Zero-variance of perturbation Hamiltonian density in perturbed spin systems

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Abstract

We study effects of perturbation Hamiltonian to quantum spin systems which can include quenched disorder. Model-independent inequalities are derived, using an additional artificial disordered perturbation. These inequalities enable us to prove that the variance of the perturbation Hamiltonian density vanishes in the infinite volume limit even if the artificial perturbation is switched off. This theorem is applied to spontaneous symmetry breaking phenomena in a disordered classical spin model, a quantum spin model without disorder and a disordered quantum spin model.

1   Introduction

We study quantum spin systems on a lattice V_N which consists of N sites. A spin operator S_j^p ($p = x, y, z$) at a site $j \in V_N$ on a Hilbert space $\mathcal{H} := \bigotimes_{j \in V_N} \mathcal{H}_j$ is defined by a tensor product of the spin matrix acting on $\mathcal{H}_j \simeq \mathbb{C}^{2S+1}$ and unities, where S is an arbitrary fixed half integer. These operators are self-adjoint and satisfy the commutation relations

$$[S_j^x, S_k^y] = i\delta_{j,k}S_j^z, \qquad [S_j^y, S_k^z] = i\delta_{j,k}S_j^x, \qquad [S_j^z, S_k^x] = i\delta_{j,k}S_j^y, \tag{1}$$

and the spin at each site $i \in V_N$ has a fixed magnitude

$$\sum_{a=x,y,z} (S_j^a)^2 = S(S+1)\mathbf{1}. \tag{2}$$

We define an unperturbed Hamiltonian $H_N(\mathbf{S})$ first. $\mathcal{P}(V_N)$ denotes the collection of all subsets of V_N. Let $C_N \subset \mathcal{P}(V_N)$ be a collection of interaction ranges and let $\mathbf{J} = (J_X)_{X \in C_N}$ be a sequence of real valued random variables with a finite expectation $\mathbb{E}|J_X|$ for each $X \in C_N$, where $\mathbb{E}$ denotes the expectation over $\mathbf{J}$ for its functions. $\mathbf{S}_X$ denotes a sequence of spin operators $(S_j^p)_{j \in X, p=x,y,z}$ on a subset X and φ is a self-adjoint valued function of $\mathbf{S}_X$. We consider a model defined by the following Hamiltonian with C_N, φ and $\mathbf{J}$

$$H_N(\mathbf{S}, \mathbf{J}) := \sum_{X \in C_N} J_X \varphi(\mathbf{S}_X). \tag{3}$$

One can assume a symmetry of the Hamiltonian $H_N(\mathbf{S}, \mathbf{J})$, if one is interested in symmetry breaking phenomena. To detect a spontaneous symmetry breaking, long-range order of order operator $h_N(\mathbf{S})$ is utilized in the symmetric Gibbs state. Although the symmetric Gibbs state with long-range order is mathematically well defined, such state is unstable due to strong fluctuation and it cannot be realized. On the other hand, it is believed that a perturbed Gibbs state with infinitesimal symmetry breaking Hamiltonian is stable and realistic. Consider a perturbed Hamiltonian as a function of spin operators $\mathbf{S} = (S_j^p)_{j \in V_N, p=x,y,z}$

$$H := H_N(\mathbf{S}, \mathbf{J}) - N\lambda h_N(\mathbf{S}), \tag{4}$$

where $h_N(\mathbf{S})$ is a bounded operator and $\lambda \in \mathbb{R}$. Assume upper bounds on $h_N(\mathbf{S})$

$$\|h_N(\mathbf{S})\| \leq C_h, \tag{5}$$

where C_h is a positive constant independent of the system size N. The operator norm is defined by $\|O\|^2 := \sup_{(\phi,\phi)=1}(O\phi, O\phi)$ for an arbitrary linear operator O on $\mathcal{H}$. To study spontaneous symmetry

breaking, one can regard $h_N(\mathbf{S})$ as an order operator which breaks the symmetry. For instance, h_N is a spin density

$$h_N(\mathbf{S}) = \frac{1}{N} \sum_{j \in V_N} S_j^z. \tag{6}$$

Define Gibbs state with the Hamiltonian (4). For $\beta > 0$, the partition function is defined by

$$Z_N(\beta, \lambda, \mathbf{J}) := \mathrm{Tr}\, e^{-\beta H_N(\mathbf{S}, \mathbf{J}) + \beta N \lambda h_N(\mathbf{S})} \tag{7}$$

where the trace is taken over the Hilbert space $\mathcal{H}$. Let f be an arbitrary function of spin operators. The expectation of f in the Gibbs state is given by

$$\langle f(\mathbf{S}) \rangle_\lambda = \frac{1}{Z_N(\beta, \lambda, \mathbf{J})} \mathrm{Tr}\, f(\mathbf{S}) e^{-\beta H_N(\mathbf{S}, \mathbf{J}) + \beta N \lambda h_N(\mathbf{S})}. \tag{8}$$

Define the following function from the partition function

$$\psi_N(\beta, \lambda, \mathbf{J}) := \frac{1}{N} \log Z_N(\beta, \lambda, \mathbf{J}), \tag{9}$$

and its expectation

$$p_N(\beta, \lambda) := \mathbb{E} \psi_N(\beta, \lambda, \mathbf{J}). \tag{10}$$

where $\mathbb{E}$ denotes the expectation over $\mathbf{J}$. The function $-\frac{N}{\beta} \psi_N$ is called free energy of the sample in statistical physics.

In the present paper, we require the following three assumptions on the Gibbs state defined by the perturbed Hamiltonian (4).

Assumption 1 *The infinite volume limit of the function p_N*

$$p(\beta, \lambda) = \lim_{N \nearrow \infty} p_N(\beta, \lambda), \tag{11}$$

exists for each $(\beta, \lambda) \in (0, \infty) \times \mathbb{R}$.

Assumption 2 *The variance of ψ_N vanishes in the infinite volume limit*

$$\lim_{N \nearrow \infty} \mathbb{E}[\psi_N(\beta, \mathbf{J}, \lambda) - p_N(\beta, \lambda)]^2 = 0,$$

for each $(\beta, \lambda) \in (0, \infty) \times \mathbb{R}$

Assumption 3 *The following commutator of the perturbation operator h_N and the Hamiltonian vanishes in the infinite volume limit*

$$\lim_{N \nearrow \infty} \|[h_N(\mathbf{S}), [H(\mathbf{S}, \mathbf{J}), h_N(\mathbf{S})]]\| = 0, \tag{12}$$

for an arbitrarily fixed sequence $\mathbf{J}$.

In the present paper, we prove the following main theorem for an arbitrary spin model with the Hamiltonian (4) satisfying Assumptions 1, 2 and 3.

Theorem 1.1 *Consider a quantum spin model defined by the Hamiltonian (4) satisfying Assumptions 1, 2 and 3. The expectation of the perturbation operator*

$$\lim_{N \nearrow \infty} \mathbb{E} \langle h_N(\mathbf{S}) \rangle_\lambda, \tag{13}$$

exists in the infinite volume limit for almost all λ and its variance in the Gibbs state and the distribution of disorder vanishes

$$\lim_{N \nearrow \infty} \mathbb{E} \langle (h_N(\mathbf{S}) - \mathbb{E} \langle h_N(\mathbf{S}) \rangle_\lambda)^2 \rangle_\lambda = 0, \tag{14}$$

in the infinite volume limit for almost all $\lambda \in \mathbb{R}$.

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Theorem 1.1 implies also the existence of the following infinite volume limit for almost all $\lambda \in \mathbb{R}$

$$\lim_{N \nearrow \infty} \mathbb{E}\langle h_N(\mathbf{S})^2 \rangle_\lambda = (\lim_{N \nearrow \infty} \mathbb{E}\langle h_N(\mathbf{S}) \rangle_\lambda)^2. \tag{15}$$

The perturbation operator h_N is self-averaging in the perturbed model. Although the claim of Theorem 1.1 is physically quite natural and physicists believe it by their experiences supported by lots of examples, it has never been proved rigorously in general setting. In section 2, we prove Theorem 1.1. In section 3, we apply Theorem 1.1 to spontaneous symmetry breaking phenomena in several examples.

2 Proof

In this section, we introduce an extra perturbation Hamiltonian with a quenched disorder to prove Theorem 1.1. Consider the following perturbed Hamiltonian

$$H = H_N(\mathbf{S}, \mathbf{J}) - (N\lambda + N^\alpha \mu g) h_N(\mathbf{S}). \tag{16}$$

where g is a standard Gaussian random variable with $\mu \in \mathbb{R}$, and $\alpha \in (0, 1)$. The introduced random variable g is artificial and our final goal is to study the model at $\mu = 0$. In this section, the symbol $\mathbb{E}$ denotes the expectation over all random variables $\mathbf{J}, g$. and $\mathbb{E}_g$ denotes that over only g. In this section, we regard the following functions for the Hamiltonian (16)

$$Z_N(\beta, \lambda, \mathbf{J}, \mu g) := \mathrm{Tr}\, e^{-\beta H_N(\mathbf{S},\mathbf{J})+\beta(N\lambda+N^\alpha \mu g)h_N(\mathbf{S})}, \tag{17}$$

and

$$\psi_N(\beta, \lambda, \mathbf{J}, \mu g) := \frac{1}{N} \log Z_N(\beta, \lambda, \mathbf{J}, \mu g), \tag{18}$$

as functions of $(\beta, \lambda, \mathbf{J}, \mu g)$ The expectation of ψ_N is

$$p_N(\beta, \lambda, \mu) := \mathbb{E}\psi_N(\beta, \lambda, \mathbf{J}, \mu g). \tag{19}$$

For an arbitrary function f, of spin operators. the Gibbs expectation of f is

$$\langle f(\mathbf{S}) \rangle_{\lambda, \mu g} = \frac{1}{Z_N(\beta, \lambda, \mathbf{J}, \mu g)} \mathrm{Tr} f(\mathbf{S}) e^{-\beta H_N(\mathbf{S},\mathbf{J})+\beta(N\lambda+N^\alpha \mu g)h_N(\mathbf{S})}. \tag{20}$$

Here, we introduce a fictitious time $t \in [0, 1]$ and define a time evolution of operators with the Hamiltonian. Let O be an arbitrary self-adjoint operator, and we define an operator valued function $O(t)$ of $t \in [0, 1]$ by

$$O(t) := e^{-tH} O e^{tH}. \tag{21}$$

Furthermore, we define the Duhamel product of time independent operators $O_1, O_2, \cdots, O_k$ by

$$(O_1, O_2, \cdots, O_k)_{\lambda, \mu g} := \int_{[0,1]^k} dt_1 \cdots dt_k \langle \mathrm{T}[O_1(t_1)O_2(t_2)\cdots O_k(t_k)] \rangle_{\lambda, \mu g}, \tag{22}$$

where the symbol T is a multilinear mapping of the chronological ordering. If we define a partition function with arbitrary self adjoint operators $O_1, \cdots, O_k$ and real numbers $x_1, \cdots, x_k$

$$Z(x_1, \cdots, x_k) := \mathrm{Tr} \exp \beta \left[-H + \sum_{i=1}^k x_i O_i \right], \tag{23}$$

the Duhamel product of k operators represents the k-th order derivative of the partition function [5, 8, 22]

$$\beta^k (O_1, \cdots, O_k)_{\lambda, \mu g} = \frac{1}{Z} \frac{\partial^k Z}{\partial x_1 \cdots \partial x_k}. \tag{24}$$

Furthermore, a truncated Duhamel product is defined by

$$\beta^k (O_1; \cdots; O_k)_{\lambda, \mu g} = \frac{\partial^k}{\partial x_1 \cdots \partial x_k} \log Z. \tag{25}$$

The following lemma can be shown in the standard convexity argument to obtain the Ghirlanda-Guerra identities [1, 4, 7, 15, 16, 20, 24] in classical and quantum disordered systems. The proof can be done on the basis of of convexity of functions ψ_N, p_N, p and their almost everywhere differentiability and Assumptions 1 and 2.

Lemma 2.1 *For almost all $\lambda \in \mathbb{R}$, the infinite volume limit*

$$\frac{\partial p}{\partial \lambda}(\beta, \lambda, 0) = \lim_{N \nearrow \infty} \beta \mathbb{E}\langle h_N(\mathbf{S})\rangle_{\lambda, 0} \tag{26}$$

exists and the following variance vanishes

$$\lim_{N \nearrow \infty} [\mathbb{E}\langle h_N(\mathbf{S})\rangle_{\lambda, 0}^2 - (\mathbb{E}\langle h_N(\mathbf{S})\rangle_{\lambda, 0})^2] = 0, \tag{27}$$

for each $\beta \in (0, \infty)$.

Proof. In this proof, regard $p_N(\lambda)$ $p(\lambda)$ and $\psi_N(\lambda)$ as functions of λ for lighter notation. Define the following functions

$$w_N(\epsilon) := \frac{1}{\epsilon}[|\psi_N(\lambda + \epsilon) - p_N(\lambda + \epsilon)| + |\psi_N(\lambda - \epsilon) - p_N(\lambda - \epsilon)| + |\psi_N(\lambda) - p_N(\lambda)|]$$

$$e_N(\epsilon) := \frac{1}{\epsilon}[|p_N(\lambda + \epsilon) - p(\lambda + \epsilon)| + |p_N(\lambda - \epsilon) - p(\lambda - \epsilon)| + |p_N(\lambda) - p(\lambda)|], \tag{28}$$

for $\epsilon > 0$. Assumption 1 and Assumption 2 on ψ_N give

$$\lim_{N \nearrow \infty} \mathbb{E}w_N(\epsilon) = 0, \quad \lim_{N \nearrow \infty} e_N(\epsilon) = 0, \tag{29}$$

for any $\epsilon > 0$. Since ψ_N, p_N and p are convex functions of λ, we have

$$\frac{\partial \psi_N}{\partial \lambda}(\lambda) - \frac{\partial p}{\partial \lambda}(\lambda) \le \frac{1}{\epsilon}[\psi_N(\lambda + \epsilon) - \psi_N(\lambda)] - \frac{\partial p}{\partial \lambda} \tag{30}$$

$$\le \frac{1}{\epsilon}[\psi_N(\lambda + \epsilon) - p_N(\lambda + \epsilon) + p_N(\lambda + \epsilon) - p_N(\lambda) + p_N(\lambda) - \psi_N(\lambda)$$

$$- p(\lambda + \epsilon) + p(\lambda + \epsilon) + p(\lambda) - p(\lambda)] - \frac{\partial p}{\partial \lambda}(\lambda) \tag{31}$$

$$\le \frac{1}{\epsilon}[|\psi_N(\lambda + \epsilon) - p_N(\lambda + \epsilon)| + |p_N(\lambda) - \psi_N(\lambda)| + |p_N(\lambda + \epsilon) - p(\lambda + \epsilon)|$$

$$+ |p_N(\lambda) - p(\lambda)|] + \frac{1}{\epsilon}[p(\lambda + \epsilon) - p(\lambda)] - \frac{\partial p}{\partial \lambda}(\lambda) \tag{32}$$

$$\le w_N(\epsilon) + e_N(\epsilon) + \frac{\partial p}{\partial \lambda}(\lambda + \epsilon) - \frac{\partial p}{\partial \lambda}(\lambda). \tag{33}$$

As in the same calculation, we have

$$\frac{\partial \psi_N}{\partial b\lambda}(\lambda) - \frac{\partial p}{\partial \lambda}(\lambda) \ge \frac{1}{\epsilon}[\psi_N(\lambda) - \psi_N(\lambda - \epsilon)] - \frac{\partial p}{\partial \lambda}(\lambda) \tag{34}$$

$$\ge -w_N(\epsilon) - e_N(\epsilon) + \frac{\partial p}{\partial \lambda}(\lambda - \epsilon) - \frac{\partial p}{\partial \lambda}(\lambda). \tag{35}$$

Then,

$$\mathbb{E}\left|\frac{\partial \psi_N}{\partial \lambda}(\lambda) - \frac{\partial p}{\partial \lambda}(\lambda)\right| \le \mathbb{E}w_N(\epsilon) + e_N(\epsilon) + \frac{\partial p}{\partial \lambda}(\lambda + \epsilon) - \frac{\partial p}{\partial \lambda}(\lambda - \epsilon). \tag{36}$$

Convergence of p_N in the infinite volume limit implies

$$\lim_{N \nearrow \infty} \mathbb{E}\left|\beta\langle h_N(\mathbf{S})\rangle_{\lambda, 0} - \frac{\partial p}{\partial \lambda}(\lambda)\right| \le \frac{\partial p}{\partial \lambda}(\lambda + \epsilon) - \frac{\partial p}{\partial \lambda}(\lambda - \epsilon), \tag{37}$$

The right hand side vanishes, since the convex function$p(\lambda)$ is continuously differentiable almost everywhere and $\epsilon > 0$ is arbitrary. Therefore

$$\lim_{N \nearrow \infty} \mathbb{E}\left|\beta\langle h_N(\mathbf{S})\rangle_{\lambda, 0} - \frac{\partial p}{\partial \lambda}(\lambda)\right| = 0. \tag{38}$$

for almost all λ. Jensen's inequality gives

$$\lim_{N \nearrow \infty} \left|\mathbb{E}\beta\langle h_N(\mathbf{S})\rangle_{\lambda, 0} - \frac{\partial p}{\partial \lambda}(\lambda)\right| = 0. \tag{39}$$

This implies the first equality (26). Since the $p(\lambda)$ is continuously differentiable almost everywhere in $\mathbb{R}$, these equalities imply also

$$\lim_{N \nearrow \infty} \mathbb{E}|\langle h_N(\mathbf{S})\rangle_{\lambda,0} - \mathbb{E}\langle h_N(\mathbf{S})\rangle_{\lambda,0}| = 0. \tag{40}$$

The bound on $h_N(\mathbf{S})$ concludes the following limit

$$\lim_{N \nearrow \infty} \mathbb{E}(\langle h_N(\mathbf{S})\rangle_{\lambda,0} - \mathbb{E}\langle h_N(\mathbf{S})\rangle_{\lambda,0})^2 \le 2C_h \lim_{N \nearrow \infty} \mathbb{E}|\langle h_N(\mathbf{S})\rangle_{\lambda,0} - \mathbb{E}\langle h_N(\mathbf{S})\rangle_{\lambda,0}| = 0. \tag{41}$$

This completes the proof. $\square$

Note that Lemma 2.1 guarantees the existence of the following infinite volume limit for almost all $\lambda \in \mathbb{R}$

$$\lim_{N \nearrow \infty} \mathbb{E}\langle h_N(\mathbf{S})\rangle_{\lambda,0}^2 = (\lim_{N \nearrow \infty} \mathbb{E}\langle h_N(\mathbf{S})\rangle_{\lambda,0})^2 \tag{42}$$

Lemma 2.2 *Let $f(\mathbf{S})$ be a function of spin operators which is self-adjoint and bounded by a constant C_f independent of N*

$$\|f(\mathbf{S})\| \le C_f. \tag{43}$$

For any $(\beta, \lambda, \mu) \in [0, \infty) \times \mathbb{R}^2$, any positive integer N and k, arbitrarily fixed $\mathbf{J}$, an upper bound on the following k-th order derivative is given by

$$\left|\mathbb{E}_g \frac{\partial^k}{\partial \lambda^k} \langle f(\mathbf{S})\rangle_{\lambda,\mu g}\right| \le \sqrt{k!} C_f |\mu|^{-k} N^{k(1-\alpha)}, \tag{44}$$

where $\mathbb{E}_g$ denotes the expectation only over the standard Gaussian random variable g.

Proof. Let g, g' be i.i.d. standard Gaussian random variables and define a function with a parameter $u \in [0, 1]$

$$G(u) := \sqrt{u} g + \sqrt{1-u} g'. \tag{45}$$

Define a generating function χ_f of the parameter $u \in [0, 1]$ for f by

$$\chi_f(u) := \mathbb{E}_g[\mathbb{E}'_g\langle f(\mathbf{S})\rangle_{\lambda,\mu G(u)}]^2, \tag{46}$$

where $\mathbb{E}'_g$ is expectation over only g' and $\mathbb{E}_g$ is expectation over only random variables g and g'. This generating function χ_f is a generalization of a function introduced by Chatterjee [3]. First we prove the following formula

$$\frac{d^k \chi_f}{du^k}(u) = N^{2(\alpha-1)k} \mu^{2k} \mathbb{E}_g \left[\mathbb{E}'_g \frac{\partial^k}{\partial \lambda^k} \langle f(\mathbf{S})\rangle_{\lambda,\mu G(u)}\right]^2. \tag{47}$$

The following inductivity for a positive integer k proves this formula.
For $k = 1$, the first derivative of χ_f is

$$\begin{aligned}
\chi'_f(u) &= N^\alpha \beta \mu \mathbb{E}_g \mathbb{E}'_g \langle f(\mathbf{S})\rangle_{\lambda,\mu G(u)} \mathbb{E}'_g \left(\frac{g}{\sqrt{u}} - \frac{g'}{\sqrt{1-u}}\right)(f(\mathbf{S}); h_N(\mathbf{S}))_{\lambda,\mu G(u)} \\
&= N^\alpha \beta \mu \mathbb{E}_g \left[\frac{1}{\sqrt{u}} \frac{\partial}{\partial g} \mathbb{E}'_g \langle f(\mathbf{S})\rangle_{\lambda,\mu G(u)} \mathbb{E}'_g (f(\mathbf{S}); h_N(\mathbf{S}))_{\lambda,\mu G(u)} \right. \\
&\quad \left. - \mathbb{E}'_g \langle f(\mathbf{S})\rangle_{\lambda,\mu G(u)} \mathbb{E}'_g \frac{1}{\sqrt{1-u}} \frac{\partial}{\partial g'} (f(\mathbf{S}); h_N(\mathbf{S}))_{\lambda,\mu G(u)}\right] \tag{48} \\
&= N^{2\alpha} \beta^2 \mu^2 \mathbb{E}_g [\mathbb{E}'_g (f(\mathbf{S}); h_N(\mathbf{S}))_{\lambda,\mu G(u)}]^2 \tag{49} \\
&= N^{2(\alpha-1)} \mu^2 \mathbb{E}_g \left[\mathbb{E}'_g \frac{\partial}{\partial \lambda} \langle f(\mathbf{S})\rangle_{\lambda,\mu G(u)}\right]^2, \tag{50}
\end{aligned}$$

where integration by parts over g and g' has been used. If the validity of the formula (47) is assumed for an arbitrary positive integer k, then (47) for $k+1$ can be proved using integration by parts. The formula (47) shows that k-th derivative of $\chi_f(u)$ is positive semi-definite for any k, then it is a monotonically increasing function of u. From Taylor's theorem, there exists $v \in (0, u)$ such that

$$\chi_f(u) = \sum_{n=0}^{k-1} \frac{u^n}{n!} \chi_f^{(n)}(0) + \frac{u^k}{k!} \chi_f^{(k)}(v). \tag{51}$$

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Each term in this series is bounded from the above by

$$\chi_f(1) = \mathbb{E}_g \langle f(\mathbf{S}) \rangle^2_{\lambda, \mu g} \le \|f\|^2 \le C_f^2. \tag{52}$$

The definition of $G(u)$ and the formula (47) give

$$N^{2(\alpha-1)k} \mu^{2k} \left[\mathbb{E}_g \frac{\partial^k}{\partial \lambda^k} \langle f(\mathbf{S}) \rangle_{\lambda, \mu g} \right]^2 = N^{2(\alpha-1)k} \mu^{2k} \mathbb{E}_g \left[\mathbb{E}'_g \frac{\partial^k}{\partial \lambda^k} \langle f(\mathbf{S}) \rangle_{\lambda, \mu G(0)} \right]^2$$

$$= \frac{d^k \chi_f}{du^k}(0) \le k! \chi_f(1) \le k! C_f^2. \tag{53}$$

This completes the proof. $\square$

Lemma 2.3 *The infinite volume limit*

$$p(\beta, \lambda, \mu) := \lim_{N \nearrow \infty} p_N(\beta, \lambda, \mu) \tag{54}$$

exists for each $(\alpha, \beta, \lambda, \mu) \in (0,1) \times (0,\infty) \times \mathbb{R}^2$, and $p(\beta, \lambda, \mu) = p(\beta, \lambda, 0)$.

Proof. Integration of the derivative function of p_N over the interval $(0, \mu)$ and integration by parts with respect to g give

$$|p_N(\beta, \lambda, \mu) - p_N(\beta, \lambda, 0)| = \left| \int_0^\mu dx \frac{\partial}{\partial x} p_N(\beta, \lambda, x) \right| \tag{55}$$

$$= \left| \int_0^\mu dx \mathbb{E} N^{\alpha-1} \beta g \langle h_N(\mathbf{S}) \rangle_{\lambda, xg} \right| \tag{56}$$

$$= \left| \int_0^\mu dx \mathbb{E} N^{2\alpha-1} \beta^2 x (h_N(\mathbf{S}); h_N(\mathbf{S}))_{\lambda, xg} \right| \tag{57}$$

$$= \left| \int_0^\mu dx x N^{2\alpha-2} \beta \frac{\partial}{\partial \lambda} \mathbb{E} \langle h_N(\mathbf{S}) \rangle_{\lambda, xg} \right| \tag{58}$$

$$\le N^{2\alpha-2} \beta \left| \int_0^\mu dx |x| \frac{\partial}{\partial \lambda} \mathbb{E} \langle h_N(\mathbf{S}) \rangle_{\lambda, xg} \right| \tag{59}$$

$$\le N^{\alpha-1} \beta C_h \left| \int_0^\mu dx \right| = N^{\alpha-1} \beta C_h |\mu|. \tag{60}$$

We have used Lemma 2.2. The right hand side vanishes in the infinite volume limit can be taken for $\alpha < 1$, and this completes the proof. $\square$

Lemma 2.3 and Assumption 1 guarantee the existence of $p(\beta, \lambda, \mu)$ for each $(\beta, \lambda, \mu) \in (0, \infty) \times \mathbb{R}$. Next, we prove an identity between expectation values of an arbitrary bounded operator for $\mu \ne 0$ and $\mu = 0$ in a method similar to that used in Ref [17].

Lemma 2.4 *Let f be a bounded function of spin operators whose infinite volume limit*

$$\lim_{N \nearrow \infty} \mathbb{E} \langle f(\mathbf{S}) \rangle_{\lambda, 0} \tag{61}$$

exists at $\mu = 0$. Then the infinite volume limit at $\mu \ne 0$ exists for $\alpha < 1$ and for almost all $\lambda \in \mathbb{R}$, and

$$\lim_{N \nearrow \infty} \mathbb{E} \langle f(\mathbf{S}) \rangle_{\lambda, \mu g} = \lim_{N \nearrow \infty} \mathbb{E} \langle f(\mathbf{S}) \rangle_{\lambda, 0}. \tag{62}$$

Proof. Integration of the derivative function over the interval $(0, \mu)$ for an arbitrary $\mu \in \mathbb{R}$ gives

$$\mathbb{E} \langle f(\mathbf{S}) \rangle_{\lambda, \mu g} - \mathbb{E} \langle f(\mathbf{S}) \rangle_{\lambda, 0} = \int_0^\mu dx \frac{\partial}{\partial x} \mathbb{E} \langle f(\mathbf{S}) \rangle_{\lambda, xg} \tag{63}$$

$$= \int_0^\mu dx \mathbb{E} N^\alpha \beta g (f(\mathbf{S}); h_N(\mathbf{S}))_{\lambda, xg} \tag{64}$$

$$= \int_0^\mu dx \mathbb{E} N^{2\alpha} \beta^2 x (f(\mathbf{S}); h_N(\mathbf{S}); h_N(\mathbf{S}))_{\lambda, xg} \tag{65}$$

$$= \int_0^\mu dx x N^{2(\alpha-1)} \frac{\partial^2}{\partial \lambda^2} \mathbb{E} \langle f(\mathbf{S}) \rangle_{\lambda, xg}. \tag{66}$$

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Integration over an arbitrary interval of λ and Lemma 2.2 imply

$$\left| \int_a^b d\lambda [\mathbb{E}\langle f(\mathbf{S})\rangle_{\lambda,\mu g} - \mathbb{E}\langle f(\mathbf{S})\rangle_{\lambda,0}] \right| \tag{67}$$

$$= \left| \int_0^\mu dx x N^{2(\alpha-1)} \left[\frac{\partial}{\partial b} \mathbb{E}\langle f(\mathbf{S})\rangle_{b,xg} - \frac{\partial}{\partial a} \mathbb{E}\langle f(\mathbf{S})\rangle_{a,xg} \right] \right| \tag{68}$$

$$\leq \left| \int_0^\mu dx |x| N^{2(\alpha-1)} \left[\left| \frac{\partial}{\partial b} \mathbb{E}\langle f(\mathbf{S})\rangle_{b,xg} \right| + \left| \frac{\partial}{\partial a} \mathbb{E}\langle f(\mathbf{S})\rangle_{a,xg} \right| \right] \right| \tag{69}$$

$$\leq 2N^{\alpha-1} \left| \int_0^\mu dx C_f \right| = 2N^{\alpha-1} C_f |\mu|. \tag{70}$$

The right hand side vanishes in the infinite volume limit for $\alpha < 1$. Since the integration interval (a, b) is arbitrary, the integrand in the left hand side vanishes for almost all λ in the infinite volume limit. This completes the proof. $\square$

Lemma 2.5 *The infinit volume limit of the following function*

$$\lim_{N \nearrow \infty} \mathbb{E}\langle h_N(\mathbf{S})\rangle_{\lambda,\mu g}^2 \tag{71}$$

exists for $\alpha < 1$ for almost all $\lambda \in \mathbb{R}$, and it is identical to that at $\mu = 0$

$$\lim_{N \nearrow \infty} \mathbb{E}\langle h_N(\mathbf{S})\rangle_{\lambda,\mu g}^2 = \lim_{N \nearrow \infty} \mathbb{E}\langle h_N(\mathbf{S})\rangle_{\lambda,0}^2. \tag{72}$$

Proof. Integration of the derivative function over the interval $(0, \mu)$ for an arbitrary $\mu \in \mathbb{R}$ gives

$$\mathbb{E}\langle h_N(\mathbf{S})\rangle_{\lambda,\mu g}^2 - \mathbb{E}\langle h_N(\mathbf{S})\rangle_{\lambda,0}^2 = \int_0^\mu dx \frac{\partial}{\partial x} \mathbb{E}\langle h_N(\mathbf{S})\rangle_{\lambda,xg}^2 \tag{73}$$

$$= 2 \int_0^\mu dx \mathbb{E} N^\alpha \beta g(h_N(\mathbf{S}); h_N(\mathbf{S}))_{\lambda,xg} \langle h_N(\mathbf{S})\rangle_{\lambda,xg} \tag{74}$$

$$= 2 \int_0^\mu dx \mathbb{E} N^\alpha \beta \frac{\partial}{\partial g} (h_N(\mathbf{S}); h_N(\mathbf{S}))_{\lambda,xg} \langle h_N(\mathbf{S})\rangle_{\lambda,xg} \tag{75}$$

$$= 2\beta \int_0^\mu dx x N^{2\alpha-1} \frac{\partial}{\partial \lambda} \mathbb{E}(h_N(\mathbf{S}); h_N(\mathbf{S}))_{\lambda,xg} \langle h_N(\mathbf{S})\rangle_{\lambda,xg} \tag{76}$$

Integration over an arbitrary interval of λ and Lemma 2.2 imply

$$\left| \int_a^b d\lambda [\mathbb{E}\langle h_N(\mathbf{S})\rangle_{\lambda,\mu g}^2 - \mathbb{E}\langle h_N(\mathbf{S})\rangle_{\lambda,0}^2] \right| \tag{77}$$

$$= \left| 2\beta \int_0^\mu dx x N^{2\alpha-1} \left[\mathbb{E}(h_N(\mathbf{S}); h_N(\mathbf{S}))_{b,xg} \langle h_N(\mathbf{S})\rangle_{b,xg} \right. \right.$$
$$\left. \left. - \mathbb{E}(h_N(\mathbf{S}); h_N(\mathbf{S}))_{a,xg} \langle h_N(\mathbf{S})\rangle_{a,xg} \right] \right| \tag{78}$$

$$\leq 2\beta \left| \int_0^\mu dx |x| N^{2\alpha-1} \left[\mathbb{E}|(h_N(\mathbf{S}); h_N(\mathbf{S}))_{b,xg}| |\langle h_N(\mathbf{S})\rangle_{b,xg}| \right. \right.$$
$$\left. \left. + \mathbb{E}|(h_N(\mathbf{S}); h_N(\mathbf{S}))_{a,xg}| |\langle h_N(\mathbf{S})\rangle_{a,xg}| \right] \right| \tag{79}$$

$$\leq 2\beta C_h \left| \int_0^\mu dx |x| N^{2\alpha-1} \left[\mathbb{E}(h_N(\mathbf{S}); h_N(\mathbf{S}))_{b,xg} + \mathbb{E}(h_N(\mathbf{S}); h_N(\mathbf{S}))_{a,xg} \right] \right| \tag{80}$$

$$= 2C_h \left| \int_0^\mu dx |x| N^{2(\alpha-1)} \left[\frac{\partial}{\partial b} \mathbb{E}\langle h_N(\mathbf{S})\rangle_{b,xg} + \frac{\partial}{\partial a} \mathbb{E}\langle h_N(\mathbf{S})\rangle_{a,xg} \right] \right| \tag{81}$$

$$\leq 4N^{\alpha-1} \left| \int_0^\mu dx C_h^2 \right| = 4N^{\alpha-1} C_h^2 |\mu|. \tag{82}$$

The right hand side vanishes in the infinite volume limit for $\alpha < 1$. Since the integration interval (a, b) is arbitrary, the integrand in the left hand side vanishes for almost all λ in the infinite volume limit. This completes the proof. $\square$

Now, we prove Theorem 1.1. Lemma 2.2 indicates that the artificial random perturbation suppresses the variance of the corresponding perturbation operator even in the weak coupling and it vanishes in the infinite volume limit. Lemmas 2.4 and 2.5 imply that the variance at $\mu = 0$ is identical to that at $\mu \neq 0$ for almost all non-random perturbation λ in the infinite volume limit. Assumption 3 is necessary to prove Theorem 1.1 for quantum systems.

Proof of Theorem 1.1
Lemma 2.2 for $f(\mathbf{S}) = h_N(\mathbf{S})$ and $k = 1$ yields

$$\mathbb{E}(h_N(\mathbf{S}); h_N(\mathbf{S}))_{\lambda,\mu g} \leq \frac{C_h}{\beta|\mu|} N^{-\alpha}, \tag{83}$$

for an arbitrary $\lambda \in \mathbb{R}$ and $\mu \neq 0$. Harris' inequality of the Bogolyubov type between the Duhamel product and the Gibbs expectation of the square of arbitrary self-adjoint operator O [14]

$$(O, O)_{\lambda,\mu g} \leq \langle O^2 \rangle_{\lambda,\mu g} \leq (O, O)_{\lambda,\mu g} + \frac{\beta}{12} \langle [O, [H, O]] \rangle_{\lambda,\mu g}, \tag{84}$$

and Assumption 3 enable us to obtain

$$\lim_{N \nearrow \infty} \mathbb{E}\langle h_N(\mathbf{S})^2 \rangle_{\lambda,\mu g} = \lim_{N \nearrow \infty} \mathbb{E}(h_N(\mathbf{S}), h_N(\mathbf{S}))_{\lambda,\mu g}. \tag{85}$$

This and the bound (83) for $\mu \neq 0$ imply

$$\lim_{N \nearrow \infty} \mathbb{E}[\langle h_N(\mathbf{S})^2 \rangle_{\lambda,\mu g} - \langle h_N(\mathbf{S}) \rangle_{\lambda,\mu g}^2] = \lim_{N \nearrow \infty} \mathbb{E}(h_N(\mathbf{S}); h_N(\mathbf{S}))_{\lambda,\mu g} = 0. \tag{86}$$

This is true also for $\mu = 0$ by Lemma 2.4 and Lemma 2.5 with by a uniform bound on h_N^2

$$\lim_{N \nearrow \infty} \mathbb{E}[\langle h_N(\mathbf{S})^2 \rangle_{\lambda,0} - \langle h_N(\mathbf{S}) \rangle_{\lambda,0}^2] = 0. \tag{87}$$

This and Lemma 2.1 give

$$\lim_{N \nearrow \infty} [\mathbb{E}\langle h_N(\mathbf{S})^2 \rangle_{\lambda,0} - (\mathbb{E}\langle h_N(\mathbf{S}) \rangle_{\lambda,0})^2] = 0. \tag{88}$$

This completes the proof of Theorem 1.1. $\square$

3 Applications to several models

3.1 Random energy model

Random energy model is a well known simple model where replica symmetry breaking appears. This model contains only $(S_i^z)_{i \in V_N}$ with spin $S = \frac{1}{2}$. In the definition of the unperturbed Hamiltonian (3), $C_N := \mathcal{P}(V_N)$ and the function φ is defined by

$$\varphi(\mathbf{S}_X) := \prod_{i \in X} \delta_{2S_i^z, 1} \prod_{j \in X^c} \delta_{2S_j^z, -1} \tag{89}$$

The possible state in H is represented in a spin configuration $\sigma = (\sigma_i)_{i \in V_N} \in \Sigma_N := \{1, -1\}^{V_N}$, which is a sequence of eigenvalues of the operators $(2S_i^z)_{i \in V_N}$. The function φ defines a natural bijection $\mathcal{P}(V_N) \to \Sigma_N$, such that $2S_i^z = 1$ for $i \in X$ and $2S_i^z = -1$ for $i \in X^c$. We identify a subset X and the corresponding spin configuration σ with this bijection. Let us represent the Hamiltonian in terms of spin configurations. Let $(J_X)_{X \in C_N}$ be i.i.d. standard Gaussian random variables in the Hamiltonian (3), and identify them to $\mathbf{J} = (J_\sigma)_{\sigma \in \Sigma_N}$. The Hamiltonian defines a partition function

$$Z_N(\beta, \mathbf{J}) := \sum_{\sigma \in \Sigma_N} \exp(-\beta H_N(\sigma)), \tag{90}$$

where the unperturbed Hamiltonian on V_N can be written in

$$H_N(\sigma) := -\sqrt{N} J_\sigma. \tag{91}$$

Consider a n-replicated random energy model whose state is given by n spin configurations $(\sigma^1, \cdots, \sigma^n) \in \Sigma_N^n$. The Hamiltonian of this model is given by

$$H_{N,n}(\sigma^1, \cdots, \sigma^n) := \sum_{a=1}^n H_N(\sigma^a). \tag{92}$$

Here we attach index V to the Hamiltonian on V_N for later convenience. This Hamiltonian is invariant under a permutation s

$$H_{N,n}(\sigma^{s(1)}, \cdots, \sigma^{s(n)}) = H_N(\sigma^1, \cdots, \sigma^n), \tag{93}$$

where $s : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, n\}$ is an arbitrary bijection. This symmetry is replica symmetry. To study the spontaneous replica symmetry breaking, consider the following symmetry breaking perturbation

$$h_N(\sigma^1, \sigma^2) := \prod_{i \in V_N} \delta_{\sigma_i^1, \sigma_i^2}. \tag{94}$$

This order parameter becomes finite if and only if two replicated spin configurations σ^1 and σ^2 are identical $(\sigma_i^1)_{i \in V_N} = (\sigma_i^2)_{i \in V_N}$, otherwise it vanishes. Note the upper bound for this operator

$$\|h_N(\sigma^1, \sigma^2)\| \leq 1. \tag{95}$$

The partition function is defined by

$$Z_{N,n}(\beta, \lambda, \mathbf{J}) := \sum_{\sigma^1, \cdots, \sigma^n} \exp[-\beta H_{N,n}(\sigma^1, \cdots, \sigma^n) + \beta N \lambda h_N(\sigma^1, \sigma^2)]. \tag{96}$$

Dfine

$$\psi_{N,n}(\beta, \lambda, \mathbf{J}) := \frac{1}{N} \log Z_{N,n}(\beta, \lambda, \mathbf{J}), \tag{97}$$

and

$$p_{N,n}(\beta, \lambda) := \mathbb{E}\psi_{N,n}(\beta, \lambda), \tag{98}$$

whose infinite volume limit is

$$p_n(\beta, \lambda) := \lim_{N \nearrow \infty} p_{N,n}(\beta, \lambda). \tag{99}$$

Guerra obtains the following explicit form [10]

$$p_n(\beta, \lambda) = \max\{np_1(\beta, 0), \ p_1(2\beta, 0) + \beta\lambda + (n-2)p_1(\beta, 0)\}, \tag{100}$$

where

$$p_1(\beta, 0) = \begin{cases} \beta\sqrt{2\log 2} & (\beta > \sqrt{2\log 2}) \\ \beta^2/2 + \log 2 & (\beta \leq \sqrt{2\log 2}). \end{cases} \tag{101}$$

This shows Assumption 1.

Assumption 2 is proved in the following lemma.

Lemma 3.1 *The variance of $\psi_{N,n}(\beta, \lambda, \mathbf{J})$ vanishes in the model defined by (97) for any positive integers N and for any $(\beta, \lambda) \in (0, \infty) \times \mathbb{R}$*

Proof. Let $(J_{\sigma^a})_{\sigma \in \Sigma_N, a=1, \cdots, n}$ $(J'_{\sigma^a})_{\sigma \in \Sigma_N, a=1, \cdots, n}$ be i.i.d. standard Gaussian random variables. Define $\mathcal{J}(u) = (\mathcal{J}_{\sigma^a}(u))_{\sigma^a \in \Sigma_N}$ of $u \in [0, 1]$ by

$$\mathcal{J}_{\sigma^a}(u) := \sqrt{u}J_{\sigma^a} + \sqrt{1-u}J'_{\sigma^a} \tag{102}$$

for each spin configuration $\sigma \in \Sigma_N$ and $a = 1, 2$. Define a function $\gamma(u)$ by

$$\gamma(u) := \mathbb{E}[\mathbb{E}'\psi_{N,n}(\beta, \lambda, \mathcal{J}(u))]^2, \tag{103}$$

where $\mathbb{E}$ $\mathbb{E}'$ denote the expectation over $\mathbf{J}$ and $\mathbf{J}'$ respectively. Its derivative is evaluated as

$$\gamma'(u) = \frac{\beta}{N}\mathbb{E}\mathbb{E}'\psi_{N,n}(\beta,\lambda,\mathcal{J}(u))\mathbb{E}'\sum_{a=1}^{n}\sum_{\sigma^1,\cdots,\sigma^n\in\Sigma_N}\left(\frac{J_{\sigma^a}}{\sqrt{u}}-\frac{J_{\sigma^a}}{\sqrt{1-u}}\right)\frac{e^{-\beta H_{N,n}(\sigma^1,\cdots,\sigma^n)}}{Z_{N,n}(\beta,\lambda,\mathcal{J}(u))}$$

$$= \frac{\beta}{\sqrt{N}}\mathbb{E}\sum_{a}^{n}\sum_{\sigma^1,\cdots,\sigma^n\in\Sigma_N}\left[\frac{1}{\sqrt{u}}\frac{\partial}{\partial J_{\sigma^a}}\mathbb{E}'\psi_{N,n}(\beta,\lambda,\mathcal{J}(u))\mathbb{E}'\frac{e^{-\beta H_{N,n}(\sigma^1,\cdots,\sigma^n)}}{Z_{N,n}(\beta,\lambda,\mathcal{J}(u))}\right.$$

$$\left.-\mathbb{E}'\psi_{N,n}(\beta,\lambda,\mathcal{J}(u))\mathbb{E}'\frac{1}{\sqrt{1-u}}\frac{\partial}{\partial J'_{\sigma^a}}\frac{e^{-\beta H_{N,n}(\sigma^1,\cdots,\sigma^n)}}{Z_{N,n}(\beta,\lambda,\mathcal{J}(u))}\right] \tag{104}$$

$$= \frac{\beta^2}{N}\mathbb{E}\sum_{\sigma^1,\cdots,\sigma^n,\tau^1,\cdots,\tau^n\in\Sigma_N}\sum_{1\leq a,b\leq n}\mathbb{E}'\delta_{\sigma^a,\tau^b}\frac{e^{-\beta H_{N,n}(\sigma^1,\cdots,\sigma^n)}}{Z_{N,n}(\beta,\lambda,\mathcal{J}(u))}\mathbb{E}'\frac{e^{-\beta H_{N,n}(\tau^1,\cdots,\tau^n)}}{Z_{N,n}(\beta,\lambda,\mathcal{J}(u))} \tag{105}$$

$$\leq \frac{\beta^2 n^2}{N}, \tag{106}$$

for any $u \in [0,1]$. Then the variance of $\psi_{N,n}(\beta,\lambda)$ is

$$\mathbb{E}\psi_{N,n}(\beta,\lambda)^2 - p_{N,n}(\beta,\lambda)^2 = \gamma(1)-\gamma(0) = \int_0^1 du\gamma'(u) \leq \frac{\beta^2 n^2}{N}, \tag{107}$$

for any positive integers N and for any $(\beta,\lambda) \in (0,\infty)\times\mathbb{R}$. $\square$

Assumption 3 is satisfied trivially, since the Hamiltonian of this model commutes with h_N. The following corollary for the perturbed random energy model is obtained from Theorem 1.1.

Corollary 3.2 *In the n replicated random energy model perturbed by the Hamiltonian (94) in the infinite volume limit, for almost all $\lambda \in \mathbb{R}$, the expectation of the perturbing operator takes the value*

$$\lim_{N\nearrow\infty}\mathbb{E}\langle h_N(\sigma^1,\sigma^2)\rangle_\lambda = 0 \text{ or } 1. \tag{108}$$

Proof. Note the relation

$$h_N(\sigma^1,\sigma^2)^2 = h_N(\sigma^1,\sigma^2). \tag{109}$$

Then, Theorem 1.1 implies

$$\lim_{N\nearrow\infty}\mathbb{E}\langle h_N(\sigma^1,\sigma^2)\rangle_\lambda(1-\mathbb{E}\langle h_N(\sigma^1,\sigma^2)\rangle_\lambda) = 0. \tag{110}$$

Therefore, $\lim_{n\to\infty}\mathbb{E}\langle h_N(\sigma^1,\sigma^2)\rangle_\lambda$ takes the value either 0 or1. $\square$

Note that this corollary is also true for an arbitrary projection operator satisfying $h_N^2 = h_N$ in other models.

It is well known that the observation of $\lim_{\lambda\searrow0}\lim_{N\nearrow\infty}\mathbb{E}\langle h_N(\sigma^1,\sigma^2)\rangle_\lambda = 1$ implies the spontaneous replica symmetry breaking. The replica symmetry breaking is also detect by the replica symmetric Gibbs state. If the replica symmetric calculation shows

$$0 < \lim_{N\nearrow\infty}\mathbb{E}\langle h_N(\sigma^1,\sigma^2)\rangle_0 < 1, \tag{111}$$

then this implies the finite variance

$$\lim_{N\nearrow\infty}[\mathbb{E}\langle h_N(\sigma^1,\sigma^2)^2\rangle_0 - (\mathbb{E}\langle h_N(\sigma^1,\sigma^2)\rangle_0)^2] > 0 \tag{112}$$

which gives an instability of the replica symmetric Gibbs state due to the large fluctuation. At the same time, this implies the non-commutativity of limiting procedure

$$\lim_{N\nearrow\infty}\lim_{\lambda\searrow0}\mathbb{E}\langle h_N(\sigma^1,\sigma^2)\rangle_\lambda \neq \lim_{\lambda\searrow0}\lim_{N\nearrow\infty}\mathbb{E}\langle h_N(\sigma^1,\sigma^2)\rangle_\lambda. \tag{113}$$

This is a typical phenomenon in spontaneous symmetry breaking.

Here, we point out an agreement between Corollary 3.2 and Guerra's result [10]. Guerra has studied the replica symmetry breaking in the random energy model as a spontaneous symmetry breaking phenomenon. For $\beta \leq \beta_c = \sqrt{2\log 2}$ and for a sufficiently small λ, $p_n(\beta,\lambda) = p_n(\beta,0)$, then Guerra's formula (100) gives

$$\lim_{\lambda\to0}\lim_{N\nearrow\infty}\mathbb{E}\langle h_N(\sigma^1,\sigma^2)\rangle_\lambda = \frac{1}{\beta}\frac{\partial p_n}{\partial\lambda} = 0. \tag{114}$$

For $\beta > \beta_c$ becomes

$$p_n(\beta, \lambda) = \begin{cases} n\beta\sqrt{\log 2} + \beta\lambda & (\lambda > 0) \\ n\beta\sqrt{\log 2} & (\lambda \leq 0), \end{cases} \tag{115}$$

which implies

$$\lim_{\lambda \searrow 0} \lim_{N \nearrow \infty} \mathbb{E}\langle h_N(\sigma^1, \sigma^2)\rangle_\lambda = \lim_{\lambda \searrow 0} \frac{1}{\beta}\frac{\partial p_n}{\partial \lambda}(\beta, \lambda) = 1, \tag{116}$$

and

$$\lim_{\lambda \nearrow 0} \lim_{N \nearrow \infty} \mathbb{E}\langle h_N(\sigma^1, \sigma^2)\rangle_\lambda = \lim_{\lambda \nearrow 0} \frac{1}{\beta}\frac{\partial p_n}{\partial \lambda}(\beta, \lambda) = 0. \tag{117}$$

Corollary 3.2 agrees with these results. The non-differentiability of $p_n(\beta, \lambda)$ at $\lambda = 0$ is pointed out also by Mukaida [?]. On the other hand, Guerra's replica symmetric calculation of the order parameter shows

$$\lim_{N \nearrow \infty} \lim_{\lambda \to 0} \mathbb{E}\langle h_N(\sigma^1, \sigma^2)\rangle_\lambda = \lim_{N \nearrow \infty} \mathbb{E}\langle h_N(\sigma^1, \sigma^2)\rangle_0 = 1 - \frac{\beta_c}{\beta} < 1. \tag{118}$$

These show the non-commutativity of two limiting procedures. Since the finite variance of h_N shows the instability of the replica symmetric Gibbs state, it is not realistic and the replica symmetry breaking Gibbs state with the vanishing variance should be realized. In this case, the identity (116) implies that two replicated spin configurations σ^1 and σ^2 are identical.

3.2 Quantum Heisenberg model without disorder

Here we study spontaneous symmetry breaking of SU(2) invariance in the antiferromagnetic quantum Heisenberg model without disorder. Let V_N be a hyper cubic lattice $V_N := [1, L]^d \cap \mathbb{Z}^d$ and bipartite, namely there exist two subsets A and B of V_N such that $V_N = A \cup B$ and $A \cap B = \phi$. The model Hamiltonian is defined by

$$H_N(\mathbf{S}) := \sum_{i \in A, j \in B} \sum_{p=x,y,z} J_{i,j} S_i^p S_j^p, \tag{119}$$

where $J_{i,j} \geq 0$ is short-ranged and translationally invariant, i.e. there exists $c \geq 1$ such that $J_{i,j} = 0$ for any $|i - j| > c$, and $J_{i+v,j+v} = J_{i,j}$ for any $i, j, v \in V_N$. Consider an antiferromagnetic order operator as a perturbation operator

$$h_N(\mathbf{S}) := \frac{1}{N}\Big(\sum_{i \in A} S_i^z - \sum_{j \in B} S_j^z\Big). \tag{120}$$

This operator is bounded by $\|h_N(\mathbf{S})\| \leq S$. Define a perturbed Hamiltonian by

$$H := H_N(\mathbf{S}) - N\lambda h_N(\mathbf{S}). \tag{121}$$

In this model, Assumption 1 is proved in a standard method to show the sequence $p_N(\beta, \lambda)$ for positive integers N becomes Cauchy for any $(\beta, \lambda) \in (0, \infty) \times \mathbb{R}$. Assumption 2 is trivial for the model without disorder and Assumption 3 is obvious for short-range interactions.

Theorem 1.1 $\lim_{N \nearrow \infty}[\langle h_N(\mathbf{S})^2\rangle_\lambda - \langle h_N(\mathbf{S})\rangle_\lambda^2] = 0$ and SU(2) invariance $\langle h_N(\mathbf{S})\rangle_0 = 0$ yield the following corollary.

Corollary 3.3 *If the SU(2) invariant Gibbs state of the antiferromagnetic quantum Heisenberg model has a long-range order*

$$\lim_{N \nearrow \infty} \langle h_N(\mathbf{S})^2\rangle_0 \neq 0, \tag{122}$$

then we have

$$\lim_{\lambda \searrow 0} \lim_{N \nearrow \infty}[\langle h_N(\mathbf{S})^2\rangle_\lambda - \langle h_N(\mathbf{S})\rangle_\lambda^2] \neq \lim_{N \nearrow \infty} \lim_{\lambda \searrow 0}[\langle h_N(\mathbf{S})^2\rangle_\lambda - \langle h_N(\mathbf{S})\rangle_\lambda^2]. \tag{123}$$

The right hand side is non-zero, since the long-range order of the Gibbs state and its SU(2) invariance recovered by taking the limit $\lambda \to 0$ first. On the other hand, Theorem 1.1 states that the left hand side vanishes. The non-commutativity of limiting procedures in Corollary 3.3 claims the spontaneous SU(2) symmetry breaking, when a long-range order exists. Koma and Tasaki have shown that the long-range order (equivalent to a finite variance of order operator) in the symmetric Gibbs state implies the

spontaneous symmetry breaking in the quantum Heisenberg model with short-range antiferromagnetic interactions [18]. They have proved

$$\sqrt{\lim_{N \nearrow \infty} \langle h_N(\mathbf{S})^2 \rangle_0} \leq \lim_{\lambda \searrow 0} \lim_{N \nearrow \infty} \langle h_N(\mathbf{S}) \rangle_\lambda. \tag{124}$$

For ferromagnetic case, the corresponding inequality was proved by Griffiths [9]. Even though the symmetric Gibbs state with the long-range order is unstable and unrealistic, it is mathematically well defined and can detect the symmetry breaking in the evaluation result of the finite variance of order operator. Recently, Tasaki has shown that the variance of order operator vanishes in the symmetry breaking ground state in the infinite volume limit [25]. Theorem 1.1 for quantum spin systems is consistent with his result.

3.3 Quantum Edwards-Anderson model

It is quite interesting whether or not, a replica symmetry breaking occurs in short-range disordered spin systems as in the Sherrington-Kirkpatrick model described by the Parisi formula [21, 23, 24]. Here, we discuss a replica symmetry breaking as a spontaneous symmetry breaking phenomenon in the quantum Edwards-Anderson model [6].

Let $(S_j^p)_{j \in V_N, p=x,y,z}$ be spin operators on a d-dimensional hyper cubic lattice $V_N := [1, L]^d \cap \mathbb{Z}^d$, where $N = |V_N| = L^d$. Let A be a bounded subset of V_N, such that $|A| \leq C$, where C is a positive constant independent of N. Define a collection of interaction ranges by

$$C_N := \{X | X = A + v \subset V_N, v \in V_N\}. \tag{125}$$

Define

$$S_X^p := \prod_{j \in X} S_j^p \tag{126}$$

for $X \in C_N$ and for $p = x, y, z$. The Hamiltonian has disordered short-range interaction

$$H_N(\mathbf{S}, \mathbf{J}) := - \sum_{X \in C_N} \sum_{p=x,y,z} J_X K^p S_X^p, \tag{127}$$

where $(J_X)_{X \in C_N}$ are i.i.d standard Gaussian random variables and positive constants $(K^p)_{p=x,y,z}$. The interaction is short-ranged and translationally invariant, where $|C_N| \leq |A|N$. Consider a n-replicated model with spin operators $(S_j^{p,1}, \cdots, S_j^{p,n})_{j \in V_N, p=x,y,z}$ and define a replica symmetric Hamiltonian

$$\sum_{a=1}^{n} H_N(\mathbf{S}^a, \mathbf{J}). \tag{128}$$

Define a spin overlap as a perturbing operator

$$h_N(\mathbf{S}^1, \mathbf{S}^2) := \frac{1}{N} \sum_{i \in V_N} S_i^{z,1} S_i^{z,2}, \tag{129}$$

which breaks the replica symmetry. Note the following bound

$$\|h_N(\mathbf{S}^1, \mathbf{S}^2)\| \leq S^2. \tag{130}$$

Consider the model defined by

$$H_{N,n}(\mathbf{S}^1, \cdots, \mathbf{S}^n, \mathbf{J}) := \sum_{a=1}^{n} H_N(\mathbf{S}^a, \mathbf{J}) - N\lambda h_N(\mathbf{S}^1, \mathbf{S}^2). \tag{131}$$

To consider replica symmetry breaking, we attach the index n to several functions in this subsection as in Subsection 3.1. In this model, Assumption 1 is proved in a standard method to show the sequence $p_{N,n}(\beta, \lambda, 0)$ for positive integers N becomes Cauchy for any $(\beta, \lambda, 0) \in (0, \infty) \times \mathbb{R}$ as in the previous model. Assumption 3 is obvious for short-range interactions. Assumption 2 is proved in the following lemma.

Lemma 3.4 *The variance of $\psi_{N,n}(\beta, \lambda, \mathbf{J})$ defined by the Hamiltonian (131) vanishes in the infinite volume limit for each $(\beta, \lambda) \in (0, \infty) \times \mathbb{R}$.*

Proof. To prove this, we employ the generating function $\gamma(u)$. Let $\mathbf{J}' := (J'_X)_{X \in C_N}$ be i.i.d. standard Gaussian random variables, and define

$$\mathcal{J}(u) := \sqrt{u}\mathbf{J} + \sqrt{1-u}\mathbf{J}' \tag{132}$$

with $u \in [0.1]$. Define a generating function

$$\gamma(u) := \mathbb{E}[\mathbb{E}'\psi_{N,n}(\beta, \lambda, \mathcal{J}(u))]^2, \tag{133}$$

where $\mathbb{E}'$ stands for the expectation over only $\mathbf{J}'$. Its derivative in u is evaluated in the integration by parts

$$
\begin{aligned}
\gamma'(u) &= \mathbb{E}\mathbb{E}'\psi_{N,n}(\beta, \lambda, \mathcal{J}(u)) \sum_{X \in C_N} \mathbb{E}'\Big(\frac{J_X}{\sqrt{u}} - \frac{J'_X}{\sqrt{1-u}}\Big)\frac{\partial\psi_{N,n}}{\partial\mathcal{J}_X} \tag{134} \\
&= \sum_{X \in C_N} \mathbb{E}\Big[\frac{1}{\sqrt{u}}\frac{\partial}{\partial J_X}\mathbb{E}'\psi_{N,n}(\beta, \lambda, \mathcal{J}(u))\frac{\partial\psi_{N,n}}{\partial\mathcal{J}_X} \tag{135} \\
&\qquad - \mathbb{E}'\psi_{N,n}(\beta, \lambda, \mathcal{J}(u))\mathbb{E}'\frac{1}{\sqrt{1-u}}\frac{\partial}{\partial J'_X}\frac{\partial\psi_{N,n}}{\partial\mathcal{J}_X}\Big] \tag{136} \\
&= \sum_{X \in C_N} \mathbb{E}\Big[\mathbb{E}'\frac{\partial\psi_{N,n}}{\partial\mathcal{J}_X}\Big]^2 \tag{137} \\
&= \frac{\beta^2}{N^2} \sum_{X \in C_N} \mathbb{E}\Big(\mathbb{E}' \sum_{p=x,y,z} \sum_{a=1}^{n} K^p \langle S_X^{p,a}\rangle_u\Big)^2 \tag{138} \\
&\leq \frac{\beta^2|A|n^2 S^{2|A|}}{N}\Big(\sum_{p=x,y,z} K^p\Big)^2, \tag{139}
\end{aligned}
$$

where we denote

$$\langle f(\mathbf{S}^1, \cdots, \mathbf{S}^n)\rangle_u := \frac{1}{Z_{N,n}(\beta, \lambda, \mathcal{J}(u))}\mathrm{Tr}[f(\mathbf{S}^1, \cdots, \mathbf{S}^n)e^{-\beta H_{N,n}(\mathbf{S}^1, \cdots, \mathbf{S}^n, \mathcal{J}(u))}],$$

for the Gibbs expectation of an arbitrary function f of spin operators. The variance of ψ_N is given by

$$\mathbb{E}\psi_{N,n}(\beta, \lambda)^2 - p_{N,n}(\beta, \lambda)^2 = \int_0^1 du\,\gamma'(u) \leq \frac{\beta^2|A|n^2 S^{2|A|}}{N}\Big(\sum_{p=x,y,z} K^p\Big)^2. \tag{140}$$

Then the variance of $\psi_{N,n}(\beta, \lambda)$ vanishes in the infinite volume limit for arbitrary $(\beta, \lambda) \in (0, \infty) \times \mathbb{R}$. $\square$

Chatterjee's definition of replica symmetry breaking [2] is the following finite variance of spin overlap in the replica symmetric Gibbs state with $\lambda = \mu = 0$

$$\lim_{N \nearrow \infty} \mathbb{E}\langle(h_N(\mathbf{S}^1, \mathbf{S}^2) - \mathbb{E}\langle h_N(\mathbf{S}^1, \mathbf{S}^2)\rangle_0)^2\rangle_0 > 0. \tag{141}$$

Theorem 1.1 gives

$$\lim_{\lambda \searrow 0} \lim_{N \nearrow \infty} \mathbb{E}\langle(h_N(\mathbf{S}^1, \mathbf{S}^2) - \mathbb{E}\langle h_N(\mathbf{S}^1, \mathbf{S}^2)\rangle_\lambda)^2\rangle_\lambda = 0, \tag{142}$$

then this yields the following corollary.

Corollary 3.5 *If the replica symmetry breaking defined by Chatterjee occurs in the model defined by the Hamiltonian (128), the following limiting procedures do not commute*

$$\lim_{N \nearrow \infty} \lim_{\lambda \searrow 0} \mathbb{E}\langle(h_N(\mathbf{S}^1, \mathbf{S}^2) - \mathbb{E}\langle h_N(\mathbf{S}^1, \mathbf{S}^2)\rangle_\lambda)^2\rangle_\lambda \neq \lim_{\lambda \searrow 0} \lim_{N \nearrow \infty} \mathbb{E}\langle(h_N(\mathbf{S}^1, \mathbf{S}^2) - \mathbb{E}\langle h_N(\mathbf{S}^1, \mathbf{S}^2)\rangle_\lambda)^2\rangle_\lambda. \tag{143}$$

Ref. [17] indicates that the variance of the order operator (129) vanishes by the disordered replica symmetry breaking perturbation

$$\sum_{i \in V_N} (\nu g_i + \lambda)S_i^{z,1} S_i^{z,2}, \tag{144}$$

with Gaussian random variables g_i and constants $(\lambda, \nu) \in \mathbb{R}$. Even for $\nu = 0$, however, Theorem 1.1 implies that the variance of the order operator (129) vanishes for almost all $\lambda \in \mathbb{R}$.

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