

Most general cubic-order Horndeski Lagrangian allowing for scaling solutions and the application to dark energy

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In cubic-order Horndeski theories where a scalar field ϕ is coupled to nonrelativistic matter with a field-dependent coupling $Q(\phi)$, we derive the most general Lagrangian having scaling solutions on the isotropic and homogenous cosmological background. For constant Q including the case of vanishing coupling, the corresponding Lagrangian reduces to the form $L = Xg_2(Y) - g_3(Y)\square\phi$, where $X = -\partial_\mu\phi\partial^\mu\phi/2$ and g_2, g_3 are arbitrary functions of $Y = Xe^{\lambda\phi}$ with constant λ . We obtain the fixed points of the scaling Lagrangian for constant Q and show that the ϕ -matter-dominated-epoch (ϕ MDE) is present for the cubic coupling $g_3(Y)$ containing inverse power-law functions of Y . The stability analysis around the fixed points indicates that the ϕ MDE can be followed by a stable critical point responsible for the cosmic acceleration. We propose a concrete dark energy model allowing for such a cosmological sequence and show that the ghost and Laplacian instabilities can be avoided even in the presence of the cubic coupling.

I. INTRODUCTION

Two decades have passed since the discovery of the late-time cosmic acceleration [1, 2], but the origin of this phenomenon is still unknown. The simplest candidate for dark energy is the cosmological constant Λ [3], but it is still a challenging problem to relate the vacuum energy arising from particle physics with the observed dark energy scale. In the Λ -Cold-Dark-Matter (Λ CDM) model, there have been also tensions between the values of the Hubble constant H_0 constrained from the Cosmic Microwave Background (CMB) [4, 5] and low-redshift measurements [6]. It is worthwhile to pursue alternative possibilities for realizing the cosmic acceleration and study whether they can be better fitted with observational data over the Λ CDM model.

A scalar field ϕ with an associated potential energy $V(\phi)$, dubbed quintessence, is one of the candidates for dark energy [7–12]. For example, the late-time cosmic acceleration can be driven by runaway potentials like the exponential potential $V(\phi) = V_0e^{-\lambda\phi}$ [9, 13–15] and the inverse power-law potential $V(\phi) = V_0\phi^{-p}$ ($p > 0$) [16, 17]. Since the potential energy increases toward the past, quintessence can alleviate the small energy-scale problem of the cosmological constant.

In particular, if the field density ρ_ϕ scales in the same manner as the background matter density ρ_m , such a scalar field can be compatible with the energy scale related to particle physics. The cosmological solution along which the ratio ρ_ϕ/ρ_m remains constant is known as a scaling solution [13–15, 18–28]. In quintessence, the exponential potential $V(\phi) = V_0e^{-\lambda\phi}$ gives rise to the scaling solution for $\lambda^2 > 3(1 + w_m)$, where w_m is the matter equation of state. This solution can be responsible for the scaling radiation and matter eras. Since the cosmic acceleration occurs for $\lambda^2 < 2$, we require the modification of the potential at late time to exit from the scaling

matter era. For instance, this is possible by taking into account another shallow exponential potential [29–32].

In the presence of a direct coupling $Q(\phi)$ between the scalar field ϕ and nonrelativistic matter, there exists another type of scaling solutions called the ϕ -matter-dominated-epoch (ϕ MDE) for quintessence with the exponential potential [33, 34]. Extending the analysis to k-essence [35–37] for constant Q , it was shown that the Lagrangian with scaling solutions is restricted to be of the form $L = Xg_2(Y)$, where $X = -\partial_\mu\phi\partial^\mu\phi/2$ and g_2 is an arbitrary function of $Y = Xe^{\lambda\phi}$ [38–40]. The derivation of the scaling k-essence Lagrangian is also possible for the field-dependent coupling $Q(\phi)$ [41].

Quintessence and k-essence belong to subclasses of most general scalar-tensor theories with second-order equations of motion—dubbed Horndeski theories [42]. If we apply Horndeski theories to dark energy and impose the condition that the speed of gravitational waves on the cosmological background is equivalent to that of light (as constrained from the GW170817 event [43] together with the electromagnetic counterpart [44]), the Lagrangian is of the form $L = G_2(\phi, X) - G_3(\phi, X)\square\phi + G_4(\phi)R$, where G_4 depends on ϕ alone with the Ricci scalar R , and G_2, G_3 are functions of ϕ and X [45–51].

In cubic-order Horndeski theories with $G_4 = \text{constant}$, the Lagrangian with scaling solutions was derived in Ref. [52] for the ϕ -dependent coupling $Q(\phi)$ (see also Ref. [53]). They imposed a particular ansatz (Eq. (4.5) of Ref. [52]) for deriving the scaling Lagrangian, in addition to the choice of a specific form of the coupling $Q(\phi) = 1/(c_1\phi + c_2)$. For constant Q , this led to the scaling Lagrangian of the form $L = Xg_2(Y) - g_3(Y)\square\phi$ with $g_3(Y) = a_1Y + a_2Y^2$, where a_1 and a_2 are constants.

On the other hand, the recent study of Ref. [54] showed that there exists a scaling solution for the cubic coupling $g_3(Y) = A \ln Y$, where A is a constant, anticipating the fact that the scaling solution may be present for a more

general cubic coupling than that derived in Ref. [52]. Indeed, for constant couplings Q and G_4 , it has been found that the cubic Lagrangian $-g_3(Y)\square\phi$ with an arbitrary function $g_3(Y)$ can allow for the existence of scaling solutions [55].

In this paper, we derive the most general cubic-order Horndeski Lagrangian with scaling solutions for a field-dependent coupling $Q(\phi)$. We show that, in the presence of the cubic Lagrangian $-G_3(\phi, X)\square\phi$, the coupling is constrained to be of the form $Q(\phi) = 1/(c_1\phi + c_2)$ for the existence of scaling solutions. For constant Q , the scaling Lagrangian reduces to the form $L = Xg_2(Y) - g_3(Y)\square\phi$, which is in agreement with the result of Ref. [55]. Moreover, our analysis encompasses the vanishing coupling ($Q = 0$) as a special case.

In the presence of the nonvanishing coupling constant Q , there exists a ϕ MDE for the models in which the functions $g_2(Y)$ and $dg_3(Y)/dY$ contain inverse powers Y^{-n} ($n > 0$), e.g., $g_2(Y) = c_0 + c_1/Y$ and $g_3(Y) = d_1 \ln Y - d_2/Y$, where c_0, c_1, d_1, d_2 are constants. The ϕ MDE is characterized by the scaling solution with the field density parameter Ω_ϕ affected by Q and d_1 . This can lead to interesting cosmological solutions with the scaling saddle matter era followed by the late-time cosmic acceleration. The analysis in Ref. [52] overlooked the presence of ϕ MDE in cubic Horndeski theories, as it is not present for the function $g_3(Y) = a_1Y + a_2Y^2$, while in Ref. [55], the authors did not consider a concrete cubic coupling $g_3(Y)$ with the ϕ MDE.

We will show that, for the nonvanishing constant Q , there exist viable dark energy models with the ϕ MDE in cubic-order Horndeski theories. We study the background cosmological dynamics by paying particular attention to the evolution of the field density parameter Ω_ϕ and the dark energy equation of state w_ϕ . Unlike the Λ CDM model, Ω_ϕ does not need to be very much smaller than the background density parameters at early time. Moreover, it is possible to avoid the ghost and Laplacian instabilities during the cosmic expansion history from the radiation era to today. Unlike the analysis of Ref. [54], the Lagrangian does not need to be modified at late time to give rise to the cosmic acceleration.

This paper is organized as follows. In Sec. II, we formulate the coupled dark energy scenario in cubic-order Horndeski theories in terms of the Schutz-Sorkin action. In Sec. III, the Lagrangian allowing for scaling solutions is generally derived for the field-dependent matter coupling $Q(\phi)$. In Sec. IV, we obtain the fixed points for the scaling Lagrangian with constant Q and show the existence of ϕ MDE for particular choices of $g_3(Y)$. In Sec. V, we discuss the stability of the fixed points in the presence of nonrelativistic matter ($w_m = 0$). In Sec. VI, we propose a concrete model of dark energy and study the dynamics of late-time cosmic acceleration preceded by the ϕ MDE. We conclude in Sec. VII.

Throughout the paper, we use the units where the speed of light c , the reduced Planck constant \hbar , and the reduced Planck mass M_{pl} are equivalent to 1.

II. BACKGROUND EQUATIONS IN CUBIC HORNDESKI THEORIES

Let us consider the cubic-order Horndeski theories given by the action [56]

$$\mathcal{S} = \int d^4x \sqrt{-g} \left(\frac{R}{2} + L \right) + \mathcal{S}_m(\phi, g_{\mu\nu}), \quad (2.1)$$

where g is the determinant of metric tensor $g_{\mu\nu}$, R is the Ricci scalar, and

$$L = G_2(\phi, X) - G_3(\phi, X)\square\phi. \quad (2.2)$$

The functions G_2 and G_3 depend on ϕ and $X = -\partial_\mu\phi\partial^\mu\phi/2$. The Lagrangian (2.2) belongs to the subclass of Horndeski theories with second-order equations of motion [42, 57–59]. For this cubic-order Horndeski theory, the speed of gravitational waves c_t on the cosmological background is equivalent to that of light [58, 60], so the model is consistent with the bound on c_t constrained from the GW170817 event [43].

For the matter action \mathcal{S}_m , we consider a barotropic perfect fluid coupled to the scalar field ϕ . Such a coupled system of matter and ϕ can be described by a Schutz-Sorkin action [61–64] with the matter density ρ_m coupled to ϕ . In this case, the Schutz-Sorkin action is given by

$$\mathcal{S}_m = - \int d^4x [\sqrt{-g} \rho_m(n, \phi) + J^\mu \nabla_\mu \ell], \quad (2.3)$$

where ρ_m depends on the fluid number density n as well as on ϕ , and ℓ is a scalar quantity with the covariant derivative operator ∇_μ , and J^μ is a four vector related to n , as

$$n = \sqrt{\frac{J^\mu J^\nu g_{\mu\nu}}{g}}. \quad (2.4)$$

Since we are now considering scalar-tensor theories, we do not need to take into account vector degrees of freedom in the action (2.3).

We derive the background equations of motion on the flat Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime given by the line element

$$ds^2 = -N^2(t)dt^2 + a^2(t)\delta_{ij}dx^i dx^j, \quad (2.5)$$

where $N(t)$ is a lapse, and $a(t)$ is a scale factor. From Eq. (2.4), the background fluid number density n_0 is related to J^0 , as $J^0 = n_0 a^3$. Then, the matter action (2.3) reduces to

$$\mathcal{S}_m = - \int d^4x a^3 [N\rho_m(n_0, \phi) + n_0 \dot{\ell}], \quad (2.6)$$

where a dot represents a derivative with respect to the cosmic time t . Varying the action (2.6) with respect to ℓ , we find that $n_0 a^3 = \text{constant}$, i.e.,

$$\dot{n}_0 + 3Hn_0 = 0, \quad (2.7)$$

where $H = \dot{a}/a$ is the Hubble expansion rate. This relation corresponds to the conservation of total fluid number.

We vary the total action (2.1) with respect to N and a , and finally set $N = 1$. This process leads to the following equations of motion:

$$3H^2 = \rho_\phi + \rho_m, \quad (2.8)$$

$$2\dot{H} = -\rho_\phi - P_\phi - \rho_m - P_m, \quad (2.9)$$

where

$$\rho_\phi = \dot{\phi}^2 G_{2,X} - G_2 - \dot{\phi}^2 (G_{3,\phi} - 3H\dot{\phi}G_{3,X}), \quad (2.10)$$

$$P_\phi = G_2 - \dot{\phi}^2 (G_{3,\phi} + \ddot{\phi}G_{3,X}), \quad (2.11)$$

$$P_m = -n_0\dot{\ell} - \rho_m, \quad (2.12)$$

with the notations $G_{i,\phi} = \partial G_i / \partial \phi$ and $G_{i,X} = \partial G_i / \partial X$. We note that ρ_ϕ and P_ϕ correspond to the field density and pressure, respectively. Variation of the matter action with respect to n_0 leads to $\dot{\ell} = -\rho_{m,n}$, where $\rho_{m,n} = \partial \rho_m / \partial n_0$. Then, the matter pressure (2.12) is expressed as

$$P_m = n_0 \rho_{m,n} - \rho_m. \quad (2.13)$$

Taking the time derivative of $\rho_m = \rho_m(n_0, \phi)$, we obtain

$$\dot{\rho}_m = \rho_{m,n}\dot{n}_0 + Q(\phi)\rho_m\dot{\phi}, \quad (2.14)$$

where

$$Q(\phi) \equiv \frac{\rho_{m,\phi}}{\rho_m}. \quad (2.15)$$

Substituting Eq. (2.7) into Eq. (2.14) and using Eq. (2.13), it follows that

$$\dot{\rho}_m + 3H(1 + w_m)\rho_m = Q(\phi)\rho_m\dot{\phi}, \quad (2.16)$$

where $w_m = P_m/\rho_m$. Varying the action (2.1) with respect to ϕ , the scalar field obeys

$$\dot{\rho}_\phi + 3H(1 + w_\phi)\rho_\phi = -Q(\phi)\rho_m\dot{\phi}, \quad (2.17)$$

where $w_\phi = P_\phi/\rho_\phi$. The coupling $Q(\phi)$ quantifies the interaction between matter and ϕ . For $Q(\phi) = \text{constant}$, the background matter density can be expressed in the form $\rho_m = \tilde{\rho}_m(n_0)e^{Q\phi}$, where $\tilde{\rho}_m$ is a function of n_0 .

From Eq. (2.8), the density parameters $\Omega_\phi = \rho_\phi/(3H^2)$ and $\Omega_m = \rho_m/(3H^2)$ obey the relation

$$\Omega_\phi + \Omega_m = 1. \quad (2.18)$$

From Eq. (2.9), we have

$$\frac{\dot{H}}{H^2} = -\frac{3}{2}(1 + w_{\text{eff}}), \quad w_{\text{eff}} = w_\phi\Omega_\phi + w_m\Omega_m, \quad (2.19)$$

being w_{eff} the effective equation of state.

Solving Eqs. (2.9) and (2.17) for $\ddot{\phi}$ and \dot{H} , we obtain

$$\begin{aligned} \ddot{\phi} &= [9G_{3,X}^2 H \dot{\phi}^5 + 3G_{3,X}(G_{2,X} - 2G_{3,\phi})\dot{\phi}^4 - 6G_{3,X\phi} H \dot{\phi}^3 + 2(G_{3,\phi\phi} - G_{2,X\phi} - 9G_{3,X}H^2)\dot{\phi}^2 \\ &\quad - 6H(G_{2,X} - 2G_{3,\phi})\dot{\phi} + 2G_{2,\phi} + 3(\rho_m + P_m)G_{3,X}\dot{\phi}^2 - 2Q\rho_m]/q_s, \\ \dot{H} &= [-9G_{3,X}G_{3,XX}H^2\dot{\phi}^6 - 3\{G_{2,XX}G_{3,X} + G_{3,XX}(G_{2,X} - 2G_{3,\phi})\}H\dot{\phi}^5 + \{(G_{2,X} - 2G_{3,\phi})(G_{3,X\phi} - G_{2,XX}) \\ &\quad + G_{3,X}(G_{3,\phi\phi} - G_{2,X\phi}) - 27H^2G_{3,X}^2\}\dot{\phi}^4 - 12(G_{2,X} - 2G_{3,\phi})G_{3,X}H\dot{\phi}^3 + (G_{2,\phi}G_{3,X} + 4G_{2,X}G_{3,\phi} \\ &\quad - G_{2,X}^2 - 4G_{3,\phi}^2)\dot{\phi}^2 - (\rho_m + P_m)(G_{2,X} + G_{2,XX}\dot{\phi}^2 - 2G_{3,\phi} + 6G_{3,X}H\dot{\phi} - G_{3,X\phi}\dot{\phi}^2 + 3G_{3,XX}H\dot{\phi}^3) \\ &\quad - Q\rho_m G_{3,X}\dot{\phi}^2]/q_s, \end{aligned} \quad (2.20)$$

where

$$q_s \equiv 3G_{3,X}^2\dot{\phi}^4 + 6G_{3,XX}H\dot{\phi}^3 + 2(G_{2,XX} - G_{3,X\phi})\dot{\phi}^2 + 12G_{3,X}H\dot{\phi} + 2(G_{2,X} - 2G_{3,\phi}). \quad (2.22)$$

The background dynamics is known by solving Eqs. (2.16), (2.20) and (2.21) together with the constraint Eq. (2.8).

III. LAGRANGIAN ALLOWING FOR SCALING SOLUTIONS

The scaling solution is characterized by a nonvanishing constant ratio Ω_ϕ/Ω_m , so that both Ω_ϕ and Ω_m are constant from Eq. (2.18). Moreover, we would like to consider the case in which the field equation of state $w_\phi = P_\phi/\rho_\phi$ as well as w_m do not vary in time in the

scaling regime. Then, from Eq. (2.19), both w_{eff} and \dot{H}/H^2 are constant.

In Sec. III A, we first obtain the Lagrangian with scaling solutions for constant Q and show that this agrees with the result recently found in Ref. [55]. This analysis also accommodates the vanishing coupling ($Q = 0$) as a special case. In Sec. III B, we derive the scaling Lagrangian for more general cases in which the coupling Q depends on ϕ .

A. Constant Q (including $Q = 0$)

For scaling solutions, both ρ_ϕ and ρ_m are in proportion to H^2 . Then, all the terms on the left hand side of Eq. (2.17) are in proportion to H^3 . For constant Q , the compatibility with the right hand side of Eq. (2.17) shows that $\dot{\phi} \propto H$, i.e.,

$$\frac{\dot{\phi}}{H} = \alpha, \quad (3.1)$$

where α is a constant. The relation (3.1) is also consistent with Eq. (2.16). On using the scaling relation $\dot{\rho}_\phi/\rho_\phi = \dot{\rho}_m/\rho_m$ for $Q \neq 0$, it follows that $\alpha = 3\Omega_\phi(w_m - w_\phi)/Q$.

While Eq. (3.1) has been derived for the nonvanishing constant Q , there are scaling solutions satisfying the condition (3.1) even for $Q = 0$. For example, the canonical term $G_2 = X$ gives rise to the contribution $\dot{\phi}^2/2$ to ρ_ϕ . Existence of this term in Eq. (2.8) is consistent with the relation (3.1). Now, we search for scaling solutions obeying the relation (3.1) for an arbitrary constant α .

For the realization of scaling solutions, we consider the case in which each term in ρ_ϕ and P_ϕ is proportional to H^2 . Since G_2 is one of such terms, we require that $G_2 \propto H^2$. This relation translates to

$$\frac{\dot{G}_2}{HG_2} = 2\frac{\dot{H}}{H^2} = -3(1 + w_{\text{eff}}). \quad (3.2)$$

On using Eq. (3.1), the derivative $\dot{G}_2 = G_{2,\phi}\dot{\phi} + G_{2,X}\dot{X}$ is expressed as

$$\dot{G}_2 = H[\alpha G_{2,\phi} - 3(1 + w_{\text{eff}})XG_{2,X}]. \quad (3.3)$$

Then, Eq. (3.2) reduces to

$$XG_{2,X} - \frac{1}{\lambda}G_{2,\phi} - G_2 = 0, \quad (3.4)$$

where

$$\lambda \equiv \frac{3(1 + w_{\text{eff}})}{\alpha}. \quad (3.5)$$

The partial differential Eq. (3.4) is integrated to give

$$G_2(\phi, X) = Xg_2(Y), \quad (3.6)$$

where g_2 is an arbitrary function of

$$Y \equiv Xe^{\lambda\phi}. \quad (3.7)$$

From Eq. (3.1), the evolution of ϕ along the scaling solution is given by

$$\phi = \alpha \ln a + \phi_0, \quad (3.8)$$

where ϕ_0 is a constant. For $w_{\text{eff}} = \text{constant}$, the integrated solution to Eq. (2.19) reads

$$H = \frac{2}{3(1 + w_{\text{eff}})(t - t_0)}, \quad a \propto (t - t_0)^{\frac{2}{3(1 + w_{\text{eff}})}}, \quad (3.9)$$

where t_0 is a constant. Since $X \propto H^2 \propto (t - t_0)^{-2}$ and $e^{\lambda\phi} \propto a^{3(1 + w_{\text{eff}})} \propto (t - t_0)^2$ for scaling solutions, the quantity $Y = Xe^{\lambda\phi}$ does not vary in time. Taking the X derivative of Eq. (3.6), it follows that $\dot{\phi}^2 G_{2,X} = \dot{\phi}^2(g_2 + Yg_{2,Y}) \propto H^2$. Hence the term $\dot{\phi}^2 G_{2,X}$ in ρ_ϕ obeys the same scaling relation as G_2 . We have thus shown that the quadratic Lagrangian $G_2(\phi, X) = Xg_2(Y)$, which was derived by the scaling property of one of the terms in ρ_ϕ and P_ϕ , has the scaling solution. The result (3.6) coincides with that derived in Refs. [38, 39] by assuming the nonvanishing constant Q .

For the cubic coupling G_3 , the term $\dot{\phi}^2 G_{3,\phi}$ in ρ_ϕ and P_ϕ needs to be in proportion to $H^2 \propto \dot{\phi}^2$ for the existence of scaling solutions, so that $G_{3,\phi} = \text{constant}$. Taking the time derivative of this relation, it follows that

$$XG_{3,\phi X} - \frac{1}{\lambda}G_{3,\phi\phi} = 0. \quad (3.10)$$

This is integrated to give

$$G_3 = g_3(Y) + h_3(X), \quad (3.11)$$

where g_3 and h_3 are arbitrary functions of Y and X , respectively. Since $G_{3,X} = e^{\lambda\phi}g_{3,Y} + h_{3,X}$, the terms $3H\dot{\phi}^3 G_{3,X}$ and $-\dot{\phi}^2\ddot{\phi}G_{3,X}$ in ρ_ϕ and P_ϕ are both proportional to $H\dot{\phi}Y \propto H^2$. Hence the scaling relation is satisfied for any functional form of $g_3(Y)$.

On the other hand, $h_3(X)$ gives rise to the terms proportional to $H^2 X h_{3,X}$ in ρ_ϕ and P_ϕ . In order to satisfy the scaling relation, we require that $Xh_{3,X} = \text{constant}$. This is integrated to give

$$h_3(X) = c + d \ln X, \quad (3.12)$$

where c and d are constants. Then, the Lagrangian $-h_3(X)\square\phi$ is given by $(-c - d \ln Y + d\lambda\phi)\square\phi$, where we used the relation $\ln X = \ln Y - \lambda\phi$. The first term $-c\square\phi$ is just a total derivative. The second term $-(d \ln Y)\square\phi$ can be absorbed into $-g_3(Y)\square\phi$. The third term $d\lambda\phi\square\phi$ is equivalent to $2d\lambda X$ up to a boundary term, so it can be absorbed into the Lagrangian (3.6) by choosing $g_2(Y) = 2d\lambda$. Then, the cubic interaction satisfying the scaling relation is simply expressed as

$$G_3(\phi, X) = g_3(Y). \quad (3.13)$$

From the above discussion, the scaling solution with $\dot{\phi}/H = \text{constant}$ exists for the Lagrangian

$$L = Xg_2(Y) - g_3(Y)\square\phi, \quad (3.14)$$

where Y is given by Eq. (3.7). This result is valid not only for the nonvanishing Q but also for $Q = 0$. In Ref. [52], the authors obtained the cubic Lagrangian of the form $g_3(Y) = a_1 Y + a_2 Y^2$ by assuming a specific relation in the process of deriving the scaling Lagrangian. As it is clear from the above discussion, any Y -dependent cubic coupling $g_3(Y)$, besides the Lagrangian $Xg_2(Y)$, gives rise to scaling solutions. This is in agreement with the recent result of Ref. [55].

We also showed that the scaling Lagrangian can be expressed in the form $L = Xg_2(Y) - [g_3(Y) - d\lambda\phi]\square\phi$. If we choose $g_2(Y) = 1 - V_0/Y$ and $g_3(Y) = A \ln Y$, it accommodates the scaling model studied in Ref. [54], i.e., the Lagrangian $L = X - V_0 e^{-\lambda\phi} - A \ln(X e^{\lambda\phi})\square\phi$ with $\lambda = \lambda(1 - d/A)$.

B. Field-dependent coupling $Q(\phi)$

In this section, we derive the Lagrangian with scaling solutions for a field-dependent nonvanishing coupling $Q(\phi)$. Employing the scaling relation $\dot{\rho}_\phi/\rho_\phi = \dot{\rho}_m/\rho_m$ in Eqs. (2.16) and (2.17), it follows that

$$\dot{\phi} = 3\Omega_\phi(w_m - w_\phi)\frac{H}{Q(\phi)}, \quad (3.15)$$

and

$$\frac{\dot{\rho}_\phi}{H\rho_\phi} = \frac{\dot{\rho}_m}{H\rho_m} = -3(1 + w_{\text{eff}}). \quad (3.16)$$

The field pressure $P_\phi = w_\phi\rho_\phi \propto \rho_\phi$ also obeys the same relation as Eq. (3.16), i.e., $\dot{P}_\phi/(HP_\phi) = -3(1 + w_{\text{eff}})$. This amounts to the scaling behavior $P_\phi \propto \rho_\phi \propto H^2$.

The Lagrangian (2.2) contains the term G_2 , which is also present in ρ_ϕ and P_ϕ . After integrating by parts the cubic Lagrangian $-G_3(\phi, X)\square\phi$ in the action (2.1), the resulting Lagrangian contains the same terms which are present in P_ϕ . Thus, we search for the Lagrangian L allowing for the same scaling property as ρ_ϕ and P_ϕ , i.e.,

$$\frac{\dot{L}}{HL} = -3(1 + w_{\text{eff}}), \quad (3.17)$$

or equivalently, $L \propto H^2$. After deriving the Lagrangian satisfying the condition (3.17), we need to confirm whether each term in ρ_ϕ and P_ϕ obeys the scaling relation.

Since L depends on ϕ, X , and $\square\phi$, it follows that

$$\frac{\partial \ln L}{\partial \phi} \frac{\dot{\phi}}{H} + \frac{\partial \ln L}{\partial \ln X} \frac{\dot{X}}{HX} + \frac{\partial \ln L}{\partial \ln \square\phi} \frac{(\dot{\square\phi})}{H\square\phi} = -3(1 + w_{\text{eff}}). \quad (3.18)$$

From Eq. (3.15), the field derivative $X = \dot{\phi}^2/2$ is proportional to $H^2/Q^2(\phi)$. Then, we have

$$\frac{\dot{X}}{HX} = -3(1 + w_{\text{eff}}) \left(1 + \frac{2Q_{,\phi}}{\lambda Q^2} \right), \quad (3.19)$$

where

$$\tilde{\lambda} \equiv \frac{1 + w_{\text{eff}}}{\Omega_\phi(w_m - w_\phi)}. \quad (3.20)$$

Similarly, the term $\square\phi = -\ddot{\phi} - 3H\dot{\phi}$ is expressed as

$$\square\phi = -\frac{3(1 - w_{\text{eff}}^2)P_\phi}{2\tilde{\lambda}w_\phi\Omega_\phi Q} + \frac{3(1 + w_{\text{eff}})^2 Q_{,\phi}P_\phi}{\tilde{\lambda}^2 w_\phi \Omega_\phi Q^3}. \quad (3.21)$$

Taking the time derivative of $\square\phi$, we find

$$\frac{(\dot{\square\phi})}{H\square\phi} = -3(1 + w_{\text{eff}})(1 + \mathcal{F}), \quad (3.22)$$

where

$$\mathcal{F} = \frac{\tilde{\lambda}(w_{\text{eff}} - 1)Q^2 Q_{,\phi} - 2(w_{\text{eff}} + 1)(QQ_{,\phi} - 3Q_{,\phi}^2)}{\tilde{\lambda}Q^2[\tilde{\lambda}(w_{\text{eff}} - 1)Q^2 + 2(w_{\text{eff}} + 1)Q_{,\phi}]}. \quad (3.23)$$

Substituting Eqs. (3.15), (3.19), and (3.22) into Eq. (3.18), we obtain

$$\frac{\partial \ln L}{\partial \phi} \frac{1}{\tilde{\lambda}Q} - \frac{\partial \ln L}{\partial \ln X} \left(1 + \frac{2Q_{,\phi}}{\tilde{\lambda}Q^2} \right) - \frac{\partial \ln L}{\partial \ln \square\phi} (1 + \mathcal{F}) = -1. \quad (3.24)$$

Plugging the Lagrangian (2.2) into Eq. (3.24), we find that the functions G_2 and G_3 need to satisfy the following relations

$$\left(1 + \frac{2Q_{,\phi}}{\tilde{\lambda}Q^2} \right) XG_{2,X} - \frac{1}{\tilde{\lambda}Q} G_{2,\phi} - G_2 = 0, \quad (3.25)$$

$$\left(1 + \frac{2Q_{,\phi}}{\tilde{\lambda}Q^2} \right) XG_{3,X} + \mathcal{F}G_3 - \frac{1}{\tilde{\lambda}Q} G_{3,\phi} = 0. \quad (3.26)$$

From Eq. (3.25), we obtain the following integrated solution

$$G_2(\phi, X) = Q^2(\phi)X\tilde{g}_2(\tilde{Y}), \quad (3.27)$$

where \tilde{g}_2 is an arbitrary function of

$$\tilde{Y} = Q^2(\phi)X e^{\tilde{\lambda}\psi}, \quad (3.28)$$

and

$$\psi = \int Q(\phi) d\phi. \quad (3.29)$$

The integrated solution to Eq. (3.26) is given by

$$G_3(\phi, X) = \frac{Q(\phi)\tilde{g}_3(\tilde{Y})}{1 + \mu Q_{,\phi}(\phi)/Q^2(\phi)}, \quad (3.30)$$

where \tilde{g}_3 is an arbitrary function of \tilde{Y} , and

$$\mu = \frac{2(w_{\text{eff}} + 1)}{\tilde{\lambda}(w_{\text{eff}} - 1)}. \quad (3.31)$$

Now, we will confirm whether each term in ρ_ϕ and P_ϕ is in proportion to H^2 . On using the relations (3.9) and

(3.15) in the scaling regime, the quantity (3.29) is given by

$$\psi = \frac{2}{\lambda} \ln(t - t_0) + \psi_0, \quad (3.32)$$

where ψ_0 is a constant. Then, it follows that $\tilde{Y} \propto Q^2(\phi)\dot{\phi}^2(t - t_0)^2 \propto H^2(t - t_0)^2 = \text{constant}$. Hence the term G_2 in ρ_ϕ and P_ϕ has the dependence $G_2 \propto Q^2(\phi)\dot{\phi}^2 \propto H^2$. Similarly, the term $\dot{\phi}^2 G_{2,X}$ in ρ_ϕ is proportional to $\dot{\phi}^2 Q^2(\phi)(\tilde{g}_2 + \tilde{Y}\tilde{g}_{2,\tilde{Y}}) \propto H^2$.

For the cubic coupling (3.30), its X derivative is given by $G_{3,X} = X^{-1}Q(\phi)\tilde{Y}\tilde{g}_{3,\tilde{Y}}/(1 + \mu Q_{,\phi}/Q^2)$. Then, only for $Q_{,\phi}/Q^2 = \text{constant}$, the term $3H\dot{\phi}^3 G_{3,X}$ in ρ_ϕ is in proportion to H^2 . Integration of this relation leads to

$$Q(\phi) = \frac{1}{c_1\phi + c_2}, \quad (3.33)$$

where c_1 and c_2 are constants. In this case, the other terms $-\dot{\phi}^2 G_{3,\phi}$ and $-\dot{\phi}^2 \ddot{\phi} G_{3,X}$ in ρ_ϕ and P_ϕ are also proportional to H^2 .

In Ref. [52], the coupling (3.33) was a priori assumed for simplicity to derive the Lagrangian with scaling solutions. Here, we showed that the coupling is restricted to be of this form to realize the exact scaling properties of ρ_ϕ and P_ϕ associated with the cubic function $G_3(\phi, X)$. Absorbing the constant in the denominator of Eq. (3.30) into the definition of $\tilde{g}_3(\tilde{Y})$, the cubic coupling with scaling solutions can be expressed as

$$G_3(\phi, X) = Q(\phi)\tilde{g}_3(\tilde{Y}), \quad (3.34)$$

where

$$\tilde{Y} = Q(\phi)^{2-\tilde{\lambda}/c_1} X, \quad (3.35)$$

which is valid for $c_1 \neq 0$.

The coupling (3.33) includes the case of constant Q (i.e., $c_1 = 0$). In this case, we have $\psi = Q\phi$ and hence the argument in the functions \tilde{g}_2 and \tilde{g}_3 reduces to $\tilde{Y} = Q^2 X e^{\lambda Q\phi}$. Instead of $\tilde{\lambda}$, we define

$$\lambda \equiv \tilde{\lambda}Q = \frac{Q(1 + w_{\text{eff}})}{\Omega_\phi(w_m - w_\phi)}, \quad (3.36)$$

as well as $Y = X e^{\lambda\phi}$, and absorb Q into the definitions of \tilde{g}_2 and \tilde{g}_3 . Then, the scaling Lagrangian can be written in the form

$$L = X g_2(Y) - g_3(Y) \square \phi, \quad (3.37)$$

which coincides with Eq. (3.14).

IV. FIXED POINTS FOR THE DYNAMICAL SYSTEM

In this section, we derive fixed points for the theories given by the Lagrangian (3.37) in presence of the constant

coupling Q . Defining the dimensionless variables:

$$x \equiv \frac{\dot{\phi}}{\sqrt{6}H}, \quad y \equiv \frac{e^{-\lambda\phi/2}}{\sqrt{3}H}, \quad (4.1)$$

the quantity $Y = X e^{\lambda\phi}$ can be expressed as

$$Y = \frac{x^2}{y^2}. \quad (4.2)$$

As we showed in Sec. III A, x and Y are constant along the scaling solution, so y is also constant from Eq. (4.2).

The dimensionless variables x and y obey the differential equations

$$x' = x(\epsilon_\phi - \epsilon_h), \quad (4.3)$$

$$y' = -y \left(\frac{\sqrt{6}}{2} \lambda x + \epsilon_h \right), \quad (4.4)$$

where a prime represents a derivative with respect to $N = \ln a$, and

$$\epsilon_\phi \equiv \frac{\ddot{\phi}}{H\dot{\phi}}, \quad \epsilon_h \equiv \frac{\dot{H}}{H^2}. \quad (4.5)$$

The time derivatives $\ddot{\phi}$ and \dot{H} are known by substituting $G_2 = X g_2(Y)$, $G_3 = g_3(Y)$, and their ϕ, X derivatives into Eqs. (2.20) and (2.21). From Eq. (2.10), the field density parameter $\Omega_\phi = \rho_\phi/(3H^2)$ is given by

$$\Omega_\phi = x^2 (g_2 + 2Y g_{2,Y}) - 2Y g_{3,Y} x \left(\lambda x - \sqrt{6} \right). \quad (4.6)$$

The fixed points of the dynamical system (4.3)-(4.4) can be derived by setting $x' = 0$ and $y' = 0$. Since the variables x and y are constant on fixed points, the quantity $Y = x^2/y^2$ and the functions $g_2(Y), g_3(Y)$ do not vary in time. The scaling solution discussed in Sec. III obeys the following relations

$$\epsilon_\phi = \epsilon_h = -\frac{\sqrt{6}}{2} \lambda x_c. \quad (4.7)$$

Here and in the following, we use the subscript “ c ” for the variables x, y, Y in the case where they are associated with critical points of the dynamical system. On using Eqs. (2.20) and (2.21), we obtain the following relation from Eq. (4.7):

$$\left[2(Q + \lambda)x_c - \sqrt{6}(1 + w_m) \right] \left[\sqrt{6}\lambda x_c - 3(1 + g_2 x_c^2) \right] = 0. \quad (4.8)$$

There are two fixed points satisfying Eq. (4.7). In the following, we will discuss each of them in turn.

A. Point (a): Scaling solution

One of the solutions to Eq. (4.8) is given by

$$x_c = \frac{\sqrt{6}(1 + w_m)}{2(Q + \lambda)}. \quad (4.9)$$

This corresponds to scaling solutions discussed in Sec. III. Indeed, the field density parameter Ω_ϕ and the equation of state w_ϕ reduce, respectively, to

$$\Omega_\phi = \frac{[2Q(Q + \lambda) + 3(1 + w_m)g_2](1 + w_m)}{2w_m(Q + \lambda)^2}, \quad (4.10)$$

$$w_\phi = \frac{3w_m(1 + w_m)g_2}{3(1 + w_m)g_2 + 2Q(Q + \lambda)}, \quad (4.11)$$

which are both constants. In the limit $Q \rightarrow 0$, we have $w_\phi \rightarrow w_m$ and hence $\rho_\phi \propto \rho_m \propto a^{-3(1+w_m)}$. As we mentioned in Sec. III A, the scaling solution exists not only for $Q \neq 0$ but also for $Q = 0$.

We note that there are the following relations

$$(1 - w_m)g_2 - 2w_m Y_c g_{2,Y} - \frac{2Q(Q + \lambda) - 6Y_c g_{3,Y} w_m (2Q + \lambda - w_m \lambda)}{3(1 + w_m)}, \quad (4.12)$$

$$G_{2,X} = g_2 + Y_c g_{2,Y}, \quad (4.13)$$

which can be used to express g_2 and $g_{2,Y}$ in terms of $G_{2,X}$ and $g_{3,Y}$. Then, the density parameter (4.10) is expressed as

$$\Omega_\phi = [Q(Q + \lambda) + 3(1 + w_m)G_{2,X} + 3(2Q + \lambda - w_m \lambda)Y_c g_{3,Y}]/(Q + \lambda)^2, \quad (4.14)$$

which explicitly shows the cubic-coupling contribution to Ω_ϕ . For $g_3 = 0$, Eq. (4.14) recovers the result derived in Ref. [40]. Taking the limit $Q \rightarrow 0$ in Eq. (4.14), we obtain

$$\Omega_\phi \rightarrow \frac{3}{\lambda^2} [(1 + w_m)G_{2,X} + \lambda Y_c g_{3,Y}(1 - w_m)]. \quad (4.15)$$

The quintessence with an exponential potential, which is given by the Lagrangian $G_2 = X - V_0 e^{-\lambda\phi}$, i.e., $g_2(Y) = 1 - V_0/Y$, corresponds to the density parameter $\Omega_\phi = 3(1 + w_m)/\lambda^2$ [15].

The effective equation of state $w_{\text{eff}} = -1 - 2\epsilon_h/3$ reduces to

$$w_{\text{eff}} = \frac{w_m \lambda - Q}{Q + \lambda}. \quad (4.16)$$

If $Q = 0$, then w_{eff} is equivalent to w_m . For $Q \neq 0$, the fixed point (a) can lead to the cosmic acceleration under the condition $(w_m \lambda - Q)/(Q + \lambda) < -1/3$. If we use this solution for the late-time cosmic acceleration with w_{eff} close to -1 , the coupling $|Q|$ needs to be larger than the order $|\lambda|$. Since the CMB observations place the upper bound $|Q| < \mathcal{O}(0.1)$ [34, 65], it is generally difficult to realize the scaling accelerated era characterized by $\Omega_\phi \simeq 0.7$ preceded by the scaling ϕ MDE [41]. Hence we will not employ the fixed point (a) for the late-time cosmic acceleration.

In the limit $Q \rightarrow 0$, the fixed point (a) can be used for the scaling radiation and matter eras characterized by $w_{\text{eff}} = w_\phi = w_m$. As we will show in Sec. V, this scaling solution is typically a stable attractor for $\Omega_\phi < 1$, so it does not exit from the scaling matter era. If we want

to realize the epoch of cosmic acceleration preceded by the scaling matter fixed point (a), the Lagrangian (3.37) needs to be modified in a suitable way at late time. In Refs. [29–31, 54], the authors took into account an additional scalar potential for achieving this purpose.

In this paper, we will pursue yet another possibility for realizing the cosmic acceleration preceded by the scaling ϕ MDE without modifying the Lagrangian (3.37).

B. Point (b): Scalar-field domination

The other solution to Eq. (4.8) corresponds to

$$g_2 = \frac{\sqrt{6}\lambda x_c - 3}{3x_c^2}. \quad (4.17)$$

From Eq. (4.7), we also obtain

$$g_{2,Y} = \frac{(\sqrt{6} - \lambda x_c)(\sqrt{6} - 6x_c Y_c g_{3,Y})}{6x_c^2 Y_c}. \quad (4.18)$$

The field density parameter (4.6) reduces to

$$\Omega_\phi = 1, \quad (4.19)$$

which means that the fixed point (b) satisfying the conditions (4.17) and (4.18) is the scalar-field dominated point.

On point (b), the effective equation of state and the field equation of state are equivalent to each other, such that

$$w_{\text{eff}} = w_\phi = -1 + \frac{\sqrt{6}}{3}\lambda x_c. \quad (4.20)$$

The cosmic acceleration occurs under the condition

$$\lambda x_c < \frac{\sqrt{6}}{3}. \quad (4.21)$$

From Eqs. (4.17) and (4.18), we obtain the following relation

$$\sqrt{6}G_{2,X}x_c = \lambda + (\sqrt{6}\lambda x_c - 6)Y_c g_{3,Y}. \quad (4.22)$$

Then, Eq. (4.20) can be expressed as

$$w_{\text{eff}} = w_\phi = -1 + \frac{\lambda^2 + \lambda(\sqrt{6}\lambda x_c - 6)Y_c g_{3,Y}}{3G_{2,X}}. \quad (4.23)$$

In the limit $\lambda \rightarrow 0$, we have $w_{\text{eff}} = w_\phi \rightarrow -1$. Hence, for λ close to 0, the fixed point (b) can be used for the late-time cosmic acceleration. When $\lambda \neq 0$, the cubic coupling g_3 contributes to w_{eff} and w_ϕ , whose property was shown in Ref. [54] for a specific choice of $g_3(Y)$. For given functions of $g_2(Y)$ and $g_3(Y)$, we can solve Eqs. (4.17) and (4.18) for x_c and Y_c , so that w_{eff} and w_ϕ are known accordingly.

	x_c	Ω_ϕ	w_{eff}	Stability
(a)	$\frac{\sqrt{6}(1+w_m)}{2(Q+\lambda)}$	Eq. (4.14)	$\frac{w_m\lambda-Q}{Q+\lambda}$	Stable for $\frac{2Q+\lambda}{Q+\lambda} > 0$, $\Omega_\phi < 1$, $g_{2,Y} > 0$
(b)	known from Eqs. (4.17)-(4.18)	1	$-1 + \frac{\sqrt{6}}{3}\lambda x_c$	Stable for $(Q+\lambda)x_c < \frac{\sqrt{6}}{2}$
(c)	$\frac{\sqrt{6}[Q-3d_1(w_m-1)]}{3(s+2d_1Q)}$	$\frac{2[c_0Q+d_1\{3s+2Q(6d_1-\lambda)\}]x_c}{\sqrt{6}(s+2d_1Q)}$	$\frac{3w_ms-2Q[Q+3d_1(1-2w_m)]}{3(s+2d_1Q)}$	Saddle for $\frac{3c_0+2Q(Q+\lambda)}{2c_0-4(Q+\lambda)d_1} > 0$
(d1)	$\frac{-\sqrt{6}d_1+\sqrt{c_0+2d_1(3d_1-\lambda)}}{c_0-2d_1\lambda}$	1	1	Unstable for $ Qx_c < \frac{\sqrt{6}}{2}$, $ \lambda x_c < \sqrt{6}$
(d2)	$\frac{-\sqrt{6}d_1-\sqrt{c_0+2d_1(3d_1-\lambda)}}{c_0-2d_1\lambda}$	1	1	Unstable for $ Qx_c < \frac{\sqrt{6}}{2}$, $ \lambda x_c < \sqrt{6}$

TABLE I. Critical points and corresponding values of x_c , Ω_ϕ , w_{eff} in the presence of a barotropic perfect fluid with the equation of state w_m . We also show the stability of fixed points for $w_m = 0$. For given functions $g_2(Y)$ and $g_3(Y)$, the variables Y_c and y_c are known by solving Eq. (4.12) for point (a) and Eqs. (4.17)-(4.18) for point (b). The points (c), (d1), (d2), which satisfy $y_c = 0$, are present for the functions $g_2(Y)$ and $g_3(Y)$ given by Eqs. (4.25) and (4.27).

C. Points (c) and (d1), (d2): Kinetic solutions

We proceed to the second class of solutions to Eq. (4.4), i.e.,

$$y_c = 0. \quad (4.24)$$

Let us consider the functions $g_2(Y)$ and $g_3(Y)$ containing the power-law functions of Y . The contributions arising from $g_2(Y)$ to ρ_ϕ and P_ϕ appear as the forms Xg_2 and $XYg_{2,Y}$, while, for $g_3(Y)$, they arise as the combination $Yg_{3,Y}$. In order to avoid the singular behavior at $Y = x^2/y^2 \rightarrow \infty$, they are constrained to be of the forms

$$g_2(Y) = c_0 + \sum_{n>0} c_n Y^{-n}, \quad (4.25)$$

$$g_{3,Y}(Y) = \sum_{n\geq 1} d_n Y^{-n}, \quad (4.26)$$

where c_0, c_n, d_n and n are constants. Integrating Eq. (4.26) with respect to Y , it follows that

$$g_3(Y) = d_1 \ln Y + \sum_{n=2} \tilde{d}_n Y^{-n+1}, \quad (4.27)$$

where $\tilde{d}_n = d_n/(-n+1)$. Here, we omitted the integration constant d_0 in $g_3(Y)$, as it does not contribute to the cosmological dynamics.

Substituting the above expressions of $g_2(Y)$, $g_3(Y)$ and their Y derivatives into Eq. (4.3) and taking the limit $Y \rightarrow \infty$, we obtain

$$\begin{aligned} x' = & -[3(s+2d_1Q)x - \sqrt{6}\{Q-3d_1(w_m-1)\}] \\ & \times [(c_0-2d_1\lambda)x^2 + 2\sqrt{6}d_1x - 1] \\ & \times [2c_0 + 4d_1(3d_1-\lambda)]^{-1}, \end{aligned} \quad (4.28)$$

where

$$s \equiv (w_m - 1)(c_0 - 2d_1\lambda). \quad (4.29)$$

The fixed point is determined by the coefficients c_0 and d_1 in Eqs. (4.25) and (4.26). From Eq. (4.28), there are the following two fixed points.

• Point (c): ϕ MDE

One of the solutions to Eq. (4.28) is given by

$$x_c = \frac{\sqrt{6}[Q-3d_1(w_m-1)]}{3(s+2d_1Q)}. \quad (4.30)$$

On this fixed point (c), we have

$$\Omega_\phi = \frac{2[c_0Q + d_1\{3s + 2Q(6d_1 - \lambda)\}]x_c}{\sqrt{6}(s + 2d_1Q)}, \quad (4.31)$$

$$w_\phi = \frac{Q[c_0 + 2d_1(6d_1w_m - Q - \lambda)] + 3w_md_1s}{Q[c_0 + 2d_1(6d_1 - \lambda)] + 3d_1s}, \quad (4.32)$$

$$w_{\text{eff}} = \frac{3w_ms - 2Q[Q + 3d_1(1 - 2w_m)]}{3(s + 2d_1Q)}. \quad (4.33)$$

This is the scaling solution along which $\Omega_\phi, w_\phi, w_{\text{eff}}$ are constant. In absence of the cubic coupling $d_1 \ln Y$ in $g_3(Y)$, we have $\Omega_\phi = w_{\text{eff}} = 2Q^2/(3c_0)$ and $w_\phi = 1$ for $w_m = 0$. This is known as the ϕ MDE [33, 34], in which the dynamics of standard matter era is modified by the coupling Q .

Besides the constant c_0 in $g_2(Y)$, the function $d_1 \ln Y$ in $g_3(Y)$ gives rise to contributions to the dynamics of ϕ MDE. This is a new ϕ MDE solution corrected by the logarithmic cubic coupling $d_1 \ln Y$. The other terms on the right hand sides of Eqs. (4.25) and (4.27) do not modify the values of $\Omega_\phi, w_\phi, w_{\text{eff}}$. Taking the limit $Q \rightarrow 0$ in Eqs. (4.30) and (4.31)-(4.33), we obtain $x_c = \sqrt{6}d_1/(2d_1\lambda - c_0)$, $\Omega_\phi = 6d_1^2/(2d_1\lambda - c_0)$, and $w_\phi = w_{\text{eff}} = w_m$. This agrees with the fixed point (c) derived in Ref. [54] for the model $g_2(Y) = c_0 + c_1/Y$ and $g_3(Y) = d_1 \ln Y$. For this fixed point, the scalar sound speed squared is negative ($c_s^2 = -1/3$). As we will see in Sec. VI, the ϕ MDE with a nonvanishing coupling Q can evade this problem.

• Points (d1), (d2): Purely kinetic solutions

The other solutions to Eq. (4.28) are given by

$$x_c = \frac{-\sqrt{6}d_1 \pm \sqrt{c_0 + 2d_1(3d_1 - \lambda)}}{c_0 - 2d_1\lambda}, \quad (4.34)$$

where the plus and minus signs of x_c correspond to the fixed points (d1) and (d2), respectively. They are purely kinetic solutions, satisfying

$$\Omega_\phi = 1, \quad w_{\text{eff}} = 1, \quad w_\phi = 1, \quad (4.35)$$

which are relevant to neither radiation/matter eras nor the epoch of cosmic acceleration.

In Table I, we summarize the fixed points and their properties. In Sec. V, we will study the stability of each point.

V. STABILITY OF FIXED POINTS

To study the stability of fixed points (x_c, y_c) derived in Sec. IV, we consider small homogeneous perturbations δx and δy around them, i.e.,

$$x = x_c + \delta x, \quad y = y_c + \delta y. \quad (5.1)$$

Then, the quantity Y can be expressed as $Y = Y_c + \delta Y$, where the perturbation δY is expressed as

$$\delta Y = 2 \left(\frac{x_c}{y_c^2} \delta x - \frac{x_c^2}{y_c^3} \delta y \right). \quad (5.2)$$

From Eqs. (4.3) and (4.4), we obtain the linearized equations for δx and δy in the forms

$$\begin{pmatrix} \delta x' \\ \delta y' \end{pmatrix} = \mathcal{M} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix}, \quad (5.3)$$

where \mathcal{M} is a 2×2 matrix given by

$$\mathcal{M} = \begin{pmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} \end{pmatrix}_{(x=x_c, y=y_c)} \equiv \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}. \quad (5.4)$$

The eigenvalues of \mathcal{M} are

$$\mu_\pm = \alpha_1 (1 \pm \sqrt{1 - \alpha_2}), \quad (5.5)$$

where

$$\alpha_1 = \frac{a_{11} + a_{22}}{2}, \quad \alpha_2 = \frac{4(a_{11}a_{22} - a_{12}a_{21})}{(a_{11} + a_{22})^2}. \quad (5.6)$$

If both μ_+ and μ_- are negative or μ_\pm have negative real parts, then the fixed point is stable. When either μ_+ or μ_- is positive, while the other is negative, it corresponds to a saddle point. If both μ_+ and μ_- are positive, the fixed point is an unstable node.

In the following, we consider nonrelativistic matter characterized by

$$w_m = 0, \quad (5.7)$$

as a background fluid.

A. Point (a)

The fixed point (a) corresponds to the scaling solution characterized by $x_c = \sqrt{6}/[2(Q + \lambda)]$. We first use Eqs. (4.12) and (4.13) to eliminate g_2 from Eq. (4.14). The term $g_{3,Y}$ can be expressed in terms of Ω_ϕ and $g_{2,Y}$. Then, we obtain α_1 and α_2 in Eq. (5.5), as

$$\alpha_1 = -\frac{3(2Q + \lambda)}{4(Q + \lambda)}, \quad (5.8)$$

$$\alpha_2 = \frac{24(1 - \Omega_\phi)g_{2,Y}}{q_s y_c^2 (2Q + \lambda)^2}, \quad (5.9)$$

where q_s is defined by Eq. (2.22). The term $g_{2,Y}$ appearing in α_2 has been replaced with q_s . The condition for the absence of scalar ghosts in the small-scale limit corresponds to

$$q_s > 0, \quad (5.10)$$

which is the same as that derived in Refs. [60, 66] for $Q = 0$.

The stability of point (a) is ensured for $\alpha_1 < 0$ and $\alpha_2 > 0$. Since $\Omega_m > 0$, the field density parameter should be in the range $\Omega_\phi < 1$. Then, the fixed point (a) is stable under the conditions

$$\frac{2Q + \lambda}{Q + \lambda} > 0, \quad \Omega_\phi < 1, \quad g_{2,Y} > 0. \quad (5.11)$$

The first and third conditions do not depend on g_3 . From Eq. (4.14), the second condition translates to

$$\lambda^2 > 3(G_{2,X} + \lambda Y_c g_{3,Y}) + Q(6Y_c g_{3,Y} - \lambda). \quad (5.12)$$

This means that $|\lambda|$ is generally bounded from below. If the right hand side of Eq. (5.12) is of order 1, then $|\lambda| \gtrsim \mathcal{O}(1)$. In quintessence with the exponential potential $V(\phi) = V_0 e^{-\lambda\phi}$, we have $g_2(Y) = 1 - V_0/Y$ with $V_0 > 0$ and hence $g_{2,Y} = V_0/Y^2 > 0$. The positivity of $g_{2,Y}$ also holds for the dilatonic ghost condensate model [38] given by the function $g_2(Y) = -1 + cY$ with $c > 0$.

In the uncoupled case ($Q = 0$), the first condition of Eq. (5.11) is automatically satisfied. The second condition $\Omega_\phi < 1$ translates to $\lambda^2 > 3(G_{2,X} + \lambda Y_c g_{3,Y})$. This is consistent with the stability criterion derived in Ref. [54] for the model $G_2 = X - V_0 e^{-\beta\phi}$ and $g_3 = A \ln Y$. Since $\beta = \lambda$ in the present case, the stability condition of point (a) reduces to $\lambda^2 > 3(1 + A\lambda)$.

B. Point (b)

The scalar-field dominated point (b) satisfies the relations (4.17) and (4.18). In this case, the eigenvalues μ_\pm yield

$$\mu_+ = -3 + \sqrt{6}(Q + \lambda)x_c, \quad (5.13)$$

$$\mu_- = -3 + \frac{\sqrt{6}}{2}\lambda x_c, \quad (5.14)$$

which agree with those derived in Ref. [40] in absence of the cubic coupling g_3 . If the point (b) is responsible for the cosmic acceleration, we require that $w_{\text{eff}} = -1 + \sqrt{6}\lambda x_c/3 < -1/3$, i.e., $\lambda x_c < \sqrt{6}/3$ and hence $\mu_- < -2$. Then, the stability of point (b) is ensured for

$$(Q + \lambda)x_c < \frac{\sqrt{6}}{2}. \quad (5.15)$$

On using Eqs. (4.17) and (4.18) with $G_{2,X} = g_2 + Y_c g_{2,Y}$, the variable x_c can be expressed as

$$x_c = \frac{\lambda - 6Y_c g_{3,Y}}{\sqrt{6}(G_{2,X} - \lambda Y_c g_{3,Y})}. \quad (5.16)$$

Then, the inequality (5.15) reads

$$\frac{\lambda^2 - 3(G_{2,X} + \lambda Y_c g_{3,Y}) - Q(6Y_c g_{3,Y} - \lambda)}{G_{2,X} - \lambda Y_c g_{3,Y}} < 0. \quad (5.17)$$

If $G_{2,X}$ dominates over $\lambda Y_c g_{3,Y}$, i.e.,

$$G_{2,X} > \lambda Y_c g_{3,Y}, \quad (5.18)$$

then the condition (5.17) translates to

$$\lambda^2 < 3(G_{2,X} + \lambda Y_c g_{3,Y}) + Q(6Y_c g_{3,Y} - \lambda), \quad (5.19)$$

which is exactly opposite to the stability condition (5.12) of point (a). This means that, if the scalar-field dominated point (b) is stable, the scaling solution (a) is not, and vice versa.

If the opposite inequality to Eq. (5.18) holds, then the term $\lambda(\sqrt{6}\lambda x_c - 6)Y_c g_{3,Y}/(3G_{2,X})$ in Eq. (4.23) exceeds the order of 1. This leads to the large deviation of w_ϕ from -1 , whose behavior is not observationally favored. Hence it is natural to consider the inequality (5.18), as it is the case for the specific model studied in Ref. [54].

For the application to dark energy studied later in Sec. VI, we resort to the point (b) as a late-time attractor with the cosmic acceleration. In this case, the scaling solution (a) is not stable, so it is irrelevant to the dark energy dynamics at late time.

C. Points (c) and (d1), (d2)

The fixed point (c) corresponds to

$$x_c = -\frac{\sqrt{6}(Q + 3d_1)}{3[c_0 - 2(Q + \lambda)d_1]}, \quad y_c = 0, \quad (5.20)$$

during the matter dominance. In this case, the eigenvalues (5.5) reduce to

$$\mu_+ = -\frac{3c_0 - 2Q^2 - 6(2Q + \lambda)d_1}{2c_0 - 4(Q + \lambda)d_1}, \quad (5.21)$$

$$\mu_- = \frac{3c_0 + 2Q(Q + \lambda)}{2c_0 - 4(Q + \lambda)d_1}. \quad (5.22)$$

In terms of μ_+ , the field density parameter (4.31) can be expressed as

$$\Omega_\phi - 1 = \frac{2\mu_+}{3} \left[1 + \frac{(2\mu_+ + 3)d_1}{Q} \right]. \quad (5.23)$$

Provided that the ratio $|d_1/Q|$ is smaller than the order 1, we have $\mu_+ < 0$ for $\Omega_\phi < 1$. If the condition

$$\mu_- = \frac{3c_0 + 2Q(Q + \lambda)}{2c_0 - 4(Q + \lambda)d_1} > 0, \quad (5.24)$$

is satisfied, the point (c) is a saddle. From the CMB observations, the coupling is constrained to be in the range $|Q| \lesssim \mathcal{O}(0.1)$ [65]. Since we are considering the case $|d_1/Q| < \mathcal{O}(1)$, we have $|d_1| \lesssim \mathcal{O}(0.1)$. If the scalar-field dominated point (b) corresponds to the late-time attractor, the quantity λ is bounded as Eq. (5.19). For $c_0 = \mathcal{O}(1)$, λ is typically smaller than the order 1, so that the condition (5.24) is satisfied. In this case, the ϕ MDE point (c) is a saddle, which is followed by the stable point (b) with the cosmic acceleration.

For the points (d1) and (d2), the eigenvalues are given by

$$\mu_+ = 3 \pm \sqrt{6}Qx_c, \quad (5.25)$$

$$\mu_- = 3 \mp \frac{\sqrt{6}}{2}\lambda x_c, \quad (5.26)$$

where the double signs are in the same orders as x_c given in Eq. (4.34). If the conditions

$$|Qx_c| < \frac{\sqrt{6}}{2}, \quad |\lambda x_c| < \sqrt{6} \quad (5.27)$$

are satisfied, these points are unstable nodes.

VI. APPLICATION TO DARK ENERGY

Let us apply the theory given by the action (3.14) to the dynamics of dark energy. We assume that the scalar field ϕ is coupled to cold dark matter (density ρ_c with vanishing pressure) with a constant coupling Q , such that

$$\dot{\rho}_c + 3H\rho_c = Q\rho_c\dot{\phi}. \quad (6.1)$$

We take into account baryons (density ρ_b with vanishing pressure) and radiation (density ρ_r and pressure $P_r = \rho_r/3$), which are both uncoupled to the field ϕ . Then, the continuity equations are given, respectively, by

$$\dot{\rho}_b + 3H\rho_b = 0, \quad (6.2)$$

$$\dot{\rho}_r + 4H\rho_r = 0. \quad (6.3)$$

A. Cubic Horndeski theories with ϕ MDE

We are interested in the cosmological sequence of the ϕ MDE followed by the cosmic acceleration driven by the

fixed point (b). For this purpose, we consider the model given by the functions:

$$g_2(Y) = 1 + \frac{c_1}{Y}, \quad (6.4)$$

$$g_3(Y) = d_1 \ln Y - \frac{d_2}{Y}, \quad (6.5)$$

which correspond to $n = 1$ in (4.25) with $c_0 = 1$ and $n = 2$ in (4.27). The corresponding Lagrangian is given by

$$L = X + c_1 e^{-\lambda\phi} - \left(d_1 \ln Y - \frac{d_2}{Y} \right) \square\phi. \quad (6.6)$$

For $d_2 = 0$, this reduces to the model studied in Ref. [54]. However, there are several important differences. First of all, we take into account a nonvanishing coupling Q , which gives rise to the existence of ϕ MDE. Secondly, the scaling fixed point (a) is not used for the early cosmological dynamics. If the point (a) is responsible for scaling radiation and matter eras, then the slope λ of exponential potential $V(\phi) = -c_1 e^{-\lambda\phi}$ needs to obey the condition (5.12). In this case, however, the stability condition (5.19) of point (b) is not satisfied, so the system does not exit from the scaling solution (a) to the epoch of cosmic acceleration driven by point (b). The authors in Ref. [54] took into account another shallow exponential potential $V_2 e^{-\lambda_2\phi}$ for achieving this purpose.

In this paper, we do not modify the scaling Lagrangian (6.6) at late time. In this case, the slope λ needs to satisfy the condition (5.19), so that the solutions finally

approach the stable fixed point (b) with cosmic acceleration. We will study whether this fixed point (b) is preceded by the ϕ MDE point (c) without having ghost and Laplacian instabilities. In the following, we study the case in which $|Q|$ and $|\lambda|$ are at most of the order 1.

In the small-scale limit, the scalar ghost is absent under the condition $q_s > 0$, where q_s is given by Eq. (2.22). Expanding the action (2.1) up to second order in scalar perturbations, one can show that the scalar propagation speed squared c_s^2 is of the same form as that derived in Refs. [60, 66], i.e.,

$$c_s^2 = \frac{\xi_s}{q_s}, \quad (6.7)$$

where

$$\xi_s = 2[(G_{3,XX}\ddot{\phi} + G_{3,X\phi})\dot{\phi}^2 + (2\ddot{\phi} + 4H\dot{\phi})G_{3,X} + G_{2,X} - 2G_{3,\phi}] - \dot{\phi}^4 G_{3,X}^2. \quad (6.8)$$

For the model (6.6), the quantities q_s and c_s^2 reduce, respectively, to

$$q_s = 2 + 4d_1(3d_1 - \lambda) + \frac{4d_2y^2}{x^4} (6d_1x^2 + 3d_2y^2 - \sqrt{6}x), \quad (6.9)$$

$$c_s^2 = [6(1 - 2d_1^2 - 2d_1\lambda)x^4 - 24d_2(d_1 + \lambda)x^2y^2 - 12d_2^2y^4 + 4\sqrt{6}x\{2d_1x^2 + d_2y^2(2 - \epsilon_\phi)\}]/(3q_sx^4), \quad (6.10)$$

where

$$\epsilon_\phi = \frac{1}{q_s} \left[6\sqrt{6}d_1(1 - 2d_1\lambda)x - 6(1 - 2d_1\lambda - 6d_1^2) - \frac{\sqrt{6}}{x} \{ [c_1\lambda - 2d_2(3 - 12d_1\lambda - \lambda^2)]y^2 + d_1(6 - 3\Omega_b - 3\Omega_c - 4\Omega_r) + Q\Omega_c \} + \frac{24d_2(\lambda + 3d_1)y^2}{x^2} - \frac{\sqrt{6}d_2(12d_2\lambda y^2 + 6 - 3\Omega_b - 3\Omega_c - 4\Omega_r)y^2}{x^3} + \frac{36d_2^2y^4}{x^4} \right]. \quad (6.11)$$

Here, we introduced the density parameters in the matter sector, as

$$\Omega_c = \frac{\rho_c}{3H^2}, \quad \Omega_b = \frac{\rho_b}{3H^2}, \quad \Omega_r = \frac{\rho_r}{3H^2}. \quad (6.12)$$

For $d_1 = 0$ and $d_2 = 0$, we have $q_s = 2$ and $c_s^2 = 1$, so there are neither ghost nor Laplacian instabilities. The cubic couplings d_1 and d_2 modify the values of q_s and c_s^2 .

Using the variables x and y defined in Eq. (4.1), the field density parameter Ω_ϕ is expressed as $\Omega_\phi = \Omega_{G_2} + \Omega_{G_3}$, where

$$\Omega_{G_2} = x^2 - c_1y^2, \quad (6.13)$$

$$\Omega_{G_3} = \frac{2}{x} (d_1x^2 + d_2y^2) (\sqrt{6} - \lambda x). \quad (6.14)$$

Since $\rho_m = \rho_c + \rho_b + \rho_r$ in Eq. (2.8), we obtain

$$\Omega_c = 1 - \Omega_b - \Omega_r - \Omega_{G_2} - \Omega_{G_3}. \quad (6.15)$$

The dark energy equation of state $w_\phi = P_\phi/\rho_\phi$ is given by

$$w_\phi = [3(1 - 2d_1\lambda)x^3 + 3xy^2(c_1 - 2d_2\lambda) - 2\sqrt{6}\epsilon_\phi(d_1x^2 + d_2y^2)]/[3(1 - 2d_1\lambda)x^3 - 3xy^2(c_1 + 2d_2\lambda) + 6\sqrt{6}(d_1x^2 + d_2y^2)]. \quad (6.16)$$

The density parameters of baryons and radiation satisfy the differential equations

$$\Omega'_b = -\Omega_b(3 + 2\epsilon_h), \quad (6.17)$$

$$\Omega'_r = -2\Omega_r(2 + \epsilon_h), \quad (6.18)$$

where

$$\epsilon_h = -\frac{1}{q_s} \left[6(1-2d_1\lambda)^2 x^2 + 12\sqrt{6}d_1(1-2d_1\lambda)x + 6\lambda(c_1d_1 - 2d_2 + 6d_1d_2\lambda)y^2 + 6d_1(6d_1 + Q\Omega_c) + (1-2d_1\lambda)(3\Omega_b + 3\Omega_c + 4\Omega_r) - \frac{24\sqrt{6}d_1d_2\lambda y^2}{x} + \frac{6d_2\{\lambda(c_1 + 2d_2\lambda)y^2 + Q\Omega_c\}y^2}{x^2} - \frac{2\sqrt{6}d_2(3\Omega_b + 3\Omega_c + 4\Omega_r)y^2}{x^3} - \frac{36d_2^2y^4}{x^4} \right]. \quad (6.19)$$

The variables x and y obey the differential Eqs. (4.3) and (4.4), where ϵ_ϕ and ϵ_h are given, respectively, by Eqs. (6.11) and (6.19).

For the above dynamical system, the fixed point relevant to the radiation-dominated epoch is

$$\begin{aligned} x_c &= -\frac{\sqrt{6}d_1}{1-2d_1\lambda}, & y_c &= 0, & \Omega_b &= 0, \\ \Omega_r &= \frac{1+2d_1(3d_1-\lambda)}{1-2d_1\lambda}. \end{aligned} \quad (6.20)$$

On this fixed point with $d_1 \neq 0$, the quantities q_s and c_s^2 reduce, respectively, to

$$q_s = 2 + 4d_1(3d_1 - \lambda), \quad (6.21)$$

$$c_s^2 = -\frac{1}{3}. \quad (6.22)$$

Since $c_s^2 < 0$, the scalar perturbation is subject to Laplacian instabilities.

For the theories with $d_2 = 0$, the scalar propagation speed squared is generally given by

$$c_s^2 = \frac{3x(1-2d_1^2-2d_1\lambda) + 4\sqrt{6}d_1}{3x(1+6d_1^2-2d_1\lambda)}. \quad (6.23)$$

For $|d_1\lambda| < \mathcal{O}(1)$, $|x|$ is the same order as $|d_1|$ around the radiation fixed point (6.20), in which case c_s^2 is negative. The only way of avoiding this instability problem is to consider the initial conditions satisfying $|d_1| \ll |x| \ll 1$, under which c_s^2 is close to 1. As long as the solutions do not approach the fixed point (6.20) during the radiation era, it is possible to avoid the Laplacian instability¹. In such cases, however, the cubic coupling $g_3(Y) = d_1 \ln Y$ needs to be suppressed relative to $g_2(Y) = 1 + c_1/Y$ even in the early radiation era. Then, after the end of the radiation era, the effect of the cubic coupling on the scalar-field dynamics can be practically negligible. Since the cosmological dynamics in such cases is indistinguishable from coupled quintessence with the exponential potential, we will not discuss the model $d_1 \neq 0$ any further.

¹ If we use the fixed point (a) for realizing the scaling radiation era, there is a viable parameter space in which neither ghost nor Laplacian instabilities are present [54]. In this case, unless the Lagrangian (3.37) is modified, the solutions do not exit from the scaling matter era to the epoch of cosmic acceleration.

B. Model with $d_1 = 0$

In the following, we study the model given by the Lagrangian (6.6) with

$$d_1 = 0. \quad (6.24)$$

In this case, the fixed point associated with the radiation-dominated epoch is

$$x_c = 0, \quad y_c = 0, \quad \Omega_b = 0, \quad \Omega_r = 1. \quad (6.25)$$

In realistic cosmology, the variables x and y do not exactly vanish during the radiation era. Substituting $\Omega_b = 0$ and $\Omega_r = 1$ into Eq. (6.16) and $w_{\text{eff}} = -1 - 2\epsilon_h/3$ and expanding them around $y_c = 0$, it follows that

$$w_\phi = 1 + 2(c_1 - Qd_2) \frac{y^2}{x^2} + \mathcal{O}(y^4), \quad (6.26)$$

$$w_{\text{eff}} = \frac{1}{3} + x^2 + \left[c_1x + 2d_2 \left\{ \sqrt{6} - (Q + \lambda)x \right\} \right] \frac{y^2}{x} + \mathcal{O}(y^4). \quad (6.27)$$

As long as $|c_1y^2/x^2| \ll 1$, $|d_2y^2/x^2| \ll 1$, and $x^2 \ll 1$, it follows that $w_\phi \simeq 1$ and $w_{\text{eff}} \simeq 1/3$.

Similarly, the expansions of Eqs. (6.9)-(6.10) around $y_c = 0$ lead to

$$q_s = 2 - 4\sqrt{6} \frac{d_2y^2}{x^3} + \mathcal{O}(y^4), \quad (6.28)$$

$$c_s^2 = 1 + \frac{2}{3} \left[8\sqrt{6} - 3(Q + 2\lambda)x \right] \frac{d_2y^2}{x^3} + \mathcal{O}(y^4). \quad (6.29)$$

Provided that $|d_2y^2/x^3| \ll 1$, we have $q_s \simeq 2$ and $c_s^2 \simeq 1$, so there are neither ghost nor Laplacian instabilities during the radiation dominance.

The fixed point (c) corresponding to the ϕ MDE is given by

$$x_c = -\frac{\sqrt{6}Q}{3}, \quad y_c = 0, \quad \Omega_c = 1 - \frac{2Q^2}{3}, \quad \Omega_r = 0, \quad (6.30)$$

with

$$\Omega_\phi = \frac{2Q^2}{3}, \quad w_\phi = 1, \quad w_{\text{eff}} = \frac{2Q^2}{3}. \quad (6.31)$$

During the ϕ MDE, we have

$$q_s = 2, \quad c_s^2 = 1, \quad (6.32)$$

and hence neither ghost nor Laplacian instabilities are present.

For the scalar-field dominated point (b) relevant to the late-time cosmic acceleration, it is difficult to derive the analytic expressions of x_c and y_c . For $d_2 = 0$, this fixed point corresponds to $x_c = \lambda/\sqrt{6}$ and $y_c = \sqrt{(\lambda^2 - 6)/(6c_1)}$. Dealing with the cubic coupling $g_3(Y) = -d_2/Y$ as a correction to the leading-order solution derived for $d_2 = 0$, we obtain

$$\begin{aligned} x_c &= \frac{\lambda}{\sqrt{6}} + \frac{(\lambda^2 - 6)^2}{\sqrt{6}c_1\lambda^2}d_2 + \mathcal{O}(d_2^2), \\ y_c &= \sqrt{\frac{\lambda^2 - 6}{6c_1}} + \mathcal{O}(d_2^2), \\ \Omega_b &= 0, \quad \Omega_r = 0, \end{aligned} \quad (6.33)$$

with $\Omega_\phi = 1$. For the validity of this solution, we require that $|d_2(\lambda^2 - 6)^2/(c_1\lambda^3)| \ll 1$. Since we are considering the positive exponential potential ($c_1 < 0$), we require that $\lambda^2 < 6$. We recall that the point (b) is stable under the condition (5.15). For $d_2 = 0$, this condition amounts to $(Q + \lambda)\lambda < 3$.

On the fixed point (b) given by Eq. (6.33), the dark energy equation of state w_ϕ and the effective equation of state w_{eff} are

$$w_\phi = w_{\text{eff}} = -1 + \frac{\lambda^2}{3} + \frac{(\lambda^2 - 6)^2}{3c_1\lambda}d_2 + \mathcal{O}(d_2^2). \quad (6.34)$$

The quantities q_s and c_s^2 can be estimated as

$$q_s = 2 + \frac{24(6 - \lambda^2)}{c_1\lambda^3}d_2 + \mathcal{O}(d_2^2), \quad (6.35)$$

$$c_s^2 = 1 - \frac{2(6 - \lambda^2)(10 - \lambda^2)}{c_1\lambda^3}d_2 + \mathcal{O}(d_2^2). \quad (6.36)$$

Provided that the cubic coupling d_2 is suppressed relative to the leading-order terms in Eqs. (6.35) and (6.36), there are neither ghost nor Laplacian instabilities.

In Fig. 1, we plot the evolution of $\Omega_\phi, \Omega_c, \Omega_b, \Omega_r$ versus $z + 1$ (where $z = 1/a - 1$ is the redshift) for the model parameters $c_1 = -1$, $d_1 = 0$, $d_2 = 10^{-3}$, $Q = 0.04$, and $\lambda = -0.5$. The corresponding variations of w_ϕ, w_{eff} and q_s, c_s^2 are also shown in Figs. 2 and 3, respectively. The radiation fixed point (6.25) is followed by the ϕ MDE (6.30) characterized by a nearly constant Ω_ϕ . Since $Q = 0.04$ in the numerical simulation of Fig. 1, the ϕ MDE corresponds to $x \simeq -0.03$ and $\Omega_\phi \simeq 0.001$. The initial condition of x in the radiation era is chosen as $x = -10^{-3}$, in which case Ω_ϕ initially decreases with the decrease of $|x|$. As we see in Fig. 1, however, Ω_ϕ starts to increase around the redshift $z = 10^6$ toward the ϕ MDE value $2Q^2/3$. In other words, even if Ω_ϕ is initially as large as the background density parameters, the solutions approach the scaling ϕ MDE with a nonnegligible dark energy density. Thus, unlike the Λ CDM model, the field density does not need to be negligibly small relative to the background density even in the early radiation-dominated epoch.

For the scaling solution during the matter era, the Planck team placed the bound $\Omega_\phi < 0.02$ (95% CL)

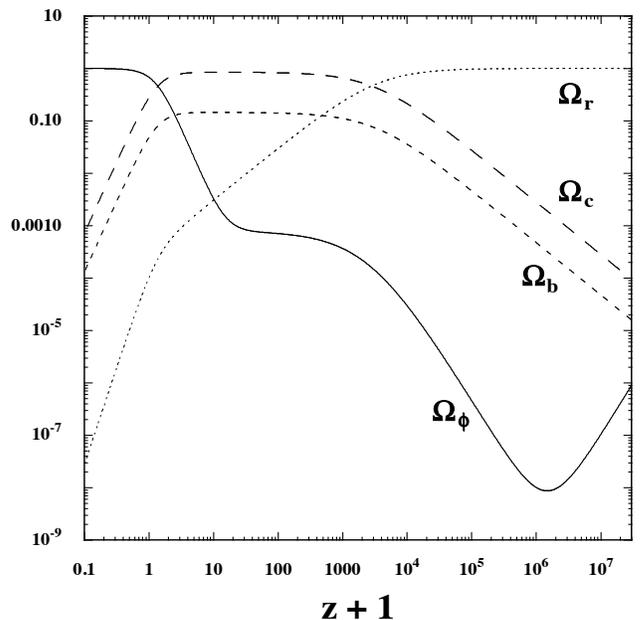


FIG. 1. Evolution of $\Omega_\phi, \Omega_c, \Omega_b, \Omega_r$ versus $z + 1$ for $c_1 = -1$, $d_1 = 0$, $d_2 = 10^{-3}$, $Q = 0.04$, and $\lambda = -0.5$. The initial conditions are chosen to be $x = -10^{-3}$, $y = 10^{-13}$, $\Omega_b = 1.5 \times 10^{-5}$, and $\Omega_r = 0.999895$ at the redshift $z = 3.18 \times 10^7$. Today's field density parameter corresponds to $\Omega_\phi^{(0)} = 0.681$.

around the redshift $z = 50$ from the measurement of CMB temperature anisotropies [65]. On using the value $\Omega_\phi = 2Q^2/3$ during the ϕ MDE, we obtain the upper limit $|Q| < 0.17$. The coupling Q used in Fig. 1 is consistent with this bound.

In Fig. 1, the conditions $|c_1 y^2/x^2| \ll 1$, $|d_2 y^2/x^2| \ll 1$, $x^2 \ll 1$, and $|d_2 y^2/x^3| \ll 1$ are satisfied during the radiation era, so that $w_\phi \simeq 1$, $w_{\text{eff}} \simeq 1/3$, $q_s \simeq 2$, and $c_s^2 \simeq 1$ from Eqs. (6.26)-(6.29). From Eqs. (6.31)-(6.32), we have $w_\phi = 1$, $w_{\text{eff}} \simeq 0.001$, $q_s = 2$, and $c_s^2 = 1$ during the ϕ MDE. Indeed, these properties can be confirmed in Figs. 2 and 3.

As we observe in Fig. 1, the field density parameter Ω_ϕ starts to increase from the ϕ MDE value $2Q^2/3$ toward the asymptotic value 1 around the redshift $z = 20$. Since the ϕ MDE under consideration corresponds to a saddle satisfying the condition (5.24), the solution finally approaches the scalar-field dominated point (b). In this case, today's density parameters (at $z = 0$) are $\Omega_\phi^{(0)} = 0.681$, $\Omega_c^{(0)} = 0.272$, $\Omega_b^{(0)} = 0.047$, and $\Omega_r^{(0)} = 1.0 \times 10^{-4}$.

In the numerical simulation of Fig. 1, the asymptotic value of x in the future is $x = -0.242$, so the stability condition (5.15) of point (b) is satisfied. Up to the order of $\mathcal{O}(d_2)$, the approximate solutions (6.33) and (6.34) give $x = -0.258$ and $w_\phi = -0.895$ for $\lambda = -0.5$, $c_1 = -1$, and $d_2 = 10^{-3}$. They are slightly different from the numerical values $x = -0.242$ and $w_\phi = -0.901$. This difference is attributed to the fact that the contribution to x arising

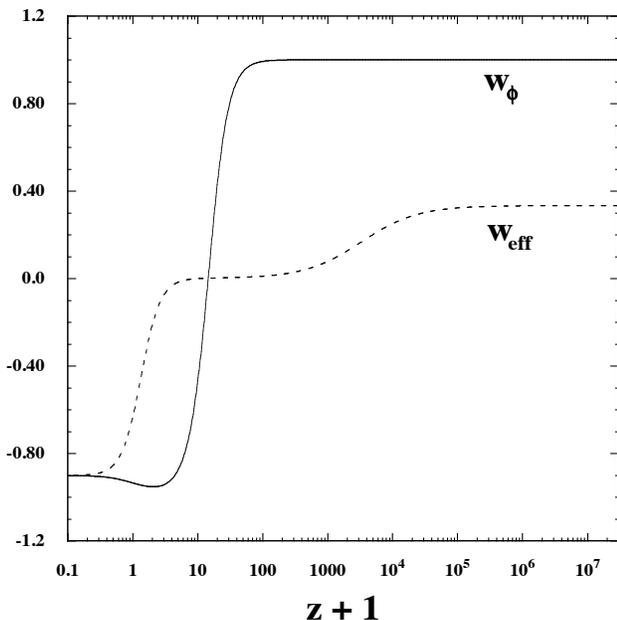


FIG. 2. Evolution of w_ϕ and w_{eff} for the same model parameters and initial conditions as those given in Fig. 1.

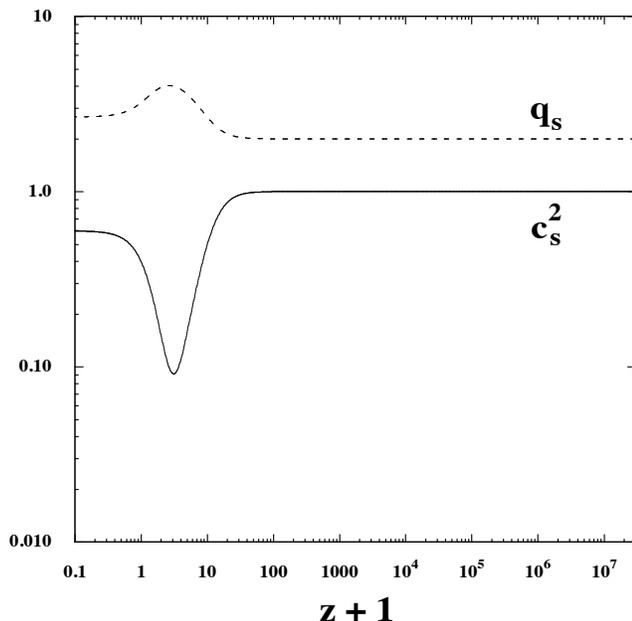


FIG. 3. Evolution of q_s and c_s^2 for the same model parameters and initial conditions as those given in Fig. 1.

from the cubic coupling d_2 is not very much smaller than the leading-order term $\lambda/\sqrt{6}$ for the model parameters used in Fig. 1.

In Fig. 2, we observe that w_ϕ starts to decrease from 1 around the end of the ϕ MDE and then temporally approaches the value close to $w_\phi \simeq -1$ at the redshift $z \approx 3$.

Then, it grows toward the asymptotic value $w_\phi = -0.901$ of point (b). The evolution of w_{eff} is quite different from w_ϕ by today, but their asymptotic values are equivalent to each other. For the model parameters used in Fig. 2, the Universe enters the stage of cosmic acceleration at the redshift $z < 0.6$.

From Eqs. (6.35) and (6.36), the values of q_s and c_s^2 on point (b) are in the ranges $q_s > 2$ and $c_s^2 < 1$ for $d_2/(c_1\lambda^3) > 0$ with $\lambda^2 \lesssim 1$. This is the case for the numerical simulation of Fig. 3, where $c_1 < 0$, $d_2 > 0$, and $\lambda < 0$. In Fig. 3, we find that c_s^2 decreases from 1 to the minimum value 0.091 around $z = 2.1$ and then it grows toward the asymptotic value 0.597. Since both q_s and c_s^2 are positive from the radiation era to the late-time accelerated attractor, there are neither ghost nor Laplacian instabilities of scalar perturbations. We note that the terms of order d_2 in Eqs. (6.35) and (6.36) give rise to contributions to q_s and c_s^2 of order 1 for the model parameters used in Fig. 3. This leads to the analytic values $q_s = 3.104$ and $c_s^2 = 0.103$ on point (b), which do not exhibit good agreement with their asymptotic values seen in Fig. 3. The approximate formulas (6.35) and (6.36) are valid only for $d_2/(c_1\lambda^3) \ll 10^{-2}$.

In the numerical simulations of Figs. 1-3, the density parameter arising from the cubic coupling is $\Omega_{G_3} = -0.018$ today, so it is by one order of magnitude smaller than the contribution $\Omega_{G_2} = 0.699$. For increasing d_2 , the contribution Ω_{G_3} to the total field density parameter Ω_ϕ tends to be larger. At the same time, the minimum value of c_s^2 gets smaller and hence it can reach the instability region $c_s^2 < 0$. When $\lambda = -0.5$, this instability occurs for $d_2/(c_1\lambda^3) \gtrsim 2 \times 10^{-2}$. For $\lambda = -\mathcal{O}(0.1)$ with positive $d_2/(c_1\lambda^3)$, the criterion for avoiding the Laplacian instability is given by $d_2/(c_1\lambda^3) \lesssim 10^{-2}$. If we consider λ closer to 0, then the upper bound on $d_2/(c_1\lambda^3)$ is generally loosened. For negative $d_2/(c_1\lambda^3)$, the sound speed squared (6.36) becomes superluminal. Moreover, the upper bound on $|d_2/(c_1\lambda^3)|$ for avoiding the instability at low redshifts tends to be severer relative to the case $d_2/(c_1\lambda^3) > 0$.

In summary, we have found an interesting scaling ϕ MDE followed by the cosmic acceleration for the model given by the Lagrangian $L = X + c_1 e^{-\lambda\phi} + (d_2/Y)\square\phi$. The cubic coupling d_2 modifies the conditions for the absence of ghost and Laplacian instabilities, but there are viable cosmological solutions without instabilities like those shown in Figs. 1-3.

VII. CONCLUSIONS

In this paper, we derived the most general Lagrangian in cubic-order Horndeski theories allowing for the existence of cosmological scaling solutions with the field-dependent coupling $Q(\phi)$. The functions G_2 and G_3 are restricted to be Eqs. (3.27) and (3.34), respectively, to realize the scaling behavior $P_\phi \propto \rho_\phi \propto H^2$. We showed that, in the presence of the cubic Lagrangian

$-G_3(\phi, X)\square\phi$, the coupling $Q(\phi)$ is constrained to be of the form $Q(\phi) = 1/(c_1\phi + c_2)$. Unlike the analysis of Ref. [52] in which the same coupling is a priori assumed, we derived its form from the scaling relation of each term in the background equations of motion.

For the constant coupling Q , which includes $Q = 0$ as a special case, the Lagrangian reduces to the form (3.37). Unlike Ref. [52] the cubic coupling is not restricted to be of the form $G_3 = a_1Y + a_2Y^2$, but G_3 is an arbitrary function $g_3(Y)$ with respect to $Y = Xe^{\lambda\phi}$. This property is consistent with the recent finding of Ref. [55].

In Sec. IV, we obtained the fixed points of the dynamical system with a matter perfect fluid for the theories given by the Lagrangian (3.37) with constant Q . Without specifying any functional forms of $g_2(Y)$ and $g_3(Y)$, we derived the two fixed points (a) and (b) corresponding to the scaling solution and the scalar-field domination, respectively. For the functions given by Eqs. (4.25) and (4.27), we showed the existence of the ϕ MDE fixed point (c) besides the purely kinetic solutions (d1) and (d2). We note that the ϕ MDE does not exist for the cubic coupling of the form $g_3(Y) = a_1Y + a_2Y^2$ derived in Ref. [52].

In Sec. V, we studied the stability of fixed points by considering homogenous perturbations around them. We showed that, if the scalar-field dominated point (b) is a stable attractor, the scaling solution (a) is not stable, and vice versa. The ϕ MDE fixed point (c) is a saddle under the condition (5.24). The ϕ MDE, which exists in the presence of the nonvanishing coupling Q , can be followed by the epoch of cosmic acceleration driven by point (b). Since the coupling is constrained to be in the range $|Q| < \mathcal{O}(0.1)$ from CMB observations, the point (a) is difficult to be used as a scaling accelerated attractor with $\Omega_\phi \simeq 0.7$. The point (a) can be applied to the scaling radiation and matter eras, but in this case the Lagrangian (3.37) needs to be modified to exit from the scaling regime to the epoch of cosmic acceleration (as studied in Ref. [54]). In this paper, we employ the scaling ϕ MDE point (c) instead of the scaling solution (a) for the matter era, without modifying the scaling Lagrangian (3.37).

In Sec. VI, we proposed a concrete model of dark energy given by the Lagrangian (6.6) with the ϕ MDE followed by the scalar-field dominated point (b). For the model with $d_1 \neq 0$, there exists the radiation-dominated fixed point (6.20), on which the scalar sound speed squared is negative ($c_s^2 = -1/3$). To avoid the Laplacian instability, the variable x initially needs to be in the range $|d_1| \ll |x| \ll 1$. In such cases it is possible to evade the instability problem, but the effect of the cubic cou-

pling on the scalar-field dynamics is practically negligible after the end of the radiation era.

On the other hand, the model with $d_1 = 0$ gives rise to interesting cosmological solutions where the cubic coupling $g_3(Y) = -d_2/Y$ provides an important contribution to the late-time dynamics of dark energy. As we observe in Fig. 1, there exists the ϕ MDE followed by the scalar-field dominated point (b). Moreover, the field density parameter Ω_ϕ does not need to be negligibly small in the early radiation era, so it alleviates the small energy-scale problem of the Λ CDM model. After the ϕ MDE the dark energy equation of state w_ϕ starts to decrease from 1, reaches a minimum close to -1 at low redshifts, and then finally approaches the value (6.34) of point (b), see Fig. 2. As we showed in Fig. 3, there exists the viable model parameter space in which the conditions for the absence of ghost and Laplacian instabilities are satisfied even in the presence of the cubic coupling.

It will be of interest to place observational constraints on the model proposed in Sec. VI by extending the EFTCAMB code developed in [67, 68]. For coupled quintessence without the cubic coupling, the likelihood analysis based on the Planck CMB data combined with the data of baryon acoustic oscillations and weak lensing showed that there is a peak around $|Q| = 0.04$ for the marginalized posterior distribution of Q [65, 69]. It remains to be seen whether this property also persists when the cubic coupling is present.

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