

Angular reduction in multiparticle matrix elements*

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Abstract

A general method for the reduction of coupled spherical harmonic products is presented. When the total angular coupling is zero, the reduction leads to an explicitly real expression in the scalar products within the unit vector arguments of the spherical harmonics. For non-scalar couplings, the reduction gives Cartesian tensor forms for the spherical harmonic products, with tensors built from the physical vectors in the original expression. The reduction for arbitrary couplings is given in closed form, making it amenable to symbolic manipulation on a computer. The final expressions do not depend on a special choice of coordinate axes, nor do they contain azimuthal quantum number summations, nor do they have complex tensor terms for couplings to a scalar. Consequently, they are easily interpretable from the properties of the physical vectors they contain.

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I. INTRODUCTION

A common occurrence in quantum mechanical calculations for multiparticle systems is the product of several spherical harmonics coming from the operators and eigenstates of particle or cluster wave functions. For example, in three-body models of ${}^6\text{Li}$, the quadrupole form factor begins with up to five spherical harmonics coupled to zero total angular momentum, each with a different argument (two each in the initial and final states, one in the quadrupole operator). There are a variety of ways to evaluate transition amplitudes and expectation values involving these products. This paper will present an alternative that can be applied to arbitrary tensor couplings. When those tensors are built from physical vectors in the problem, the method leads to scalar couplings expressed as polynomials of the scalar products of those vectors.

Several methods for handling a series of spherical harmonic couplings have been suggested in the literature. One technique applies when only three are coupled, and makes use of the freedom of choice for the orientation of the spatial axis system. One of the spherical harmonic argument vectors is aligned with the azimuthal quantization z axis, and another pair defines the xz plane. (See, for example, the paper by Balian and Brezin.¹) Putting coplanar vectors all in the xy plane also simplifies the explicit form of the spherical harmonics. In either case, a sum over azimuthal quantum numbers remains for scalar expressions. An extension of the above method takes advantage of the three-dimensional character of the underlying space. The vector argument within any spherical harmonic in a product expression is written in terms of any three independent vectors in the problem. Those spherical harmonics with an argument direction determined by a pair of vectors can be expanded as a product of spherical harmonics in each of these vectors.² Spherical harmonics with the same argument are then combined. The result of the reduction will be a sum over products of no more than three spherical harmonics in three different solid angles. The technique described above can then be applied to these remaining spherical harmonics.

For the case of a pair of coupled spherical harmonics with angular arguments determined by two different unit vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$, each spherical harmonic with high angular indices (l_1 , and l_2) coupled to a total angular momentum of low angular index (such as $L = 0, 1, 2$, or

¹ R. Balian and E. Brezin, *Nuovo Cimento B* **61**, 403 (1969).

² W. Kohn and N. Rostoker, *Phys. Rev.* **94**, 1111 (1954); see also M. Danos and L. C. Maximon, *J. Math. Phys.* **6**, 766 (1965) for further references going back to Lord Rayleigh.

3), it is possible³ to express the coupled pair in terms of a basis set of pair-coupled spherical harmonics each with minimal angular index, times Legendre functions of argument $\hat{\mathbf{a}} \cdot \hat{\mathbf{b}}$. Such results will turn out to be special cases of the method given in the following.

In this paper, we wish to present a general method for the reduction of products of spherical harmonics which we have been using for some years.⁴ When the total angular coupling is zero, the reduction leads to an explicitly real expression in the scalar dot products of the vector arguments of the original spherical harmonics. For non-scalar couplings, the reduction gives Cartesian tensor forms for the spherical harmonic products; tensors built from the physical vectors in the original problem. The advantages of the method are the following: (1) The result is readily interpretable from the known properties of the physical vectors it contains. (2) No special choice of coordinate axes are needed. (3) The final expression contains no azimuthal quantum number summations and no complex terms for couplings to a scalar. (4) The reduction for arbitrary couplings can be given in closed form, making it easily programmable in a computer calculation. As there are no spherical harmonic origin-shift expansions, numerical convergence problems associated with this re-expansion are avoided.

Section II introduces how the reduction of the scalar couplings of spherical harmonics can lead to simple results in terms of the corresponding vector dot-product expression. In Sec. III, we set up a method for transforming between Cartesian and spherical tensors. Section IV gives the general results for expanding the coupling of Cartesian tensors into an irreducible tensor sum. A by-product of this work is a general formula for the Cartesian Clebsch-Gordan coefficients. Section V shows how the Cartesian coupling can reduce arbitrarily coupled spherical harmonics with different arguments, using a few simple rules. Finally, Sec. VI summarizes our results.

³ This method has been used by J. L. Friar and G. L. Payne in two- and three-body calculations; for details, see J. L. Friar and G. L. Payne, **Phys. Rev. C** **38**, 1 (1988).

⁴ The method was originally developed by one of the authors (DRL) in conjunction with the derivation of the three-body, bound-state equations for ${}^6\text{He}$ and ${}^6\text{Li}$ [A. Ghovanlou and D. R. Lehman, **Phys. Rev. C** **9**, 1730 (1973); D. R. Lehman, M. Rai, and A. Ghovanlou, **Phys. Rev. C** **17**, 744 (1978)]. For the work on the $A = 6$ system, the method was worked out for angular-momentum values up to $l = 5$, and used by DRL and his collaborators in numerous applications since that time [for example, D. R. Lehman and M. Rajan, **Phys. Rev. C** **25**, 2743 (1982); B. F. Gibson and D. R. Lehman, **Phys. Rev. C** **29**, 1017 (1984)]. Recently, in association with our work on the ${}^6\text{Li}$ quadrupole form factor with A. Eskandarian [A. Eskandarian, D. R. Lehman, and W. C. Parke, **Phys. Rev. C** **38**, 2341 (1988)], where the method was used to obtain programmable expressions for five spherical harmonics coupled to zero, WCP generalized the method to arbitrary l and derived the irreducible decomposition of a product of two irreducible Cartesian tensors of any rank.

II. EXAMPLES OF SCALAR COUPLING REDUCTIONS

As a way of introducing the general scheme for Cartesian recoupling, consider the following expression:

$$[Y^{[2]}(\hat{\mathbf{a}}) \times [Y^{[1]}(\hat{\mathbf{c}}) \times Y^{[1]}(\hat{\mathbf{d}})]^{[2]}]^{[0]}. \quad (1)$$

We use here the angular coupling notation of Fano and Racah,⁵ i.e.,

$$[A^{[l_1]} \times B^{[l_2]}]_{m_3}^{[l_3]} = \sum_{m_1, m_2}^{m_1+m_2=m_3} \langle l_1 m_1 l_2 m_2 | l_3 m_3 \rangle A_{m_1}^{[l_1]} B_{m_2}^{[l_2]}. \quad (2)$$

The phases for the ‘contrastandard’ $Y_m^{[l]}$ spherical harmonics are fixed by

$$Y_m^{[l]} = (-i)^l Y_{lm}, \quad (3)$$

which insures that the $Y_m^{[l]}$ behave as the eigenstates of L^2 and L_z under conjugation and time reversal according to

$$\psi_m^{[l]*} = (-1)^{l+m} \psi_{-m}^{[l]}. \quad (4)$$

As emphasized by Danos, this phase choice also has the advantage of eliminating explicit phase factors in matrix element angular recoupling algebra.^{6,7}

It is widely known that the spherical harmonics $Y_{lm}(\hat{\mathbf{a}})$ can be expressed in terms of the symmetric traceless rank l tensors made from the unit vector $\hat{\mathbf{a}}$. For example, in our notation, we have in the cases of $l = 1$ and $l = 2$:

$$Y_0^{[1]}(\hat{\mathbf{a}}) = +(-i) \frac{\hat{1}}{\sqrt{4\pi}} a_3, \quad (5)$$

$$Y_{\pm 1}^{[1]}(\hat{\mathbf{a}}) = \mp (-i) \frac{\hat{1}}{\sqrt{4\pi}} \frac{1}{\sqrt{2}} (a_1 \pm ia_2),$$

and

$$Y_0^{[2]}(\hat{\mathbf{a}}) = +(-i)^2 \frac{\hat{2}}{\sqrt{4\pi}} a_{33},$$

⁵ U. Fano and G. Racah, **Irreducible Tensorial Sets** (Academic, New York, 1959), pp. 36-38.

⁶ M. Danos, **Ann. Phys.** **63**, 319 (1971); D. R. Lehman and J. S. O’Connell, “Graphical Recoupling of Angular Momenta,” **National Bureau of Standards Monograph 136** (1973), p. 12; M. Danos, V. Gillet, and M. Cauvin, **Methods in Relativistic Nuclear Physics** (North-Holland, Amsterdam, 1984), p. 59.

⁷ A. R. Edmonds, **Angular Momentum in Quantum Mechanics** (Princeton UP, Princeton, NJ, 1960), Chaps. 6 and 7.

$$Y_{\pm 1}^{[2]}(\widehat{\mathbf{a}}) = \mp (-i)^2 \frac{\widehat{2}}{\sqrt{4\pi}} \sqrt{\frac{2}{3}} (a_{13} \pm ia_{23}) , \quad (6)$$

$$Y_{\pm 2}^{[2]}(\widehat{\mathbf{a}}) = + (-i)^2 \frac{\widehat{2}}{\sqrt{4\pi}} \sqrt{\frac{1}{6}} (a_{11} - a_{22} \pm 2ia_{12}) ,$$

where

$$\widehat{l} \equiv \sqrt{2l+1} ,$$

and we define the second-rank symmetric and traceless tensor a_{ij} as

$$a_{ij} \equiv \frac{3}{2} \left(a_i a_j - \frac{1}{3} \delta_{ij} \right) . \quad (7)$$

Our irreducible Cartesian tensors of rank n , $a_{i_1 i_2 \dots i_n}$, are constructed from direct products of a unit vector, and are normalized to make contraction with that vector give the corresponding next lower rank tensor, until one reaches $\widehat{\mathbf{a}} \cdot \widehat{\mathbf{a}}$, giving unity.

The expressions Eqs. (5) and (6) for the spherical harmonics can be checked using

$$\widehat{\mathbf{a}} = \sin \theta \cos \phi \widehat{\mathbf{e}}_x + \sin \theta \sin \phi \widehat{\mathbf{e}}_y + \cos \theta \widehat{\mathbf{e}}_z \quad (8)$$

and then comparing to known forms for the spherical harmonics⁸ written in terms of the spherical angles (θ, ϕ) .

As another example, the angular components of $[Y^{[1]}(\widehat{\mathbf{c}}) \times Y^{[1]}(\widehat{\mathbf{d}})]^{[2]}$ can be expressed in terms of the components of the irreducible Cartesian tensor

$$Q(c, d)_{ij} \equiv \frac{3}{4} \left(c_i d_j + c_j d_i - \frac{2}{3} (c \cdot d) \delta_{ij} \right) . \quad (9)$$

The identity

$$[Y^{[1]}(\widehat{\mathbf{c}}) \times Y^{[1]}(\widehat{\mathbf{c}})]_m^{[2]} = \frac{\widehat{11}}{\sqrt{4\pi}} \left| \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \right| Y_m^{[2]}(\widehat{\mathbf{c}}) \quad (10)$$

is employed to determine the constant factor in the results

$$[Y^{[1]}(\widehat{\mathbf{c}}) \times Y^{[1]}(\widehat{\mathbf{d}})]_0^{[2]} = + (-i)^2 \frac{\widehat{2}}{\sqrt{4\pi}} Q(c, d)_{33} ,$$

⁸ A. R. Edmonds, *op. cit.*, Eq. (2.5.29).

$$[Y^{[1]}(\hat{\mathbf{c}}) \times Y^{[1]}(\hat{\mathbf{d}})]_{\pm 1}^{[2]} = \mp (-i)^2 \frac{\hat{2}}{\sqrt{4\pi}} \sqrt{\frac{2}{3}} (Q(c, d)_{13} \pm iQ(c, d)_{23}) , \quad (11)$$

$$[Y^{[1]}(\hat{\mathbf{c}}) \times Y^{[1]}(\hat{\mathbf{d}})]_{\pm 2}^{[2]} = + (-i)^2 \frac{\hat{2}}{\sqrt{4\pi}} \sqrt{\frac{1}{6}} (Q(c, d)_{11} - Q(c, d)_{22} \pm 2iQ(c, d)_{12}) .$$

Using the traceless nature of a_{ij} and $Q(c, d)_{ij}$, we have

$$a_{11}Q(c, d)_{22} + a_{22}Q(c, d)_{11} = a_{33}Q(c, d)_{33} - a_{11}Q(c, d)_{11} - a_{22}Q(c, d)_{22}$$

so one finds

$$\left[Y^{[2]}(\hat{\mathbf{a}}) \times [Y^{[1]}(\hat{\mathbf{c}}) \times Y^{[1]}(\hat{\mathbf{d}})]^{[2]} \right]^{[0]} = \frac{\hat{2}}{4\pi} \frac{2}{3} \sqrt{\frac{2 \cdot 3}{5 \cdot 4\pi}} \sum_{i,j} a_{ij} Q(c, d)_{ij} . \quad (12)$$

The last factor, $\sum_{i,j} a_{ij} Q(c, d)_{ij}$, is just

$$\frac{3}{2} \hat{\mathbf{a}} \cdot Q(c, d) \cdot \hat{\mathbf{a}} = \left(\frac{3}{2} \right)^2 \left\{ (a \cdot c)(a \cdot d) - \frac{1}{3}(c \cdot d) \right\} . \quad (13)$$

Aligning vector directions to help find the connection between the spherical harmonic recoupling and the corresponding contracted Cartesian tensor products will not work if the couplings have odd parity, such as in the expression

$$\left[\left[Y^{[2]}(\hat{\mathbf{a}}) \times Y^{[2]}(\hat{\mathbf{b}}) \right]^{[1]} \times [Y^{[2]}(\hat{\mathbf{c}}) \times Y^{[2]}(\hat{\mathbf{d}})]^{[1]} \right]^{[0]} .$$

Two of the couplings above produce an axial vector from the direct product of two tensors of rank two. If we define the pseudo-vector

$$R(a, b)_i \equiv \left(\frac{4}{9} \right) \sum_{jkl} \epsilon_{ijk} a_{jl} b_{kl} = a \cdot b (a \times b)_i \quad (14)$$

(ϵ_{ijk} is the completely antisymmetric tensor in three dimensions with $\epsilon_{123} = 1$), then

$$\begin{aligned} & \left[Y^{[2]}(\hat{\mathbf{a}}) \times Y^{[2]}(\hat{\mathbf{b}}) \right]_m^{[1]} \\ &= \frac{1}{\sqrt{4\pi}} \sqrt{\frac{3 \cdot 5}{2}} (-i) \sqrt{\frac{3}{4\pi}} \begin{cases} R(a, b)_3 & (m = 0) \\ \mp \frac{1}{\sqrt{2}} (R(a, b)_1 \pm R(a, b)_2) & (m = \pm 1) . \end{cases} \end{aligned} \quad (15)$$

In this odd parity case, the coefficient in the expression can be determined by the explicit Clebsch-Gordan recoupling of the spherical harmonics with total azimuthal quantum number $m = 0$. We now write the double pair coupling to zero as

$$\begin{aligned} & \left[\left[Y^{[2]}(\hat{\mathbf{a}}) \times Y^{[2]}(\hat{\mathbf{b}}) \right]^{[1]} \times \left[Y^{[2]}(\hat{\mathbf{c}}) \times Y^{[2]}(\hat{\mathbf{d}}) \right]^{[1]} \right]^{[0]} \\ &= \left(\frac{\sqrt{3}}{4\pi} \right) \left(\frac{1}{\sqrt{4\pi}} \sqrt{\frac{3 \cdot 5}{2}} \right)^2 \sum_i R(a, b)_i R(a, b)_i . \end{aligned} \quad (16)$$

The summed factor above becomes

$$(\hat{\mathbf{a}} \cdot \hat{\mathbf{b}})(\hat{\mathbf{d}} \cdot \hat{\mathbf{e}}) \left[(\hat{\mathbf{a}} \times \hat{\mathbf{b}}) \cdot (\hat{\mathbf{c}} \times \hat{\mathbf{d}}) \right] = (\hat{\mathbf{a}} \cdot \hat{\mathbf{b}})(\hat{\mathbf{d}} \cdot \hat{\mathbf{e}}) \left[(\hat{\mathbf{a}} \cdot \hat{\mathbf{c}})(\hat{\mathbf{b}} \cdot \hat{\mathbf{d}}) - (\hat{\mathbf{a}} \cdot \hat{\mathbf{d}})(\hat{\mathbf{b}} \cdot \hat{\mathbf{c}}) \right] .$$

In Sec. V, we show that expressions such as Eqs. (12) and (16) can be written by inspection for arbitrary couplings.

III. GENERAL TRANSFORMATION BETWEEN IRREDUCIBLE SPHERICAL AND CARTESIAN TENSORS

In this section, we will find a covariant connection between spherical and Cartesian tensor components of arbitrary rank. This will lead to the generalization of the results of Sec. II to arbitrary couplings of spherical harmonics. To establish our notation, we first review the connection between the generators of rotations and angular momentum. An orthonormal basis $\hat{\mathbf{e}}_i$, in Euclidean three-space can be defined through the infinitesimal displacements in that space by

$$d\mathbf{r} = \sum_i dx_i \hat{\mathbf{e}}_i . \quad (17)$$

In a coordinate transformed frame, they become

$$\tilde{\mathbf{e}}'_i = \sum_j \frac{dx_j}{dx'_i} \hat{\mathbf{e}}_j . \quad (18)$$

The condition

$$\sum_{i,j} \frac{dx'_k}{dx_i} \frac{dx'_l}{dx_j} \delta_{ij} = \delta_{kl} \quad (19)$$

makes the transformation a rotation. For infinitesimal orthogonal transformations, Eqs. (18) and (19) give

$$\hat{\mathbf{e}}'_i = \sum_j \left(\delta_{ij} + \sum_k \epsilon_{ijk} n_k \delta\theta \right) \hat{\mathbf{e}}_j , \quad (20)$$

where \hat{n} is a unit vector along the axis of rotation in the right-hand sense and $\delta\theta$ is the rotation angle. Taking \mathcal{R} to be an element of the rotation group, an infinitesimal rotation can be represented by

$$\mathcal{R} = \left(\mathcal{I} + i \sum_k S_k n_k \delta\theta \right) . \quad (21)$$

Comparing with Eq. (20), we can read the generator for infinitesimal rotations of the Cartesian basis vectors to be the standard result:

$$(S_k)_{ij} = -i\epsilon_{ijk} . \quad (22)$$

Apart from Planck's constant, these are a representation for the angular momentum operators for a spin-one field in quantum theory. However, the z component of the angular momentum operator is usually taken as diagonal with elements being the possible measured values of this S_z . The matrix S_z is diagonalized by the unitary matrix

$$(U_{mi}) = -i \begin{bmatrix} 1/\sqrt{2} & -i/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ -1/\sqrt{2} & -i/\sqrt{2} & 0 \end{bmatrix} , \quad (23)$$

giving

$$U S_z U^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} . \quad (24)$$

The vector basis set in this contrastandard spherical representation is that given by Danos⁹:

$$\hat{\mathbf{e}}_m^{[1]} = \sum_j U_{mj} \hat{\mathbf{e}}_j ,$$

⁹ M. Danos, **Ann. Phys.** **63**, 319 (1971); D. R. Lehman and J. S. O'Connell, "Graphical Recoupling of Angular Momenta," **National Bureau of Standards Monograph 136** (1973), p. 12; M. Danos, V. Gillet, and M. Cauvin, **Methods in Relativistic Nuclear Physics** (North-Holland, Amsterdam, 1984), p. 59.

$$\hat{\mathbf{e}}_{\pm}^{[1]} = \pm \frac{i}{\sqrt{2}} (\hat{\mathbf{e}}_x \pm \hat{\mathbf{e}}_y), \quad \hat{\mathbf{e}}_0^{[1]} = -i\hat{\mathbf{e}}_z. \quad (25)$$

Furthermore,

$$\hat{\mathbf{e}}_m^{[1]\dagger} \hat{\mathbf{e}}_n^{[1]} = \delta_{mn}, \quad \sum_m \hat{\mathbf{e}}_m^{[1]} \hat{\mathbf{e}}_m^{[1]\dagger} = \mathbf{1}, \quad (26)$$

where $\mathbf{1}$ is the unit dyadic operator. (Note that these basis vectors differ from those of Fano and Racah¹⁰. The Danos choice satisfies the conditions of Eq. (4), thus avoiding explicit phases when recoupling involves the angular unit vectors.)

The arbitrary phase in the unitary transformation has been taken to make the spherical basis vectors conform with the conjugation property of angular momentum eigenstates given in Eq. (4). A contrastandard spherical tensor carries a superscripted square bracket enclosing its rank index. Higher-weight spherical tensors irreducible under the rotation group can be constructed from angular couplings of the vector basis set:

$$\hat{\mathbf{e}}_m^{[l]} = [\hat{\mathbf{e}}^{[1]} \times \hat{\mathbf{e}}^{[1]} \times \hat{\mathbf{e}}^{[1]} \times \dots (l) \dots \times \hat{\mathbf{e}}^{[1]}]_m^{[l]}. \quad (27)$$

Individual pairwise couplings on the right-hand side of Eq. (27) taken in any order give the same result. This fact comes from the “stretched” form of the tensor, i.e., it has the highest rank which can be constructed from l vectors of rank 1. Explicitly, the Clebsch-Gordan products in Eq. (27) give

$$\hat{\mathbf{e}}_m^{[l]} = \left[\frac{(l-m)!(l+m)!}{l!(2l-1)!!} \right]^{1/2} \times \sum_{m'_s \text{ from } -1 \text{ to } 1} \left[\frac{1}{(1-m_1)!(1+m_1)! \dots (1-m_l)!(1+m_l)!} \right]^{1/2} \hat{\mathbf{e}}_{m_1}^{[1]} \dots \hat{\mathbf{e}}_{m_l}^{[1]} \quad (28)$$

The summation expression in Eq. (28) implicitly depends on m , since the coupled terms on the right-hand side of Eq. (27) must have their azimuthal quantum numbers add to m . The rank- l tensors $\hat{\mathbf{e}}_m^{[l]}$ satisfy

$$\hat{\mathbf{e}}_m^{[l]\dagger} \hat{\mathbf{e}}_n^{[l]} = \delta_{mn} \quad (29)$$

and

$$\sum_m \hat{\mathbf{e}}_m^{[l]} \hat{\mathbf{e}}_m^{[l]\dagger} = \mathcal{P}^{[l]}, \quad (30)$$

¹⁰ U. Fano and G. Racah, *op. cit.*, p. 21.

where $\mathcal{P}^{[l]}$ is a projection operator on rank- l Cartesian tensors which picks out only the irreducible part. The dot products which appear between higher rank tensors imply contraction over all Cartesian tensor indices.

A Cartesian tensor irreducible under the rotation group and of rank- l must be both completely symmetric in its l indices and traceless. Suppose $T_{i_1 \dots i_l}$ is such a tensor. Then a natural connection between this Cartesian tensor and its spherical representation is given by the scalar expression:

$$\begin{aligned} \mathbf{T} &= \sum_{i's} T_{i_1 i_2 \dots i_l}^{\{l\}} \hat{\mathbf{e}}_{i_1} \hat{\mathbf{e}}_{i_2} \cdots \hat{\mathbf{e}}_{i_l} \\ &= \sum_m T_m^{[l]} \hat{\mathbf{e}}_m^{[l]\dagger} = \hat{l} [T^{[l]} \times \hat{\mathbf{e}}^{[l]}]^{[0]} . \end{aligned} \quad (31)$$

Contrastandard Cartesian tensors will be denoted by putting their rank index in curly brackets. With Eq.(31) the transformation coefficients between Cartesian and spherical tensors become

$$\mathcal{U}_{m i_1 \dots i_l}^{[l]} = \hat{\mathbf{e}}_m^{[l]} \cdot \hat{\mathbf{e}}_{i_1} \hat{\mathbf{e}}_{i_2} \cdots \hat{\mathbf{e}}_{i_l} . \quad (32)$$

Thus,

$$T_{i_1 i_2 \dots i_l}^{\{l\}} = \hat{l} [T^{[l]} \times \mathcal{U}_{i_1 \dots i_l}^{[l]}]^{[0]} \quad (33)$$

and

$$T_m^{[l]} = \sum_{i's} T_{i_1 i_2 \dots i_l}^{\{l\}} \mathcal{U}_{m i_1 \dots i_l}^{[l]} . \quad (34)$$

The transformation coefficients satisfy the orthonormality conditions

$$\sum_{i's} \mathcal{U}_{m i_1 \dots i_l}^{[l]\dagger} \mathcal{U}_{n i_1 \dots i_l}^{[l]} = \delta_{mn} . \quad (35)$$

With

$$\mathcal{U}_{m3}^{[1]} = (-i) \delta_{m0} , \quad (36)$$

we find from Eq. (28),

$$\mathcal{U}_{m3 \dots 3}^{[l]} = (-i)^l \left[\frac{l!}{(2l-1)!!} \right]^{1/2} \delta_{m0} . \quad (37)$$

The coefficients $\mathcal{U}_{m i_1 \dots i_l}^{[l]}$, are completely symmetric and traceless in the Cartesian indices i_1 to i_l . Thus, they are irreducible in the space of both their spherical and Cartesian indices.

We use Eq. (37) to set the scale for normalization of Cartesian tensor components relative to spherical ones:

$$T_{3\dots 3}^{\{l\}} = i^l \left[\frac{l!}{(2l-1)!!} \right]^{1/2} T_0^{\{l\}} . \quad (38)$$

IV. CARTESIAN TENSOR RECOUPLING

A symmetric and traceless tensor of rank l can be constructed from a unit vector $\hat{\mathbf{a}}$ in the form:

$$a^{\{l\}} = \left[\frac{(2l-1)!!}{l!} \right] \sum_{r=0}^{\lfloor l/2 \rfloor} (-1)^r \left[\frac{(2l-2r-1)!!}{(2l-1)!!} \right] \{a \cdots (l-2r) \cdots a \delta \cdots (r) \cdots \delta\} . \quad (39)$$

These are the Cartesian equivalents of the spherical harmonics, which we will refer to as ‘Cartesian harmonic tensors’. In this expression, as in Eq. (27), the parenthetical value between continuation dots shows the number of repetitions of the factor shown before and after the dots. The l -Cartesian indices have been suppressed on $a^{\{l\}}$ and in each term of the summation. The δ ’s above are double-indexed Kronecker deltas. The curly brackets in the right-hand side of Eq. (39) direct that the terms inside are to be summed over all permutations of the unsymmetrized indices. For a given summation index r , there will be $\lfloor l! / (l-2r)! 2^r r! \rfloor$ such terms in the symmetrization bracket. Our choice for normalization of $a^{\{l\}}$ leads to (all vectors here, even when not marked with a caret, are unit vectors)

$$\hat{\mathbf{a}}^{\{l\}} \cdot \hat{\mathbf{b}} \cdots (l - \text{contractions}) \cdots \hat{\mathbf{b}} = P_l(\hat{\mathbf{a}} \cdot \hat{\mathbf{b}}) , \quad (40)$$

where $\hat{\mathbf{b}}$ is a second unit vector and $P_l(\hat{\mathbf{a}} \cdot \hat{\mathbf{b}})$ is the Legendre polynomial. As an example, Eq. (39) for $l = 3$ becomes

$$a_{i_1 i_2 i_3}^{\{3\}} = \frac{5}{2} a_{i_1} a_{i_2} a_{i_3} - \frac{1}{2} (a_{i_1} \delta_{i_2 i_3} + a_{i_2} \delta_{i_3 i_1} + a_{i_3} \delta_{i_1 i_2}) . \quad (41)$$

By using Eq. (38) and

$$Y_0^{\{l\}}(\hat{\mathbf{a}}) = (-i)^l \frac{\hat{l}}{\sqrt{4\pi}} P_l(\hat{\mathbf{a}} \cdot \hat{\mathbf{e}}_3) , \quad (42)$$

it follows that the irreducible Cartesian tensors defined by Eq. (39) are related to the Cartesian transformed spherical harmonics by

$$Y^{\{l\}}(\hat{\mathbf{a}}) = \frac{\hat{l}}{\sqrt{4\pi}} \left[\frac{l!}{(2l-1)!!} \right]^{1/2} a^{\{l\}}. \quad (43)$$

Now consider the coupling of two irreducible tensors of rank l_1 and l_2 . The result can be decomposed into a sum of irreducible tensors from rank $|l_1 - l_2|$ to $l_1 + l_2$. This summation is well known in the case of spherical tensors, giving a Clebsch-Gordan series. The irreducible Cartesian tensors following from this decomposition must again be completely symmetric and traceless. By explicitly constructing symmetric and traceless tensors from the products of two irreducible tensors $A^{\{l_1\}}$ and $B^{\{l_2\}}$, it is straightforward to show that the general form for decomposition of the irreducible rank l_3 Cartesian tensor is given by

$$\begin{aligned} [A^{\{l_1\}} \times B^{\{l_2\}}]^{\{l_3\}} &= C_{l_1 l_2 l_3} \left[\frac{((l_1 - l_2 + l_3)/2)! ((l_2 - l_1 + l_3)/2)!}{l_3!} \right] \times \\ &\sum_{r=0}^{\min[l_1 - k, l_2 - k]} (-1)^r 2^r \frac{(2l_3 - 2r - 1)!!}{(2l_3 - 1)!!} \{ A^{\{l_1\}} \cdot (k + r) B^{\{l_2\}} \delta \dots (r) \dots \delta \} \end{aligned} \quad (44)$$

when $l_1 + l_2 - l_3 \equiv 2k$ is even, and by

$$\begin{aligned} [A^{\{l_1\}} \times B^{\{l_2\}}]^{\{l_3\}} &= D_{l_1 l_2 l_3} \left[\frac{((l_1 - l_2 + l_3 - 1)/2)! ((l_2 - l_1 + l_3 - 1)/2)!}{l_3!} \right] \frac{1}{\sqrt{2}} \times \\ &\sum_{r=0}^{\min[l_1 - k' - 1, l_2 - k' - 1]} (-1)^r 2^r \frac{(2l_3 - 2r - 1)!!}{(2l_3 - 1)!!} \{ \epsilon : A^{\{l_1\}} \cdot (k' + r) B^{\{l_2\}} \delta \dots (r) \dots \delta \} \end{aligned} \quad (45)$$

when $l_1 + l_2 - l_3 \equiv 2k' + 1$ is odd.

In these expressions, a dot on the left side of a parenthetical value (k) and between two Cartesian tensors indicates a tensor contraction of order k . A triple of dots on each side of a parenthetical value (k) between a tensor on each side indicates one is to include a direct product of k such tensors. In addition, the colon indicates a double contraction of the form

$$(\epsilon : AB)_{i_1 i_2 \dots i_l} = \sum_{j,k} \epsilon_{i_1 j k} A_{j i_2 \dots} B_{k \dots i_l}. \quad (46)$$

In Eq. (44), terms within the curly bracket are summed over permutations of the indices

across A , B , and δ , leading to a symmetric tensor with

$$\left[\frac{l_3!}{(l_1 - k - r)!(l_2 - k - r)!2^r r!} \right] \quad (47)$$

terms for each r , while for Eq. (45), the symmetrization bracket gives

$$\left[\frac{l_3!}{(l_1 - k' - r - 1)!(l_2 - k' - r - 1)!2^r r!} \right] \quad (48)$$

terms for each r . The factors $C_{l_1 l_2 l_3}$ and $D_{l_1 l_2 l_3}$ will be determined by specializing the tensors in Eqs. (44) and (45) to ones constructed from vectors. The square-bracketed coefficients in Eqs. (44) and (45) are the inverse of the number of terms in the first symmetrization bracket of the following summation. The $(1/\sqrt{2})$ factor in Eq. (45) is inserted in anticipation of the concurrence of the C and D coefficients.

For $A^{\{l_1\}} = a^{\{l_1\}}$ and $B^{\{l_2\}} = a^{\{l_2\}}$, the right-hand side of Eq. (44) must be proportional to $a^{\{l_3\}}$. With these substitutions, the summations in Eq. (44) can be performed, giving

$$[a^{\{l_1\}} \times a^{\{l_2\}}]^{\{l_3\}} = C_{l_1 l_2 l_3} \frac{J_1!! J_2!! J_3!! (J/2)!}{l_3! (2l_3 - 1)!!} a^{\{l_3\}}, \quad (49)$$

where $J \equiv l_1 + l_2 + l_3$ and $J_i \equiv J - 2l_i - 1$, and $(-1)!! \equiv -1$.

With the spherical harmonic coupling identity

$$[Y^{[l_1]}(\hat{\mathbf{c}}) \times Y^{[l_2]}(\hat{\mathbf{c}})]_m^{[l_3]} = \frac{\widehat{l}_1 \widehat{l}_2}{\sqrt{4\pi}} \left| \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} \right| Y_m^{[l_3]}(\hat{\mathbf{c}}) \quad (50)$$

together with Eq. (43), we find

$$C_{l_1 l_2 l_3} = \widehat{l}_3 \left[\frac{(2l_1)(2l_2)(2l_3)}{(J_1 + 1)(J_2 + 1)(J_3 + 1)(J + 1)} \right]^{1/2} \quad (51)$$

For odd $l_1 + l_2 + l_3$, one can compare relation (45) for $A^{\{l_1\}} = a^{\{l_1\}}$ and $B^{\{l_2\}} = b^{\{l_2\}}$ with the corresponding spherical harmonic coupling. Using the Clebsch-Gordan coefficients for $m_3 = 0$, it follows (after some tedious algebra) that

$$D_{l_1 l_2 l_3} = C_{l_1 l_2 l_3}. \quad (52)$$

The relations (44) and (45) with (51) and (52) constitute an explicit solution for the Clebsch-Gordan coefficients in an expansion of a product of irreducible tensors in Cartesian form.

V. APPLICATIONS TO SPHERICAL HARMONIC COUPLINGS

We now are in a position to reduce any set of spherical harmonic couplings to Cartesian form. Repeated application of the pairwise coupling formula (44) and (45) will necessarily lead to a Cartesian expression in the original vectors of the problem. For couplings to a scalar, clearly the result will be a polynomial in the scalar products of these vectors, with order no greater than the smaller of the ranks of the two spherical harmonics entering with these vector arguments. If the coupling is to a pseudo-scalar, a “box” product (e.g., $\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}$) of three independent vectors must be an overall factor.

The reduction of an arbitrary series of spherical harmonic couplings proceeds as follows: For each spherical harmonic, introduce the rescaling factors shown in Eq. (43). For each pair coupling, write the appropriate Cartesian coupling as in Eq. (44) or (45). Finally, perform the indicated Cartesian tensor contractions, starting with Eq. (39) for each spherical harmonic. In this process, the traceless nature of these tensors greatly simplifies the reduction, since the Kronecker delta’s within one such tensor contracted with another irreducible tensor will vanish.

For example,¹¹ the above method can be used to show that

$$\left[Y^{[l]}(\hat{\mathbf{a}}) \times Y^{[l]}(\hat{\mathbf{b}}) \right]_m^{[1]} = \frac{(-i)}{4\pi} \left[\frac{3(2l+1)}{l(l+1)} \right]^{1/2} P'_l \left[\hat{\mathbf{a}} \times \hat{\mathbf{b}} \right]_m \quad (53)$$

and

$$\left[Y^{[l-1]}(\hat{\mathbf{a}}) \times Y^{[l]}(\hat{\mathbf{b}}) \right]_m^{[1]} = \frac{(-i)}{4\pi} \left[\frac{3}{l} \right]^{1/2} \left[P'_l \hat{\mathbf{b}}_m - \left((l-1) P_{l-2} + \hat{\mathbf{a}} \cdot \hat{\mathbf{b}} P'_{l-2} \right) \hat{\mathbf{a}}_m \right],$$

where $P_l(\hat{\mathbf{a}} \cdot \hat{\mathbf{b}})$ is the Legendre function of order l and P'_l is its derivative with respect to its argument. Similarly, higher-order couplings of the form $\left[Y^{[l_1]}(\hat{\mathbf{a}}) \times Y^{[l_2]}(\hat{\mathbf{b}}) \right]^{[L]}$ for

¹¹ See Eq. (17b) of Friar and Payne, *op. cit.* We used similar relations to derive the equations for the shell structure of the $A = 6$ ground state from three-body dynamics as given in the appendix of D. R. Lehman and W. C. Parke, **Phys. Rev. C** **28**, 364 (1983).

$L = 2, 3, \dots$ can be expressed in terms of the order- L “stretched” even or odd parity couplings of the vectors $\widehat{\mathbf{a}}$ and $\widehat{\mathbf{b}}$ times Legendre functions and their derivatives. They are most easily derived by expanding the given form in terms of an independent set of stretched couplings with unknown scalar coefficients, then contracting with each tensor of the set to form scalar relations for the coefficients.

In matrix element calculations, spherical harmonic couplings to total angular momentum of zero arise. In these cases, we have found it convenient to introduce a set of rules for generating the final scalar expression given the initial coupling. These rules result from the Cartesian recoupling formalism of the last section and are taken in a form which allows for an easy verification of each step.

The rules are as follows:

Step (1a): For each interior pair coupling of even parity, introduce the Cartesian tensor factor

$$Q_{l_1 l_2 l_3} (A, B) \equiv \left[\frac{l_1! l_2! (2l_3)! ((J_1+1)/2)! ((J_2+1)/2)!}{l_3! J_1! J_2! J_3! (J/2)!} \right] \times \quad (54)$$

$$\sum_{r=0}^{\min[l_1-k, l_2-k]} (-1)^r 2^r \frac{(2l_3-2r-1)!!}{(2l_3-1)!!} \times \{ A^{\{l_1\}} \cdot (k+r) B^{\{l_2\}} \delta \dots (r) \dots \delta \}$$

coming from the coupling in Eq.(44). As before, $J \equiv l_1 + l_2 + l_3$, $J_i \equiv J - 2l_i - 1$, $2k = l_1 + l_2 - l_3$, $(-1)!! = 1$, and the bracketed terms contain an implicit symmetrization sum, with the number of such terms given by the expression in (47). This rank- l_3 , tensor has been normalized so that when $A^{\{l_1\}} = a^{\{l_1\}}$ and $B^{\{l_2\}} = b^{\{l_2\}}$, Q reduces to $a^{\{l_3\}}$. Thus the Q 's are a natural generalization of the Cartesian harmonic tensors. Note also that

$$Q_{l_1 l_2 l_3} (a, a) \cdot (l_3) \mathbf{b} = P_{l_3} (\widehat{\mathbf{a}} \cdot \widehat{\mathbf{b}})$$

and so

$$Q_{l_1 l_2 l_3} (a, a) \cdot (l_3) \mathbf{a} = 1 .$$

Step (1b): For each interior pair coupling of odd parity, introduce the Cartesian tensor factor

$$R_{l_1 l_2 l_3} (A, B) \equiv \left[\frac{2l_1! l_2! (2l_3-1)! (J_1/2)! (J_2/2)!}{(l_3-1)! (J_1+1)! (J_2+1)! (J+1)! ((J+1)/2)!} \right] \quad (55)$$

$$\times \sum_{r=0}^{\min[l_1-k'-1, l_2-k'-1]} (-1)^r 2^r \frac{(2l_3-2r-1)!!}{(2l_3-1)!!} \{ \epsilon : A^{\{l_1\}} \cdot (k'+r) B^{\{l_2\}} \delta \dots (r) \dots \delta \}$$

coming from the coupling in Eq. (45) ($J \equiv l_1 + l_2 + l_3$, $J_i \equiv J - 2l_i - 1$, $2k' + 1 = l_1 + l_2 - l_3$). As before, the bracketed term contains an implicit symmetrization sum, with the number of such terms given in (48). This tensor has been normalized so that, for $A^{\{l_1\}} = a^{\{l_1\}}$, $B^{\{l_2\}} = b^{\{l_2\}}$, and as the vector $\widehat{\mathbf{b}}$ approaches $\widehat{\mathbf{a}}$, we have

$$\lim_{b \rightarrow a} \frac{|R_{l_1 l_2 l_3}(a, b) \cdot (l_3 - 1) \mathbf{a}|}{|\widehat{\mathbf{a}} \times \widehat{\mathbf{b}}|} = 1 \quad (56)$$

Step (2a) : For even parity couplings, introduce a factor

$$\begin{aligned} q_{l_1 l_2 l_3} &\equiv \frac{\widehat{l}_1 \widehat{l}_2}{\sqrt{4\pi}} \left| \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} \right| \\ &= \frac{\widehat{l}_1 \widehat{l}_2}{\sqrt{4\pi}} \left[\frac{J_1!! J_2!! J_3!! (J/2)!}{((J_1 + 1)/2)! ((J_2 + 1)/2)! ((J_3 + 1)/2)! (J + 1)!!} \right]^{1/2}. \end{aligned} \quad (57)$$

Our normalization for Q makes q the same factor which one would ordinarily use in coupling spherical harmonics with identical arguments.

Step (2b) : For odd parity couplings, introduce a factor

$$\begin{aligned} r_{l_1 l_2 l_3} &\equiv \frac{\widehat{l}_1 \widehat{l}_2}{\sqrt{4\pi}} \left[\frac{l_1 (l_1 + 1) l_2 (l_2 + 1)}{2l_3 \cdot 2l_3} \right]^{1/2} \left| \begin{pmatrix} l_1 & l_2 & l_3 \\ 1 & -1 & 0 \end{pmatrix} \right| \\ &= \frac{\widehat{l}_1 \widehat{l}_2}{2l_3 \sqrt{4\pi}} \left[\frac{(J_1 + 1)!! (J_2 + 1)!! (J_3 + 1)!! ((J + 1)/2)!}{(J_1/2)! (J_2/2)! (J_3/2)! J!!} \right]^{1/2}. \end{aligned} \quad (58)$$

Step (3) : For the final $L \times L$ coupling to 0, use a factor

$$S \equiv \frac{\widehat{L}}{4\pi} \frac{L!}{(2L - 1)!!} \quad (59)$$

and fully contract the final pair of Cartesian tensors. The factors in Eqs. (12) and (16) have been arranged to exhibit these steps.

As another example, consider the fourfold coupling

$$\left[\left[Y^{[2]}(\widehat{\mathbf{a}}) Y^{[2]}(\widehat{\mathbf{b}}) \right]^{[2]} \times \left[Y^{[1]}(\widehat{\mathbf{c}}) \times Y^{[3]}(\widehat{\mathbf{d}}) \right]^{[2]} \right]^{[0]}. \quad (60)$$

Performing each step on the above coupling (from right to left), we have

$$\frac{\sqrt{5}}{4\pi} \cdot \frac{2}{3} \cdot \sqrt{\frac{3 \cdot 3}{5 \cdot 4\pi}} \sqrt{\frac{2 \cdot 5}{7 \cdot 4\pi}} Q_{222}(a, b) : Q_{132}(c, d), \quad (61)$$

where

$$Q_{222}(a, b) = \frac{9}{4} \left\{ \hat{\mathbf{a}} \cdot \hat{\mathbf{b}} \left(\hat{\mathbf{a}}\hat{\mathbf{b}} + \hat{\mathbf{b}}\hat{\mathbf{a}} - \frac{2}{3}\hat{\mathbf{a}} \cdot \hat{\mathbf{b}}\delta \right) - \frac{2}{3} \left(\hat{\mathbf{a}}\hat{\mathbf{a}} - \frac{1}{3}\delta \right) - \frac{2}{3} \left(\hat{\mathbf{b}}\hat{\mathbf{b}} - \frac{1}{3}\delta \right) \right\} \quad (62)$$

and

$$Q_{132}(c, d) = \frac{5}{2} \left\{ \hat{\mathbf{c}} \cdot \hat{\mathbf{d}} \left(\hat{\mathbf{d}}\hat{\mathbf{d}} - \frac{1}{3}\delta \right) + \frac{1}{5} \left(\hat{\mathbf{c}}\hat{\mathbf{d}} + \hat{\mathbf{d}}\hat{\mathbf{c}} - \frac{2}{3}\hat{\mathbf{c}} \cdot \hat{\mathbf{d}}\delta \right) \right\}. \quad (63)$$

Making the last contraction to a scalar is simplified by noting that all contractions of the Kronecker deltas in $Q_{222}(a, b)$ with $Q_{132}(c, d)$ must vanish. The surviving terms for $Q_{222}(a, b) : Q_{132}(c, d)$ are (4/7) times the terms

$$\begin{aligned} & -5 \left(\hat{\mathbf{a}} \cdot \hat{\mathbf{d}} \right)^2 \left(\hat{\mathbf{c}} \cdot \hat{\mathbf{d}} \right) + 15 \left(\hat{\mathbf{a}} \cdot \hat{\mathbf{b}} \right) \left(\hat{\mathbf{c}} \cdot \hat{\mathbf{d}} \right) \left(\hat{\mathbf{a}} \cdot \hat{\mathbf{d}} \right) \left(\hat{\mathbf{b}} \cdot \hat{\mathbf{d}} \right) \\ & -5 \left(\hat{\mathbf{b}} \cdot \hat{\mathbf{d}} \right)^2 \left(\hat{\mathbf{c}} \cdot \hat{\mathbf{d}} \right) - 3 \left(\hat{\mathbf{a}} \cdot \hat{\mathbf{b}} \right)^2 \left(\hat{\mathbf{c}} \cdot \hat{\mathbf{d}} \right) + 2 \left(\hat{\mathbf{c}} \cdot \hat{\mathbf{d}} \right) \\ & \quad + 2 \left(\hat{\mathbf{a}} \cdot \hat{\mathbf{c}} \right) \left(\hat{\mathbf{a}} \cdot \hat{\mathbf{d}} \right) - 3 \left(\hat{\mathbf{a}} \cdot \hat{\mathbf{b}} \right) \left(\hat{\mathbf{a}} \cdot \hat{\mathbf{c}} \right) \left(\hat{\mathbf{b}} \cdot \hat{\mathbf{d}} \right) \\ & \quad - 3 \left(\hat{\mathbf{a}} \cdot \hat{\mathbf{b}} \right) \left(\hat{\mathbf{a}} \cdot \hat{\mathbf{d}} \right) \left(\hat{\mathbf{b}} \cdot \hat{\mathbf{c}} \right) + 2 \left(\hat{\mathbf{b}} \cdot \hat{\mathbf{c}} \right) \left(\hat{\mathbf{b}} \cdot \hat{\mathbf{d}} \right), \end{aligned} \quad (64)$$

Inserting into Eq. (61) gives the final answer for the fourfold coupling shown in the Eq. (A9).

Evidently the above procedure will work for spherical harmonics of arbitrarily high rank and argument, coupled to each other any number of times. The algorithm is susceptible to algebraic coding within a reasonably sophisticated algebraic manipulation program.

In the Appendix of this paper, we give results for a selection of spherical harmonic couplings as a reference and as a check of the implementation of our method.

VI. CONCLUSIONS

Although a large body of work covers angular coupling of irreducible tensors, explicit results for the coupling of Cartesian tensors of arbitrary rank have not been available. For many physical applications, using Cartesian coupling has some distinct advantages over the corresponding spherical case. We have shown that the Cartesian coupling of spherical

harmonics can be performed in a straightforward manner, following a well-defined procedure. The results are relatively simple and easy to interpret. Specifically, a simple algorithm permits one to write down directly a scalar expression for the coupling to zero of any number of spherical harmonics in terms of the unit vectors involved. We also note that leaving the coupling to a numerical calculation of azimuthal sums can introduce significant numerical errors when many intermediate terms should add to zero, but do not because of numerical truncations. This difficulty does not arise when resultant analytic forms are first calculated, as in this paper, before numerics are programmed.

Note added in proof: After this manuscript was submitted, R. F. Snider brought to our attention earlier work on irreducible Cartesian tensors that the reader may find useful.¹²

VII. ACKNOWLEDGMENT

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Appendix A: Appendix

In this appendix, we give examples of the reduction described in the paper for some commonly found spherical harmonic couplings to a scalar. The results serve to show the simplicity of the expressions, to exhibit their usefulness for physical interpretations in terms of the initial vector directions contained in the spherical harmonics, and to act as reference.

[Note: We have suppressed bold facing and vector hats in the following. Non-the-less, the letters a, b, c, d, \dots should be taken as vectors of unit length.]

$$\begin{aligned} & \left[Y^{[2]}(a) \times \left[Y^{[1]}(b) \times Y^{[1]}(c) \right]^{[2]} \right]^{[0]} = 3/(16\pi^{3/2}) \\ & \{ 5(a \cdot c)^2(b \cdot c) \\ & - 2(a \cdot b)(a \cdot c) \\ & - (b \cdot c) \}. \end{aligned} \tag{A1}$$

¹² J. A. R. Coope, R. F. Snider, and F. R. McCourt, **J. Chem. Phys.** **43**, 2269 (1965); J. A. R. Coope and R. F. Snider, **J. Math. Phys.** **11**, 1003 (1970); J. A. R. Coope, *ibid.* **11**, 1591 (1970).

$$\begin{aligned}
& \left[[Y^{[2]}(a) \times Y^{[2]}(b)]^{[2]} \times Y^{[2]}(c) \right]^{[0]} = 5/(8\sqrt{14}\pi^{3/2}) \\
& \{9(a \cdot b)(a \cdot c)(b \cdot c) \\
& -3(a \cdot b)^2 \\
& -3(a \cdot c)^2 \\
& -3(b \cdot c)^2 \\
& +2\}.
\end{aligned} \tag{A2}$$

$$\begin{aligned}
& \left[Y^{[1]}(a) \times [Y^{[1]}(b) \times Y^{[2]}(c)]^{[1]} \right]^{[0]} = \sqrt{3}/(8\sqrt{2}\pi^{3/2}) \\
& \{3(a \cdot c)(b \cdot c) \\
& -(a \cdot b)\}.
\end{aligned} \tag{A3}$$

$$\begin{aligned}
& \left[[Y^{[1]}(a) \times Y^{[1]}(b)]^{[2]} \times [Y^{[1]}(c) \times Y^{[1]}(d)]^{[2]} \right]^{[0]} = 3/(8\sqrt{2}\pi^2) \\
& \{3(a \cdot c)(b \cdot d) \\
& +3(b \cdot c)(a \cdot d) \\
& -2(a \cdot b)(c \cdot d)\}.
\end{aligned} \tag{A4}$$

$$\begin{aligned}
& \left[[Y^{[1]}(a) \times Y^{[1]}(b)]^{[2]} \times [Y^{[1]}(c) \times Y^{[3]}(d)]^{[2]} \right]^{[0]} = 3\sqrt{15}/(80\sqrt{2}\pi^2) \\
& \{5(a \cdot c)(b \cdot c)(c \cdot d) \\
& -(a \cdot c)(b \cdot d) \\
& -(b \cdot c)(a \cdot d) \\
& -(c \cdot d)(a \cdot b)\}.
\end{aligned} \tag{A5}$$

$$\begin{aligned}
& \left[[Y^{[1]}(a) \times Y^{[3]}(b)]^{[2]} \times [Y^{[1]}(c) \times Y^{[3]}(d)]^{[2]} \right]^{[0]} = 3\sqrt{50}/(160\pi^2) \\
& \{-10(c \cdot d)(a \cdot d)(b \cdot d) \\
& +25(c \cdot d)(b \cdot d)^2(a \cdot b) \\
& -3(c \cdot d)(a \cdot b) \\
& +2(a \cdot c)(b \cdot d) \\
& +2(b \cdot c)(a \cdot d) \\
& -10(b \cdot c)(b \cdot d)(a \cdot b)\}.
\end{aligned} \tag{A6}$$

$$\begin{aligned}
& \left[[Y^{[2]}(a) \times Y^{[2]}(b)]^{[1]} \times [Y^{[1]}(c) \times Y^{[1]}(d)]^{[1]} \right]^{[0]} = 3\sqrt{15}/(32\pi^2) \\
& (a \cdot b)(a \cdot c) \\
& \{(b \cdot d) \\
& -(b \cdot c)(a \cdot d)\}.
\end{aligned} \tag{A7}$$

$$\begin{aligned}
& \left[[Y^{[2]}(a)Y^{[2]}(b)]^{[2]} \times [Y^{[1]}(c) \times Y^{[1]}(d)]^{[2]} \right]^{[0]} = \sqrt{15}/(32\sqrt{7}\pi^2) \\
& \{-6(c \cdot d)(a \cdot b)^2 \\
& +4(c \cdot d) \\
& -6(a \cdot c)(a \cdot d) \\
& +9(a \cdot c)(b \cdot d)(a \cdot b) \\
& +9(b \cdot c)(a \cdot d)(a \cdot b) \\
& -6(b \cdot c)(b \cdot d)\}.
\end{aligned} \tag{A8}$$

$$\begin{aligned}
& \left[[Y^{[2]}(a) \times Y^{[2]}(b)]^{[2]} \times [Y^{[1]}(c) \times Y^{[3]}(d)]^{[2]} \right]^{[0]} = 3\sqrt{5}/(16\sqrt{14}\pi^2) \\
& \{-5(c \cdot d)(a \cdot d)^2 \\
& +15(c \cdot d)(a \cdot d)(b \cdot d)(a \cdot b) \\
& -5(c \cdot d)(b \cdot d) \\
& -3(c \cdot d)(a \cdot b)^2 \\
& +2(c \cdot d) \\
& +2(a \cdot c)(a \cdot d) \\
& -3(a \cdot c)(b \cdot d)(a \cdot b) \\
& -3(b \cdot c)(a \cdot d)(a \cdot b) \\
& +2(b \cdot c)(b \cdot d)\}.
\end{aligned} \tag{A9}$$

$$\begin{aligned}
& \left[\left[\left[Y^{[2]}(a) \times Y^{[1]}(b) \right]^{[1]} \times Y^{[2]}(c) \right]^{[1]} \left[\times Y^{[2]}(d) \times Y^{[2]}(e) \right]^{[1]} \right]^{[0]} = 15\sqrt{3}/(64\sqrt{2}\pi^{5/2}) \\
& (a \cdot b)(d \cdot e) \\
& \{ 3(a \cdot c)(b \cdot d)(c \cdot e) \\
& -3(a \cdot c)(b \cdot e)(c \cdot d) \\
& -3(a \cdot d)(b \cdot c)(c \cdot e) \\
& +2(a \cdot d)(b \cdot e) \\
& +3(a \cdot e)(b \cdot c)(c \cdot d) \\
& -2(a \cdot e)(b \cdot d) \}.
\end{aligned} \tag{A10}$$

$$\begin{aligned}
& \left[\left[\left[Y^{[2]}(a) \times Y^{[2]}(b) \right]^{[1]} \times Y^{[2]}(c) \right]^{[2]} \times \left[Y^{[2]}(d) \times Y^{[2]}(e) \right]^{[2]} \right]^{[0]} = 15\sqrt{15}/(64\sqrt{14}\pi^{5/2}) \\
& (a \cdot b) \\
& \{ -2(a \cdot c)(b \cdot d)(c \cdot d) \\
& +3(a \cdot c)(b \cdot d)(c \cdot e)(d \cdot e) \\
& +3(a \cdot c)(b \cdot e)(c \cdot d)(d \cdot e) \\
& -2(a \cdot c)(b \cdot e)(c \cdot e) \\
& +2(a \cdot d)(b \cdot c)(c \cdot d) \\
& -3(a \cdot d)(b \cdot c)(c \cdot e)(d \cdot e) \\
& -3(a \cdot e)(b \cdot c)(c \cdot d)(d \cdot e) \\
& +2(a \cdot e)(b \cdot c)(c \cdot e) \}.
\end{aligned}$$

(A11)

$$\left[\left[\left[Y^{[2]}(a) \times Y^{[2]}(b) \right]^{[2]} \times Y^{[2]}(c) \right]^{[1]} \times \left[Y^{[2]}(d) \times Y^{[2]}(e) \right]^{[1]} \right]^{[0]} = 15\sqrt{15}/(64\sqrt{14}\pi^{5/2})$$

$(d \cdot e)$

$$\begin{aligned} & \{3(a \cdot b)(a \cdot c)(b \cdot d)(c \cdot e) \\ & -3(a \cdot b)(a \cdot c)(b \cdot e)(c \cdot d) \\ & +3(a \cdot b)(a \cdot d)(b \cdot c)(c \cdot e) \\ & -3(a \cdot b)(a \cdot e)(b \cdot c)(c \cdot d) \\ & -2(a \cdot c)(a \cdot d)(c \cdot e) \\ & +2(a \cdot c)(a \cdot e)(c \cdot d) \\ & -2(b \cdot c)(b \cdot d)(c \cdot e) \\ & +2(b \cdot c)(b \cdot e)(c \cdot d)\}. \end{aligned}$$

(A12)

$$\left[\left[\left[Y^{[2]}(a) \times Y^{[2]}(b) \right]^{[2]} \times Y^{[2]}(c) \right]^{[2]} \times \left[Y^{[2]}(d) \times Y^{[2]}(e) \right]^{[2]} \right]^{[0]} = 25/(448\sqrt{14}\pi^{5/2})$$

$$\begin{aligned} & \{36(a \cdot b)^2(c \cdot d)^2 - 108(a \cdot b)^2(c \cdot d)(c \cdot e)(d \cdot e) + 36(a \cdot b)^2(c \cdot e)^2 + 72(a \cdot b)^2(d \cdot e)^2 \\ & -48(a \cdot b)^2 - 108(a \cdot b)(a \cdot c)(b \cdot c)(d \cdot e)^2 + 72(a \cdot b)(a \cdot c)(b \cdot c) \\ & -54(a \cdot b)(a \cdot c)(b \cdot d)(c \cdot d) + 81(a \cdot b)(a \cdot c)(b \cdot d)(c \cdot e)(d \cdot e) \\ & +81(a \cdot b)(a \cdot c)(b \cdot e)(c \cdot d)(d \cdot e) - 54(a \cdot b)(a \cdot c)(b \cdot e)(c \cdot e) \\ & -54(a \cdot b)(a \cdot d)(b \cdot c)(c \cdot d) + 81(a \cdot b)(a \cdot d)(b \cdot c)(c \cdot e)(d \cdot e) \\ & +36(a \cdot b)(a \cdot d)(b \cdot d) - 54(a \cdot b)(a \cdot d)(b \cdot e)(d \cdot e) \\ & +81(a \cdot b)(a \cdot e)(b \cdot c)(c \cdot d)(d \cdot e) - 54(a \cdot b)(a \cdot e)(b \cdot c)(c \cdot e) \\ & -54(a \cdot b)(a \cdot e)(b \cdot d)(d \cdot e) + 36(a \cdot b)(a \cdot e)(b \cdot e) \\ & +36(a \cdot c)^2(d \cdot e)^2 - 24(a \cdot c)^2 + 36(a \cdot c)(a \cdot d)(c \cdot d) \\ & -54(a \cdot c)(a \cdot d)(c \cdot e)(d \cdot e) - 54(a \cdot c)(a \cdot e)(c \cdot d)(d \cdot e) \\ & +36(a \cdot c)(a \cdot e)(c \cdot e) - 12(a \cdot d)^2 + 36(a \cdot d)(a \cdot e)(d \cdot e) - 12(a \cdot e)^2 \\ & +36(b \cdot c)^2(d \cdot e)^2 - 24(b \cdot c)^2 + 36(b \cdot c)(b \cdot d)(c \cdot d) \\ & +54(b \cdot c)(b \cdot d)(c \cdot e)(d \cdot e) - 54(b \cdot c)(b \cdot e)(c \cdot d)(d \cdot e) \\ & +36(b \cdot c)(b \cdot e)(c \cdot e) - 12(b \cdot d)^2 + 36(b \cdot d)(b \cdot e)(d \cdot e) - 12(b \cdot e)^2 \\ & -24(c \cdot d)^2 + 72(c \cdot d)(c \cdot e)(d \cdot e) - 24(c \cdot e)^2 - 48(d \cdot e)^2 + 32\}. \end{aligned}$$

(A13)

$$\begin{aligned}
& \left[\left[[Y^{[2]}(a) \times Y^{[2]}(b)]^{[1]} \times Y^{[2]}(c) \right]^{[1]} \times [Y^{[1]}(d) \times Y^{[1]}(e)]^{[1]} \right]^{[0]} = 3\sqrt{15}/(64\sqrt{2}\pi^{5/2}) \\
& (a \cdot b) \\
& \{3(a \cdot c)(b \cdot d)(c \cdot e) \\
& -3(a \cdot c)(b \cdot e)(c \cdot d) \\
& -3(a \cdot d)(b \cdot c)(c \cdot e) \\
& +2(a \cdot d)(b \cdot e) \\
& +3(a \cdot e)(b \cdot c)(c \cdot d) \\
& -2(a \cdot e)(b \cdot d)\}.
\end{aligned} \tag{A14}$$

$$\begin{aligned}
& \left[\left[[Y^{[2]}(a) \times Y^{[2]}(b)]^{[1]} \times Y^{[2]}(c) \right]^{[2]} \times [Y^{[1]}(d) \times Y^{[1]}(e)]^{[2]} \right]^{[0]} = 9\sqrt{5}/(64\sqrt{2}\pi^{5/2}) \\
& (a \cdot b) \\
& \{(a \cdot c)(b \cdot d)(c \cdot e) \\
& +(a \cdot c)(b \cdot e)(c \cdot d) \\
& -(a \cdot d)(b \cdot c)(c \cdot e) \\
& -(a \cdot e)(b \cdot c)(c \cdot d)\}.
\end{aligned} \tag{A15}$$

$$\begin{aligned}
& \left[\left[[Y^{[2]}(a) \times Y^{[2]}(b)]^{[2]} \times Y^{[2]}(c) \right]^{[1]} \times [Y^{[1]}(d) \times Y^{[1]}(e)]^{[1]} \right]^{[0]} = 15\sqrt{3}/(64\sqrt{14}\pi^{5/2}) \\
& \{3(a \cdot b)(a \cdot c)(b \cdot d)(c \cdot e) \\
& -3(a \cdot b)(a \cdot c)(b \cdot e)(c \cdot d) \\
& +3(a \cdot b)(a \cdot d)(b \cdot c)(c \cdot e) \\
& -3(a \cdot b)(a \cdot e)(b \cdot c)(c \cdot d) \\
& -2(a \cdot c)(a \cdot d)(c \cdot e) \\
& +2(a \cdot c)(a \cdot e)(c \cdot d) \\
& -2(b \cdot c)(b \cdot d)(c \cdot e) \\
& +2(b \cdot c)(b \cdot e)(c \cdot d)\}.
\end{aligned} \tag{A16}$$

$$\begin{aligned}
& \left[\left[[Y^{[2]}(a) \times Y^{[2]}(b)]^{[2]} \times Y^{[2]}(c) \right]^{[2]} \times [Y^{[1]}(d) \times Y^{[1]}(e)]^{[2]} \right]^{[0]} = 5\sqrt{3}/(448\sqrt{2}\pi^{5/2}) \\
& \{-36(a \cdot b)^2(c \cdot d)(c \cdot e) \\
& +24(a \cdot b)^2(d \cdot e) \\
& -36(a \cdot b)(a \cdot c)(b \cdot c)(d \cdot e) \\
& +27(a \cdot b)(a \cdot c)(b \cdot d)(c \cdot e) \\
& +27(a \cdot b)(a \cdot c)(b \cdot e)(c \cdot d) \\
& +27(a \cdot b)(a \cdot d)(b \cdot c)(c \cdot e) \\
& -18(a \cdot b)(a \cdot d)(b \cdot e) \\
& +27(a \cdot b)(a \cdot e)(b \cdot c)(c \cdot d) \\
& -18(a \cdot b)(a \cdot e)(b \cdot d) \\
& +12(a \cdot c)^2(d \cdot e) \\
& -18(a \cdot c)(a \cdot d)(c \cdot e) \\
& -18(a \cdot c)(a \cdot e)(c \cdot d) \\
& +12(a \cdot d)(a \cdot e) \\
& +12(b \cdot c)^2(d \cdot e) \\
& -18(b \cdot c)(b \cdot d)(c \cdot e) \\
& -18(b \cdot c)(b \cdot e)(c \cdot d) \\
& +12(b \cdot d)(b \cdot e) \\
& +24(c \cdot d)(c \cdot e) \\
& -16(d \cdot e)\}.
\end{aligned} \tag{A17}$$

$$\left[\left[\left[Y^{[2]}(a) \times Y^{[2]}(b) \right]^{[1]} \times Y^{[2]}(c) \right]^{[2]} \times \left[Y^{[1]}(d) \times Y^{[3]}(e) \right]^{[2]} \right]^{[0]} = 3\sqrt{15}/(64\pi^{5/2})$$

$$(a \cdot b)$$

$$\{ -(a \cdot c)(b \cdot d)(c \cdot e)$$

$$-(a \cdot c)(b \cdot e)(c \cdot d)$$

$$+5(a \cdot c)(b \cdot e)(c \cdot e)(d \cdot e)$$

$$+(a \cdot d)(b \cdot c)(c \cdot e)$$

$$+(a \cdot e)(b \cdot c)(c \cdot d)$$

$$-5(a \cdot e)(b \cdot c)(c \cdot e)(d \cdot e) \}.$$

(A18)

$$\begin{aligned}
& \left[\left[[Y^{[2]}(a) \times Y^{[2]}(b)]^{[2]} \times Y^{[2]}(c) \right]^{[2]} \times [Y^{[1]}(d) \times Y^{[3]}(e)]^{[2]} \right]^{[0]} = 15/(448\pi^{5/2}) \\
& \{12(a \cdot b)^2(c \cdot d)(c \cdot e) \\
& -30(a \cdot b)^2(c \cdot e)^2(d \cdot e) \\
& +12(a \cdot b)^2(d \cdot e) \\
& -18(a \cdot b)(a \cdot c)(b \cdot c)(d \cdot e) \\
& -9(a \cdot b)(a \cdot c)(b \cdot d)(c \cdot e) \\
& -9(a \cdot b)(a \cdot c)(b \cdot e)(c \cdot d) \\
& +45(a \cdot b)(a \cdot c)(b \cdot e)(c \cdot e)(d \cdot e) \\
& -9(a \cdot b)(a \cdot d)(b \cdot c)(c \cdot e) \\
& +6(a \cdot b)(a \cdot d)(b \cdot e) \\
& -9(a \cdot b)(a \cdot e)(b \cdot c)(c \cdot d) \\
& +45(a \cdot b)(a \cdot e)(b \cdot c)(c \cdot e)(d \cdot e) \\
& +6(a \cdot b)(a \cdot e)(b \cdot d) \\
& -30(a \cdot b)(a \cdot e)(b \cdot e)(d \cdot e) \\
& +6(a \cdot c)^2(d \cdot e) \\
& +6(a \cdot c)(a \cdot d)(c \cdot e) \\
& +6(a \cdot c)(a \cdot e)(c \cdot d) \\
& -30(a \cdot c)(a \cdot e)(c \cdot e)(d \cdot e) \\
& -4(a \cdot d)(a \cdot e) \\
& +10(a \cdot e)^2(d \cdot e) \\
& +6(b \cdot c)^2(d \cdot e) \\
& +6(b \cdot c)(b \cdot d)(c \cdot e) \\
& +6(b \cdot c)(b \cdot e)(c \cdot d) \\
& -30(b \cdot c)(b \cdot e)(c \cdot e)(d \cdot e) \\
& -4(b \cdot d)(b \cdot e) \\
& +10(b \cdot e)^2(d \cdot e) \\
& -8(c \cdot d)(c \cdot e) \\
& +20(c \cdot e)^2(d \cdot e) \\
& -8(d \cdot e)\}.
\end{aligned} \tag{A19}$$

$$\begin{aligned}
& \left[\left[[Y^{[1]}(a) \times Y^{[2]}(b)]^{[2]} \times Y^{[2]}(c) \right]^{[2]} \times [Y^{[1]}(d) \times Y^{[1]}(e)]^{[2]} \right]^{[0]} = 3\sqrt{3}/(64\sqrt{2}\pi^{5/2}) \\
& \{3(a \cdot c)(b \cdot d)(c \cdot e) \\
& -3(a \cdot c)(b \cdot e)(c \cdot d) \\
& -3(a \cdot d)(b \cdot c)(c \cdot e) \\
& +2(a \cdot d)(b \cdot e) \\
& +3(a \cdot e)(b \cdot c)(c \cdot d) \\
& -2(a \cdot e)(b \cdot d)\}.
\end{aligned} \tag{A20}$$

$$\begin{aligned}
& \left[\left[[Y^{[1]}(a) \times Y^{[1]}(b)]^{[2]} \times Y^{[2]}(c) \right]^{[2]} \times [Y^{[1]}(d) \times Y^{[1]}(e)]^{[2]} \right]^{[0]} = 3/(64\sqrt{14}\pi^{5/2}) \\
& \{-12(a \cdot b)(c \cdot d)(c \cdot e) \\
& +8(a \cdot b)(d \cdot e) \\
& -12(a \cdot c)(b \cdot c)(d \cdot e) \\
& +9(a \cdot c)(b \cdot d)(c \cdot e) \\
& +9(a \cdot c)(b \cdot e)(c \cdot d) \\
& +9(a \cdot d)(b \cdot c)(c \cdot e) \\
& -6(a \cdot d)(b \cdot e) \\
& +9(a \cdot e)(b \cdot c)(c \cdot d) \\
& -6(a \cdot e)(b \cdot d)\}.
\end{aligned} \tag{A21}$$

$$\begin{aligned}
& \left[\left[[Y^{[1]}(a) \times Y^{[1]}(b)]^{[1]} \times Y^{[2]}(e) \right]^{[2]} \times [Y^{[1]}(d) \times Y^{[3]}(e)]^{[2]} \right]^{[0]} = 3\sqrt{3}/(64\pi^{5/2}) \\
& \{-(a \cdot c)(b \cdot d)(c \cdot e) \\
& -(a \cdot c)(b \cdot e)(c \cdot d) \\
& +5(a \cdot e)(b \cdot e)(c \cdot e)(d \cdot e) \\
& +(a \cdot d)(b \cdot c)(c \cdot e) \\
& +(a \cdot e)(b \cdot c)(c \cdot d) \\
& -5(a \cdot e)(b \cdot c)(c \cdot e)(d \cdot e)\}.
\end{aligned} \tag{A22}$$

$$\begin{aligned}
& \left[\left[[Y^{[1]}(a) \times Y^{[1]}(b)]^{[2]} \times Y^{[1]}(e) \right]^{[2]} \times [Y^{[1]}(d) \times Y^{[3]}(e)]^{[2]} \right]^{[0]} = 3\sqrt{3}/(64\sqrt{7}\pi^{5/2}) \\
& \{4(a \cdot b)(c \cdot d)(c \cdot e) \\
& -10(a \cdot b)(c \cdot e)^2(d \cdot e) \\
& +4(a \cdot b)(d \cdot e) \\
& -6(a \cdot c)(b \cdot c)(d \cdot e) \\
& -3(a \cdot c)(b \cdot d)(c \cdot e) \\
& -3(a \cdot c)(b \cdot e)(c \cdot d) \\
& +15(a \cdot c)(b \cdot e)(c \cdot e)(d \cdot e) \\
& -3(a \cdot d)(b \cdot c)(c \cdot e) \\
& +2(a \cdot d)(b \cdot e) \\
& -3(a \cdot e)(b \cdot c)(c \cdot d) \\
& +15(a \cdot e)(b \cdot c)(c \cdot e)(d \cdot e) \\
& +2(a \cdot e)(b \cdot d) \\
& -10(a \cdot e)(b \cdot e)(d \cdot e)\}.
\end{aligned} \tag{A23}$$

$$\begin{aligned}
& \left[\left[[Y^{[1]}(a) \times Y^{[3]}(b)]^{[2]} \times Y^{[2]}(e) \right]^{[1]} \times [Y^{[1]}(d) \times Y^{[1]}(e)]^{[1]} \right]^{[0]} = 3\sqrt{3}/(64\pi^{5/2}) \\
& \{5(a \cdot b)(b \cdot c)(b \cdot d)(c \cdot e) \\
& -5(a \cdot b)(b \cdot c)(b \cdot e)(c \cdot d) \\
& -(a \cdot c)(b \cdot d)(c \cdot e) \\
& +(a \cdot c)(b \cdot e)(c \cdot d) \\
& -(a \cdot d)(b \cdot c)(c \cdot e) \\
& +(a \cdot e)(b \cdot c)(c \cdot d)\}.
\end{aligned} \tag{A24}$$

$$\begin{aligned}
& \left[\left[[Y^{[1]}(a) \times Y^{[3]}(b)]^{[2]} \times Y^{[2]}(c) \right]^{[2]} \times [Y^{[1]}(d) \times Y^{[1]}(e)]^{[2]} \right]^{[0]} = 3\sqrt{3}/(64\sqrt{7}\pi^{5/2}) \\
& \{ -10(a \cdot b)(b \cdot c)^2(d \cdot e) \\
& +15(a \cdot b)(b \cdot c)(b \cdot d)(c \cdot e) \\
& +15(a \cdot b)(b \cdot c)(b \cdot e)(c \cdot d) \\
& -10(a \cdot b)(b \cdot d)(b \cdot e) \\
& -6(a \cdot b)(c \cdot d)(c \cdot e) \\
& +4(a \cdot b)(d \cdot e) \\
& +4(a \cdot c)(b \cdot c)(d \cdot e) \\
& -3(a \cdot c)(b \cdot d)(c \cdot e) \\
& -3(a \cdot c)(b \cdot e)(c \cdot d) \\
& -3(a \cdot d)(b \cdot c)(c \cdot e) \\
& +2(a \cdot d)(b \cdot e) \\
& -3(a \cdot e)(b \cdot c)(c \cdot d) \\
& +2(a \cdot e)(b \cdot d) \}.
\end{aligned} \tag{A25}$$

$$\begin{aligned}
& \left[\left[[Y^{[1]}(a) \times Y^{[3]}(b)]^{[2]} \times Y^{[2]}(c) \right]^{[2]} \times [Y^{[1]}(d) \times Y^{[3]}(e)]^{[2]} \right]^{[0]} = 3/(32\sqrt{14}\pi^{5/2}) \\
& \{-15(a \cdot b)(b \cdot e)^2(d \cdot e) \\
& -15(a \cdot b)(b \cdot c)(b \cdot d)(c \cdot e) \\
& -15(a \cdot b)(b \cdot c)(b \cdot e)(c \cdot d) \\
& +75(a \cdot b)(b \cdot c)(b \cdot e)(c \cdot e)(d \cdot e) \\
& +10(a \cdot b)(b \cdot d)(b \cdot e) \\
& -25(a \cdot b)(b \cdot e)^2(d \cdot e) \\
& +6(a \cdot b)(c \cdot d)(c \cdot e) \\
& -15(a \cdot b)(c \cdot e)^2(d \cdot e) \\
& +6(a \cdot b)(d \cdot e) \\
& +6(a \cdot c)(b \cdot c)(d \cdot e) \\
& +3(a \cdot c)(b \cdot d)(c \cdot e) \\
& +3(a \cdot c)(b \cdot e)(c \cdot d) \\
& -15(a \cdot c)(b \cdot e)(c \cdot e)(d \cdot e) \\
& +3(a \cdot d)(b \cdot c)(c \cdot e) \\
& -2(a \cdot d)(b \cdot e) \\
& +3(a \cdot e)(b \cdot c)(c \cdot d) \\
& -15(a \cdot e)(b \cdot c)(c \cdot e)(d \cdot e) \\
& -2(a \cdot e)(b \cdot d) \\
& +10(a \cdot e)(b \cdot e)(d \cdot e)\}.
\end{aligned} \tag{A26}$$