INTERRELATION OF THE EQUATION OF RSJ MODEL OF JOSEPHSON JUNCTION AND THE SPECIAL DOUBLE CONFLUENT HEUN EQUATION

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Abstract. An explicit representation of the maps interconnecting the sets of solutions to the special double confluent Heun equation and the equation of the RSJ model of overdamped Josephson junction in case of shifted sinusoidal bias is given. The approach leans on specific properties of eigenfunctions of a remarkable linear operator acting on functions holomorphic on the universal cover of the punctured complex plane. The functional equation the eigenfunctions noted obey is derived. The matrix form of the monodromy transformation they manifest is given.

Introduction

In a sense, the non-linear first-order ordinary differential equation

$$\dot{\varphi}(t) + \sin\varphi(t) = B + A\cos\omega t, \tag{1}$$

where the symbols A, B, ω denote real constants, stands out in the dispersed totality of particular instances of differential equations due to its emerging in a number of problems of physics, mechanics, dynamical systems theory, geometry [1, 2, 3]. Perhaps most frequently this equation and its generalizations appear in investigations concerning with theoretical study of dynamics of Josephson junctions [4, 5]. Eq. (1) seems to be the most simple equation (or, at least, should be considered among the most simple ones) which is able to properly embody the so called phase lock effect utilized in many devices built upon capabilities of the Josephson effect. The latter was theoretically predicted in 1962 and was recognized in an experiment reported in 1963. Thereafter, in 1968, a heuristic model of behavior of a Josephson junction incorporated in a circuit with given properties was proposed [6, 7] which is currently referred to as RSJ (or sometimes as RCSJ) model. Eq. (1) follows from it in the limiting case of a small effective junction capacitance under conditions when its effect is negligible. Besides, the right-hand side of (1) corresponds to excitation ("bias") of a Josephson junction by a controllable DC (described by the dimensionless parameter B) combined with an also controllable sinusoidal AC of the dimensionless frequency ω, of the fixed (zero) initial phase, and of the given amplitude (characterized by the dimensionless parameter A).

Eq. (1) suits well for a fast and accurate numerical integration and is thus convenient for application in numerical simulations. At the same time, perhaps somewhat surprisingly, "the pure mathematics" associated with it proves to be fairly profound and is definitely of considerable interest. Several approaches can be here employed while the most efficient one starts with an appropriate complexifcation of the equation in question. We consider below this step in details and establish equivalence (mentioned for the first time in Ref. [16]) of Eq. (1) to a double confluent Heun equation. The latter, in turn, can be further explored by the methods of complex analysis and the theory of linear differential equations in the complex domain.

Transition of Eq. (1) to Complex Domain

To begin with, let us notice that the right-hand side of (1) does not depend on φ and is periodic in the free real variable t. We embody this periodicity in the circular motion coupled to the varying real t in the complex plane $\mathbb{C}$ around zero. In other words, denoting a generic point in $\mathbb{C}$ as z we associate $z = e^{\mathrm{i}\omega t}$ to processes described by the function φ obeying Eq. (1).

Similarly, the left-hand side of the equation, involving all the entries of φ, is invariant with respect to the shifts $\varphi \Leftarrow \varphi \pm 2\pi$. It suggest us to utilize in complex domain the exponent $\Phi = e^{\mathrm{i}\varphi}$ instead of the original φ. The next natural step is to consider Φ as a holomorphic function of z and assume that the above equality takes place *on the unit circle*, i.e. when $z = e^{\mathrm{i}\omega t}$. In other words, it is assumed that for real t it holds $\Phi(e^{\mathrm{i}\omega t}) = e^{\mathrm{i}\varphi(t)}$. At the same time, for generic z, the function $\Phi(z)$ becomes the analytic continuation of its instantiation on the above circle, being therefore not pointwise representable through values of φ. The next and also last action in the constructing of the transformation we search for is the selecting of a differential equation constraining holomorphic $\Phi(z)$ in such a way that, when restricted to the unit circle, it would turn into Eq. (1), provided the above identifications are taken into account. Such an equation can easily be found. It reads

$$z^2\Phi' = (2\mathrm{i}\omega)^{-1} z\,(1-\Phi^2) + \big(\ell\, z + \mu(z^2+1)\big)\Phi. \tag{2}$$

Here $\ell = B/\omega, \mu = A/2/\omega$ are the new but cognate constant parameters.

The non-linear ODE (2) belongs to the Riccati' family. It is well known that all these equations are convertible to certain linear second-order ODEs. We are going to employ such an equivalence leading, in our case, to a double confluent Heun equation. However, we make here use of an indirect method for its derivation which is based on inspection of properties of a remarkable linear operator $\mathcal{L}_C$ defined in the next section and having, at first glance, no relation to the equation of RSJ model.

Operator $\mathcal{L}_C$

Let us consider the linear operator $\mathcal{L}_C$ which sends a holomorphic function E of the complex argument z to the function $\mathcal{L}_C[E]$ of the same argument as follows

$$\mathcal{L}_C : E(z) \mapsto \mathcal{L}_C[E](z) \;=\; 2\,\omega\, z^{-\ell-1}\Big\lfloor_{z\Leftarrow z^{-1}} \big(E'(z) - \mu E(z)\big). \tag{3}$$

Here the symbols ℓ, μ, ω denote the constant parameters which are, for now, considered arbitrary except for the claim of fulfillment of the non-degeneracy conditions $\mu \neq 0 \neq \omega$. The mark $\lfloor_{z\Leftarrow z^{-1}}$ indicates the operation of replacement of the variable in the expression situated to the right of it. The function $\mathcal{L}_C[E](z)$ is obviously holomorphic in the correspondingly transformed domain, provided the latter does not contain zero.

It is natural to adopt as the domain $\mathbf{\Omega}$ of the operator $\mathcal{L}_C$ some set (say, a linear space) of functions which is preserved under its action. This assumption requires of the domain Θ of the members of $\mathbf{\Omega}$ to be invariant with respect to the map $\mathsf{C} : z \mapsto z^{-1}$ or, at least, to produce a nonempty intersection $\mathsf{C}\,\Theta \cap \Theta \neq \varnothing$. The punctured complex plane $\mathbb{C}^* = \mathbb{C} \setminus 0$ is an example of such a domain. For the sake of definiteness, we shall utilize it, provisionally, in the role of Θ. This turns out to be not the best solution but later on we shall become able to specify a more appropriate Θ realization.

The following statement holds true.

Lemma 1. *The squared (composed with itself) transformation $\mathcal{L}_C$ preserves its argument, i.e.*

$$\mathcal{L}_C \circ \mathcal{L}_C[E] = E, \tag{4}$$

if and only if the function $E = E(z)$ obeys the equation

$$z^2 E'' + \big((\ell+1)z + \mu(1-z^2)\big)E' + (-\mu(\ell+1)z + \lambda)E = 0, \tag{5}$$

$$\text{where } \lambda = (2\omega)^{-2} - \mu^2. \tag{6}$$

Proof. The above assertion immediately follows from the identity

$$\mathcal{L}_C \circ \mathcal{L}_C[E](z) \quad \equiv \quad E(z) - (2\omega)^2 \cdot \mathrm{lhs}(5), \tag{7}$$

where 'lhs(5)' stands for the left-hand side expression of Eq. (5) considered as a function of z. In particular, the equality (6) also follows from a straightforward computation verifying Eq. (7) by means of expansion of its left-hand side. □

It has to be noted that the ordinary second order linear homogeneous differential equation (5) belongs to the family of so called double confluent Heun equations (often referred to as DCHE or similarly). They are discussed in Ref.s [8, 9]; see also the online resource [10] and Ref. [11] for more recent bibliography. A generic DCHE is identified by *four* constant parameters while Eq. (5) involves only *three* ones. Accordingly, it was suggested to name Eq. (5) the *special* double confluent Heun equation (which may be referred as sDCHE, accordingly) and the term is here adopted for definiteness as well.

The clarification of relationship between the equations (5) and (1) can be built upon the study of the eigenfunctions of the operator (3). The principal point is here that any such eigenfunction is automatically a solution to Eq. (5) (for the appropriate value of the parameter λ). Indeed, the following statements holds true:

Lemma 2. *An eigenfunction of the operator $\mathcal{L}_C$ with eigenvalue $\nu \neq 0$ obeys Eq.* (5) *with $\lambda = \nu^2/(2\omega)^2 - \mu^2$. If the parameter link* (6) *is met then $\nu^2 = 1$.*

Proof. The lemma assertion follows in obvious way from the same identity (7). □

Remark 1. Omitting above the case $\nu = 0$, we are not at risk to forfeit any non-trivial relationship since, obviously, the only eigenfunction of $\mathcal{L}_C$ with null eigenvalue is the exponent $E(z) = \exp(\mu z)$ which can at any moment be taken into account, if necessary.

Motivated by the two above lemmas, we may adopt the set of solutions to Eq. (5), which constitutes a 2-dimensional linear space, as the functional space $\mathbf{\Omega}$ on which the action of the operator $\mathcal{L}_C$ has to be considered.

Remark 2. In canonical representation (i.e. when resolved with respect to the higher derivative) the linear differential equation (5) suffers of the only singularity situated at $z = 0$. Its solutions are thus holomorphic everywhere except at zero. The singularity of the equation at the center of $\mathbb{C}$ is irregular. However solutions holomorphic thereat may, in principle, exist. This can occur only on some special subset of "tuned" constant parameters of lower dimension, see Ref.s [12, 13, 14]. Moreover, even on it, only a single (unique up to a constant factor) solution is regular at zero whereas all other ones are not. It means that, when considering the common domain Θ of functions constituting $\mathbf{\Omega}$, one must remove from it the center $z = 0$. Then, starting from the complex plane, the punctured one $\mathbb{C}^* = \mathbb{C} \setminus 0$ arises. It is not simply connected and as a consequence a generic solution E to Eq. (5), excluding the mentioned exceptional cases of regularity at zero, can not live on it. The point is that the analytic continuation of a generic solution to Eq. (5) along non-homotopic curves evading zero may produce different values at the point where they meet, leading therefore to a multi-valued function. The non-uniqueness arises here since the genuine domain for solutions to Eq. (5) is not a subset of $\mathbb{C}$ (such as $\mathbb{C}^*$) but a Riemann surface reducing here to *the universal cover* $\tilde{\mathbb{C}}^*$ of $\mathbb{C}^*$. This surface is diffeomorphic to $\mathbb{C}$, the covering projection $\Pi : \tilde{\mathbb{C}}^* \simeq \mathbb{C} \overset{\exp}{\mapsto} \mathbb{C}^*$ being realized by the natural exponential function.
As a consequence, when lifted to $\tilde{\mathbb{C}}^*$, the map

$$\mathsf{C} : z \mapsto z^{-1} \tag{8}$$

involved in the transformation (3) in the form of replacement of the free variable loses the uniqueness of its "implementation". Indeed, C may now have only a single fixed point. Hence one has either $\mathsf{C}(1) = 1$ and $\mathsf{C}(z) \neq z$ for all the other points z of the E domain including the (lifts of) -1, or it holds $\mathsf{C}(-1) = -1$ and $\mathsf{C}(z) \neq z$ otherwise. There is therefore no unaffected point playing role of -1 in the former case (the first C "implementation") and similarly for $+1$ in the latter case (for the alternative "implementation" of the map C).
These subtleties go beyond the scope of the present notes, however. In order to focus on the principal points of the relationship in question, we restrict our consideration to *a subset* (subdomain) of the genuine domain of functions verifying Eq. (5). Namely, we consider it to be the open set obtained from $\mathbb{C}^*$ by a removal of the ray of negative reals, $\Theta^* = \mathbb{C}^* \setminus \mathbb{R}_{<0}$. The resulting subdomain is simply connected and any function holomorphic in it, including solutions to Eq. (5), is single-valued. Besides, the behavior of the transformation C remains (locally) "standard" and claims no precautions, all this at a price of the dropping out from consideration the value $-1 \notin \Theta^*$ of the argument z as well as all the other negative real numbers.

Let us consider now the properties of the operator $\mathcal{L}_C$ in more details and show how they enables one to establish the explicit form of the relationship of the equations (5) and (2) of which the latter is directly related, in turn, to Eq. (1).

Eigenfunctions of the Operator $\mathcal{L}_C$ and Their Properties

Let the equation (5) with fixed parameters ℓ, λ, μ such that $\lambda + \mu^2 \neq 0$ be given. Then one can resolve Eq. (6) with respect to ω (the scaling parameter in (3)), i.e. select it obeying the equation

$$4\omega^2(\lambda + \mu^2) = 1.$$

Given such ω, we define the operator $\mathcal{L}_C$ by the formula (3) treated "as it stands" in vicinity of $z = 1$ and assume that it acts on the linear space $\mathbf{\Omega}$ of solutions to Eq. (5). In view of the lemma 2, an eigenfunction of the operator $\mathcal{L}_C$ belonging to $\mathbf{\Omega}$ may only correspond to either the eigenvalue $+1$ or to the eigenvalue -1. We denote such eigenfunctions (if they exist) by the the symbols $E_{\{+\}}$ and $E_{\{-\}}$, respectively.

The following simple but important statements hold true.

Lemma 3.

- *If a solution $E = E(z)$ to Eq. (5) is an eigenfunction of the operator $\mathcal{L}_C$ then it solves the Cauchy problem for this equation posed at $z = 1$ with the initial data obeying one of the two constraints*

$$E'(1) = (\pm(2\omega)^{-1} + \mu)E(1). \tag{9}$$

  These correspond to the eigenvalues ± 1, respectively.
- *The eigenfunctions $E_{\{\pm\}}$, if exist, obey the functional equation*

$$E_{\{+\}}(z)E_{\{-\}}(1/z) + E_{\{-\}}(z)E_{\{+\}}(1/z) = 2\, e^{\mu(z+1/z-2)} E_{\{+\}}(1)E_{\{-\}}(1). \tag{10}$$

Corollary 4.

- *$E_{\{\pm\}}(1) \neq 0$ for any eigenfunction of the operator $\mathcal{L}_C$.*
- *There may exist not more than two, up to constant factors, eigenfunctions of the operator $\mathcal{L}_C$; their eigenvalues are distinct and amount to ± 1.*

Accordingly, the two eigenfunctions $E_{\{\pm\}}$ are linearly independent and hence provide the basis of the linear space $\mathbf{\Omega}$ of solutions to Eq. (5).

Lemma proof. In accordance with lemma 2, each eigenfunction of the operator $\mathcal{L}_C$ verifies Eq. (5). Next, by definition, the property of being an eigenfunction of $\mathcal{L}_C$ with the eigenvalue either $+1$ or -1 is equivalent to the equalities

$$E'_{\{\pm\}}(z) = \pm(2\omega)^{-1} z^{-\ell-1} E_{\{\pm\}}(1/z) + \mu E_{\{\pm\}}(z), \tag{11}$$

respectively. Evaluating them at $z = 1$, one obtains Eq.s (9).

Further, considering Eq. (10), let us denote as $U = U(z)$ the difference of its left- and right-hand sides. Computing its derivative and eliminating the derivatives $E'_{\{\pm\}}$ by means of the equations (11), the equation $U' = \mu \cdot (z - 1/z) \cdot U$ arises. Since $U(1) = 0$, obviously, this linear homogeneous first order ODE forces U to coincide with its trivial null solution implying $U(z) \equiv 0$. The lemma is proven. □

Remark 3. *Yet another quite predictable constraint which the eigenfunctions $E_{\{\pm\}}$ obey reads*

$$E'_{\{+\}}(z)E_{\{-\}}(z) - E_{\{+\}}(z)E'_{\{-\}}(z) = \omega^{-1} z^{-\ell-1} e^{\mu(z+1/z-2)} E_{\{+\}}(1)E_{\{-\}}(1)$$

It follows from consideration of the Wronskian for Eq. (5) which applies since the eigenfunctions $E_{\{\pm\}}$ verify the latter.

Explicit Representations of Eigenfunctions of $\mathcal{L}_C$

As it has been mentioned, the eigenfunctions of the operator $\mathcal{L}_C$ can be utilized for description of the space of solutions to Eq. (5). However, we should show, at first, that they do exist. We remind that the defining property of the functions $E_{\{\pm\}}$ is equivalent to the claim of fulfillment of one of the equations (11). The latter are however not the "classical" ODEs since an unknown function involved therein is invoked with two distinct arguments. Hence the corresponding standard theorem of existence of solutions of ODE is here not directly applicable and a separate proof of existence of eigenfunctions of the operator $\mathcal{L}_C$ has to be given. To that end, let us consider the following

Lemma 5. *Let the holomorphic function $\Phi = \Phi(z)$ defined in a simply connected vicinity of the point $z = 1$ obey the Riccati equation (cf. Eq. (2))*

$$z\Phi' + (2\mathrm{i}\omega)^{-1}(\Phi^2 - 1) = (\ell + \mu(z + z^{-1}))\Phi, \tag{12}$$

and the holomorphic function $\Psi = \Psi(z)$ obey the (subsidiary) decoupled linear homogeneous first order ODE

$$2\mathrm{i}\omega z\Psi' = (\Phi + \Phi^{-1})\Psi. \tag{13}$$

Let also

$$|\Phi(1)| = 1 \text{ and } \Psi(1) = 1. \tag{14}$$

Then the expressions

$$\begin{aligned} E_{\{\pm\}}(z) = 2^{-1}e^{\mu(z+1/z-2)/2}z^{-\ell/2}\times \\ \left(\frac{1 \pm \mathrm{i}}{\sqrt{2}}(\Psi(z)\Phi(z))^{1/2} + \frac{1 \mp \mathrm{i}}{\sqrt{2}}(\Psi(1/z)/\Phi(1/z))^{1/2}\right) \end{aligned} \tag{15}$$

determine the two eigenfunctions of the operator $\mathcal{L}_C$ with eigenvalues ± 1, respectively, provided neither of them is the identically zero function. In the exceptional case pointed out, another function from the pair (15) is still a proper (non-trivial) eigenfunction of $\mathcal{L}_C$.

Proof outline. To verify the asserted property of a function $E_{\{\pm\}}(z)$ (where one among the two sign symbols has been chosen and fixed), one has to compute its derivative and to examine the fulfillment of the corresponding equation among Eq.s (11). In our case, utilizing the equations (12) and (13), the aforementioned derivative is expressed in terms of products of the same functions Φ and Ψ with the same arguments z and $1/z$ which are involved in the definition (15). Subsequent algebraic simplification establishes the identical vanishing of the coefficients in front of all the remaining products of Ψ and Φ. □

The existence of the functions Φ and Ψ in vicinity of the point $z = 1$ is ensured by the wellknown theorem of existence of local solution of Cauchy problem for ordinary differential equations. In case of Eq.s (12) and (13), one can state even more according to the following

Lemma 6. *Let the parameters ℓ, μ, ω be real and $\omega > 0$. In case of initial conditions obeying the constraints (14), the solution $\Phi(z)$, $\Psi(z)$ of the Cauchy problem for the system of equations (12) and (13) exists in some vicinity of the "punctured unit circle"*

$$\grave{S}^1 = \{z \in \mathbb{C}, |z| = 1, z \neq -1\}, \tag{16}$$

both functions $\Phi(z)$, $\Psi(z)$ having also no zeros therein.

Proof. Let us restrict Eq. (12) to the unit circle embedded into $\mathbb{C}$ and parameterized by means of the substitutions

$$z \Leftarrow e^{\mathrm{i}\omega t},\ \Phi(z) \Leftarrow e^{\mathrm{i}\varphi(t)},\ t \in \Xi = (-\pi\omega^{-1}, \pi\omega^{-1}) \subset \mathbb{R}. \tag{17}$$

Then we obtain exactly Eq. (1) with the parameters

$$A = 2\omega\mu,\ B = \omega\ell. \tag{18}$$

Similar conversion of Eq. (13) leads to the equation

$$\dot{P}(t) = \cos\varphi(t),$$

where the function $P(t)$ is related to the original unknown $\Psi(z)$ through the equation

$$e^{P(t)} = \Psi(e^{\mathrm{i}\omega t}).$$

For any real A, B, and ω, Eq. (1) is solvable on any segment of the real axis for any real initial data $\varphi(t_0) = \varphi_0$ set up at any prescribed real t_0. Moreover, the corresponding solution is a real-analytic function. Accordingly, let some real φ_0 be fixed and let the real-analytic function $\varphi(t)$ verify Eq. (1) on the segment Ξ, obeying the initial condition $\varphi(0) = \varphi_0$. Let us also introduce the real-analytic function $P(t) = \int_0^t \cos\varphi(\tilde{t})\, d\tilde{t}$ on the same domain Ξ.

The analytic continuation of the map $(\mathbb{C} \supset \mathbb{R} \supset)\, \Xi \ni t \mapsto e^{\mathrm{i}\omega t} \in {}^{\backprime}S^1 (\subset \mathbb{C}^*)$ establishes the holomorphic diffeomorphism of some vicinity of the segment Ξ to a vicinity of "the punctured unit circle" ${}^{\backprime}S^1$ (16), the former being in smooth bijection with the latter. The holomorphic functions Φ and Ψ arising as the induced pullbacks of analytic continuations of the real analytic functions $e^{\mathrm{i}\varphi(t)}$ and $e^{P(t)}$, respectively, verify Eq.s (12), (13). By definition, they have no zeros on ${}^{\backprime}S^1$; moreover, $|\Phi| = 1$ whereas Ψ is real and strictly positive therein. Hence there exist no their zeros in some vicinity of ${}^{\backprime}S^1$ as well. Besides, in accordance with definitions and the posing of the Cauchy problem for the function φ, it holds

$$\Phi(1) = e^{\mathrm{i}\varphi_0},\ \Psi(1) = 1 \tag{19}$$

(where φ_0 can be chosen arbitrary real). Eq.s (14) are thus also fulfilled. The lemma is proven. ☐

Remark 4. The non-uniqueness of the square root function involved in Eq. (15) is to be eliminated by means of the assignment to the functions $\Phi^{1/2}$, $\Phi^{-1/2}$, and $\Psi^{1/2}$ (the pullbacks of) the analytic continuations of the functions $\exp\frac{\mathrm{i}}{2}\varphi(t)$, $\exp\frac{-\mathrm{i}}{2}\varphi(t)$, and $\exp\frac{1}{2}\int_0^t \cos\varphi(\tilde{t})d\tilde{t}$, respectively.

Remark 5. The requirement of the above lemma claiming of the constant parameters to be real is motivated by Eq.s (18), in which the constants A, B, ω are constrained by their meaning inferred from physical or geometrical problems in which Eq. (1) is utilized. Similarly, the variable t is there interpreted as a (rescaled dimensionless) time or length. While maintaining contact with applications, we assume below the above reality conditions to be fulfilled throughout. At the same time, it is worth noting that the existence results (and most formulas evading application of complex conjugation) remain valid, at least, for sufficiently small variations of the parameters shifting them from the real axis to $\mathbb{C}$.

We see that any solution to Eq. (1) generates a pair of eigenfunctions of the operator $\mathcal{L}_C$ which are defined by Eq.s (15) in terms of the functions $\Phi(z)$ and $\Psi(z)$ the above lemma operates with. However, one of them (not both, though) may prove to be identical zero. To clarify conditions of appearance of such a "pathology", we need the following property of the eigenfunctions of $\mathcal{L}_C$.

Lemma 7. *Let us define the sequence of pairs of functions* $\{a_k(z), b_k(z)\}$, $k = 1, 2, \ldots$, *holomorphic everywhere except zero, by means of the following recurrent scheme:*

$$a_1 = \mu,\ b_1 = \pm(2\omega)^{-1}z^{-\ell-1}; \tag{20}$$

$$\begin{aligned} a_{k+1} &= \mu a_k \mp (2\omega)^{-1}z^{\ell-1}b_k + a_k', \\ b_{k+1} &= \pm(2\omega)^{-1}z^{-\ell-1}a_k - \mu z^{-2}b_k + b_k'. \end{aligned} \tag{21}$$

Let also the functions $E_{\{\pm\}}$ *obey the equations* $\mathcal{L}_C E_{\{\pm\}} = \pm E_{\{\pm\}}$. *Then their derivatives admit the following representations:*

$$\frac{d^k}{dz^k}E_{\{\pm\}}(z) = a_k(z)E_{\{\pm\}}(z) + b_k(z)E_{\{\pm\}}(1/z),\ k = 1, 2, \ldots \tag{22}$$

In particular, it holds

$$\frac{d^k}{dz^k}E_{\{\pm\}}(1) = (a_k(1) + b_k(1))E_{\{\pm\}}(1),\ k = 1, 2, \ldots$$

Proof. Let us apply the mathematical induction. The induction base, the case $k = 1$, reduces to the equality which, in view of (20), is equivalent just to the corresponding equation $\mathcal{L}_C E_{\{\pm\}} = \pm E_{\{\pm\}}$ fulfilled by construction. Next, let us compute the derivative of the both sides of Eq. (22) for some fixed k, eliminating afterwards $E_{\{\pm\}}'$ on the right by means of Eq. (22) get with $k = 1$, and eliminating the derivatives a_k', b_k' with the help of Eq.s (21). As it can be shown by a straightforward computation, the result reduces to the same equation (22) in which the index k is replaced by $k + 1$. The induction step has thus been carried out and the lemma proof is accomplished. □

Corollary 8. *The function* $E_{\{\pm\}}(z)$ *defined by Eq.* (15) *is the identically zero function if and only if* $E_{\{\pm\}}(1) = 0$.

We apply the corollary 8 to clarification of the conditions leading to identically zero function $E_{\{\pm\}}$ defined by Eq. (15). Indeed, substituting therein $z = 1$ and taking into account Eq.s (19), one gets

$$E_{\{\pm\}}(1) = \mp \sin \tfrac{1}{2}(\varphi_0 \mp \pi/2).$$

Hence one of the functions $E_{\{+\}}$ and $E_{\{-\}}$ can, indeed, be identical zero and this takes place if and only if $\varphi_0 = \pi/2 \,(\mathrm{mod}\ \pi)$.

Remark 6.

- The varying of the initial value $\varphi_0 = \varphi(0)$ of a solution to Eq. (1) results in appearance of some additional constant factors. This is the only distinction of the functions $E_{\{\pm\}}$, obtained by means of Eq.s (15), from the "fiducial" ones corresponding to, say, $\varphi_0 = 0$. Besides, with respect to the case $\varphi_0 = 0$, the absolute values of these $\varphi(0)$-dependent factors do not exceed 1.

- In case of the identical vanishing of one of the functions $E_{\{\pm\}}$, the corresponding sum in brackets in Eq.s (15) vanishes. Then the same sum but with the opposite choice of the signs amounts to twice its first summand. Accordingly, the following factorized representation of the nontrivial eigenfunction $E_{\{\}}$ still produced by one of Eq.s (15) arises:

$$E_{\{\}} \propto (e^{\mu(z+1/z-2)} z^{-\ell}\Psi(z)\Phi(z))^{1/2}.$$

As we have mentioned, this situation occurs if $\varphi_0 = \pi/2 \,(\mathrm{mod}\ \pi)$.

Resuming, we have our first key

Theorem 9. *Let a solution $\varphi(t)$ to the equation* (1) *on the segment $\Xi = (-\pi\omega^{-1}, \pi\omega^{-1})$ be given. Then the analytic continuations of the functions $\exp(\mathrm{i}\varphi(t))$ and $\exp(\int_0^t \cos\varphi(t)dt)$ from Ξ to some vicinity of Ξ in $\mathbb{C}$, converted by means of the transformation* (17) *to the functions $\Phi(z)$ and $\Psi(z)$ holomorphic in the corresponding vicinity of the punctured circle* (16)*, determine therein the two solutions $E_{\{\pm\}} = E_{\{\pm\}}(z)$ to Eq.* (5) *by means of the formulas* (15)*. The functions $E_{\{\pm\}}$ are linearly independent unless one of them is the identically zero function that takes place if and only if either $\varphi(0) = \pi/2 \,(\mathrm{mod}\ 2\pi)$ (leading to $E_{\{+\}}(z) \equiv 0$) or $\varphi(0) = -\pi/2 \,(\mathrm{mod}\ 2\pi)$ (leading to $E_{\{-\}}(z) \equiv 0$, respectively). In case of linear independence the functions $E_{\{\pm\}}$ constitute the basis of the space $\mathbf{\Omega}$ of solutions to Eq.* (5)*.*
The functions $E_{\{\pm\}}$ are also the eigenfunctions with eigenvalues ± 1, respectively, of the linear operator $\mathcal{L}_C$ defined by Eq. (3)*; $\mathcal{L}_C$ is, thus, represented in the basis $\{E_{\{+\}}, E_{\{-\}}\}$ by the diagonal matrix* $\mathrm{diag}(1,-1)$*. The linear space $\mathbf{\Omega}$ is invariant with respect to the operator $\mathcal{L}_C$ which acts on it as an involutive automorphism.*

Corollary 10. *The eigenfunctions of the operator $\mathcal{L}_C$ with eigenvalues ± 1 are exactly the non-trivial solutions to Eq.* (5) *which obey the initial data constraint* (9)*.*

Remark 7. Since the operator $\mathcal{L}_C$ is involutive any non-trivial solution to Eq. (5) is either its eigenfunction itself or the expressions $const \cdot (E \pm \mathcal{L}_C E)$ constitute a pair of such eigenfunctions which are linearly independent. Adjusting the above factor *const*, they can be made real (self-conjugated, see the next section).

Self-Conjugation Property of Eigenfunctions of the Operator $\mathcal{L}_C$

The explicit formulas for eigenfunctions of the operator $\mathcal{L}_C$ enables one an easy establishing of their invariance with respect to the complex conjugation. However, the analogous relations for the functions Φ and Ψ involved in $E_{\{\pm\}}$ definition (15) have to be derived beforehand. To that end, let us introduce the following auxiliary working definition.

Definition. *Let $\Upsilon(z)$ be any function holomorphic in some connected and simply connected open subset of $\mathbb{C}$ containing the point $z = 1$. We shall name the function*

$$\tilde{\Upsilon}(z) = \overline{\Upsilon(1/\overline{z})} \qquad (23)$$

dual to the function $\Upsilon(z)$.

Remark 8. The above definition obviously implies that

- The function dual to a holomorphic function is also holomorphic in some open set containing the point $z = 1$; the intersection of the domains of Υ and $\tilde{\Upsilon}$ is open, non-empty, and also contains 1.
- "The duality map" $\tilde{}: \Upsilon \mapsto \tilde{\Upsilon}$ is involutive; in particular, the function $\Upsilon(z)$ is, in turn, dual to the function $\tilde{\Upsilon}(z)$.

Lemma 11. *Let the holomorphic function* $\Phi = \Phi(z)$ *be a solution to Eq.* (12) *obeying the constraint* $|\Phi(1)| = 1$ *(cf. Eq.s* (14)*). Then*

$$\Phi(z)\tilde{\Phi}(z) = 1. \tag{24}$$

To prove the lemma, we note first that the function $\tilde{\Phi} = \tilde{\Phi}(z)$ dual to solution $\Phi(z)$ to Eq. (12) obeys the equation

$$z\tilde{\Phi}' + (\mathrm{i}2\omega)^{-1}(\tilde{\Phi}^2 - 1) = -(\ell + \mu(z + z^{-1}))\tilde{\Phi}. \tag{25}$$

Then a straightforward computation shows that, as a consequence of (12) and (25), it holds

$$\frac{d}{dz}(\Phi(z)\tilde{\Phi}(z) - 1) = (-2\mathrm{i}\omega z)^{-1}(\Phi(z) + \tilde{\Phi}(z))(\Phi(z)\tilde{\Phi}(z) - 1). \tag{26}$$

Now let us introduce an auxiliary sequence of functions δ_n (in fact, polynomials) of the three arguments z, Φ, and $\tilde{\Phi}$ which all are regarded here, for a time, as free complex variables. (It is worth noting that the functions δ_n depends also on the parameters ℓ, μ, ω but these their arguments will be suppressed for the sake of the symbolism simplicity.) The functions δ_n are defined by means of the following recurrent scheme:

$$\delta_1 = z(\Phi + \tilde{\Phi}), \tag{27}$$

$$\begin{aligned} \delta_{n+1} = {} & (\Phi + \tilde{\Phi} + 4\mathrm{i}\omega n)\delta_n - 2\mathrm{i}\omega z^2 \frac{\partial \delta_n}{\partial z} \\ & + (z(\Phi^2 - 1) - 2\mathrm{i}\omega(\ell z + \mu(z^2 + 1))\Phi)\frac{\partial \delta_n}{\partial \Phi} \\ & + (z(\tilde{\Phi}^2 - 1) + 2\mathrm{i}\omega(\ell z + \mu(z^2 + 1))\tilde{\Phi})\frac{\partial \delta_n}{\partial \tilde{\Phi}}, n = 1, 2, \cdots \end{aligned} \tag{28}$$

We utilize them for introduction of the functions

$$\Lambda_n(z, \Phi, \tilde{\Phi}) = (-2\mathrm{i}\omega z^2)^{-n}\delta_n(z, \Phi, \tilde{\Phi})(\Phi\tilde{\Phi} - 1), \ n = 1, 2, \cdots . \tag{29}$$

Lemma 12. *Under the conditions of the lemma 11, it holds*

$$\frac{d}{dz}\Lambda_n(z, \Phi(z), \tilde{\Phi}(z)) = \Lambda_{n+1}(z, \Phi(z), \tilde{\Phi}(z)), n = 1, 2 \cdots . \tag{30}$$

Proof. It is easy to show that, in view of Eq. (12) and Eq. (25), the above assertion is equivalent to Eq. (27) for $n = 1$ and to Eq. (28) for $n > 1$. □

Corollary 13. *Under the conditions of the lemma 11, it holds*

$$\frac{d^n}{dz^n}(\Phi(z)\tilde{\Phi}(z) - 1) = \Lambda_n(z, \Phi(z), \tilde{\Phi}(z)), \ n = 1, 2, \cdots . \tag{31}$$

Proof. In case $n = 1$ the above equation follows from Eq.s (26) and (27), and the definition (29). It is extended to higher derivative orders $n = 2, 3, \cdots$ by means of the mathematical induction based on Eq. (30). □

Corollary 14. *Under the conditions of the lemma 11, all the derivatives of the function* $\Phi(z)\tilde{\Phi}(z) - 1$ *vanish at the point* $z = 1$.

Proof. In accordance with Λ_n definition (29) and Eq. (31), for any $n = 1, 2, \cdots$ the derivative $d^n(\Phi(z)\tilde{\Phi}(z) - 1)/dz^n$ factorizes into a function holomorphic in vicinity of the point $z = 1$ times the function $\Phi(z)\tilde{\Phi}(z) - 1$ itself. The latter is zero at the unity (since $\Phi(1)\tilde{\Phi}(1) = |\Phi(1)|^2 = 1$); accordingly, the above multiple derivative is zero thereat as well. Thus all such derivatives at $z = 1$ are null. □

Proof of the lemma 11 . Since the function $\Phi(z)\tilde{\Phi}(z) - 1$ is analytic at the point $z = 1$, the above corollary implies its identical vanishing and thus the validity of the assertion of the lemma 11. □

Similarly to above, let us consider how the function $\tilde{\Psi} = \tilde{\Psi}(z)$ dual to solution $\Psi = \Psi(z)$ to Eq. (13) is related to Ψ. A straightforward computation establishes the fulfillment of the equation

$$2\mathrm{i}\omega z\tilde{\Psi}' = (\tilde{\Phi} + \tilde{\Phi}^{-1})\tilde{\Psi}. \tag{32}$$

As a consequence, it holds

$$\frac{d}{dz}(\Psi - \tilde{\Psi}) = (4\mathrm{i}\omega z)^{-1}(\Phi + \tilde{\Phi})(\Psi - \tilde{\Psi}), \tag{33}$$

provided the functions $\Phi = \Phi(z)$ and $\tilde{\Phi} = \tilde{\Phi}(z)$ (mutually dual) obey Eq. (24). Let us notice now that, as the functions Φ and $\tilde{\Phi}$ are given, Eq. (33) can be regarded as a linear homogeneous first order ODE for the holomorphic function $\delta = \delta(z) = \Psi(z) - \tilde{\Psi}(z)$ which is correctly defined in the intersection of the domains of the functions Φ and $\tilde{\Phi}$ (with zero removed, if necessary). As a consequence, one may claim that the function δ either has no zeros in its domain or is the identically zero function. But if the function $\Psi(z)$ complies with "the initial condition" (19) then $\tilde{\Psi}(1) = 1$ as well implying $\delta(1) = 0$. Thus $\delta(z) \equiv 0$ at least in a connected vicinity of the point $z = 1$. We have therefore proven the following

Theorem 15. *Let the functions $\Phi(z)$ and $\Psi(z)$ be holomorphic in some connected and simply connected open subset of $\mathbb{C}^*$ containing the point $z = 1$, obeying therein the system of equations* (12), (13)*; let the constraints* (14) *be also fulfilled. Then the equation* (24) *and the equation*

$$\tilde{\Psi}(z) = \Psi(z) \tag{34}$$

hold true.

The lemma 6 and the above theorem lead to the following

Corollary 16. *Under the conditions of the theorem 15, it holds $|\Phi| = 1$ and $\mathsf{Im}\,\Psi = 0$ on "the punctured unit circle"* (16).

Proof. Since $\overline{z} = z^{-1}$ on the unit circle in $\mathbb{C}$, the assertions to be proven follow from Eq.s (24) and (34). □

Remark 9. We have shown, in particular, that any holomorphic functions Φ, Ψ obeying conditions of the theorem 15 determine the smooth real valued functions $\varphi(t), P(t)$ verifying the equations (1) and (19), respectively.

Now a short straightforward computation leaning on Eq.s (24) and (34) proves the following

Theorem 17. *Let the functions $\Phi(z)$ and $\Psi(z)$ obey the system of equations* (12), (13) *and the constraints* (14). *Then the functions $E_{\{+\}}(z)$ and $E_{\{-\}}(z)$ defined by Eq.s* (15) *are real (self-conjugated), i.e. obey the constraints*

$$\overline{E_{\{\pm\}}(\overline{z})} = E_{\{\pm\}}(z). \tag{35}$$

Representation of general solution to the equation of RSJ model in terms of solutions to special double confluent Heun equation

Having outlined the way of constructing of solutions to Eq. (5) from solutions to Eq. (1), we proceed with description of the inverse relationship. It can be expressed in the form of the following

Theorem 18. *Let the holomorphic functions $E_{\{+\}}(z)$ and $E_{\{-\}}(z)$ be the real (self-conjugated, see Eq.* (35)*) eigenfunctions of the operator $\mathcal{L}_C$ defined by Eq.* (3) *with the corresponding eigenvalues ± 1; let also α be an arbitrary real constant. We define the holomorphic functions $\Phi(z)$ and $\Theta(z)$ as follows:*

$$\Phi(z) = -\mathrm{i}z^l \frac{\cos(\frac{1}{2}\alpha)E_{\{+\}}(z) + \mathrm{i}\sin(\frac{1}{2}\alpha)E_{\{-\}}(z)}{\cos(\frac{1}{2}\alpha)E_{\{+\}}(1/z) - \mathrm{i}\sin(\frac{1}{2}\alpha)E_{\{-\}}(1/z)}, \tag{36}$$

$$\Theta(z) = -\mathrm{i}\frac{\cos(\frac{1}{2}\alpha){E_{\{+\}}}^2(1)E_{\{-\}}(z) + \mathrm{i}\sin(\frac{1}{2}\alpha){E_{\{-\}}}^2(1)E_{\{+\}}(z)}{E_{\{+\}}(1)E_{\{-\}}(1)\big(\cos(\frac{1}{2}\alpha)E_{\{+\}}(z) + \mathrm{i}\sin(\frac{1}{2}\alpha)E_{\{-\}}(z)\big)}. \tag{37}$$

Then

- *the continuous function $\varphi(t)$ of the real variable t determined by the equation*
$$e^{\mathrm{i}\varphi(t)} = \Phi(e^{\mathrm{i}\omega t}) \tag{38}$$
is well defined, real valued, smooth and verifying Eq. (1)*;*
- *the functions $P(t)$ and $Q(t)$ defined as follows*
$$P(t) = -\log(-\mathsf{Im}\,\Theta(e^{\mathrm{i}\omega t})),\ Q(t) = \mathsf{Re}\,\Theta(e^{\mathrm{i}\omega t})$$
are well defined, real valued, smooth and are related to the function $\varphi(t)$ by the subsequent quadratures as follows
$$P(t) = \int_0^t \cos\varphi(\tilde{t})\,d\,\tilde{t},\ Q(t) = \int_0^t e^{-P(\tilde{t})}\sin\varphi(\tilde{t})\,d\,\tilde{t}. \tag{39}$$

Remark 10. In view of the lemma 5, the both functions $E_{\{\pm\}}(z)$ obey Eq. (5) and one learns from lemmas 5 and 6 that they always exist. Hence, the functions $\Phi(z)$ and $\Theta(z)$, as well as the functions $\varphi(t), P(t), Q(t)$ which they give rise to, are built (and always can be built) upon solutions of this equation.

Theorem proof. Let us notice that since the functions $E_{\{\pm\}}(z)$ obey a linear homogeneous second order differential equation with coefficients holomorphic everywhere except at zero (Eq. (5) times z^{-2}), they are themselves holomorphic everywhere except, perhaps, at zero. Besides, in accord with the corollary 8, $E_{\{+\}}(1) \neq 0 \neq E_{\{-\}}(1)$ that eliminates the source of an a priori conceivable fault of the definition (37).

Now let us consider the identity

$$\begin{aligned} &\mathrm{i}e^{\mathrm{i}\varphi(t)}\big(\dot{\varphi}(t) + \sin\varphi(t) - \omega(\ell + 2\mu\cos\omega t)\big) \equiv \big(e^{\mathrm{i}\varphi(t)} - \Phi(e^{\mathrm{i}\omega t})\big)^{\bullet} \\ &+ \Big(2^{-1}(e^{\mathrm{i}\varphi(t)} + \Phi(e^{\mathrm{i}\omega t})) - \mathrm{i}\omega(\ell + 2\mu\cos\omega t)\Big)\big(e^{\mathrm{i}\varphi(t)} - \Phi(e^{\mathrm{i}\omega t})\big), \end{aligned} \tag{40}$$

which takes place for arbitrary smooth function $\varphi(t)$ and which is proven by means of straightforward computation taking into account the Φ definition (36) and Eq.s (11).

Thus it follows from (40) that if Eq. (38) is fulfilled then $\varphi(t)$ verifies Eq. (1) with $A = 2\omega\mu, B = \omega\ell$ (cf. Eq.s (18)).

Further, let us note that since the functions $E_{\{\pm\}}(z)$ are real, one obtains in case of a real α the following equalities:

$$\overline{\Phi(z)} = \mathrm{i}\bar{z}^l \frac{\cos(\frac{1}{2}\alpha)E_{\{+\}}(\bar{z}) - \mathrm{i}\sin(\frac{1}{2}\alpha)E_{\{-\}}(\bar{z})}{\cos(\frac{1}{2}\alpha)E_{\{+\}}(1/\bar{z}) + \mathrm{i}\sin(\frac{1}{2}\alpha)E_{\{-\}}(1/\bar{z})} \equiv \Phi(1/\bar{z})^{-1}.$$

For $z = e^{\mathrm{i}\omega t}$ and real t, it holds $1/\bar{z} = z$. Accordingly, one infers from above that $\overline{\Phi(e^{\mathrm{i}\omega t})} = \Phi(e^{\mathrm{i}\omega t})^{-1}$ and, consequently, $|\Phi(e^{\mathrm{i}\omega t})| = 1$. Then Eq. (38) yields $|e^{\mathrm{i}\varphi(t)}| = 1$, and the real-valued smooth function $\varphi(t)$ is determined in terms of the logarithm of the non-zero smooth function $\Phi(e^{\mathrm{i}\omega t})$ in the standard way. The first assertion of the theorem is therefore proven.

Addressing now the second assertion, let us introduce, in addition to the function $\Theta(z)$, the function $\tilde{\Theta}(z)$ as follows:

$$\tilde{\Theta}(z) = \mathrm{i}\frac{\cos(\frac{1}{2}\alpha)E_{\{+\}}{}^2(1)E_{\{-\}}(1/z) - \mathrm{i}\sin(\frac{1}{2}\alpha)E_{\{-\}}{}^2(1)E_{\{+\}}(1/z)}{E_{\{+\}}(1)E_{\{-\}}(1)\big(\cos(\frac{1}{2}\alpha)E_{\{+\}}(1/z) - \mathrm{i}\sin(\frac{1}{2}\alpha)E_{\{-\}}(1/z)\big)}. \tag{41}$$

The functions $\Theta = \Theta(z)$ and $\tilde{\Theta} = \tilde{\Theta}(z)$ obey the following system of the two linear homogeneous first order differential equations

$$\mathrm{i}\omega z\Theta' = -\Phi^{-1}(\Theta - \tilde{\Theta}), \tag{42}$$

This is the direct consequence of definitions and Eq.s (11).

A straightforward verification also based on definitions shows that for real eigenfunctions $E_{\{\pm\}}$ (and for real constant α) it holds $\overline{\Theta(z)} = \tilde{\Theta}(1/\bar{z})$, i.e. the function $\tilde{\Theta}$ defined by means of a separate formula (41) is actually dual to the function Θ (see Eq. (23)). As a consequence, it holds $\tilde{\Theta}(e^{\mathrm{i}\omega t}) = \overline{\Theta(e^{\mathrm{i}\omega t})}$. Then Eq.s (42) yield the equation

$$\frac{d}{dt}\Theta(e^{\mathrm{i}\omega t}) = -\Phi(e^{\mathrm{i}\omega t})^{-1}\big(\Theta(e^{\mathrm{i}\omega t}) - \overline{\Theta(e^{\mathrm{i}\omega t})}\big).$$

Separating its real and imaginary parts and taking into account Eq. (38), one gets

$$\frac{d}{dt}\mathsf{Re}\,\Theta(e^{\mathrm{i}\omega t}) = -\,\mathsf{Im}\,\Theta(e^{\mathrm{i}\omega t})\sin\varphi(t),$$
$$\frac{d}{dt}\mathsf{Im}\,\Theta(e^{\mathrm{i}\omega t}) = -\,\mathsf{Im}\,\Theta(e^{\mathrm{i}\omega t})\cos\varphi(t).$$

In case of a given real valued function $\varphi(t)$, the latter equation determining $\mathsf{Im}\,\Theta$ can be integrated by means of a quadrature. Then the former one is integrated by means of another quadrature. The integration constants are fixed making use of the initial conditions $\mathsf{Re}\,\Theta(e^{\mathrm{i}\omega t})|_{t=0} = \mathsf{Re}\,\Theta(1) = 0$, $\mathsf{Im}\,\Theta(e^{\mathrm{i}\omega t})|_{t=0} = \mathsf{Im}\,\Theta(1) = -1$ which follow from the Θ definition (37) evaluated at the point $z = 1$. The ultimate result of the integrations is just the formulas (39). The theorem proof has been accomplished. □

Let us note that for $t = 0$ Eq.s (38) and (36) are equivalent to the equation

$$E_{\{-\}}(1)\sin(\tfrac{1}{2}\varphi(0) - \tfrac{\pi}{4})\sin(\tfrac{1}{2}\alpha) + E_{\{+\}}(1)\cos(\tfrac{1}{2}\varphi(0) - \tfrac{\pi}{4})\cos(\tfrac{1}{2}\alpha) = 0. \tag{43}$$

Obviously, it is solvable with respect to the angular parameter α for any given real $\varphi(0)$ (recall that the values of the functions $E_{\{\pm\}}$ are real when their argument is real and $E_{\{\pm\}}(1) \neq 0$). Conversely, for any $\alpha \in [0, 2\pi)$ some "initial data" $\varphi_0 = \varphi(0) \in [0, 2\pi)$ obeying Eq. (43) can be found. We obtain, therefore, the following

Corollary 19. *Eq.s* (38)*,* (36) *enable one to obtain any solution to Eq.* (1)*, representing it in terms of solutions to Eq.* (5)*.*

Conclusion

We have here shown that any solution to the equation (1), utilized for the modeling of dynamics of a Josephson junction, can be converted to solutions to Eq. (5) by means of a quadrature and analytic continuation of two real analytic functions (theorem 9). Moreover, in a generic case, a basis of the space of solutions to Eq. (5) can then be produced and it is constituted by the eigenfunctions of the operator $\mathcal{L}_C$ (defined by Eq. (3)); moreover, these are real (self-conjugated, see theorem 17).

Conversely, let the two real eigenfunctions of the operator $\mathcal{L}_C$ with eigenvalues $+1$ and -1 be given. Then all the solutions to Eq. (1) can be obtained making use of the formulas (36) and (38) (theorem 18, corollary 19). A similar formula, Eq. (37), yields explicit representations of the integrals (39) which are involved in the criterion of the so called phase-lock [15, 16], the remarkable property manifested under certain conditions by solutions to Eq. (1) [4, 5].

In total, the relationships indicated above establish the explicit 1-to-1 correspondence between solutions spaces of Eq. (5) and Eq. (1), essentially, because ambiguity still retained can be considered trivial.

It is also worth noting that the eigenfunctions of the operator $\mathcal{L}_C$ (as well as this operator on its own, of course) are the important tools proving to be efficient in investigation of various problems related to sDCHE (5). In particular, the following explicit matrix representation $\mathbf{M}$ of *the monodromy transformation*[1] of its space of solution with respect to the basis $\{E_{\{+\}}, E_{\{-\}}\}$ can be obtained[2]:

$$\mathbf{M} = e^{4\mu}\big(2E_{\{+\}}(1)E_{\{-\}}(1)\big)^{-1} \times \tag{44}$$
$$\begin{pmatrix} E_{\{+\}}(\overset{\curvearrowleft}{-1})E_{\{-\}}(\overset{\curvearrowleft}{-1}) + E_{\{+\}}(\overset{\curvearrowright}{-1})E_{\{-\}}(\overset{\curvearrowright}{-1}) & E_{\{+\}}(\overset{\curvearrowleft}{-1})^2 - E_{\{+\}}(\overset{\curvearrowright}{-1})^2 \\ E_{\{-\}}(\overset{\curvearrowleft}{-1})^2 - E_{\{-\}}(\overset{\curvearrowright}{-1})^2 & E_{\{+\}}(\overset{\curvearrowleft}{-1})E_{\{-\}}(\overset{\curvearrowleft}{-1}) + E_{\{+\}}(\overset{\curvearrowright}{-1})E_{\{-\}}(\overset{\curvearrowright}{-1}) \end{pmatrix}.$$

Here the symbols $\overset{\curvearrowleft}{-1}$ and $\overset{\curvearrowright}{-1}$ denote the preimages of $-1 \in \mathbb{C}^*$ in the Riemann surface $\tilde{\mathbb{C}}^*$, the domain of generic solutions to Eq. (5), "branching" over $\mathbb{C}^*$ around "the axis" passing through the removed zero (see the remark 2). More exactly, these preimages of -1 are selected as the closest ones to the preimage of 1, the fixed point of lifting of the transformation (8), which we may denote just as 1. Of them, $\overset{\curvearrowleft}{-1}$ is reached from 1 along an arc passed in the counterclockwise direction while for $\overset{\curvearrowright}{-1}$ similar arc is directed clockwise. The above formula shows, in particular, that the diagonal elements of $\mathbf{M}$ are real and coincide while the off-diagonal ones are pure imaginary. It also follows from the equation (10) that $\det \mathbf{M} = 1$. The two eigenvalues of the matrix (44) coincide if and only if one of its off-diagonal elements vanishes and this observation can be utilized as the base of yet another criterion of

[1]Here the monodromy transformation sends a solution to Eq. (5) to another its solution which is obtained from the former by means of point-wise analytic continuation along counterclockwise oriented full circle arcs encircling the singular center $z = 0$. On the set of solutions to Eq. (1), the monodromy transformation of solutions to Eq. (5) is converted to the map $\varphi(t) \mapsto \mathcal{M}\varphi(t) = \varphi(t + 2\pi/\omega)$. Given $\mathbf{M}$, the making use of Eq.s (15), (36), (37), etc enables one to obtain an explicit representation of this transformation.

[2]The formula (44) had been derived in case of integer orders ℓ. The cases of other ℓ require additional examination.

the phase-lock behavior for solutions to Eq. (1), this time referring to properties of eigenfunctions of the operator $\mathcal{L}_C$.

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