^{N-1}$ sigma model on $\mathbb{R}\times S^1$ with a twisted boundary condition. In particular, we show that the semiclassical bion contribution have the imaginary ambiguity, which is consistent with that of the infrared renormalon. We start with the $\mathcal{N}=(2,2)$ supersymmetric (SUSY) model with a SUSY breaking deformation parameter $\delta\epsilon$. Generalization to non-supersymmetric cases is also discussed by introducing $n_{\rm F}$ copies of the fermionic degrees of freedom. We derive the real and complex bion solutions by solving the complexified equations of motion derived from the holomorphic action. The summation over the Kaluza-Klein (KK) modes of the quantum fluctuations around the bion configurations correctly renormalizes the coupling constant in the bion effective action and gives the dimensionally transmuted dynamical mass scale $\Lambda_{\mathbb{C}P^{N-1}}$. Based on this renormalized effective action, we compute the semiclassical bion contributions to the vacuum energy. In the undeformed case ($\delta\epsilon=0$), we find that the bion contribution vanishes due to a cancellation between the real and complex bions when the fermion number $n_{\rm F}$ satisfies the condition $1+N(n_{\rm F}-1)/2\in\mathbb{Z}$. This shows that the complex saddle point solutions play an important role to ensure $E=0$ in the vacuum of the supersymmetric model ($n_{\rm F}=1$). When the above condition for $n_{\rm F}$ is not satisfied, there are non-vanishing contributions to the vacuum energy with imaginary ambiguities. Even when the condition for $n_{\rm F}$ is satisfied, non-trivial bion contributions with the imaginary ambiguities appear once the deformation parameter $\delta\epsilon$ is turned on. These imaginary ambiguities are in agreement with the expected renormalon ambiguity of the Borel resummation of the perturbation series. This implies that the bion solutions can be identified as the infrared renormalon in the 2D field theory.

This paper is constructed as follows: In Sec. II, the complex bion solutions in the $\mathbb{C}P^1$ model on $\mathbb{R}\times S^1$ are derived. In Sec. III, the renormalized effective action on the quasi-moduli space of the bion solution is obtained. Based on the effective action, we calculate the contribution of the bions and compare it with the renormalon imaginary ambiguity. In Sec. IV, the calculation for the $\mathbb{C}P^1$ model is extended to the $\mathbb{C}P^{N-1}$ models. Sec. V is devoted to a summary and discussion. In Appendix. A, we illustrate the concept of the quasi-moduli space and valley solution using a simple zero-dimensional model. In Appendices B and C, the detailed calculations of one-loop determinants are summarized for the $\mathbb{C}P^1$ and $\mathbb{C}P^{N-1}$ models, respectively. The Lefschetz thimble integral is summarized in Appendix D.

II. $\mathbb{C}P^1$ SIGMA MODEL AND BION SOLUTIONS

In the present and next sections, we investigate bions in the 2D $\mathbb{C}P^1$ sigma model on $\mathbb{R}\times S^1$ with emphasis on their relevance to the renormalon problem. We derive their semiclassical contributions in both supersymmetric and non-supersymmetric cases. The procedure here will be generalized to the

$\mathbb{C}P^{N-1}$ model in Sec. IV.

A. $\mathbb{C}P^1$ sigma model on $\mathbb{R}\times S^1$

Let us consider the 2D $\mathbb{C}P^1$ sigma model on $\mathbb{R}\times S^1$. For convenience, we start with the 2D $\mathcal{N}=(2,2)$ supersymmetric model. The discussion in this section can also be generalized to the non-supersymmetric models with $n_{\rm F}$ copies of fermions[1]. The bosonic degree of freedom φ (the inhomogeneous coordinate of the target space $\mathbb{C}P^1$) and the fermionic degrees of freedom (ψ_l,ψ_r) form a chiral multiplet of the 2D $\mathcal{N}=(2,2)$ supersymmetry. The Lagrangian takes the form

$$\mathcal{L} \;=\; \frac{2}{g^2}\left[G\left(\partial\varphi\,\bar\partial\bar\varphi+\bar\partial\varphi\,\partial\bar\varphi-\bar\psi_l\mathcal{D}\psi_l-\bar\psi_r\bar{\mathcal{D}}\psi_r\right)+\frac{1}{(1+|\varphi|^2)^4}\psi_l\bar\psi_l\psi_r\bar\psi_r\right]+\mathcal{L}_{\rm top}, \tag{II.1}$$

where ∂ and $\bar\partial$ are the derivatives with respect to the Euclidean spacetime coordinates $z=x+iy$ and $\bar z=x-iy$

$$\partial\equiv\frac{1}{2}(\partial_x-i\partial_y),\qquad \bar\partial\equiv\frac{1}{2}(\partial_x+i\partial_y), \tag{II.2}$$

and $\mathcal{D}$ and $\bar{\mathcal{D}}$ are the pullbacks of the covariant derivatives onto 2D spacetime

$$\mathcal{D}\psi_l\equiv\left(\partial+\Gamma\partial\varphi\right)\psi_l,\qquad \bar{\mathcal{D}}\psi_r\equiv\left(\bar\partial+\Gamma\bar\partial\varphi\right)\psi_r. \tag{II.3}$$

Here Γ denotes the Christoffel symbol for the $\mathbb{C}P^1$ Fubini-Study metric G

$$\Gamma\equiv G^{-1}\frac{\partial}{\partial\varphi}G=-\frac{2\bar\varphi}{1+|\varphi|^2},\qquad G\equiv\frac{1}{(1+|\varphi|^2)^2}. \tag{II.4}$$

The topological θ-term $\mathcal{L}_{\rm top}$ is given by

$$\mathcal{L}_{\rm top} \;=\; \frac{i\theta}{\pi}G(\partial\varphi\bar\partial\bar\varphi-\bar\partial\varphi\partial\bar\varphi). \tag{II.5}$$

For the moment, the coupling constant is denoted as g^2 in (II.1) and we will regard it as the renormalized coupling g_R^2 when we discuss the renormalized bion effective action in Sec. III B.

This model admits 1/2 Bogomol'nyi-Prasad-Sommerfield (BPS) instanton solutions satisfying the BPS equation $\bar\partial\varphi=0$ [123]. They are characterized by non-trivial values of the topological charge. For an instanton solution with topological charge k, the Euclidean action is given by

$$S\,\big|_{k\text{-instanton}}=2\pi k i\tau,\qquad \tau\equiv\frac{\theta}{2\pi}-\frac{i}{g^2}. \tag{II.6}$$

[1] When the models with $n_{\rm F}$ copies of fermions are reduced to quantum mechanics by compactification, they are called a quasi-exactly-solvable models [28, 35] and enjoy a number of similar properties as the supersymmetric model, even though they are not supersymmetric.

Throughout this paper, we regard x and y as the (non-compact) Euclidean time and the (compact) spatial coordinate, respectively. Since the spatial direction y is compactified on S^1 with the radius R, we must specify boundary conditions for the fields. Here we impose the following common twisted boundary condition for all the fields

$$\varphi(y + 2\pi R) = e^{2\pi i m R}\varphi(y), \qquad \psi_{l,r}(y + 2\pi R) = e^{2\pi i m R}\psi_{l,r}(y), \tag{II.7}$$

where R is the radius of S^1 and m is the twist angle (in unit of 2π) satisfying $0 < mR < 1$. It is well known that this twisted boundary condition works as the nontrivial holonomy for the global symmetry in the compactified direction [95, 97–99]. We can find the potential of the 2D $\mathbb{C}P^1$ model with the twisted boundary condition by evaluating the action for the lightest mode in the KK expansion $\varphi(x, y) = \varphi_0 e^{imy}$ with constant φ_0

$$V = \frac{m^2}{g^2}\frac{|\varphi_0|^2}{(1 + |\varphi_0|^2)^2}. \tag{II.8}$$

This potential exhibits two degenerate discrete minima at $\varphi_0 = 0$ (north pole) and $\varphi_0 = \infty$ (south pole), in contrast to the vacuum manifold $\mathbb{C}P^1$ of the untwisted model.

In the presence of the nontrivial background holonomy, the BPS instanton solution decomposes into fractional instantons. Each fractional instanton also satisfies the BPS equation and its explicit form is given by [97–102]

$$\varphi_k = a\, e^{m(x+iy)}, \qquad a \equiv e^{-mx_0 + i\phi}, \tag{II.9}$$

where a is a complex moduli parameter corresponding to the position $x_0 = -(\log|a|)/m$ and internal phase $\phi = \arg a$ of the fractional instanton. We note that this fractional instanton can be viewed as a BPS kink solution connecting the two discrete vacua. Since the topological charge of a fractional instanton is smaller than that of an ordinary integer instanton, it may give a leading order non-perturbative contribution to some physical quantities. The contribution of a single fractional instanton is of order

$$|\exp(-S_{\mathrm{k}})| = \exp\left(-\frac{2\pi m R}{g^2}\right), \tag{II.10}$$

since the action of the fractional instanton is $\mathrm{Re}\, S_k = 2\pi m R/g^2$. However, such a non-perturbative contribution cannot be seen in physical quantities such as the vacuum energy. This is because the path integral for the partition function in the zero temperature limit[2], from which the vacuum energy can be obtained, receives contributions only from configurations approaching the same field value at $x \to \pm\infty$. Therefore, the lowest order non-perturbative effect is given by a fractional instanton-antiinstanton pair. Such a composite configuration is called the bion [124–130]. It is notable that the bion configurations have no topological charge ($S_{\mathrm{top}} = 0$), and tend to decay into the vacuum configuration. However, it

[2] The zero temperature limit corresponds to the limit $\beta \to \infty$, where β is the period of the Euclidean time x.

becomes an approximate solution of the equation of motion when the constituent fractional instanton and anti-instanton are well separated. In the next subsection, instead of dealing with such an approximate solution of bion, we introduce a deformation parameter so that the equation of motion admits an exact bion solution.

B. A deformation and exact single bion solution

To analyze the bion configuration and its contribution to the path integral, it is convenient to consider the following operator which is proportional to the height function

$$\Delta\mu \;=\; m\frac{1-|\varphi|^2}{1+|\varphi|^2}, \tag{II.11}$$

where Δ is the Laplacian on $\mathbb{C}P^1$ and $\mu \equiv m|\varphi|^2/(1+|\varphi|^2)$ is the moment map for the $U(1)$ symmetry $\varphi \to e^{i\alpha}\varphi$. To calculate the generating function for $\Delta\mu$, we introduce the source term

$$\delta\mathcal{L} \;=\; -\frac{\delta\epsilon}{2\pi R}\,\Delta\mu, \tag{II.12}$$

and evaluate the path integral for the partition function in the presence of $\delta\mathcal{L}$

$$Z(\delta\epsilon) = \int \mathcal{D}\varphi \, \exp\left(-S\right), \qquad S \equiv \int d^2x \, (\mathcal{L} + \delta\mathcal{L}) \,. \tag{II.13}$$

We compactify the Euclidean time direction $x \sim x+\beta$ with the periodic boundary condition, and take a decompactification limit $\beta \to \infty$. The vacuum expectation value $\langle \Delta\mu \rangle$ can be obtained by differentiating $Z(\delta\epsilon)$ with respect to the parameter $\delta\epsilon$

$$\langle \Delta\mu \rangle \;=\; \lim_{\delta\epsilon\to 0}\lim_{\beta\to\infty}\frac{1}{\beta}\frac{\partial}{\partial\delta\epsilon}\log Z(\delta\epsilon), \tag{II.14}$$

where $\langle \mathcal{O} \rangle$ denotes the expectation value of an operator $\mathcal{O}$ evaluated in one of the two vacua (corresponding to the classical vacuum $\varphi = 0$). Since $\Delta\mu$ is not invariant under the SUSY transformation, the addition of $\delta\mathcal{L}$ can be regarded as a SUSY breaking deformation of the Lagrangian[3]. In the deformed model, the vacuum energy can be expanded around the SUSY point $\delta\epsilon = 0$ as

$$E(\delta\epsilon) \;=\; -\lim_{\beta\to\infty}\frac{1}{\beta}\log Z(\delta\epsilon) \;=\; E^{(0)} + E^{(1)}\delta\epsilon + E^{(2)}\delta\epsilon^2 + \cdots, \tag{II.15}$$

where the expansion coefficients are given by

$$E^{(0)} = 0, \quad E^{(1)} = -\langle\Delta\mu\rangle, \quad E^{(2)} = -\frac{1}{2\pi R}\int d^2x \left[\langle \Delta\mu(x)\Delta\mu(0)\rangle - \langle\Delta\mu\rangle^2\right], \quad \cdots. \tag{II.16}$$

[3] This deformation is motivated by the form of the potential induced when the Hilbert space is projected to the fermion number eigenspace in $\mathbb{C}P^1$ supersymmetric quantum mechanics [35].

The zeroth order expansion coefficient $E^{(0)}$ vanishes due to the unbroken supersymmetry at $\delta\epsilon = 0$. The first order expansion coefficient $E^{(1)}$ is given by the vacuum expectation value $\langle \Delta\mu \rangle$, which implies that $\langle \Delta\mu \rangle$ can also be interpreted as the response of the vacuum energy to the small SUSY breaking deformation. In general, the n-th order coefficients correspond to the n-point functions of $\Delta\mu$ integrated over the spacetime coordinates.

To calculate $Z(\delta\epsilon)$, let us find saddle point solutions of the deformed action S by solving the Euclidean equation of motion. The deformation term (II.11) causes a splitting of the two degenerate vacua of the undeformed model in such a way that only $\varphi = 0$ remains the global minimum of the potential. This implies that only saddle point solutions satisfying the boundary condition $\varphi \to 0$ at the spatial infinity $x \to \pm\infty$ can contribute to the partition function in the limit $\beta \to \infty$. The simplest solution which satisfies this boundary condition is a single bion configuration, whose explicit form is given by

$$\varphi = \frac{\omega}{\sqrt{\omega^2 - m^2}} \frac{e^{imy + i\phi_0}}{\sinh \omega(x - x_0)}, \tag{II.17}$$

where arbitrary constants x_0 and ϕ_0 are moduli parameters corresponding to the overall position and phase and ω is the mass of the scalar field fluctuation around $\varphi = 0$

$$\omega \equiv m\sqrt{1 + \frac{g^2 \delta\epsilon}{\pi m R}}. \tag{II.18}$$

This bion solution can be viewed as a bound state of fractional instantons with opposite topological charges. We can see this more clearly by rewriting the solution as

$$\varphi = e^{imy} \left(e^{-\omega(x - x_-^{\rm rb}) - i\phi_-^{\rm rb}} + e^{\omega(x - x_+^{\rm rb}) - i\phi_+^{\rm rb}} \right)^{-1}, \tag{II.19}$$

where the position and phase of the fractional instanton $(x_+^{\rm rb}, \phi_+^{\rm rb})$ and those of the fractional anti-instanton $(x_-^{\rm rb}, \phi_-^{\rm rb})$ are given by

$$x_\pm^{\rm rb} = x_0 \pm \frac{1}{2\omega} \log \frac{4\omega^2}{\omega^2 - m^2}, \qquad e^{i\phi_\pm^{\rm rb}} = \pm e^{i\phi_0}. \tag{II.20}$$

The superscript "rb" indicates that these are the values of the parameters corresponding to the "real bion". In the weak coupling limit, which we will consider in the subsequent sections, the relative distance $|x_+^{\rm rb} - x_-^{\rm rb}|$ becomes large and diverges as $|x_+^{\rm rb} - x_-^{\rm rb}| \sim \frac{1}{m} \log \frac{1}{g^2}$. In such a situation, we can see that the bion solution is approximately given by a superposition of fractional instanton and anti-instanton

$$\varphi \sim \begin{cases} e^{\,m(z - x_-^{\rm rb}) + i\phi_-^{\rm rb}} & \text{for } x \approx x_-^{\rm rb} \\ \infty & \text{for } x \approx x_0 \\ e^{-m(\bar{z} - x_+^{\rm rb}) + i\phi_+^{\rm rb}} & \text{for } x \approx x_+^{\rm rb} \end{cases}. \tag{II.21}$$

In the semiclassical method, we need to take into account of all possible saddle point solutions not only in the original configuration space but also in the complexified field space. The $\mathbb{C}P^1$ sigma model can be complexified by regarding $(\varphi, \bar{\varphi})$ as independent holomorphic coordinates $(\varphi, \tilde{\varphi})$ of the complexified target space $(\mathbb{C}P^1)^{\mathbb{C}} \cong SU(2)^{\mathbb{C}}/U(1)^{\mathbb{C}} \cong SL(2,\mathbb{C})/\mathbb{C}^* \cong T^*\mathbb{C}P^1$. Then we can obtain another single bion solution [25]

$$\varphi = \frac{\omega}{\sqrt{\omega^2 - m^2}} \frac{ie^{imy+i\phi_0}}{\cosh \omega(x - x_0)}, \qquad \tilde{\varphi} = \frac{\omega}{\sqrt{\omega^2 - m^2}} \frac{ie^{imy-i\phi_0}}{\cosh \omega(x - x_0)}. \tag{II.22}$$

This is not a proper solution before the complexification since $\tilde{\varphi}$ is not the complex conjugate of φ, so that the saddle point solution (II.22) is called "the complex bion", whereas (II.17) is called "the real bion". The complex bion solution can also be rewritten into the same form as the real bion in Eq. (II.19) but with the following complex values of the relative separation and phase

$$x_\pm^{\rm cb} = x_0 \pm \frac{1}{2\omega} \left(\log \frac{4\omega^2}{\omega^2 - m^2} + i\pi \right), \qquad e^{i\phi_\pm^{\rm cb}} = e^{i\phi_0}. \tag{II.23}$$

The superscript "cb" stands for the "complex bion". In the next section, we will derive the semiclassical contribution from these bion solutions by calculating the associated one-loop determinants and quasi-moduli integrals.

III. NON-PERTURBATIVE BION CONTRIBUTION TO PARTITION FUNCTION

In this section, we calculate the non-perturbative contributions of the real and complex bions. We focus only on the leading non-perturbative contributions in the weak coupling limit, and hence we always ignore irrelevant terms in the limit $g \to 0$ in the following.

A. Quasi-moduli space of single bion configuration

To calculate the bion contributions to the partition function, we need to complexify the configuration space and evaluate the path integral along an appropriate path integral contour emanating from each bion saddle point. Although, in principle, such a contour can be determined by the Lefschetz thimble method, it is not easy to apply it in the infinite dimensional configuration space. Instead, let us consider a reduction of the degrees of freedom from the infinite dimensional field space to a finite dimensional subspace called "the quasi-moduli space."

In the weak coupling limit $g \to 0$, almost all massive modes can be integrated out and their contribution can eventually be expressed as one-loop determinants. However, there are four modes which become massless in the limit $g \to 0$. Two of them are the exact zero modes associated with the two moduli parameters: the overall position and phase (x_0, ϕ_0). The others are called the quasi zero modes, corresponding to the relative position and phase of the constituent fractional instantons. To evaluate the integral along such "nearly flat directions" (flat directions in the limit of $g \to 0$), let us define "valley solution" $\varphi_B(\eta)$ (quasi-solution) [131–135] as a bion ansatz satisfying the following properties:

- $\varphi_B(\eta)$ is parameterized by the positions and phases of the constituent fractional instantons $\eta^\alpha =$

$(x_-, \phi_-, x_+, \phi_+)$,

- $\varphi_B(\eta)$ becomes the exact real and complex bion solutions when these quasi-moduli parameters η^α are at the saddle point values (II.20) and (II.23).

- $\varphi_B(\eta)$ satisfies the equation of motion up to a linear combination of $\partial\varphi_B/\partial\eta^\alpha$. In other words, it is a solution of the equation

$$\left.\frac{\delta S}{\delta\varphi}\right|_{\varphi=\varphi_B} = G\overline{A^\alpha\frac{\partial\varphi_B}{\partial\eta^\alpha}}. \tag{III.1}$$

Here, the metric G is the $\mathbb{C}P^1$ Fubini-Study metric, and the coefficients of the linear combination A^α can be determined by taking the inner product of Eq. (III.1) and $\partial_\alpha\varphi_B$

$$A^\alpha = \overline{\frac{\partial S_{\text{eff}}}{\partial\eta^\beta}}\, g^{\bar{\beta}\alpha}, \tag{III.2}$$

where $S_{\text{eff}}(\eta)$ is the bion effective action and $g_{\alpha\bar{\beta}}$ is the induced metric

$$S_{\text{eff}}(\eta) = S[\varphi_B], \qquad g_{\alpha\bar{\beta}} = \int d^2x\, G\frac{\partial\varphi_B}{\partial\eta^\alpha}\overline{\frac{\partial\varphi_B}{\partial\eta^\beta}}. \tag{III.3}$$

We call the set of valley solutions "the quasi-moduli space $\mathcal{M}$". The quasi-moduli parameters η^α can be regarded as coordinates of $\mathcal{M}$ and $g_{\alpha\bar{\beta}}$ is the metric on $\mathcal{M}$. Roughly speaking, the quasi-moduli space is a valley of the action, where the gradient of S is tangent to the valley. In Appendix A, the concept of the quasi-moduli space is explained in more detail by using an example of a simple zero dimensional model.

To evaluate the path integral, let us decompose the bosonic degree of freedom into the bion background $\varphi_B(\eta)$ (parametrized by the quasi-moduli parameters η^α) and a fluctuation field $\delta\varphi$ which is orthogonal to the quasi-zero modes

$$\varphi = \varphi_B(\eta) + \delta\varphi, \qquad \int d^2z\, G\overline{\frac{\partial\varphi}{\partial\eta^\alpha}}\delta\varphi = 0. \tag{III.4}$$

Then the path integral decomposes into that for $\delta\varphi$ and the quasi-moduli integral over $\mathcal{M}$. Note that $\delta\varphi$ stands for all the modes which remain massive in the weak coupling limit $g \to 0$. It is convenient to redefine the bosonic fluctuation[4] and fermionic fields as

$$\delta\varphi = g(1+|\varphi_B|^2)\,\xi, \qquad \psi_{l,r} = g(1+|\varphi_B|^2)\,\chi_{l,r}. \tag{III.5}$$

Thanks to the definition of the valley solution (III.1) and the orthogonality condition in (III.4), no linear

[4] It would be more convenient to use the Riemann normal coordinates [136] or Kähler normal coordinates [137, 138] for higher loop computations

term of the fluctuation fields appears in the action expanded in powers of g

$$S = S_{\text{eff}}(\eta) - \int d^2z \Big[\mathcal{L}_{\text{fluc}}(\xi, \bar{\xi}) + 2\bar{\chi}_l \nabla \chi_l + 2\bar{\chi}_r \bar{\nabla} \chi_r \Big] + \mathcal{O}(g^2), \tag{III.6}$$

where $S_{\text{eff}}(\eta) = S[\varphi_B(\eta)]$ is the bion effective action and $\mathcal{L}_{\text{fluc}}(\xi, \bar{\xi})$ is a quadratic term of the bosonic fluctuation, whose variation gives the following linearized equation of motion for the fluctuation

$$\Delta_B \begin{pmatrix} \xi \\ \bar{\xi} \end{pmatrix} = 0, \tag{III.7}$$

with

$$\Delta_B = \begin{pmatrix} \{\nabla, \bar{\nabla}\} & -4\varphi_B \mathcal{E} \\ -4\overline{\varphi_B \mathcal{E}} & \{\bar{\nabla}^*, \nabla^*\} \end{pmatrix} + \frac{1}{(1+|\varphi_B|^2)^2} \begin{pmatrix} \partial_i \varphi_B \partial_i \bar{\varphi}_B & -\partial_i \varphi_B \partial_i \varphi_B \\ -\partial_i \bar{\varphi}_B \partial_i \bar{\varphi}_B & \partial_i \bar{\varphi}_B \partial_i \varphi_B \end{pmatrix} + \mathcal{O}(g^2). \tag{III.8}$$

Here we have defined $\mathcal{E}$ as[5]

$$\mathcal{E} = \frac{\mathcal{D}\bar{\partial}\varphi_B}{1+|\varphi_B|^2} = \frac{1}{1+|\varphi_B|^2} \left(\partial - \frac{2\bar{\varphi}_B}{1+|\varphi_B|^2} \partial \varphi_B \right) \bar{\partial} \varphi_B. \tag{III.9}$$

The differential operators ∇, $\bar{\nabla}$, ∇^* and $\bar{\nabla}^*$ are given by

$$\nabla = \partial + 2iA_z, \qquad \bar{\nabla} = \bar{\partial} + 2iA_{\bar{z}}, \qquad \nabla^* = \partial - 2iA_z, \qquad \bar{\nabla}^* = \bar{\partial} - 2iA_{\bar{z}}, \tag{III.10}$$

with $A_z = (A_x - iA_y)/2$ and $A_{\bar{z}} = (A_x + iA_y)/2$ defined as[6]

$$A_i = \frac{i}{2} \frac{\bar{\varphi}_B \partial_i \varphi_B - \varphi_B \partial_i \bar{\varphi}_B}{1+|\varphi_B|^2}. \tag{III.11}$$

It is notable that the expression (III.8) is valid for arbitrary values of $\delta\epsilon$ as long as we work in the weak coupling limit $g^2 \to 0$ since all the $\delta\epsilon$ dependence is included in the $\mathcal{O}(g^2)$ term.

Let S_{q} be the quantum correction induced by the bosonic and fermionic fluctuations

$$\exp(-S_{\text{q}}) = \int \mathcal{D}\Phi \mathcal{D}\Psi \, \exp\big[-(S - S_{\text{eff}}) \big], \tag{III.12}$$

where $\mathcal{D}\Phi\mathcal{D}\Psi$ denotes the path integral measure for the fluctuation. In principle, this can be evaluated by the standard perturbation expansion. In particular, the leading order term is given by the one-loop determinants

$$\exp(-S_{\text{q}}) = \frac{\det \Delta_F}{\det' \Delta_B} \big[1 + \mathcal{O}(g^2) \big]. \tag{III.13}$$

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5 $\mathcal{E} = 0$ is the equation of motion without the deformation term.

6 A_i are auxiliary gauge fields in the gauged linear sigma model realization of the $\mathbb{C}P^1$ sigma model, and the corresponding field strength is the topological charge density.

$\det \Delta_F$ is the fermionic one-loop determinant and $\det' \Delta_B$ is the bosonic one-loop determinant excluding the quasi-zero modes. Then the single bion contribution to the partition function can be rewritten as

$$Z_1 = \int_{\mathcal{M}} dv \, \exp\left(-S_{\text{eff}} - S_{\text{q}}\right). \tag{III.14}$$

where dv denotes the volume form on the quasi-moduli space $\mathcal{M}$.

B. Single bion effective action and renormalization

Let us first calculate the classical effective bion action $S_{\text{eff}}(\eta) = S[\varphi_B(\eta)]$ in the weak coupling limit. As we can see from Eqs. (II.20) and (II.23), the saddle points run away to infinity as $g \to 0$. To focus on the vicinity of the saddle points in the configuration space, we take the weak coupling limit as

$$g \to 0 \quad \text{with fixed} \quad \delta x_r \;\equiv\; x_+ - x_- - \frac{1}{\omega} \log \frac{4\omega^2}{\omega^2 - m^2}, \tag{III.15}$$

where δx_r is the deviation of the relative distance from the value at the real bion saddle point: $\delta x_r = 0$ and $\delta x_r = \pi i/\omega$ for the real and complex bions, respectively (see Eqs. (II.20) and (II.23)). Since the relative distance $|x_+ - x_-|$ is always large in this limit, we can regard

$$\exp(-\omega|x_+ - x_-|\,) \sim \mathcal{O}(g^2). \tag{III.16}$$

In the weak coupling limit, the valley equation (III.1) can be solved as

$$\varphi_B(\eta) \;=\; e^{imy} \left(e^{-\omega(x - x_-) - i\phi_-} + e^{\omega(x - x_+) - i\phi_+} \right)^{-1}. \tag{III.17}$$

We can show that this satisfies the valley equation in the weak coupling limit by substituting $\varphi_B(\eta)$ into the equation of motion $\delta S/\delta\varphi$ and by checking that it is of order $\mathcal{O}(g^2)$ everywhere on $\mathbb{R} \times S^1$. It is notable that this bion ansatz becomes the exact bion solution (II.17) or (II.22) when the quasi-moduli parameters $\eta^\alpha = (x_\pm, \phi_\pm)$ sit at the saddle point (II.20) or at (II.23), respectively. As in the case of the saddle point configuration, $\varphi_B(\eta)$ can be viewed as a superposition of fractional instantons (II.21).

By substituting the bion ansatz (III.17) into the original action with the deformation term, we obtain the single bion effective action[7]

$$S_{\text{eff}}(\eta) = \frac{4\pi R}{g^2} \Big[m - 2m \cos(\phi_+ - \phi_-) e^{-m(x_+ - x_-)} \Big] + 2\delta\epsilon m (x_+ - x_-) + \mathcal{O}\left(g^2\right). \tag{III.18}$$

The first term in $[\cdots]$ is the asymptotic value of the classical bion action, the second term in $[\cdots]$ is the interaction between the constituent fractional instantons at large separation, and the term proportional

[7] It is notable that the effective action contains a subleading term of order $\mathcal{O}(1)$ even though it is derived from the leading order valley solution (III.17). This is justified because the subleading correction $\delta\varphi_B$ does not contribute to the subleading term in S_{eff}, thanks to the definition of the valley solution (III.1) and the orthogonality between $\delta\varphi_B$ and the quasi-zero modes $\partial\varphi_B/\partial\eta^\alpha$.

to $(x_+ - x_-)$ represents the confining potential due to the deformation potential. As mentioned above, we have neglected terms of order $\mathcal{O}(e^{-2m(x_+-x_-)}/g^2)$ since the leading order contribution of the quasi-moduli integral comes from the vicinity of the saddle points, where $\mathcal{O}(e^{-2m(x_+-x_-)}/g^2) \sim \mathcal{O}(g^2)$.

Next, let us integrate out the bosonic and fermionic fluctuations in Eq. (III.6). By evaluating the Gaussian integral for the fluctuations, the single bion contribution to the partition function can be rewritten into the quasi-moduli integral

$$Z_1 = \int_{\mathcal{M}} d^4\eta \, \mathcal{J} \, \frac{\det \Delta_F}{\det' \Delta_B} \exp\left(-S_{\text{eff}}\right) + \mathcal{O}(g^2), \tag{III.19}$$

where $\mathcal{J}$ is volume factor associated with the metric of the quasi-moduli space, $\det \Delta_F$ is the fermionic one-loop determinant, and $\det' \Delta_B$ is the bosonic one-loop determinant excluding the quasi-zero modes. The divergent integral with respect to the center of mass position $x_+ + x_-$ can be regularized by compactifying the x-direction as $x \sim x + \beta$. Then, the single bion contribution to the vacuum energy can be written as

$$E_1 \;\equiv\; -\lim_{\beta\to\infty} \frac{1}{\beta}\frac{Z_1}{Z_0} \;=\; -2\pi \int_{\mathcal{M}_r} dx_r d\phi_r \, \mathcal{J} \, \exp\left(-S_R\right), \tag{III.20}$$

where $\mathcal{M}_r$ is the space of relative quasi-moduli parameterized by $x_r \equiv x_+ - x_-$ and $\phi_r \equiv \phi_+ - \phi_-$. The quantum corrections from the fluctuations is included into the renormalized effective action

$$S_R(x_r, \phi_r) \;\equiv\; S_{\text{eff}}(x_r, \phi_r) - \log \frac{\det \Delta_F}{\det \Delta_F^0} + \log \frac{\det' \Delta_B}{\det \Delta_B^0}, \tag{III.21}$$

where $\det \Delta_F^0$ and $\det \Delta_B^0$ are the one-loop determinants around the trivial saddle point $\varphi = 0$.

Now let us calculate the one-loop determinants. Since the background bion ansatz (III.17) is independent of the compactified coordinate y (except the twisting factor dictated by the twisted boundary condition), it is convenient to decompose the bosonic and fermionic fluctuations into infinite towers of KK modes. The total contribution from the non-zero bosonic KK modes takes the form

$$\log \frac{\det' \Delta_B}{\det \Delta_B^0}\bigg|_{\text{KK}} \;=\; 2\sum_{n=1}^{\infty}\left[X_n + Y_n \cos\phi_r \, e^{-m x_r} + \mathcal{O}(g^2)\right], \tag{III.22}$$

where X_n and Y_n are respectively calculated in Appendices B 1 and B 2 as

$$X_n = \log \frac{\frac{n}{R} - m}{\frac{n}{R} + m}, \qquad Y_n = \frac{4mR}{n} + \mathcal{O}(n^{-2}). \tag{III.23}$$

By using the zeta function regularization shown in (B.16) and (B.19), we find that

$$\sum_{n=1}^{\infty} X_n = -2mR \, \log R\Lambda_0 + \log \frac{\Gamma(1+mR)}{\Gamma(1-mR)}, \qquad \sum_{n=1}^{\infty} Y_n = 4mR \log R\Lambda_0 + \cdots, \tag{III.24}$$

where Λ_0 is a parameter corresponding to the cutoff scale and $\cdots$ denotes terms without Λ_0-dependence, which give only subleading contributions in the weak coupling limit. The Λ_0-dependent terms in the one-loop determinant can be absorbed into the bare coupling constant, *i.e.* they are canceled by appropriate counter terms. The renormalized effective action can be written in terms of the dynamical scale Λ defined by

$$\Lambda = \Lambda_0 \, \exp\left(-\frac{\pi}{g_0^2} - \frac{i}{2}\theta_0\right), \qquad \left(\tau_0 = \frac{1}{2\pi i}\log\frac{\Lambda_0^2}{\Lambda^2}\right), \tag{III.25}$$

where the symbols with subscript 0 denote the bare parameters. In the following, we explicitly denote the coupling constant g which has been used above as the renormalized coupling constant g_R at the scale $1/R$

$$\frac{1}{g_R^2} = -\frac{1}{\pi}\log|R\Lambda| \qquad \left(= \frac{1}{g_0^2} - \frac{1}{\pi}\log|R\Lambda_0|\right), \tag{III.26}$$

and interpret that the UV divergent terms in the one-loop determinant are canceled by the corresponding counter terms in the renormalized perturbation theory.

The contributions of the fermionic KK modes with positive and negative KK momenta cancel out (see Appendix B 3 for details)

$$\log \frac{\det \Delta_F}{\det \Delta_F^0}\bigg|_{\mathrm{KK}} = 0. \tag{III.27}$$

The contributions of the bosonic and fermionic KK zero modes (and the volume factor $\mathcal{J}$) are essentially the same as in the 1D case [25]

$$\mathcal{J}\,\frac{\det' \Delta_B}{\det \Delta_B^0}\bigg|_{n=0} \approx \left(\frac{4m^2R}{g_R^2}\right)^2 + \cdots, \qquad \frac{\det \Delta_F}{\det \Delta_F^0} \approx e^{-2mx_r} + \cdots, \tag{III.28}$$

where $\cdots$ denotes terms which are irrelevant in the weak coupling limit $g_R^2 \to 0$.

Finally, by combining the one-loop determinants (III.22), (III.27) and (III.28), we obtain the renormalized effective bion action (III.21), from which we find that the integrand for the quasi-moduli integral (III.20) is given by

$$\mathcal{J}\exp(-S_R) \; = \; \left(\frac{4m^2R}{g_R^2}\right)^2 \exp\left(-X + \frac{8\pi mR}{g_R^2}\cos\phi_r\, e^{-mx_r} - 2m\epsilon x_r\right) + \cdots, \tag{III.29}$$

where ϵ is the effective deformation parameter, which is shifted due to the fermionic contribution

$$\epsilon \equiv 1 + \delta\epsilon, \tag{III.30}$$

and X is the constant term

$$X \; \equiv \; \frac{4\pi mR}{g_R^2} + 2\log\frac{\Gamma(1+mR)}{\Gamma(1-mR)}. \tag{III.31}$$

This implies that the single bion contribution is of order

$$E_1 \;\propto\; |R\Lambda|^{4mR} \;=\; e^{-4mR\pi/g_R^2(R)}. \tag{III.32}$$

Here we denote g_R^2 as $g_R^2(R)$ to emphasize that it is renormalized at the energy scale $\sim 1/R$. The renormalized effective action S_R can be rewritten in terms of $\log|R\Lambda|$ by replacing $1/g_R^2$ with $-\frac{1}{\pi}\log|R\Lambda|$. We note that the emergence of the renormalized coupling (or the dynamical scale Λ) is a consequence of the renormalization procedure in the semiclassical calculation of bion contributions. *The renormalized coupling constant is NOT inserted by hand but it naturally appears in the semiclassical calculation.*

C. Lefschetz thimble analysis and imaginary ambiguity

Next let us calculate the single bion contribution to the vacuum energy E_1 by evaluating the quasi-moduli integral

$$E_1 \;=\; -2\pi\left(\frac{4m^2R}{g_R^2}\right)^2 e^{-X}\int dx_r d\phi_r\,\exp\left(\frac{8\pi mR}{g_R^2}\cos\phi_r\,e^{-mx_r} - 2m\epsilon x_r\right) + \cdots\,, \tag{III.33}$$

where $\cdots$ denotes irrelevant terms in the weak coupling limit. Since this is essentially the same integral as in the case of the $\mathbb{C}P^1$ quantum mechanics, we can apply the Lefschetz thimble method [139–151] in the same manner as in the 1D case [25] to evaluate the quasi-moduli integral. As shown in Ref. [25] and briefly summarized in Appendix D, S_R has saddle points corresponding to the real and complex bions and their contributions to E_1 contain an imaginary ambiguity due to the Stokes phenomena

$$E_1 \;=\; -2m\frac{\Gamma(\epsilon)}{\Gamma(1-\epsilon)}\left[\frac{4\pi mR}{g_R^2}\frac{\Gamma(1-mR)}{\Gamma(1+mR)}\right]^2\left(\frac{4\pi mR}{g_R^2}e^{\pm\frac{\pi i}{2}}\right)^{-2\epsilon}|R\Lambda|^{4mR} + \cdots\,, \tag{III.34}$$

where the upper (lower) sign corresponds to the positive (negative) imaginary part given to g_R^2 in order to avoid the coupling constant being on the Stokes line. For the $\mathbb{Z}_2$-twisted boundary condition ($m = 1/(2R)$), the result is of order $|R\Lambda|^2$, which is the expected order of the well-known imaginary ambiguity from the infrared renormalon.

The resurgence theory states that there is a cancellation between the imaginary part of the non-perturbative contribution $\mathrm{Im}\,E_{\text{n.p.}}$ and that of the perturbation series $\mathrm{Im}\,E_{\text{pert}}$. Since the single bion imaginary part $\mathrm{Im}\,E_1$ is the dominant term in $\mathrm{Im}\,E_{\text{n.p.}}$, it cancels the leading order part of $\mathrm{Im}\,E_{\text{pert}}$. This implies that the difference of $\mathrm{Im}\,E_{\text{pert}}$ for positive and negative $\mathrm{Im}\,g_R$ has the following asymptotic behavior in the weak coupling limit $g_R \to 0$

$$\Delta\mathrm{Im}\,E_{\text{pert}} \;\sim\; -\Delta\mathrm{Im}\,E_1 \;=\; -4\pi m\left[\frac{1}{\Gamma(1-\epsilon)}\frac{\Gamma(1-mR)}{\Gamma(1+mR)}\right]^2\left(\frac{4\pi mR}{g_R^2}\right)^{2(1-\epsilon)}|R\Lambda|^{4mR}. \tag{III.35}$$

From this imaginary part, we can read off the large order behavior of the perturbation series as

$$E_{\rm pert} \;=\; \sum_{n=0}^{\infty} a_n \left(\frac{g_R^2}{4\pi mR}\right)^n \quad \text{with} \quad a_n \to -2m\left[\frac{1}{\Gamma(1-\epsilon)}\frac{\Gamma(1-mR)}{\Gamma(1+mR)}\right]^2 \Gamma(n+2(1-\epsilon)). \qquad \text{(III.36)}$$

This large order behavior of the perturbation series is the prediction of the resurgence theory. In the limit $R \to 0$ with fixed $g_{\rm 1d}^2 \equiv g_R^2/(2\pi R)$, this system reduces to the $\mathbb{C}P^1$ quantum mechanics and it has been shown that the 1D limit of Eq. (III.36) is the correct large order behavior in quantum mechanics [31]. It would be interesting to check if the result (III.36) obtained by the resurgence argument gives the consistent large order behavior of the perturbation series also in the 2D case.

Next, let us consider the generalization to the case with $n_{\rm F}$ copies of fermions. Since in this case, the fermionic contribution to the bion effective action is multiplied by $n_{\rm F}$, the single bion contribution E_1 can be obtained by redefining the constant ϵ in Eq. (III.34) as

$$\epsilon \;\equiv\; 1+\delta\epsilon \;\to\; \epsilon \;\equiv\; n_{\rm F}+\delta\epsilon. \qquad \text{(III.37)}$$

Expanding E_1 around $\epsilon = n_{\rm F} = 1,\, 2,\, \cdots$, we find that

$$E_1 \;=\; C|R\Lambda|^{4mR}\left[\delta\epsilon + 2\left(\psi(n_{\rm F}) - \log\frac{4\pi mR}{g_R^2} \mp \frac{\pi i}{2}\right)\delta\epsilon^2 + \cdots\right], \qquad \text{(III.38)}$$

where $\psi(n_{\rm F}) \equiv \partial_\epsilon \log\Gamma(\epsilon)\,|_{\epsilon=n_{\rm F}} = -\gamma + \sum_{r=1}^{n_{\rm F}-1}\frac{1}{r}$ is the digamma function and

$$C \equiv -2m\left[\Gamma(n_{\rm F})\frac{\Gamma(1-mR)}{\Gamma(1+mR)}\left(\frac{4\pi mR}{g_R^2}\right)^{1-n_{\rm F}}\right]^2. \qquad \text{(III.39)}$$

Eq. (III.38) implies that the single bion contribution vanishes for $\delta\epsilon = 0$. In the supersymmetric case $n_{\rm F} = 1$, this is consistent with the fact that the vacuum energy vanishes. The absence of the non-perturbative correction at $\epsilon = n_{\rm F} = 1,\, 2,\, \cdots$ is due to the cancellation between the real and complex bion saddle points[8]. This cancellation happens because of a particular high symmetry at these points, whereas the non-perturbative corrections exist ubiquitously anywhere away from these particular points. We can see from Eq. (III.38) that a non-trivial non-perturbative correction and an imaginary ambiguity appear in the leading and subleading order terms in the small $\delta\epsilon$ expansion, respectively. We emphasize that for the $\mathbb{Z}_2$-twisted boundary condition, the imaginary ambiguity is of order $|R\Lambda|^2$, which is consistent with that from the infrared renormalon. This result indicates that the renormalon ambiguity can be canceled by the bion contribution, and hence the bion could be identified as the infrared renormalon.

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[8] This phenomenon can also be seen in some quantum mechanical models in which the so-called quasi-exact-solvability plays an important role [28, 35].

IV. GENERALIZATION TO $\mathbb{C}P^{N-1}$ MODEL

In this section, we consider the generalization of the analysis in the previous section to the $\mathbb{C}P^{N-1}$ model. We compute the single bion contribution to the vacuum energy by embedding the single bion solutions of the $\mathbb{C}P^1$ model, calculating the one-loop determinant and evaluating the quasi-moduli integral.

A. Embedding single bion solution

Let us consider the 2D $\mathcal{N}=(2,2)$ $\mathbb{C}P^{N-1}$ sigma model described by the Lagrangian

$$\mathcal{L} \;=\; \frac{2}{g^2}\left[G_{a\bar{b}}\left(\partial\varphi^a\bar{\partial}\bar{\varphi}^{\bar{b}}+\bar{\partial}\varphi^a\partial\bar{\varphi}^{\bar{b}}-\bar{\psi}_l^{\bar{b}}\,\mathcal{D}\psi_l^a-\bar{\psi}_r^{\bar{b}}\,\bar{\mathcal{D}}\psi_r^a\right)+\frac{1}{2}R_{a\bar{b}c\bar{d}}\,\psi_l^a\bar{\psi}_l^{\bar{b}}\psi_r^c\bar{\psi}_r^{\bar{d}}\right]+\mathcal{L}_{\rm top}, \tag{IV.1}$$

where $\mathcal{L}_{\rm top}$ is the topological term

$$\mathcal{L}_{\rm top}=\frac{i\theta}{\pi}G_{a\bar{b}}\left(\partial\varphi^a\bar{\partial}\bar{\varphi}^{\bar{b}}-\bar{\partial}\varphi^a\partial\bar{\varphi}^{\bar{b}}\right), \tag{IV.2}$$

$G_{a\bar{b}}$ $(a,\bar{b}=1,\cdots N-1)$ is the standard Fubini-Study metric

$$G_{a\bar{b}}=\frac{\partial^2}{\partial\varphi^a\partial\bar{\varphi}^{\bar{b}}}\log\left(1+\sum_{c=1}^{N-1}|\varphi^c|^2\right), \tag{IV.3}$$

$\mathcal{D}$ and $\bar{\mathcal{D}}$ are pullbacks of the covariant derivative

$$\mathcal{D}\psi_l^a=\partial\psi_l^a+\Gamma^a_{bc}\partial\varphi^b\psi_l^c,\qquad\qquad \bar{\mathcal{D}}\psi_r^a=\bar{\partial}\psi_r^a+\Gamma^a_{bc}\bar{\partial}\varphi^b\psi_r^c, \tag{IV.4}$$

and Γ^a_{bc}, $\bar{\Gamma}^{\bar{a}}_{\bar{b}\bar{c}}$ and $R_{a\bar{b}c\bar{d}}$ are the Christoffel symbol and curvature tensor

$$\Gamma^a_{bc}=G^{\bar{d}a}\partial_bG_{c\bar{d}},\qquad \bar{\Gamma}^{\bar{a}}_{\bar{b}\bar{c}}=G^{\bar{a}d}\partial_{\bar{b}}G_{d\bar{c}},\qquad R^{\bar{a}}{}_{\bar{b}c\bar{d}}=\partial_c\bar{\Gamma}^{\bar{a}}_{\bar{b}\bar{d}}. \tag{IV.5}$$

As in the case of the $\mathbb{C}P^1$ model, we impose the twisted boundary conditions

$$\varphi^a(y+2\pi R)=e^{2\pi i m_a R}\,\varphi^a(y),\qquad\qquad \psi^a_{l,r}(y+2\pi R)=e^{2\pi i m_a R}\,\psi^a_{l,r}(y), \tag{IV.6}$$

and introduce the following deformation potential which breaks the $\mathcal{N}=(2,2)$ supersymmetry

$$\delta\mathcal{L}\;\equiv\;-\frac{\delta\epsilon}{N\pi R}\Delta\mu\;=\;\frac{\delta\epsilon}{\pi R}\left(\mu-\frac{1}{N}\sum_{a=1}^{N-1}m_a\right),\qquad\qquad \mu\equiv\sum_{a=1}^{N-1}\frac{m_a|\varphi^a|^2}{1+\sum_{c=1}^{N-1}|\varphi^c|^2}, \tag{IV.7}$$

where Δ is the Laplacian on the $\mathbb{C}P^{N-1}$ and μ is the moment map of the $U(1)$ symmetry used for the twisted boundary condition in Eq. (IV.6). Again, this deformation is inspired by the term induced by fermions in quantum mechanics [35] . With this potential term, we can embed the single bion configuration φ_B in Eq. (III.17) as a valley solution in the $\mathbb{C}P^{N-1}$ model. For example, φ_B can be embedded into the b-th component as

$$\varphi^a = 0 \quad (a \neq b), \qquad \varphi^b = \varphi_B = e^{im_b y}\left(e^{-\omega_b(x-x_-)-i\phi_-} + e^{\omega_b(x-x_+)-i\phi_+}\right)^{-1}, \tag{IV.8}$$

where we have defined the parameter ω_b by replacing the parameter ω in (II.18) as

$$\omega \;\rightarrow\; \omega_b \equiv m_b\sqrt{1+\frac{g^2\delta\epsilon}{\pi m_b R}}. \tag{IV.9}$$

As in the $\mathbb{C}P^1$ case, the valley solution satisfies the equation of motion of the deformed $\mathbb{C}P^{N-1}$ sigma model if the quasi-moduli parameters $x_\pm$ are adjusted to the values at the saddle points: for the real bion, for instance,

$$x_\pm^{\rm rb} = x_0 \pm \frac{1}{2\omega_b}\log\frac{4\omega_b^2}{\omega_b^2-m_b^2}, \qquad e^{i\phi_\pm^{\rm rb}} = \pm e^{i\phi_0}. \tag{IV.10}$$

The classical bion effective action $S_{\rm eff}(\eta)$ in the weak coupling limit (III.15) now becomes

$$S_{{\rm eff},b}(\eta) \;=\; \frac{4\pi m_b R}{g^2}\left[1-2\cos(\phi_+-\phi_-)e^{-m_b(x_+-x_-)}\right] + 2\delta\epsilon m_b|x_+-x_-| + \mathcal{O}\left(g^2\right), \tag{IV.11}$$

where we have assumed that $e^{-m_b(x_+-x_-)} \sim \mathcal{O}(g^2)$ as in the $\mathbb{C}P^1$ case.

Generalizing (III.20) for the $\mathbb{C}P^1$ case, the single bion contribution of the $\mathbb{C}P^{N-1}$ model is given as a sum over the bion backgrounds as

$$E_1 \;\equiv\; -\lim_{\beta\to\infty}\frac{1}{\beta}\frac{Z_1}{Z_0} \;=\; -2\pi\sum_{b=1}^{N-1}\int_{\mathcal{M}_{r,b}} dx_r d\phi_r\, \mathcal{J}_b \exp\left(-S_{R,b}\right), \tag{IV.12}$$

where $\mathcal{J}_b$ is the volume factor of the relative quasi-moduli space $\mathcal{M}_{r,b}$ and $S_{R,b}$ is the renormalized effective action including all the contributions from the fluctuations

$$S_{R,b} \;\equiv\; S_{{\rm eff},b} + \sum_a\left(\log\frac{\det'\Delta_B}{\det\Delta_B^0}\bigg|_a - \log\frac{\det\Delta_F}{\det\Delta_F^0}\bigg|_a\right). \tag{IV.13}$$

In the next subsections, we discuss the renormalization of the bion effective action due to the fluctuations around the bion background (IV.8).

B. One-loop determinants

Let us consider the bosonic and fermionic fluctuations around the bion configuration in Eq. (IV.8). We can show that the fluctuations of the fields in the b-th component $(\varphi^b, \psi_l^b, \psi_r^b)$ (the same component as the bion background) give the identical contribution as in the $\mathbb{C}P^1$ case.

The one-loop determinant for the bosonic fluctuation $\delta\varphi^a$ $(a \neq b)$ can be calculated by the KK expansion. The contributions from the KK zero modes have been calculated in Ref. [35]

$$\log \frac{\det' \Delta_B}{\det \Delta_B^0}\bigg|_{n=0,\,a} = \log\left(\frac{m_a - m_b}{m_a}\right) - m_b x_r + \mathcal{O}(g^2). \tag{IV.14}$$

As shown in Appendix C, the total contribution of the KK modes is given by

$$\sum_{n\neq 0} \log \frac{\det' \Delta_B}{\det \Delta_B^0}\bigg|_{n,\,a} = \sum_{n=1}^{\infty}\Big[X_{n,a} + Y_{n,a}\cos\phi_r\, e^{-m_b x_r} + \mathcal{O}(g^2)\Big], \tag{IV.15}$$

where $X_{n,a}$ and $Y_{n,a}$ are given by (see Appendices C 1 and C 2, respectively for details)

$$X_{n,a} \equiv \log \frac{\frac{n}{R} + (m_a - m_b)}{\frac{n}{R} - (m_a - m_b)} \frac{\frac{n}{R} - m_a}{\frac{n}{R} + m_a}, \qquad Y_{n,a} \equiv \frac{4 m_b R}{n} + \mathcal{O}(n^{-2}), \tag{IV.16}$$

respectively. The zeta function regularization gives

$$\sum_{n=1}^{\infty} X_{n,a} = -2 m_b R \log R\Lambda_0 - \log \frac{\Gamma\left(1 + (m_a - m_b)R\right)}{\Gamma\left(1 - (m_a - m_b)R\right)} \frac{\Gamma\left(1 - m_a R\right)}{\Gamma\left(1 + m_a R\right)}, \tag{IV.17}$$

$$\sum_{n=1}^{\infty} Y_{n,a} = 4 m_b R \log R\Lambda_0 + \cdots, \tag{IV.18}$$

where $\cdots$ denotes irrelevant terms in the weak-coupling limit.

The one-loop determinant of the fermionic fluctuations can also be calculated by the KK decomposition. As in the $\mathbb{C}P^1$ case, the contributions from the non-zero modes cancel out and only the KK zero modes contribute to the determinant. In the weak coupling limit, the total contribution from the fermionic fluctuations is given by (see Appendix. C 3)

$$\sum_{a=1}^{N-1} \log \frac{\det \Delta_F}{\det \Delta_F^0}\bigg|_{a} = -N m_b x_r + \mathcal{O}(g^2). \tag{IV.19}$$

C. Contribution to partition function

The one-loop determinants of the b-th component fields can be obtained from Eqs. (III.22)-(III.24) and (III.28) by replacing $m \to m_b$. Combining it with the classical bion effective action (IV.11) and the one-loop determinants of the a-th components ($a \neq b$) given in Eqs. (IV.14), (IV.15) and (IV.19), we find that the integrand of the quasi-moduli integral (IV.12) is given by

$$\mathcal{J}_b \exp\left(-S_{R,b}\right) = \mathcal{A}_b \left(\frac{4 m_b^2 R}{g_R^2}\right)^2 \exp\left(-\frac{4\pi m_b R}{g_R^2}\right) \exp\left(\frac{8\pi m_b R}{g_R^2}\cos\phi_r\, e^{-m_b x_r} - 2 m_b \epsilon x_r\right) + \cdots, \tag{IV.20}$$

where we have defined the constant $\mathcal{A}_b$ as

$$\mathcal{A}_b \equiv \left[\frac{\Gamma(1-m_bR)}{\Gamma(1+m_bR)}\right]^2 \prod_{a\neq b} \frac{m_a}{m_a-m_b} \frac{\Gamma\left(1+(m_a-m_b)R\right)}{\Gamma\left(1-(m_a-m_b)R\right)} \frac{\Gamma\left(1-m_aR\right)}{\Gamma\left(1+m_aR\right)}. \tag{IV.21}$$

The parameter ϵ is the the effective deformation parameter defined in the same way as in the $\mathbb{C}P^1$ model[9]

$$\epsilon \equiv 1+\delta\epsilon. \tag{IV.22}$$

Note that g_R^2 in the renormalized effective action Eq. (IV.20) is the coupling constant renormalized at $1/R$, which appeared as a result of the renormalization procedure. The renormalized effective action can also be written in terms of Λ related to g_R^2 as

$$\frac{1}{g_R^2} = -\frac{N}{2\pi}\log|R\Lambda| \qquad \left(= \frac{1}{g_0^2} - \frac{N}{2\pi}\log|R\Lambda_0| \right). \tag{IV.23}$$

Since the renormalized effective action in Eq. (IV.20) has the same form as that in the $\mathbb{C}P^1$ case, we can apply the Lefschetz thimble method to the quasi-moduli integral (IV.12) as in the previous case. Using the results of the thimble integral summarized in Appendix D, we finally obtain the single bion contribution to the vacuum energy

$$E_1 = -\frac{\Gamma(\epsilon)}{\Gamma(1-\epsilon)} e^{\mp\pi i\epsilon} \sum_{b=1}^{N-1} 2m_b\mathcal{A}_b \left(\frac{4\pi m_b R}{g_R^2}\right)^{2(1-\epsilon)} |R\Lambda|^{2Nm_bR} + \cdots, \tag{IV.24}$$

where the upper (lower sign) corresponds to the positive (negative) imaginary part given to g_R^2 in order to avoid the Stokes line in the parameter space. Again, the single bion contribution vanishes when the deformation parameter $\delta\epsilon$ is turned off ($\epsilon = 1$). This is consistent with the fact that the vacuum energy is exactly zero for a supersymmetric vacuum. A similar phenomenon can be seen in the model with n_{F} copies of fermionic degrees of freedom. For $n_{\mathrm{F}} = 1, 2, \cdots$, the one-loop determinant from the fermion fluctuations (IV.19) is multiplied by n_{F} so that the result (IV.24) can be easily generalized to the case of n_{F} copies of fermion by redefining ϵ as

$$\epsilon \equiv 1+\delta\epsilon \quad \to \quad \epsilon \equiv 1+\delta\epsilon + \frac{N}{2}(n_{\mathrm{F}}-1). \tag{IV.25}$$

In the case of quantum mechanics, the general theorem states [28, 35] that the vacuum energy vanishes when this ϵ becomes a positive integer. Also in the 2D case, Eq. (IV.24) implies that E_1 vanishes for $\epsilon \in \mathbb{Z}_{\geq 0}$. Hence in the absence of the deformation $\delta\epsilon$, the single bion contribution vanishes for any n_{F} when N is even. On the other hand, when N is odd, there is a non-trivial non-perturbative correction for even n_{F}. Such a case provides a simple example of a non-trivial bion correction and resurgence structure

[9] It would be helpful to comment that the linear term $-2m_bx_r$ in Eq. (IV.20) emerges due to the bosonic KK zero modes in Eq. (IV.14) and the fermionic KK zero modes in Eq. (IV.19) as $(N-2)m_bx_r - Nm_bx_r = -2m_bx_r$.

without the deformation parameter $\delta\epsilon$.

When $E_1 = 0$ in the absence of the deformation, we need to consider expansion in powers of the deformation parameter $\delta\epsilon$ in order to see non-vanishing non-perturbative contributions

$$E_1 = \sum_{b=1}^{N-1} C_b |R\Lambda|^{2Nm_bR} \left[\delta\epsilon + 2\left(\psi(\epsilon_0) - \log\frac{4\pi m_b R}{g_R^2} \mp \frac{\pi i}{2}\right)\delta\epsilon^2 + \cdots\right], \tag{IV.26}$$

where $\psi(\epsilon_0)$ is the digamma function, $\epsilon_0 \equiv \epsilon\,|_{\delta\epsilon=0} \in \mathbb{Z}_{\geq 0}$ and

$$C_b = -2m_b\mathcal{A}_b\left[\Gamma(\epsilon_0)\left(\frac{4\pi m_b R}{g_R^2}\right)^{1-\epsilon_0}\right]^2. \tag{IV.27}$$

Although the imaginary ambiguity disappears at the leading order of the expansion, the higher order expansion coefficients have non-trivial imaginary parts with ambiguous signs. Since these ambiguities in the non-perturbative bion contribution should be canceled by the Borel resummation ambiguity of the perturbation series, the large order behavior of the perturbation series can be determined from this single bion contribution as in the case of the $\mathbb{C}P^1$ model.

For the $\mathbb{Z}_N$-symmetric boundary condition, which is realized by setting the parameters m_a ($a = 1, \cdots, N-1$) as $m_a = \frac{a}{N}\frac{1}{R}$, the leading term of the result Eq. (IV.26) is of order $|R\Lambda|^2$. This is consistent with the renormalon contribution $|\Lambda|^2 \propto e^{-2\pi/(g_R^2 N)}$. This result again indicates that the renormalon ambiguity can be canceled by the bion contribution, and hence the bion could be identified as the infrared renormalon.

V. SUMMARY AND DISCUSSION

In this paper, we have calculated the semiclassical contributions from the bion saddle points in the $\mathbb{C}P^{N-1}$ models on $\mathbb{R}\times S^1$ with twisted boundary conditions, with emphasis on its consistency with the infrared renormalon. We have discussed the bion contributions in the 2D $\mathcal{N} = (2,2)$ supersymmetric model and its non-supersymmetric generalization with n_{F} copies of fermions and the deformation parameter $\delta\epsilon$, including the cases corresponding to quasi-exactly-solvable model upon the dimensional reduction to quantum mechanics. We have derived the bion solutions composed of a pair of fractional instanton and anti-instanton by promoting that of the $\mathbb{C}P^{N-1}$ quantum mechanics to the 2D system. We discussed the renormalization of the effective action on the quasi-moduli space of the bion configurations, which is parametrized by the relative distance and phase between the fractional instanton and anti-instanton. The quantum fluctuation around the bion background renormalizes the coupling constant in the effective action, leading to the emergence of the dynamical scale. From the renormalized effective action, we obtained the bion contribution to the vacuum energy. Although the vacuum energy vanishes for the SUSY and quasi-exactly-solvable cases, we have shown that there are non-vanishing bion contributions exhibiting the structure expected from the resurgence theory by expanding the vacuum energy in powers of the deformation parameter $\delta\epsilon$. We showed that the imaginary ambiguity in the bion contributions is consistent with the expected infrared renormalon ambiguity. This is the first result revealing the explicit relation between the bion contribution and the infrared renormalon ambiguity in quantum field theories.

One of topics left for future works is the large-N limit with the $\mathbb{Z}_N$ twisted boundary condition. Since the $\mathbb{Z}_N$ symmetric phase has been shown to be continuous [120] as the compactification radius is increased, it would be interesting to study whether the bion contribution survives in the large-N limit. Studying the large radius limit will make the relation between the bion and the renormalon more direct.

In this paper we have not considered the twisted masses for the chiral multiplets, which can be introduced without breaking the $\mathcal{N} = (2, 2)$ supersymmetry. The twisted masses give a potential term proportional to a squared norm of a linear combination of the Killing vectors for the holomorphic isometries. Although there is no essential change in the single bion solutions, such potential terms can modify the one-loop determinants. It would be interesting to discuss how the bion contributions and imaginary ambiguities are modified in the presence of the twisted masses.

We here comment on the full trans-series and complex multi-bion solutions. In the $\mathbb{C}P^{N-1}$ quantum mechanics, the multi-bion contributions are building blocks of the full trans-series of physical quantities. Such multi-bion solutions give non-perturbative contributions also in the present field theoretical case. However, we should remember that they are not enough in the 2D $\mathbb{C}P^{N-1}$ quantum field theory. In addition to them, there are bion configurations composed of instanton and anti-instanton each of which has an integer topological charge. Such configurations also contribute to the full trans-series and it is quite possible that they may become more important as we increase the compactification radius R.

Since the $\mathbb{C}P^1$ manifold can be embedded into any Kähler manifolds of the form of coset spaces G/H, our work can be generalized to 2D $\mathcal{N} = 2$ SUSY nonlinear sigma models on Kähler G/H manifolds and their SUSY breaking deformations.

In 4D gauge theory such as Yang-Mills and QCD with an appropriate compactification, we may be able to take a similar procedure to derive contributions from bion configurations. One of the important questions in these theories is what are quasi-moduli and whether one can perform the quasi-moduli integral. Another question is whether bions are complex solutions of the complexified gauge theory. Future works are devoted to the investigation on these questions.

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Appendix A: An example of quasi-moduli space

In this appendix, we illustrate the concept of the quasi-moduli space by using an example of a simple zero dimensional model. Let us consider the perturbation expansion of the following integral on $\mathbb{R}^2$

$$Z \equiv \int d^2\phi \, \exp\left(-S/g^2\right) \quad \text{with} \quad S \equiv S_0 + g^2 S_2, \quad S_0 \equiv \Big[(\phi^1)^2 + (\phi^2)^2 - 1\Big]^2, \quad S_2 \equiv 2\epsilon(\phi^1)^2. \tag{A.1}$$

The leading order part S_0 is invariant under the rotation and has degenerate minima corresponding to the spontaneously broken rotational symmetry

$$\left.\frac{\partial S_0}{\partial \phi^i}\right|_{\phi=\varphi_0} = 0 \quad \Longrightarrow \quad \begin{pmatrix} \varphi_0^1 \\ \varphi_0^2 \end{pmatrix} \equiv \begin{pmatrix} \cos\eta \\ \sin\eta \end{pmatrix}. \tag{A.2}$$

This flat direction is lifted by the symmetry breaking term S_2, so that η can be viewed as the pseudo-Nambu-Goldstone mode and only two points ($\eta = \pm\frac{\pi}{2}$) remain the discrete minima of S. Since the symmetry breaking term vanishes in the weak coupling limit $g \to 0$, the parameter η can also be viewed as the quasi-modulus, which becomes exact flat direction for $g = 0$. Let us determine the quasi-moduli space, which is described by the embedding $\eta \to \phi^i = \varphi^i(\eta)$ satisfying the valley equation[10]

$$0 = P \left.\begin{pmatrix} \partial_{\phi^1} S \\ \partial_{\phi^2} S \end{pmatrix}\right|_{\phi=\varphi(\eta)}, \tag{A.3}$$

where P is the projection operator onto the "massive" direction

$$P \;=\; \mathbf{1}_2 - \frac{1}{(\partial_\eta \varphi^i)^2} \begin{pmatrix} \partial_\eta \varphi^1 \\ \partial_\eta \varphi^2 \end{pmatrix} \begin{pmatrix} \partial_\eta \varphi^1 & \partial_\eta \varphi^2 \end{pmatrix}. \tag{A.4}$$

Starting from $\varphi_0^i(\eta)$, we can perturbatively solve the valley solution. Let $\xi^i(\eta)$ be the deviation from $\varphi_0^i(\eta)$ satisfying the orthogonality condition

$$\begin{pmatrix} \varphi^1(\eta) \\ \varphi^2(\eta) \end{pmatrix} \;=\; \begin{pmatrix} \varphi_0^1(\eta) + \xi^1(\eta) \\ \varphi_0^2(\eta) + \xi^2(\eta) \end{pmatrix}, \qquad \xi_i \, \partial_\eta \varphi_0^i = 0. \tag{A.5}$$

[10] This valley equation can be obtained from Eqs. (III.1) and (III.2) for the $\mathbb{C}P^1$ model by replacing $G \to \delta_{ij}$ and $g^{\bar{\beta}\alpha} \to g^{\eta\eta} = 1/(\partial_\eta \varphi^i)^2$.

Expanding ξ^i as $\xi^i(\eta) = g^2\xi_2^i(\eta) + g^4\xi_4^i(\eta) + \cdots$, the valley equation can be solved order-by-order

$$h\begin{pmatrix}\xi_2^1\\ \xi_2^2\end{pmatrix} = -4\cos^2\eta\begin{pmatrix}\cos\eta\\ \sin\eta\end{pmatrix} \quad\Longrightarrow\quad \begin{pmatrix}\xi_2^1\\ \xi_2^2\end{pmatrix} = -\frac{1}{2}\epsilon\cos^2\eta\begin{pmatrix}\cos\eta\\ \sin\eta\end{pmatrix},$$
$$h\begin{pmatrix}\xi_4^1\\ \xi_4^2\end{pmatrix} = 2(3\cos^2\eta - 4)\cos^2\eta\begin{pmatrix}\cos\eta\\ \sin\eta\end{pmatrix} \quad\Longrightarrow\quad \begin{pmatrix}\xi_2^1\\ \xi_2^2\end{pmatrix} = \frac{1}{4}(3\cos^2\eta - 4)\cos^2\eta\begin{pmatrix}\cos\eta\\ \sin\eta\end{pmatrix},$$
$$\vdots$$

where h is the Hessian of S_0

$$h = 8\begin{pmatrix}\cos^2\eta & \sin\eta\cos\eta\\ \sin\eta\cos\eta & \sin^2\eta\end{pmatrix}. \tag{A.6}$$

Actually, in the case of this simple example, we can exactly solve the valley equation without expansion. By setting

$$\begin{pmatrix}\varphi^1(\eta)\\ \varphi^2(\eta)\end{pmatrix} = r(\eta)\begin{pmatrix}\cos\eta\\ \sin\eta\end{pmatrix}, \tag{A.7}$$

the valley equation can be rewritten as

$$0 = \frac{4rX}{r^2 + (\partial_\eta r)^2}\begin{pmatrix}\partial_\eta(r\sin\eta)\\ -\partial_\eta(r\cos\eta)\end{pmatrix}, \quad \text{with} \quad X \equiv g^2\epsilon\cos\eta\,\partial_\eta(r\sin\eta) + r(r^2-1). \tag{A.8}$$

The solution of $X = 0$ satisfying $\lim_{g\to 0} r = 1$ is an ellipse

$$r(\eta) = \sqrt{\frac{1-g^2\epsilon}{1-g^2\epsilon\sin^2\eta}}, \qquad \left(\frac{1}{1-g^2\epsilon}(\varphi^1)^2 + (\varphi^2)^2 = 1\right). \tag{A.9}$$

On this quasi-moduli space, the effective action is given by

$$S_{\text{eff}}(\eta) = S\,|_{\phi=\varphi(\eta)} = \left[r(\eta)^2 - 1\right]^2 - 2(1-g^2\epsilon)\left[r(\eta)^2 - 1\right]. \tag{A.10}$$

By changing the valuables from (ϕ^1, ϕ^2) to $(\eta, \delta\varphi)$

$$\begin{pmatrix}\phi^1\\ \phi^2\end{pmatrix} = \begin{pmatrix}\varphi^1(\eta)\\ \varphi^2(\eta)\end{pmatrix} + \frac{g\,\delta\varphi}{\sqrt{(\partial_\eta\varphi^1)^2 + (\partial_\eta\varphi^2)^2}}\begin{pmatrix}\partial_\eta\varphi^2\\ -\partial_\eta\varphi^1\end{pmatrix}, \tag{A.11}$$

the original integral can be rewritten as

$$Z = \int_0^{2\pi} d\eta\,\sqrt{g_{\eta\eta}}\,\exp\left(-\frac{S_{\text{eff}}}{g^2} - S_q\right), \tag{A.12}$$

where S_q is given by

$$\exp\left(-S_q\right) = g\int d\delta\varphi\,\frac{\sqrt{\tilde{g}_{\eta\eta}}}{\sqrt{g_{\eta\eta}}}\exp\left(-\frac{S_{\rm fluc}}{g^2}\right), \qquad S_{\rm fluc}\equiv S-S_{\rm eff}, \tag{A.13}$$

with

$$\frac{\sqrt{\tilde{g}_{\eta\eta}}}{\sqrt{g_{\eta\eta}}} = \frac{\sqrt{(\partial_\eta\phi^1)^2+(\partial_\eta\phi^2)^2}}{\sqrt{(\partial_\eta\varphi^1)^2+(\partial_\eta\varphi^2)^2}} = 1+\frac{g\,\delta\varphi}{1-g^2\epsilon}\left(\frac{r^2}{\sqrt{r^2+(\partial_\eta r)^2}}\right)^3. \tag{A.14}$$

Thanks to the definition of the valley solution, there is no linear term of $\delta\varphi$ in $S_{\rm fluc}$

$$\frac{S_{\rm fluc}}{g^2} = \delta\varphi^2\left[\left(g\,\delta\varphi+\frac{2r^2}{\sqrt{r^2+(\partial_\eta r)^2}}\right)^2+\frac{2r^2(\partial_\eta r)^2}{r^2+(\partial_\eta r)^2}\right] = 4\delta\varphi^2+\mathcal{O}(g), \tag{A.15}$$

so that we can integrate out $\delta\varphi$ by expanding the integrand in powers of g^2 as in the standard perturbation theory

$$\exp\left(-S_q\right) = \frac{\sqrt{\pi}g}{2}\left[1+\frac{g^2\epsilon}{2}\cos^2\eta+\mathcal{O}(g^4)\right]. \tag{A.16}$$

Then the integrand for Z can be expanded as

$$Z = \frac{\sqrt{\pi}g}{2}\int_0^{2\pi}d\eta\,\left[1+g^2\epsilon^2\cos^4\eta+\mathcal{O}(g^4)\right]\exp\left(-2\epsilon\cos^2\eta\right). \tag{A.17}$$

Evaluating the quasi-moduli integral, we obtain the perturbation series of Z

$$Z = \pi^{\frac{3}{2}}g\,e^{-\epsilon}I_0(\epsilon)\left[1+\frac{g^2\epsilon}{4}\left(2\epsilon-(1+2\epsilon)\frac{I_1(\epsilon)}{I_0(\epsilon)}\right)+\mathcal{O}(g^4)\right], \tag{A.18}$$

where $I_0(\epsilon)$ denotes the modified Bessel function of the first kind. We can check that the perturbation series derived in this way is consistent with that which can be obtained from the Borel resummed form of Z

$$Z=\frac{\pi}{2}\int_0^\infty dt\,e^{-\frac{t}{g^2}}\frac{f\left(1+\sqrt{t}\right)+f\left(1-\sqrt{t}\right)}{\sqrt{t}}, \qquad f(z)=e^{-\epsilon z}I_0(\epsilon z), \tag{A.19}$$

which can be obtained by the change of variables $(\phi^1,\phi^2)=r(t)\,(\cos\eta,\sin\eta)$ with

$$r(t)=\begin{cases}\sqrt{1+\sqrt{t}} & \text{for } |\phi^i|^2>1\\ \sqrt{1-\sqrt{t}} & \text{for } |\phi^i|^2<1\end{cases}, \tag{A.20}$$

and the η integration.

Appendix B: One-loop determinants around single bion background in $\mathbb{C}P^1$ model

In this appendix, we calculate the one-loop determinant around the single bion ansatz (III.17) in the $\mathbb{C}P^1$ model. For simplicity, we fix the center of mass position and overall phase and set $x_\pm = \pm x_r/2$ and $\phi_\pm = \pm\phi_r/2$. We will use the following theorem to calculate functional determinants (see, e.g., Appendix B of Ref. [25]). Let $\xi^\pm$ be functions such that

$$(-\partial_x^2 + V)\,\xi^\pm = 0, \qquad \lim_{x\to\pm\infty} \xi^\pm = e^{\mp Mx}, \qquad (M^2 \equiv V(x\to\pm\infty)). \tag{B.1}$$

Then the functional determinant of $-\partial_x^2 + V$ is given by

$$\frac{\det(-\partial_x^2 + \ V\ )}{\det(-\partial_x^2 + M^2)} = \lim_{x\to\mp\infty} e^{\pm Mx}\xi^\pm. \tag{B.2}$$

1. Bosonic one-loop determinant in the KK decomposition

Let us first consider the functional determinant of Δ_B defined in Eq. (III.8). Since the background is independent of y except for the twist factor, it is convenient to use the KK expansion for the bosonic fluctuation

$$\xi = \sum_{n=-\infty}^{\infty} \xi_n(x)\, e^{i\left(\frac{n}{R}+m\right)y}. \tag{B.3}$$

Then the functional determinant $\det\Delta_B$ decomposes into an infinite product of the contributions from the KK modes $\det\Delta_B|_n$. Since the contribution from the KK zero mode is essentially the same as the 1D case [25], we focus on the non-zero mode ($n \neq 0$) in the following.

As explained in Sec. III B, we consider the weak coupling limit keeping the background bion ansatz in the vicinity of the saddle points. In other words, we fix the deviation from the saddle point

$$\delta x_r \ \equiv\ x_r - \frac{1}{\omega}\log\frac{4\omega^2}{\omega^2 - m^2}, \tag{B.4}$$

and take the weak coupling limit $g \to 0$. In this limit, the operator Δ_B in Eq. (III.8) becomes a diagonal matrix, i.e. the equations for ξ and $\bar{\xi}$ become independent. For the n-th KK mode of ξ, the leading order part of Δ_B is given by

$$\Delta_B|_n = \begin{cases} \left[\partial_x + \frac{n}{R} - m\tanh(md_-)\right]\left[\partial_x - \frac{n}{R} + m\tanh(md_-)\right] + \mathcal{O}(g^2) & \text{for } x \approx -x_r/2 \\ \left[\partial_x + \frac{n}{R} - m\right]\left[\partial_x - \frac{n}{R} + m\right] + \mathcal{O}(g^2) & \text{for } x \approx 0 \\ \left[\partial_x - \frac{n}{R} - m\tanh(md_+)\right]\left[\partial_x + \frac{n}{R} + m\tanh(md_+)\right] + \mathcal{O}(g^2) & \text{for } x \approx x_r/2 \end{cases}, \tag{B.5}$$

where $d_\pm$ are the distances from the constituent fractional instantons

$$d_\pm \equiv x \mp \frac{x_r}{2}. \tag{B.6}$$

By applying the theorem (B.2), we can determine $\det \Delta_B|_n$, from the solution of the differential equation $\Delta_B|_n\, \xi = 0$. Solving the equation in each region, we obtain the solution as

$$\xi = \begin{cases} \frac{1}{2}\, e^{\frac{n}{R}x - \frac{mx_r}{2}}\, \mathrm{sech}(md_-) \Big[a_1 - a_2 F_-(x) \Big] & + \mathcal{O}(g^2) \quad \text{for } x \approx -x_r/2 \\ b_1\, e^{\left(\frac{n}{R}-m\right)x} + b_2\, e^{-\left(\frac{n}{R}-m\right)x} & + \mathcal{O}(g^2) \quad \text{for } x \approx 0 \\ \frac{1}{2}\, e^{-\frac{n}{R}x - \frac{mx_r}{2}}\, \mathrm{sech}(md_+) \Big[c_1 + c_2 F_+(x) \Big] & + \mathcal{O}(g^2) \quad \text{for } x \approx x_r/2 \end{cases}, \tag{B.7}$$

where $F_\pm(x)$ are given by

$$F_\pm(x) \;=\; 8\left(\frac{n}{R} + m\right) e^{mx_r} \int dx\, e^{\pm \frac{2n}{R}x} \cosh^2(md_\pm). \tag{B.8}$$

Let us consider the KK modes with $n > 0$. The solution ξ^- which degreases exponentially for $x \to -\infty$ can be obtained by setting $a_1 = 1,\ a_2 = 0$ in the general solution (B.7). Then, by connecting the solutions in the neighboring regions, we can determine the coefficients b_1 and c_2 as

$$b_1 = e^{-mx_r}, \qquad c_2 = \frac{\frac{n}{R} - m}{\frac{n}{R} + m} e^{-2mx_r}. \tag{B.9}$$

From the asymptotic form of ξ_- for large x, we find that $\det \Delta_B|_n$ for $n > 0$ is given by

$$\left.\frac{\det \Delta_B}{\det \Delta_B^0}\right|_n = \lim_{x\to\infty} e^{-\left(\frac{n}{R}+m\right)x}\, \xi^- \;=\; c_2 \;=\; \frac{\frac{n}{R} - m}{\frac{n}{R} + m} e^{-2mx_r}. \tag{B.10}$$

For $n < 0$, the solution ξ^- which degreases exponentially for $x \to -\infty$ can be obtained by setting $a_1 = 0,\ a_2 = 1$. Connecting the solution, we can determine b_2 and c_1 as

$$b_2 = \frac{\frac{n}{R} + m}{\frac{n}{R} - m} e^{mx_r}, \qquad c_1 = \frac{\frac{n}{R} + m}{\frac{n}{R} - m} e^{2mx_r}. \tag{B.11}$$

Therefore, $\det \Delta_B|_n$ for $n < 0$ is given by

$$\left.\frac{\det \Delta_B}{\det \Delta_B^0}\right|_n = \lim_{x\to\infty} e^{\left(\frac{n}{R}+m\right)x}\, \xi^- \;=\; c_1 \;=\; \frac{\frac{n}{R} + m}{\frac{n}{R} - m} e^{2mx_r}. \tag{B.12}$$

Combining the contributions from the positive and negative KK modes in Eqs. (B.10) and (B.12), we find that the total contribution of bosonic KK modes is given by

$$\sum_{n=1}^{\infty} \left[\log \left.\frac{\det \Delta_B}{\det \Delta_B^0}\right|_n + \log \left.\frac{\det \Delta_B}{\det \Delta_B^0}\right|_{-n} \right] \;=\; 2 \sum_{n=1}^{\infty} \log \frac{\frac{n}{R} - m}{\frac{n}{R} + m}. \tag{B.13}$$

Since this infinite sum is divergent, let us consider the zeta function regularization

$$\sum_{n=1}^{\infty} \log \frac{\frac{n}{R} - m}{\frac{n}{R} + m} = \lim_{s \to 0} \frac{\partial}{\partial s} \left[\sum_{n=1}^{\infty} \left(\frac{\Lambda_0}{\frac{n}{R} + m} \right)^s - \sum_{n=1}^{\infty} \left(\frac{\Lambda_0}{\frac{n}{R} - m} \right)^s \right], \tag{B.14}$$

where Λ_0 is an arbitrary parameter which can be identified with a UV cutoff scale. Using the Hurwitz zeta function $\zeta(s, z)$, which satisfies

$$\zeta(s, z) = \sum_{n=0}^{\infty} \frac{1}{(z + l)^s}, \qquad \zeta(0, z) = -z + \frac{1}{2}, \qquad \lim_{s \to 0} \frac{\partial}{\partial s} \zeta(s, z) = \log \frac{\Gamma(z)}{\sqrt{2\pi}}, \tag{B.15}$$

we obtain the following regularized KK mode contribution with the cutoff dependence

$$\begin{aligned} \sum_{n=1}^{\infty} \log \frac{\frac{n}{R} - m}{\frac{n}{R} + m} &= \lim_{s \to 0} \partial_s \Big[(R\Lambda_0)^s \Big\{ \zeta(s, 1 + mR) - \zeta(s, 1 - mR) \Big\} \Big] \\ &= -2mR \log R\Lambda_0 + \log \frac{\Gamma(1 + mR)}{\Gamma(1 - mR)}, \end{aligned} \tag{B.16}$$

This result gives the first equality in Eq. (III.24).

2. Large KK momentum expansion

Here, we discuss the UV divergence of the bosonic one-loop determinant in more detail by using the large KK momentum expansion. For large KK momentum (large n), the bosonic one-loop determinant can be expanded as

$$\left. \frac{\det \Delta_B}{\det \Delta_B^0} \right|_n = 1 + \frac{1}{n} A + \mathcal{O}(n^{-2}). \tag{B.17}$$

The expansion coefficient A determines the UV divergent part as

$$\sum_{n=1}^{\infty} \log \left. \frac{\det \Delta_B}{\det \Delta_B^0} \right|_n = A \log R\Lambda_0 + \{\text{UV finite terms}\}, \tag{B.18}$$

where we have used the relation

$$\sum_{n=1}^{\infty} \frac{1}{n} = \log R\Lambda_0 + \gamma, \tag{B.19}$$

which can be obtained by differentiating Eq. (B.16) with respect to m and setting $m = 0$. Eq. (B.18) implies that if we are interested only in the UV divergent part, it is sufficient to calculate the constant A by using the large KK momentum expansion.

By expanding the fluctuation as

$$\xi(x,y) = \sum_{n=-\infty}^{\infty} \xi_n(x)\, e^{i\left(\frac{n}{R}+m\right)y}, \qquad \bar{\xi}(x,y) = \sum_{n=-\infty}^{\infty} \tilde{\xi}_n(x)\, e^{i\left(\frac{n}{R}-m\right)y}, \tag{B.20}$$

the equation for the bosonic fluctuations become

$$\Delta_B \begin{pmatrix} \xi \\ \bar{\xi} \end{pmatrix} \;\to\; \Delta_B|_n \begin{pmatrix} \xi_n \\ \tilde{\xi}_n \end{pmatrix}, \tag{B.21}$$

where $\Delta_B|_n$ is the operator which can be obtained by replacing ∇, $\bar{\nabla}$, ∇^* and $\bar{\nabla}^*$ in Eq. (III.8) as

$$\nabla \;\to \frac{1}{2}\left(\partial_1 + \frac{n}{R} + m\right) + 2iA_z, \qquad \bar{\nabla} \;\to \frac{1}{2}\left(\partial_1 - \frac{n}{R} - m\right) + 2iA_{\bar{z}}, \tag{B.22}$$

$$\nabla^* \to \frac{1}{2}\left(\partial_1 + \frac{n}{R} - m\right) - 2iA_z, \qquad \bar{\nabla}^* \to \frac{1}{2}\left(\partial_1 - \frac{n}{R} + m\right) - 2iA_{\bar{z}}. \tag{B.23}$$

By generalizing the theorem (B.2), the functional determinant of $\Delta_B|_n$ can be calculated as follows (see Appendix B of Ref. [25]). For $n > 0$, let Ξ_n be a 2-by-2 matrix (a linearly independent pair of solutions of the linearized equation) such that

$$\Delta_B|_n\, \Xi_n = 0, \qquad \Xi_n = \begin{pmatrix} e^{\left(\frac{n}{R}+m\right)x} & 0 \\ 0 & e^{\left(\frac{n}{R}-m\right)x} \end{pmatrix} \exp W_n, \tag{B.24}$$

where W is a 2-by-2matrix which converges in the limit $x \to -\infty$

$$\lim_{x\to-\infty} W_n = \text{const.} \tag{B.25}$$

Then the one-loop determinant can be written as

$$\log \frac{\det \Delta_B}{\det \Delta_B^0}\bigg|_n = \int_{-\infty}^{\infty} dx\, \partial_x \log\det \exp W_n = \int_{-\infty}^{\infty} dx\, \mathrm{Tr}\, \partial_x W_n. \tag{B.26}$$

By expandong W_n in powers of $1/n$ as

$$W_n = W_{n,0} + \frac{1}{n} W_{n,1} + \frac{1}{n^2} W_{n,2} + \cdots, \tag{B.27}$$

we can recursively determine $\partial_x W_{n,k}$ by solving $\Delta_B|_n\, \Xi_n = 0$ order-by-order. Then we can show that

$$\int_{-\infty}^{\infty} dx\, \mathrm{Tr}\, \partial_x W_n = -\frac{2R}{n} \int_{-\infty}^{\infty} dx\, \frac{|\partial_x \varphi_B|^2 + m^2 |\varphi_B|^2}{(1+|\varphi_B|^2)^2} + \mathcal{O}(n^{-2}). \tag{B.28}$$

The KK modes with $n < 0$ gives the same contribution to the divergent part. As in the case of the classical bion effective action (see footnote 7), we can evaluate the integral in Eq. (B.28) up to the subleading order in the weak coupling limit. By using the zeta function regularization in Eq. (B.19), we find that

$$\sum_{n=1}^{\infty}\left[\log\frac{\det\Delta_B}{\det\Delta_B^0}\bigg|_n+\log\frac{\det\Delta_B}{\det\Delta_B^0}\bigg|_{-n}\right] \;=\; -4mR\log R\Lambda_0\left[1-2\cos\phi_r\,e^{-mx_r}+\mathcal{O}(g^4)\right]+\cdots. \quad (B.29)$$

This correctly renormalizes the coupling constant g in the effective action. From this expression, we can read the second equality in Eq. (III.24).

3. Fermionic one-loop determinant

Next, let us consider the fermionic one-loop determinant in the bion background. For a single pair of (χ_l, χ_r) in Eq. (III.6), the one-loop determinant is given by

$$\det\Delta_F=\det(\nabla\bar{\nabla})=\det(\bar{\nabla}\nabla), \qquad \det\Delta_F^0=\det(\partial\bar{\partial}). \quad (B.30)$$

Note that both ∇ and $\bar{\nabla}$ defined in Eq. (III.10) have no zero mode in the bion background. It is convenient to expand the fluctuations into the KK modes as

$$\chi_l(x,y) \;=\; e^{-2i\int^x dxA_x}\sum_{n=-\infty}^{\infty}e^{iM_ny}\chi_{l,n}(x), \qquad \chi_r(x,y) \;=\; e^{-2i\int^x dxA_x}\sum_{n=-\infty}^{\infty}e^{iM_ny}\chi_{r,n}(x), \quad (B.31)$$

where M_n is the KK mass

$$M_n=\frac{n}{R}+m. \quad (B.32)$$

Since each KK sector is an eigen subspace of the operators Δ_F and Δ_F^0, we can decompose the determinants as

$$\log\frac{\det\Delta_F}{\det\Delta_F^0}=\sum_{n=-\infty}^{\infty}\log\frac{\det\Delta_F}{\det\Delta_F^0}\bigg|_n. \quad (B.33)$$

In the n-th KK sector, the explicit form of the operators are given by

$$\nabla\bar{\nabla}\big|_n=\frac{1}{4}\left(\partial_x+2A_y+M_n\right)\left(\partial_x-2A_y-M_n\right), \qquad \partial\bar{\partial}=\frac{1}{4}\left(\partial_x^2-M_n^2\right). \quad (B.34)$$

Let us calculate the determinant by using the theorem (B.2). Let $\chi_\pm$ be the solutions of $\nabla\bar{\nabla}\big|_n\chi_\pm=0$ with the following asymptotic behaviors

$$\chi_\pm\to\begin{cases} e^{\mp|M_n|x} & \text{for } x\to\pm\infty \\ \mathcal{C}^\pm e^{\mp|M_n|x}+\mathcal{D}^\pm e^{\pm|M_n|x} & \text{for } x\to\pm\infty \end{cases}. \quad (B.35)$$

Then the ratio of the determinants is given by

$$\left.\frac{\det(\nabla\bar{\nabla})}{\det(\partial\bar{\partial})}\right|_n = \mathcal{C}^+ = \mathcal{C}^-. \tag{B.36}$$

The solution of $\bar{\nabla}\chi = 0$ gives χ_- for $n \geq 0$ and χ_+ for $n < 0$

$$\chi_- = e^{M_n x}\exp\left[2\int_{-\infty}^{x} dx'\, A_y(x')\right] \quad (\text{for } n \geq 0), \tag{B.37}$$

$$\chi_+ = e^{M_n x}\exp\left[2\int_{\infty}^{x} dx'\, A_y(x')\right] \quad (\text{for } n < 0). \tag{B.38}$$

Therefore the determinant is given by

$$\left.\frac{\det(\nabla\bar{\nabla})}{\det(\partial\bar{\partial})}\right|_n = \exp\left[\pm 2\int_{-\infty}^{\infty} dx\, A_y(x)\right] \quad (\text{for } n \geq 0 \text{ and } n < 0 \text{ respectively}). \tag{B.39}$$

It follows that all the fermionic contributions cancel out except for the KK zero mode

$$\log\frac{\det\Delta_F}{\det\Delta_F^0} = \sum_{n=-\infty}^{\infty}\log\left.\frac{\det(\nabla\bar{\nabla})}{\det(\partial\bar{\partial})}\right|_n = 2\int_{-\infty}^{\infty} dx\, A_y(x) = -2mx_r + \mathcal{O}(g^2). \tag{B.40}$$

This result gives Eqs. (III.27) and (III.28).

Appendix C: One-loop determinants around single bion background in $\mathbb{C}P^{N-1}$ model

In this appendix, we discuss the one-loop determinants around the single bion backgrounds in the $\mathbb{C}P^{N-1}$ model in Eq. (IV.8). As in the $\mathbb{C}P^1$ case, we fix the center of mass position and overall phase and set $x_\pm = \pm x_r/2$ and $\phi_\pm = \pm\phi_r/2$.

1. Bosonic one-loop determinant in the KK decomposition

When the single bion ansatz of the $\mathbb{C}P^1$ model is embedded in the b-th component field φ^b, the fluctuation of φ^b gives the same determinant as in the $\mathbb{C}P^1$ case in Eqs. (III.22)-(III.24) and (III.28) with m replaced by m_b. For $\delta\varphi^a$ $(a \neq b)$, it is convenient to redefine the fields as

$$\delta\varphi^a = g\sqrt{1+|\varphi_B|^2}\,\exp\left(-i\int^x dx' A_x(x')\right)\xi^a, \tag{C.1}$$

and expand the normalized fluctuation ξ^a into the KK modes

$$\xi^a = \sum_{n=-\infty}^{\infty} e^{i\left(\frac{n}{R}+m_a\right)y}\,\xi^a_n. \tag{C.2}$$

Then the linearized equation for the n-th KK mode of ξ^a becomes

$$0 \;=\; \Delta_B|_{n,a}\,\xi^a_n \;=\; \left[\partial_x^2 - \left(\frac{n}{R} + m_a + A_y\right)^2 + \frac{|\partial_x \varphi_B|^2 + m_b^2|\varphi_B|^2}{(1+|\varphi_B|^2)^2}\right]\xi^a_n + \mathcal{O}(g^2). \tag{C.3}$$

The leading order part of the operator $\Delta_B|_{n,a}$ is given by

$$\Delta_B|_{n,a} = \begin{cases} \left[\partial_x + M_{n,a} - \frac{m_b}{1+e^{-2m_b d_-}}\right]\left[\partial_x - M_{n,a} + \frac{m_b}{1+e^{-2m_b d_-}}\right] + \mathcal{O}(g^2) & \text{for } x \approx -x_r/2 \\ \left[\partial_x + M_{n,a} - m_b\right]\left[\partial_x - M_{n,a} + m_b\right] + \mathcal{O}(g^2) & \text{for } x \approx 0 \\ \left[\partial_x - M_{n,a} + \frac{m_b}{1+e^{2m_b d_+}}\right]\left[\partial_x + M_{n,a} - \frac{m_b}{1+e^{2m_b d_+}}\right] + \mathcal{O}(g^2) & \text{for } x \approx x_r/2 \end{cases}, \tag{C.4}$$

where $M_{n,a}$ is the KK mass and $d_\pm$ are the distances from the fractional instantons

$$M_{n,a} = \frac{n}{R} + m_a, \qquad d_\pm = x \mp \frac{x_r}{2}. \tag{C.5}$$

The leading order solution of $\Delta_B|_{n,a}\,\xi^a_n$ can be obtained by connecting the solutions in the neighboring regions. For $n \geq 0$, the decreasing solution as $x \to -\infty$ is given by

$$\xi^a_n \;=\; e^{M_{n,a}x} \times \begin{cases} (1+e^{2m_b d_-})^{-\frac{1}{2}} & \text{for } x \approx -x_r/2 \\ \exp\left[-m_b d_-\right] + \cdots & \text{for } x \approx 0 \\ (1+e^{-2m_b d_+})^{-\frac{1}{2}}\left(\frac{M_{n,a}-m_b}{M_{n,a}} e^{-m_b x_r} + e^{-2m_b x}\right) + \cdots & \text{for } x \approx x_r/2 \end{cases}. \tag{C.6}$$

Similarly, for $n < 0$ the decreasing solution as $x \to \infty$ is given by

$$\xi^a_n \;=\; e^{M_{n,a}x} \begin{cases} (1+e^{2m_b d_-})^{-\frac{1}{2}} \frac{M_{n,a}}{M_{n,a}-m_b} e^{m_b x_r} + \cdots & \text{for } x \approx -x_r/2 \\ \frac{M_{n,a}}{M_{n,a}-m_b} e^{-m_b d_+} + \cdots & \text{for } x \approx 0 \\ (1+e^{-2m_b d_+})^{-\frac{1}{2}}\left(1 + \frac{M_{n,a}}{M_{n,a}-m_b} e^{-2m_b d_+}\right) & \text{for } x \approx x_r/2 \end{cases}. \tag{C.7}$$

From the asymptotic behavior of these solutions, the one-loop determinant can be read off as

$$\log \frac{\det \Delta_B}{\det \Delta_B^0}\bigg|_{n,a} \;\approx\; \mp\left[\log \frac{\frac{n}{R} + m_a}{\frac{n}{R} + m_a - m_b} + m_b x_r\right], \tag{C.8}$$

where $-$ is for $n \geq 0$ and $+$ is for $n < 0$. Therefore, the contribution of the a-th component of the bosonic fluctuation to the one-loop determinant is given by

$$\log \frac{\det \Delta_B}{\det \Delta_B^0}\bigg|_{a} \;=\; \sum_{n=1}^{\infty}\left[\log \frac{\frac{n}{R} + m_a - m_b}{\frac{n}{R} - m_a + m_b} - \log \frac{\frac{n}{R} + m_a}{\frac{n}{R} - m_a}\right] - \log \frac{m_a}{m_a - m_b} - m x_r. \tag{C.9}$$

By using the zeta function regularization in Eq. (B.14), we find

$$\log \frac{\det \Delta_B}{\det \Delta_B^0}\bigg|_a = -2m_b R \log R\Lambda_0 - \log \frac{\Gamma(1+(m_a-m_b)R)}{\Gamma(1-(m_a-m_b)R)}\frac{\Gamma(1-m_aR)}{\Gamma(1+m_aR)}\frac{m_a}{m_a-m_b} - m x_r. \quad \text{(C.10)}$$

This gives Eq. (IV.14) and the first term of Eq. (IV.15).

2. Large KK momentum expansion

To see the UV divergence in more detail, let us consider the large KK momentum expansion. The solution of the linearized equation (C.3) which decreases as $x \to -\infty$ can be obtained by setting

$$\xi_n^a = \exp\left[\pm\left(\frac{n}{R}+m_a\right)x + \sum_{k=0}^{\infty}\frac{1}{n^k}W_{n,a,k}\right], \quad \text{(C.11)}$$

and expanding the equation in powers of n. Here $+$ is for $n \geq 0$ and $-$ is for $n < 0$. We can show that $\partial_x W_{n,a,k}$ satisfy

$$\partial_x W_{n,a,0} = \pm A_y, \qquad \partial_x W_{n,a,1} = \mp\frac{R}{2}\left[\frac{|\partial_x\varphi_B|^2 + m_b^2|\varphi_B|^2}{(1+|\varphi_B|^2)^2} + m_b\partial_x\left(\frac{1}{1+|\varphi_B|^2}\right)\right], \quad \cdots. \quad \text{(C.12)}$$

By using the theorem (B.2), we find that the total contribution of the KK modes is given by

$$\log \frac{\det \Delta_B}{\det \Delta_B^0}\bigg|_{a,\mathrm{KK}} = \sum_{k=0}^{\infty}\sum_{n=1}^{\infty}\int_{-\infty}^{\infty} dx \left[\frac{1}{n^k}\partial_x W_{n,a,k} + \frac{1}{(-n)^k}\partial_x W_{-n,a,k}\right]. \quad \text{(C.13)}$$

For $k=1$, the summation with respect to n is divergent. By applying the zeta function regularization (B.19), we find that the divergent part is given by

$$\begin{aligned}\log \frac{\det \Delta_B}{\det \Delta_B^0}\bigg|_{a,\mathrm{KK}} &= -R\log R\Lambda_0 \int_{-\infty}^{\infty} dx \frac{|\partial_x\varphi_B|^2 + m_b^2|\varphi_B|^2}{(1+|\varphi_B|^2)^2} + \cdots \\ &= -2m_bR\log R\Lambda_0\Big[1 - 2\cos\phi_r e^{-m_b x_r} + \mathcal{O}(g^4)\Big] + \cdots. \quad \text{(C.14)}\end{aligned}$$

This divergence consistently renormalizes by the coupling constant in the classical bion effective action. From this result, we can read off the second term of Eq. (IV.15).

3. Fermionic one-loop determinant

Next, let us calculate the fermionic one-loop determinant for a single pair of fermions (ψ_l^a, ψ_r^a). It is convenient to redefine the fermionic fields as

$$\psi_{l,r}^a = g(1+|\varphi_B|^2)^{\frac{q_a}{2}}\chi_{l,r}^a, \quad \text{with} \quad q_a = \begin{cases} 2 & \text{for } a=b \\ 1 & \text{for } a\neq b \end{cases}. \quad \text{(C.15)}$$

Then the linearized equations for the fermionic fluctuations become

$$\left(\partial + iq_a A_z\right)\chi_l^a = 0, \qquad \left(\bar{\partial} + iq_a A_{\bar{z}}\right)\chi_r^a = 0, \tag{C.16}$$

The fermionic one-loop determinant can be calculated in an analogous way as in the case of the $\mathbb{C}P^1$ model. Generalizing the formula (B.40) to each component, we find that the fermionic contributions to the determinant is given by

$$\sum_{a=1}^{N-1} \log \frac{\det \Delta_F}{\det \Delta_F^0}\bigg|_a \;=\; \sum_{a=1}^{N-1} q_a \int_{-\infty}^{\infty} dx\, A_y(x) \;=\; -N m_b x_r + \mathcal{O}(g^2). \tag{C.17}$$

This gives the fermionic one-loop determinant in Eq. (IV.19).

Appendix D: Lefschetz thimble integral

In this appendix, we summarize the procedure to evaluate the quasi-moduli integral by means of the Lefschetz thimble method following the argument of Ref. [25]. The quasi-moduli integrals discussed in this paper take the form

$$[\mathcal{I}\bar{\mathcal{I}}] \;\equiv\; \int_{-\infty}^{\infty} dx_r \int_{-\pi}^{\pi} d\phi_r\, e^{-S}, \qquad S(x_r,\phi_r) \;\equiv\; -\frac{8\pi m R}{g_R^2}\cos\phi_r\, e^{-m x_r} + 2m\epsilon x_r + \mathcal{O}(g^2)\,. \tag{D.1}$$

We first complexify g_R^2 as $g_R^2 \to g_R^2 e^{i\theta}$ to avoid the Stokes line. The variables x_r and ϕ_r are also complexified as $x_r = x_R + i x_I \in \mathbb{C}$, $\phi_r = \phi_R + i\phi_I \in \mathbb{C}$. By solving the equations $\partial_{x_r} S = 0$ and $\partial_{\phi_r} S = 0$, we find that the saddle points are labeled by an integer $\sigma \in \mathbb{Z}$

$$x_\sigma \;=\; \frac{1}{m}\log\left(\frac{4\pi m R}{\epsilon g_R^2}\right) + \frac{i}{m}(\sigma\pi - \theta), \qquad \phi_\sigma \;=\; -(\sigma - 1)\pi \pmod{2\pi}\,, \tag{D.2}$$

where $\sigma = 0$ and $\sigma = \pm 1$ corresponds to the real and complex bions, respectively. The thimbles $\mathcal{J}_\sigma$ and the dual thimbles $\mathcal{K}_\sigma$ associated with these saddle points are obtained by solving the flow equations

$$\frac{dx_r}{dt} = \frac{1}{2m}\overline{\frac{\partial S}{\partial x_r}}, \qquad \frac{d\phi_r}{dt} = \frac{m}{2}\overline{\frac{\partial S}{\partial \phi_r}}, \tag{D.3}$$

where the coefficients in the right hand sides of these equations are determined by the metric of the quasi-moduli space. By solving these equations, we find that the thimbles $\mathcal{J}_\sigma$ are the planes specified by

$$m x_I = \sigma\pi - \theta, \qquad \phi_R = -(\sigma - 1)\pi\,, \tag{D.4}$$

while the dual thimbles $\mathcal{K}_\sigma$ are specified by the equations

$$m x_R - \phi_I = \log\left[\frac{4\pi m R}{\epsilon g_R^2}\frac{\sin\mathcal{Y}}{\mathcal{Y}}\right], \qquad m x_R + \phi_I = \log\left[\frac{4\pi m R}{\epsilon g_R^2}\frac{\sin\tilde{\mathcal{Y}}}{\tilde{\mathcal{Y}}}\right], \tag{D.5}$$

where $\mathcal{Y}$ and $\tilde{\mathcal{Y}}$ are given by

$$\mathcal{Y} \equiv m x_I + \phi_R - \pi + \theta, \quad \tilde{\mathcal{Y}} \equiv m x_I - \phi_R - (2\sigma - 1)\pi + \theta, \quad -\pi \leq \mathcal{Y} \leq \pi, \quad -\pi \leq \tilde{\mathcal{Y}} \leq \pi. \tag{D.6}$$

By looking into these thimbles and dual thimbles, we find that the intersection numbers (n_{-1}, n_0, n_1) of the original integration contour and the dual thimbles $\mathcal{K}_{-1}$, $\mathcal{K}_1$ and $\mathcal{K}_0$ are given by

$$(n_{-1}\,,\, n_0\,,\, n_1) \;=\; \begin{cases} (\,0\,,\,-1\,,\,1\,) & \text{for } \theta > 0 \\ (\,-1\,,\,1\,,\,0\,) & \text{for } \theta < 0 \end{cases}. \tag{D.7}$$

Therefore, in the limit $\theta \to \pm 0$, the integral (D.1) has the ambiguity depending on the sign of θ

$$[\mathcal{I}\bar{\mathcal{I}}] \;=\; \begin{cases} Z_{\sigma=1} - Z_{\sigma=0} & \text{for } \theta \to +0 \\ Z_{\sigma=0} - Z_{\sigma=-1} & \text{for } \theta \to -0 \end{cases}, \tag{D.8}$$

where Z_σ denotes the integral along the thimble $\mathcal{J}_\sigma$

$$Z_\sigma \;=\; \int_{\mathcal{J}_\sigma} dx_r d\phi_r \, \exp\left[-S(x_r, \phi_r)\,\right] \;=\; \frac{i}{2m}\left(\frac{4\pi m R}{g_R^2}\right)^{-2\epsilon} e^{-2\pi i \epsilon \sigma}\, \Gamma\left(\epsilon\right)^2. \tag{D.9}$$

Therefore, the integral (D.1) is evaluated as

$$[\mathcal{I}\bar{\mathcal{I}}] \;=\; \frac{1}{m}\left(\frac{4\pi m R}{g_R^2}\right)^{-2\epsilon} \sin\epsilon\pi\, \Gamma\left(\epsilon\right)^2 \times \begin{cases} e^{-\pi i \epsilon} & \text{for } \theta = +0 \\ e^{+\pi i \epsilon} & \text{for } \theta = -0 \end{cases}. \tag{D.10}$$

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