Zero-variance of perturbation Hamiltonian density in perturbed spin systems

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Abstract

We study effects of perturbation Hamiltonian to quantum spin systems which can include quenched disorder. Model-independent inequalities are derived, using an additional artificial disordered perturbation. These inequalities enable us to prove that the variance of the perturbation Hamiltonian density vanishes in the infinite volume limit even if the artificial perturbation is switched off. This theorem is applied to spontaneous symmetry breaking phenomena in a disordered classical spin model, a quantum spin model without disorder and a disordered quantum spin model.

1 Introduction

We study quantum spin systems on a finite set $V_N := [1, N] \cap \mathbb{Z}$. A spin operator S_j^p $(p = x, y, z)$ at a site $j \in V_N$ on a Hilbert space $\mathcal{H} := \bigotimes_{j \in V_N} \mathcal{H}_j$ is defined by a tensor product of the spin matrix acting on $\mathcal{H}_j \simeq \mathbb{C}^{2S+1}$ and unities, where S is an arbitrary fixed half integer. These operators are self-adjoint and satisfies the commutation relations

$$[S_j^x, S_k^y] = i\delta_{j,k} S_j^z, \qquad [S_j^y, S_k^z] = i\delta_{j,k} S_j^x, \qquad [S_j^z, S_k^x] = i\delta_{j,k} S_j^y,$$

and the spin at each site $i \in V_N$ has a fixed magnitude

$$\sum_{a=x,y,z} (S_j^a)^2 = S(S+1)\mathbf{1}.$$

Let us consider an unperturbed Hamiltonian $H_V(\mathbf{S})$, which can include also a sequence of i.i.d. random variables $\mathbf{J} = (J_X)_{X \subset V_N}$ as quenched disorder. One can assume a symmetry of the Hamiltonian $H_V(\mathbf{S})$, if one is interested in symmetry breaking phenomena. To detect a spontaneous symmetry breaking, long-range order of order operator $h_V(\mathbf{S})$ is utilized in the symmetric Gibbs state. Although the symmetric Gibbs state with long-range order is mathematically well defined, such state is unstable due to strong fluctuation and it cannot be realized. On the other hand, it is believed that a perturbed Gibbs state with infinitesimal symmetry breaking Hamiltonian is stable and realistic. Consider a perturbed Hamiltonian as a function of spin operators $\mathbf{S} = (S_j^p)_{j \in V_N, p=x,y,z}$

$$H := H_V(\mathbf{S}, \mathbf{J}) - N\lambda h_V(\mathbf{S}), \tag{1}$$

where $h_V(\mathbf{S})$ is a bounded operator and $\lambda \in \mathbb{R}$. To study spontaneous symmetry breaking, one can regard $h_V(\mathbf{S})$ as an order operator which breaks the symmetry. Assume an upper bound on the operator $h_V(\mathbf{S})$

$$\|h_V(\mathbf{S})\| \leq C_h, \tag{2}$$

where the operator norm is defined by $\|O\| := \sup_{\phi \in \mathcal{H}} |(\phi, O\phi)|$ for an arbitrary linear operator O on $\mathcal{H}$ and C_h is a constant independent of the system size N. For instance, h_V is a spin density

$$h_V(\mathbf{S}) = \frac{1}{N} \sum_{j \in V_N} S_j^z.$$

In the present paper, to evaluate correlations of order operators in their perturbing models, we consider an extra perturbation Hamiltonian with a quenched disorder. Let us consider the following perturbed Hamiltonian

$$H = H_V(\mathbf{S}, \mathbf{J}) - (N\lambda + N^\alpha \mu g) h_V(\mathbf{S}). \tag{3}$$

where g is a standard Gaussian random variable and $\mu \in \mathbb{R}$ is a coupling constant. We choose an exponent $\alpha > 0$ to take the infinite volume limit after evaluations of physical quantities depending on the unperturbed Hamiltonian H_V. The introduced random variable g is artificial and our final goal is to study the model at $\mu = 0$. The symbol $\mathbb{E}$ denotes the expectation over all random variables $\mathbf{J}, g$.

Define Gibbs state with the Hamiltonian (1). For $\beta > 0$, the partition function is defined by

$$Z_N(\beta, \lambda, \mu g) := \mathrm{Tr} e^{-\beta H} \tag{4}$$

where the trace is taken over the Hilbert space $\mathcal{H}$. Let f be an arbitrary function of spin operators. The expectation of f in the Gibbs state is given by

$$\langle f(\mathbf{S})\rangle_{\lambda,\mu g} = \frac{1}{Z_N(\beta, \lambda, \mu g)} \mathrm{Tr} f(\mathbf{S}) e^{-\beta H}. \tag{5}$$

Define the following function from the partition function

$$\psi_N(\beta, \lambda, \mu g) := \frac{1}{N} \log Z_N(\beta, \lambda, \mu g),$$

and its expectation

$$p_N(\beta, \lambda, \mu) := \mathbb{E}\psi_N(\beta, \lambda, \mu g).$$

The function $-\frac{N}{\beta}\psi_N$ is called free energy of the sample in statistical physics.

Here, we introduce a fictitious time $t \in [0, 1]$ and define a time evolution of operators with the Hamiltonian. Let O be an arbitrary self-adjoint operator, and we define an operator valued function $O(t)$ of $t \in [0, 1]$ by

$$O(t) := e^{-tH} O e^{tH}. \tag{6}$$

Furthermore, we define the Duhamel expectation of time dependent operators $O_1(t_1), \cdots, O_k(t_k)$ by

$$(O_1, O_2, \cdots, O_k)_{\lambda,\mu g} := \int_{[0,1]^k} dt_1 \cdots dt_k \langle \mathrm{T}[O_1(t_1)O_2(t_2)\cdots O_k(t_k)]\rangle_{\lambda,\mu g},$$

where the symbol T is a multilinear mapping of the chronological ordering. If we define a partition function with arbitrary self adjoint operators $O_1, \cdots, O_k$ and real numbers $x_1, \cdots, x_k$

$$Z(x_1, \cdots, x_k) := \mathrm{Tr} \exp \beta \left[-H + \sum_{i=1}^{k} x_i O_i \right],$$

the Duhamel function of k operators represents the k-th order derivative of the partition function [5, 8, 23]

$$\beta^k (O_1, \cdots, O_k)_{\lambda,\mu g} = \frac{1}{Z} \frac{\partial^k Z}{\partial x_1 \cdots \partial x_k}.$$

Furthermore, a truncated Duhamel function is defined by

$$\beta^k (O_1; \cdots; O_k)_{\lambda,\mu g} = \frac{\partial^k}{\partial x_1 \cdots \partial x_k} \log Z.$$

In the present paper, we prove the following main theorem for an arbitrary spin model with a Hamiltonian defined by (3) at $\mu = 0$.

Theorem 1.1 *Consider a quantum spin model defined by the Hamiltonian (3) at $\mu = 0$. The expectation of the perturbation operator*

$$\lim_{N \nearrow \infty} \mathbb{E}\langle h_V(\mathbf{S})\rangle_{\lambda,0},$$

exists in the infinite volume limit for almost all λ and its variance in the Gibbs state and the distribution of disorder vanishes

$$\lim_{N \nearrow \infty} \mathbb{E}\langle (h_V(\mathbf{S}) - \mathbb{E}\langle h_V(\mathbf{S})\rangle_{\lambda,0})^2\rangle_{\lambda,0} = 0, \tag{7}$$

in the infinite volume limit for almost all $\lambda \in \mathbb{R}$.

Theorem 1.1 implies also the existence of the following infinite volume limit for almost all $\lambda \in \mathbb{R}$

$$\lim_{N \nearrow \infty} \mathbb{E}\langle h_V(\mathbf{S})^2\rangle_{\lambda,0} = (\lim_{N \nearrow \infty} \mathbb{E}\langle h_V(\mathbf{S})\rangle_{\lambda,0})^2.$$

The perturbation operator h_V is self-averaging in the perturbed model. We apply Theorem 1.1 to spontaneous symmetry breaking phenomena in several examples.

2 Proof

First, we assume some properties of the perturbed model defined by the Hamiltonian (1).
Assumption 1 *The infinite volume limit of the function* p_N

$$p(\beta,\lambda,0)=\lim_{N\nearrow\infty}p_N(\beta,\lambda,0),$$

exists for each $(\beta,\lambda)\in(0,\infty)\times\mathbb{R}$.

Assumption 2 *The variance of* ψ_N *vanishes at* $\mu=0$ *in the infinite volume limit*

$$\lim_{N\nearrow\infty}\mathbb{E}[\psi_N(\beta,\lambda,0)-p_N(\beta,\lambda,0)]^2=0,$$

for each $(\beta,\lambda)\in(0,\infty)\times\mathbb{R}$

Assumption 3 *The following commutation relation of the perturbation operator* h_V *and the Hamiltonian vanishes in the infinite volume limit*

$$\lim_{N\nearrow\infty}\|[h_V(\mathbf{S}),[H,h_V(\mathbf{S})]]\|=0.$$

The following lemma can be shown in the standard convexity argument to obtain the Ghirlanda-Guerra identities [1, 4, 7, 16, 17, 21, 25] in classical and quantum disordered systems. The proof can be done on the basis of of convexity of functions ψ_N, p_N, p and their almost everywhere differentiability.

Lemma 2.1 *For almost all* $\lambda\in\mathbb{R}$*, the infinite volume limit*

$$\frac{\partial p}{\partial\lambda}(\beta,\lambda,0)=\lim_{N\nearrow\infty}\mathbb{E}\langle h_V(\mathbf{S})\rangle_{\lambda,0} \tag{8}$$

exists and the following variance vanishes

$$\lim_{N\nearrow\infty}[\mathbb{E}\langle h_V(\mathbf{S})\rangle_{\lambda,0}^2-(\mathbb{E}\langle h_V(\mathbf{S})\rangle_{\lambda,0})^2]=0. \tag{9}$$

Proof. Regard $p_N(\lambda)$ $p(\lambda)$ and $\psi_N(\lambda)$ as functions of λ for lighter notation. Define the following functions

$$\begin{aligned}
w_N(\epsilon) &:= \frac{1}{\epsilon}[|\psi_N(\lambda+\epsilon)-p_N(\lambda+\epsilon)|+|\psi_N(\lambda-\epsilon)-p_N(\lambda-\epsilon)|+|\psi_N(\lambda)-p_N(\lambda)|]\\
e_N(\epsilon) &:= \frac{1}{\epsilon}[|p_N(\lambda+\epsilon)-p(\lambda+\epsilon)|+|p_N(\lambda-\epsilon)-p(\lambda-\epsilon)|+|p_N(\lambda)-p(\lambda)|],
\end{aligned}$$

for $\epsilon>0$. Assumption 1 and Assumption 2 on ψ_N give

$$\lim_{N\nearrow\infty}\mathbb{E}w_N(\epsilon)=0,\quad \lim_{N\nearrow\infty}e_N(\epsilon)=0, \tag{10}$$

for any $\epsilon>0$. Since ψ_N, p_N and p are convex functions of λ, we have

$$\begin{aligned}
&\frac{\partial\psi_N}{\partial\lambda}(\lambda)-\frac{\partial p}{\partial\lambda}(\lambda)\le\frac{1}{\epsilon}[\psi_N(\lambda+\epsilon)-\psi_N(\lambda)]-\frac{\partial p}{\partial\lambda}\\
&\le\frac{1}{\epsilon}[\psi_N(\lambda+\epsilon)-p_N(\lambda+\epsilon)+p_N(\lambda+\epsilon)-p_N(\lambda)+p_N(\lambda)-\psi_N(\lambda)\\
&-p(\lambda+\epsilon)+p(\lambda+\epsilon)+p(\lambda)-p(\lambda)]-\frac{\partial p}{\partial\lambda}(\lambda)\\
&\le\frac{1}{\epsilon}[|\psi_N(\lambda+\epsilon)-p_N(\lambda+\epsilon)|+|p_N(\lambda)-\psi_N(\lambda)|+|p_N(\lambda+\epsilon)-p(\lambda+\epsilon)|\\
&+|p_N(\lambda)-p(\lambda)|]+\frac{1}{\epsilon}[p(\lambda+\epsilon)-p(\lambda)]-\frac{\partial p}{\partial\lambda}(\lambda)\\
&\le w_N(\epsilon)+e_N(\epsilon)+\frac{\partial p}{\partial\lambda}(\lambda+\epsilon)-\frac{\partial p}{\partial\lambda}(\lambda).
\end{aligned}$$

As in the same calculation, we have

$$\begin{aligned}
&\frac{\partial\psi_N}{\partial b\lambda}(\lambda)-\frac{\partial p}{\partial\lambda}(\lambda)\ge\frac{1}{\epsilon}[\psi_N(\lambda)-\psi_N(\lambda-\epsilon)]-\frac{\partial p}{\partial\lambda}(\lambda)\\
&\ge-w_N(\epsilon)-e_N(\epsilon)+\frac{\partial p}{\partial\lambda}(\lambda-\epsilon)-\frac{\partial p}{\partial\lambda}(\lambda).
\end{aligned}$$

Then,

$$\mathbb{E}\Big|\frac{\partial \psi_N}{\partial \lambda}(\lambda) - \frac{\partial p}{\partial \lambda}(\lambda)\Big| \leq \mathbb{E}w_N(\epsilon) + e_N(\epsilon) + \frac{\partial p}{\partial \lambda}(\lambda+\epsilon) - \frac{\partial p}{\partial \lambda}(\lambda-\epsilon).$$

Convergence of p_N in the infinite volume limit implies

$$\lim_{N\nearrow\infty} \mathbb{E}\Big|\beta\langle h_V(\mathbf{S})\rangle_{\lambda,0} - \frac{\partial p}{\partial \lambda}(\lambda)\Big| \leq \frac{\partial p}{\partial \lambda}(\lambda+\epsilon) - \frac{\partial p}{\partial \lambda}(b\lambda-\epsilon),$$

The right hand side vanishes, since the convex function$p(\lambda)$ is continuously differentiable almost everywhere and $\epsilon > 0$ is arbitrary. Therefore

$$\lim_{N\nearrow\infty} \mathbb{E}\Big|\beta\langle h_V(\mathbf{S})\rangle_{\lambda,0} - \frac{\partial p}{\partial \lambda}(\lambda)\Big| = 0. \tag{11}$$

for almost all λ. Jensen's inequality gives

$$\lim_{N\nearrow\infty} \Big|\mathbb{E}\beta\langle h_V(\mathbf{S})\rangle_{\lambda,0} - \frac{\partial p}{\partial \lambda}(\lambda)\Big| = 0. \tag{12}$$

This implies the first equality (8). These equalities imply also

$$\lim_{N\nearrow\infty} \mathbb{E}|\langle h_V(\mathbf{S})\rangle_{\lambda,0} - \mathbb{E}\langle h_V(\mathbf{S})\rangle_{\lambda,0}| = 0.$$

The bound on $h_V(\mathbf{S})$ concludes the following limit

$$\lim_{N\nearrow\infty} \mathbb{E}(\langle h_V(\mathbf{S})\rangle_{\lambda,0} - \mathbb{E}\langle h_V(\mathbf{S})\rangle_{\lambda,0})^2 \leq 2C_h \lim_{N\nearrow\infty} \mathbb{E}|\langle h_V(\mathbf{S})\rangle_{\lambda,0} - \mathbb{E}\langle h_V(\mathbf{S})\rangle_{\lambda,0}| = 0.$$

This completes the proof. □

Note that Lemma 2.1 guarantees the existence of the following infinite volume limit for almost all $\lambda \in \mathbb{R}$

$$\lim_{N\nearrow\infty} \mathbb{E}\langle h_V(\mathbf{S})\rangle_{\lambda,0}^2 = (\lim_{N\nearrow\infty} \mathbb{E}\langle h_V(\mathbf{S})\rangle_{\lambda,0})^2$$

Lemma 2.2 *Let f be a function of spin operators bounded by constant C_f independent of N*

$$\|f(\mathbf{S})\| \leq C_f.$$

For any $(\beta,\lambda,\mu) \in [0,\infty)\times\mathbb{R}^2$, any positive integer N and k, an upper bound on the following k-th order derivative is given by

$$\Big|\mathbb{E}\frac{\partial^k}{\partial\lambda^k}\langle f(\mathbf{S})\rangle_{\lambda,\mu g}\Big| \leq \sqrt{k!}C_f\mu^{-k}N^{k(1-\alpha)}. \tag{13}$$

Proof. Let g, g' be i.i.d. standard Gaussian random variables and define a function with a parameter $u \in [0,1]$

$$G(u) := \sqrt{u}g + \sqrt{1-u}g'.$$

Define a generating function χ_f of the parameter $u \in [0,1]$ for f by

$$\chi_f(u) := \mathbb{E}[\mathbb{E}'\langle f(\mathbf{S})\rangle_{\lambda,\mu G(u)}]^2, \tag{14}$$

where $\mathbb{E}'$ is expectation over only g' and $\mathbb{E}$ is expectation over all random variables. This generating function χ_f is a generalization of a function introduced by Chatterjee [3]. First we prove the following formula

$$\frac{d^k\chi_f}{du^k}(u) = N^{2(\alpha-1)k}\mu^{2k}\mathbb{E}\Big[\mathbb{E}'\frac{\partial^k}{\partial\lambda^k}\langle f(\mathbf{S})\rangle_{\lambda,\mu G(u)}\Big]^2. \tag{15}$$

The following inductivity for a positive integer k proves this formula.
For $k=1$, the first derivative of χ_f is

$$\begin{aligned}
\chi_f'(u) &= N^\alpha\beta\mu\mathbb{E}\mathbb{E}'\langle f(\mathbf{S})\rangle_{\lambda,\mu G(u)}\mathbb{E}'\Big(\frac{g}{\sqrt{u}} - \frac{g'}{\sqrt{1-u}}\Big)(f(\mathbf{S}); h_V(\mathbf{S}))_{\lambda,\mu G(u)} \\
&= N^\alpha\beta\mu\mathbb{E}\Big[\frac{1}{\sqrt{u}}\frac{\partial}{\partial g}\mathbb{E}'\langle f(\mathbf{S})\rangle_{\lambda,\mu G(u)}\mathbb{E}'(f(\mathbf{S}); h_V(\mathbf{S}))_{\lambda,\mu G(u)} \\
&\quad -\mathbb{E}'\langle f(\mathbf{S})\rangle_{\lambda,\mu G(u)}\mathbb{E}'\frac{1}{\sqrt{1-u}}\frac{\partial}{\partial g'}(f(\mathbf{S}); h_V(\mathbf{S}))_{\lambda,\mu G(u)}\Big] \\
&= N^{2\alpha}\beta^2\mu^2\mathbb{E}[\mathbb{E}'(f(\mathbf{S}); h_V(\mathbf{S}))_{\lambda,\mu G(u)}]^2 = N^{2(\alpha-1)}\mu^2\mathbb{E}\Big[\mathbb{E}'\frac{\partial}{\partial\lambda}\langle f(\mathbf{S})\rangle_{\lambda,\mu G(u)}\Big]^2,
\end{aligned}$$

where integration by parts over g and g' has been used. If the validity of the formula (15) is assumed for an arbitrary positive integer k, then (15) for $k+1$ can be proved using integration by parts. The formula (15) shows that k-th derivative of $\chi_f(u)$ is positive semi-definite for any k, then it is a monotonically increasing function of u. From Taylor's theorem, there exists $v \in (0,u)$ such that

$$\chi_f(u) = \sum_{n=0}^{k-1} \frac{u^n}{n!} \chi_f^{(n)}(0) + \frac{u^k}{k!} \chi_f^{(k)}(v).$$

Each term in this series is bounded from the above by

$$\chi_f(1) = \mathbb{E}\langle f(\mathbf{S})\rangle_{\lambda,\mu g}^2 \leq \|f\|^2 \leq C_f^2.$$

Jensen's inequality gives

$$\begin{aligned} N^{2(\alpha-1)k}\mu^{2k}\Big[\mathbb{E}\frac{\partial^k}{\partial\lambda^k}\langle f(\mathbf{S})\rangle_{\lambda,\mu G(u)}\Big]^2 &\leq N^{2(\alpha-1)k}\mu^{2k}\mathbb{E}\Big[\mathbb{E}'\frac{\partial^k}{\partial\lambda^k}\langle f(\mathbf{S})\rangle_{\lambda,\mu G(u)}\Big]^2 \\ &\leq \frac{d^k\chi_f}{du^k}(0) \leq k!\chi_f(1) \leq k!C_f^2. \end{aligned}$$

This completes the proof. □

Lemma 2.3 *The function $p(\beta,\lambda,\mu)$ is continuous at $\mu=0$ for arbitrary $(\beta,\lambda) \in (0,\infty)\times\mathbb{R}$ and for $\alpha \leq 1$, namely*

$$\lim_{\mu\to 0} p(\beta,\lambda,\mu) = p(\beta,\lambda,0).$$

Proof. Integration of the derivative function of p_N over the interval $(0,\mu)$ gives

$$\begin{aligned} &p_N(\beta,\lambda,\mu) - p_N(\beta,\lambda,0) = \int_0^\mu dx \frac{\partial}{\partial x} p_N(\beta,\lambda,x) = \int_0^\mu dx \mathbb{E} N^{\alpha-1}\beta g\langle h_V(\mathbf{S})\rangle_{\lambda,xg} \\ &= \int_0^\mu dx \mathbb{E} N^{2\alpha-1}\beta^2 x (h_V(\mathbf{S}); h_V(\mathbf{S}))_{\lambda,xg} = \int_0^\mu dx x N^{2\alpha-2}\beta \frac{\partial}{\partial\lambda}\mathbb{E}\langle h_V(\mathbf{S})\rangle_{\lambda,xg} \\ &\leq \int_0^\mu dx N^{\alpha-1}\beta C_h = N^{\alpha-1}\beta\mu C_h. \end{aligned}$$

We have used Lemma 2.2. The infinite volume limit can be taken for $\alpha \leq 1$, and we have the continuity of p at $\mu=0$. □

Lemma 2.3 and Assumption 1 guarantee the existence of $p(\beta,\lambda,\mu)$ for each $(\beta,\lambda,\mu) \in (0,\infty)\times\mathbb{R}$.

Lemma 2.4 *The variance of $\psi_N(\beta,\lambda,\mathbf{J},\mu g)$ vanishes in the infinite volume limit for $\alpha < 1$ for each $(\beta,\lambda,\mu) \in (0,\infty)\times\mathbb{R}^2$.*

Proof. Here, we regard the function $\psi_N(\mathbf{J},\mu g)$ as the i.i.d. standard Gaussian random variables $\mathbf{J} = (J_X)_{X\subset V_N}$ and g. Define interpolating function

$$\mathcal{J}(u) := \sqrt{u}\mathbf{J} + \sqrt{1-u}\mathbf{J}', \quad G(u) := \sqrt{u}g + \sqrt{1-u}g'$$

and a generating function

$$\gamma(u,\mu) := \mathbb{E}[\mathbb{E}'\psi_N(\mathcal{J}(u),\mu G(u))]^2,$$

where $\mathbf{J}' = (J'_X)_{X\subset V_N}$ and g' are also i.i.d. standard Gaussian random variables and $\mathbb{E}'$ stands for the expectation over only $\mathbf{J}'$, and g'. Its derivative in u is evaluated as

$$\begin{aligned} \frac{\partial}{\partial u}\gamma(u,\mu) &= \frac{\beta}{N}\mathbb{E}\mathbb{E}'\psi_N\Big[\sum_{X\subset V_N}\mathbb{E}'\mathcal{J}'_X(u)\frac{\partial\psi_N}{\partial\mathcal{J}_X} + N^\alpha\mu\mathbb{E}'\Big(\frac{1}{\sqrt{u}}g - \frac{1}{\sqrt{1-u}}g'\Big)\frac{\partial\psi_N}{\partial G}\Big] \\ &= \gamma_u(u,0) + N^{2\alpha-2}\beta^2\mu^2\mathbb{E}(\mathbb{E}'\langle h_V\rangle_u)^2 \leq \gamma_u(u,0) + N^{2(\alpha-1)}\beta^2 C_h^2\mu^2. \end{aligned}$$

The variance of ψ_N is given by

$$\begin{aligned} &\mathbb{E}\psi_N(\beta,\lambda,\mu g)^2 - p_N(\beta,\lambda,\mu)^2 = \int_0^1 du\gamma_u(u,\mu) \leq \gamma(1,0) - \gamma(0,0) + N^{2(\alpha-1)}C_h^2\beta^2\mu^2 \\ &\leq \mathbb{E}\psi_N(\beta,\lambda,0)^2 - p_N(\beta,\lambda,0)^2 + N^{2(\alpha-1)}C_h^2\beta^2\mu^2. \end{aligned}$$

From Assumption 2, the variance of $\psi_N(\beta, \lambda, \mu)$ vanishes in the infinite volume limit for $\alpha < 1$ and for arbitrary $(\beta, \lambda, \mu) \in (0, \infty) \times \mathbb{R}^2$. $\square$

Lemma 2.4 shows that the perturbed model for $\mu \neq 0$ and $\alpha < 1$ has a healthy behavior as well as the original model at $\mu = 0$. Next, we prove a continuity of an expectation value of an arbitrary bounded operator at $\mu = 0$ in a method similar to that used in Ref [18].

Lemma 2.5 *Let f be a bounded function of spin operators, such that*

$$\lim_{N \nearrow \infty} \mathbb{E}\langle f(\mathbf{S})\rangle_{\lambda, \mu g}$$

exists for sufficiently small $|\mu|$. Then the following function is continuous at $\mu = 0$ in the infinite volume limit for $\alpha \leq 1$ for almost all $\lambda \in \mathbb{R}$

$$\lim_{\mu \to 0} \lim_{N \nearrow \infty} \mathbb{E}\langle f(\mathbf{S})\rangle_{\lambda, \mu g} = \lim_{N \nearrow \infty} \mathbb{E}\langle f(\mathbf{S})\rangle_{\lambda, 0}. \tag{16}$$

Proof. Integration of the derivative function over the interval $(0, \mu)$ for an arbitrary $\mu \in \mathbb{R}$ gives

$$\begin{aligned}
\mathbb{E}\langle f(\mathbf{S})\rangle_{\lambda, \mu g} - \mathbb{E}\langle f(\mathbf{S})\rangle_{\lambda, 0} &= \int_0^\mu dx \frac{\partial}{\partial x} \mathbb{E}\langle f(\mathbf{S})\rangle_{\lambda, xg} = \int_0^\mu dx \mathbb{E} N^\alpha \beta g (f(\mathbf{S}); h_V(\mathbf{S}))_{\lambda, xg} \\
= \int_0^\mu dx \mathbb{E} N^{2\alpha} \beta^2 x (f(\mathbf{S}); h_V(\mathbf{S}); h_V(\mathbf{S}))_{\lambda, xg} &= \int_0^\mu dx x N^{2(\alpha-1)} \frac{\partial^2}{\partial \lambda^2} \mathbb{E}\langle f(\mathbf{S})\rangle_{\lambda, xg}.
\end{aligned}$$

Integration over an arbitrary interval of λ and Lemma 2.2 imply

$$\begin{aligned}
&\Big| \int_a^b d\lambda [\mathbb{E}\langle f(\mathbf{S})\rangle_{\lambda, \mu g} - \mathbb{E}\langle f(\mathbf{S})\rangle_{\lambda, 0}] \Big| = \Big| \int_0^\mu dx x N^{2(\alpha-1)} \Big[\frac{\partial}{\partial b} \mathbb{E}\langle f(\mathbf{S})\rangle_{b, xg} - \frac{\partial}{\partial a} \mathbb{E}\langle f(\mathbf{S})\rangle_{a, xg} \Big] \Big| \\
&\leq \Big| \int_0^\mu dx x N^{2(\alpha-1)} \Big[\Big| \frac{\partial}{\partial b} \mathbb{E}\langle f(\mathbf{S})\rangle_{b, xg} \Big| + \Big| \frac{\partial}{\partial a} \mathbb{E}\langle f(\mathbf{S})\rangle_{a, xg} \Big| \Big] \Big| \leq 2N^{\alpha-1} \Big| \int_0^\mu dx C_f \Big| = 2N^{\alpha-1} C_f |\mu|.
\end{aligned}$$

The right hand side converges in the infinite volume limit for $\alpha \leq 1$. Since the integration interval (a, b) is arbitrary, the integrand in the left hand side vanishes for almost all λ in the limit $\mu \to 0$ after the infinite volume limit. This completes the proof. $\square$

Lemma 2.6 *The following function is continuous at $\mu = 0$ in the infinite volume limit for $\alpha \leq 1$ for almost all $\lambda \in \mathbb{R}$*

$$\lim_{\mu \to 0} \lim_{N \nearrow \infty} \mathbb{E}\langle h_V(\mathbf{S})\rangle_{\lambda, \mu g}^2 = \lim_{N \nearrow \infty} \mathbb{E}\langle h_V(\mathbf{S})\rangle_{\lambda, 0}^2. \tag{17}$$

Proof. Integration of the derivative function over the interval $(0, \mu)$ for an arbitrary $\mu \in \mathbb{R}$ gives

$$\begin{aligned}
\mathbb{E}\langle h_V(\mathbf{S})\rangle_{\lambda, \mu g}^2 - \mathbb{E}\langle h_V(\mathbf{S})\rangle_{\lambda, 0, 0}^2 &= \int_0^\mu dx \frac{\partial}{\partial x} \mathbb{E}\langle h_V(\mathbf{S})\rangle_{\lambda, xg}^2 \\
&= 2 \int_0^\mu dx \mathbb{E} N^\alpha \beta g (h_V(\mathbf{S}); h_V(\mathbf{S}))_{\lambda, xg} \langle h_V(\mathbf{S})\rangle_{\lambda, xg} \\
&= 2 \int_0^\mu dx \mathbb{E} N^\alpha \beta \frac{\partial}{\partial g} (h_V(\mathbf{S}); h_V(\mathbf{S}))_{\lambda, xg} \langle h_V(\mathbf{S})\rangle_{\lambda, xg} \\
&= 2\beta \int_0^\mu dx x N^{2\alpha-1} \frac{\partial}{\partial \lambda} \mathbb{E} (h_V(\mathbf{S}); h_V(\mathbf{S}))_{\lambda, xg} \langle h_V(\mathbf{S})\rangle_{\lambda, xg}
\end{aligned}$$

Integration over an arbitrary interval of λ and Lemma 2.2 imply

$$\begin{aligned}
&\Big| \int_a^b d\lambda [\mathbb{E}\langle h_V(\mathbf{S})\rangle_{\lambda, \mu g}^2 - \mathbb{E}\langle h_V(\mathbf{S})\rangle_{\lambda, 0, 0}^2] \Big| \\
&= \Big| 2\beta \int_0^\mu dx x N^{2\alpha-1} \Big[\mathbb{E}(h_V(\mathbf{S}); h_V(\mathbf{S}))_{b, xg} \langle h_V(\mathbf{S})\rangle_{b, xg} - \mathbb{E}(h_V(\mathbf{S}); h_V(\mathbf{S}))_{a, xg} \langle f(\mathbf{S})\rangle_{a, xg} \Big] \Big| \\
&\leq 2\beta \Big| \int_0^\mu dx x N^{2\alpha-1} \Big[\mathbb{E}|(h_V(\mathbf{S}); h_V(\mathbf{S}))_{b, xg}| |\langle h_V(\mathbf{S})\rangle_{b, xg}| + \mathbb{E}|(h_V(\mathbf{S}); h_V(\mathbf{S}))_{a, xg}| |\langle h_V(\mathbf{S})\rangle_{a, xg}| \Big] \Big| \\
&\leq 2\beta C_h \Big| \int_0^\mu dx x N^{2\alpha-1} \Big[\mathbb{E}(h_V(\mathbf{S}); h_V(\mathbf{S}))_{b, xg} + \mathbb{E}(h_V(\mathbf{S}); h_V(\mathbf{S}))_{a, xg} \Big] \Big| \\
&= 2C_h \Big| \int_0^\mu dx x N^{2(\alpha-1)} \Big[\frac{\partial}{\partial b} \mathbb{E}\langle h_V(\mathbf{S})\rangle_{b, xg} + \frac{\partial}{\partial a} \mathbb{E}\langle h_V(\mathbf{S})\rangle_{a, xg} \Big] \Big| \\
&\leq 4N^{\alpha-1} \Big| \int_0^\mu dx C_h^2 \Big| = 4N^{\alpha-1} C_h^2 |\mu|.
\end{aligned}$$

The right hand side converges in the infinite volume limit for $\alpha \leq 1$. Since the integration interval (a, b) is arbitrary, the integrand in the left hand side vanishes for almost all λ in the limit $\mu \to 0$ after the infinite volume limit. This completes the proof. $\square$

Proof of Theorem 1.1
Lemma 2.2 yields

$$\mathbb{E}(h_V(\mathbf{S}); h_V(\mathbf{S}))_{\lambda,\mu g} \leq \frac{C_h}{\beta|\mu|} N^{-\alpha}, \tag{18}$$

for arbitrary $\lambda, \mu \in \mathbb{R}$. Harris' inequality of the Bogolyubov type between the Duhamel function and the Gibbs expectation of the square of arbitrary self-adjoint operator O [13]

$$(O, O)_{\lambda,\mu g} \leq \langle O^2 \rangle_{\lambda,\mu g} \leq (O, O)_{\lambda,\mu g} + \frac{\beta}{12} \langle [O, [H, O]] \rangle_{\lambda,\mu g}, \tag{19}$$

and Assumption 3 enable us to obtain

$$\lim_{N \nearrow \infty} \mathbb{E} \langle h_V(\mathbf{S})^2 \rangle_{\lambda,\mu g} = \lim_{N \nearrow \infty} \mathbb{E}(h_V(\mathbf{S}), h_V(\mathbf{S}))_{\lambda,\mu g}.$$

This for $u = 0$ and the bound (18) imply

$$\lim_{N \nearrow \infty} \mathbb{E}[\langle h_V(\mathbf{S})^2 \rangle_{\lambda,\mu g} - \langle h_V(\mathbf{S}) \rangle_{\lambda,\mu g}^2] = \lim_{N \nearrow \infty} \mathbb{E}(h_V(\mathbf{S}); h_V(\mathbf{S}))_{\lambda,\mu g} = 0,$$

for $\mu \neq 0$. This is true also for $\mu = 0$ by Lemma 2.5 and Lemma 2.6

$$\lim_{N \nearrow \infty} \mathbb{E}[\langle h_V(\mathbf{S})^2 \rangle_{\lambda,0} - \langle h_V(\mathbf{S}) \rangle_{\lambda,0}^2] = 0.$$

Therefore, this and Lemma 2.1 give

$$\lim_{N \nearrow \infty} [\mathbb{E} \langle h_V(\mathbf{S})^2 \rangle_{\lambda,0} - (\mathbb{E} \langle h_V(\mathbf{S}) \rangle_{\lambda,0})^2] = 0. \tag{20}$$

This completes the proof of Theorem 1.1. $\square$

3 Applications to several models

3.1 Random energy model

Random energy model is a well known simple model where replica symmetry breaking appears. This model contains only $(S_i^z)_{i \in V_N}$ with spin $S = \frac{1}{2}$. The possible state is represented in a spin configuration $\sigma = (\sigma_i)_{i \in V_N} \in \Sigma_N := \{1, -1\}^{V_N}$, which is a sequence of eigenvalues of the operators $(2S_i^z)_{i \in V_N}$. The unperturbed Hamiltonian on V_N is defined by

$$H_V(\sigma) := -\sqrt{N} J_\sigma$$

where $J = (J_\sigma)_{\sigma \in \Sigma_N}$ are i.i.d. standard Gaussian random variables. The Hamiltonian defines a partition function

$$Z_N(\beta, J) := \sum_{\sigma \in \Sigma_N} \exp(\beta H(\sigma)). \tag{21}$$

Consider a n-replicated random energy model whose state is given by n spin configurations $(\sigma^1, \cdots, \sigma^n) \in \Sigma_N^n$. The Hamiltonian of this model is given by

$$H_V(\sigma^1, \cdots, \sigma^n) := \sum_{a=1}^n H_V(\sigma^a).$$

Here we attach index V to the Hamiltonian on V_N for later convenience. This Hamiltonian is invariant under a permutation s

$$H_V(\sigma^{s(1)}, \cdots, \sigma^{s(n)}) = H_V(\sigma^1, \cdots, \sigma^n),$$

where $s : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, n\}$ is an arbitrary bijection. This symmetry is replica symmetry. To study the spontaneous replica symmetry breaking, consider the following symmetry breaking perturbation

$$h_V(\sigma^1, \sigma^2) := \prod_{i \in V_N} \delta_{\sigma_i^1, \sigma_i^2}. \tag{22}$$

Note the upper bound for this operator

$$\|h_V(\sigma^1, \sigma^2)\| \le 1.$$

Define the function

$$P_N(\beta, \lambda) := \mathbb{E} \log \sum_{\sigma^1, \cdots, \sigma^n} \exp[-\beta H],$$

where the Hamiltonian is given by

$$H := H_V(\sigma^1, \cdots, \sigma^n) - N\lambda h_V(\sigma^1, \sigma^2). \tag{23}$$

The following lemma shows that the model satisfies Assumption 1. It is proved by the square root interpolation of two models [11, 12, 25].

Lemma 3.1 *In the n replicated random energy model perturbed by the Hamiltonian (22), the sequence $P_N(\beta, \lambda)$ is sub-additive for each $(\beta, \lambda) \in (0, \infty) \times \mathbb{R}$, namely for arbitrary positive integers L, M*

$$P_{L+M}(\beta, \lambda) \le P_L(\beta, \lambda) + P_M(\beta, \lambda).$$

Proof. For positive integers L, M, define $N := L + M$ and two sets $V := [1, L] \cap \mathbb{Z}$ and $W := [L + 1, L + M] \cap \mathbb{Z}$. Note $V_N = U \cup W$ and $U \cap W = \phi$. Decompose the spin configuration $\sigma = (\tau, \upsilon) \in \Sigma_N$ into two parts $\tau \in \Sigma_L$ and $\upsilon \in \Sigma_M$, and define square root interpolation of the Hamiltonians

$$\begin{aligned} H(t) &:= \sqrt{t} H_V(\sigma^1, \cdots, \sigma^n) + \sqrt{1-t} H_U(\tau^1, \cdots, \tau^n) + \sqrt{1-t} H_W(\upsilon^1, \cdots, \upsilon^n) \\ &- \; tN\lambda h_V(\sigma^1, \sigma^2) - (1-t)L\lambda h_U(\tau^1, \tau^2) - (1-t)M\lambda h_W(\upsilon^1, \upsilon^2) \end{aligned}$$

Define the following function

$$\Psi(t) := \mathbb{E} \log \sum_{\sigma^1, \cdots, \sigma^n} \exp -\beta H(t)$$

and calculate the derivative function using integration by parts with respect to all random variables and $h_V = h_U h_W$

$$\begin{aligned} \Psi'(t) &= L\beta\lambda \mathbb{E}\langle h_U(\tau^1, \tau^2)(h_W(\upsilon^1, \upsilon^2) - 1)\rangle_t \\ &+ M\beta\lambda \mathbb{E}\langle h_W(\upsilon^1, \upsilon^2)(h_U(\tau^1, \tau^2) - 1)\rangle_t \le 0, \end{aligned}$$

where $\langle \cdot \rangle_t$ denotes the Gibbs expectation defined by the Hamiltonian $H(t)$. This implies

$$P_{L+M}(\beta, \lambda) = \Psi(1) \le \Psi(0) = P_L(\beta, \lambda) + P_M(\beta, \lambda),$$

and thus the sequence P_N is sub-additive. □

Lemma 3.1 and Fekete's sub-additive lemma guarantees the infinite volume limit

$$p(\beta, \lambda, 0) = \lim_{N \nearrow \infty} \frac{P_N(\beta, \lambda)}{N}.$$

Assumption 2 is proved in the following lemma.

Lemma 3.2 *The variance of $\psi_N(\beta, \lambda, \mathbf{J})$ vanishes in the model defined by (23) for any positive integers N and for any $(\beta, \lambda) \in (0, \infty) \times \mathbb{R}$*

Proof. The derivative of the function $\gamma(u, 0)$ defined in Lemma 2.4 can be evaluated as

$$\frac{\partial}{\partial u} \gamma(u, 0) = \frac{\beta^2}{N},$$

for any $u \in [0, 1]$. Then the variance of $\psi_N(\beta, \lambda)$ is

$$\mathbb{E}\psi_N(\beta, \lambda) - p_N(\beta, \lambda) = \gamma(1, 0) - \gamma(0, 0) = \int_0^1 du \frac{\partial}{\partial u} \gamma(u, 0) = \frac{\beta^2}{N},$$

for any positive integers N and for any $(\beta, \lambda) \in (0, \infty) \times \mathbb{R}$. □

The following corollary for the perturbed random energy model is obtained from Theorem 1.1.

Corollary 3.3 *In the n replicated random energy model perturbed by the Hamiltonian (22) in the infinite volume limit, for almost all $\lambda \in \mathbb{R}$ and for $\mu = 0$ the expectation of the perturbing operator takes the value*

$$\lim_{N \nearrow \infty} \mathbb{E}\langle h_V(\sigma^1, \sigma^2)\rangle_{\lambda,0} = 0 \text{ or } 1.$$

Proof. Note the relation

$$h_V(\sigma^1, \sigma^2)^2 = h_V(\sigma^1, \sigma^2).$$

Then, Theorem 1.1 implies

$$\lim_{N \nearrow \infty} \mathbb{E}\langle h_V(\sigma^1, \sigma^2)\rangle_{\lambda,0}(1 - \mathbb{E}\langle h_V(\sigma^1, \sigma^2)\rangle_{\lambda,0}) = 0.$$

Therefore, $\lim_{n \to \infty} \mathbb{E}\langle h_V(\sigma^1, \sigma^2)\rangle_{\lambda,0}$ takes the value either 0 or1. □

Note that this corollary is also true for an arbitrary projection operator satisfying $h_V^2 = h_V$ in other models.

It is well known that the observation of $\lim_{\lambda \searrow 0} \lim_{N \nearrow \infty} \mathbb{E}\langle h_V(\sigma^1, \sigma^2)\rangle_{\lambda,0} = 1$ implies the spontaneous replica symmetry breaking. The replica symmetry breaking is also detect by the replica symmetric Gibbs state. In the replica symmetric calculation, if the replica symmetric calculation shows

$$0 < \lim_{N \nearrow \infty} \mathbb{E}\langle h_V(\sigma^1, \sigma^2)\rangle_{0,0} < 1,$$

then this implies the finite variance

$$\lim_{N \nearrow \infty} [\mathbb{E}\langle h_V(\sigma^1, \sigma^2)^2\rangle_{0,0} - (\mathbb{E}\langle h_V(\sigma^1, \sigma^2)\rangle_{0,0})^2] > 0$$

which gives an instability of the replica symmetric Gibbs state due to the large fluctuation. At the same time, this implies the non-commutativity of limiting procedure

$$\lim_{N \nearrow \infty} \lim_{\lambda \searrow 0} \mathbb{E}\langle h_V(\sigma^1, \sigma^2)\rangle_{\lambda,0,} \neq \lim_{\lambda \searrow 0} \lim_{N \nearrow \infty} \mathbb{E}\langle h_V(\sigma^1, \sigma^2)\rangle_{\lambda,0}.$$

This is typical phenomenon in spontaneous symmetry breaking. Guerra has studied the replica symmetry breaking in the random energy model as a spontaneous symmetry breaking phenomenon [10]. He has shown that the function $p(\beta, \lambda)$ for $\beta < \beta_c = \sqrt{2 \log 2}$ for $n = 2$ becomes

$$p(\beta, \lambda) = \begin{cases} 2\beta\sqrt{\log 2} + \lambda & (\lambda > 0) \\ 2\beta\sqrt{\log 2} & (\lambda \leq 0), \end{cases}$$

and the order parameter is evaluated as

$$\lim_{N \nearrow \infty} \mathbb{E}\langle h_V(\sigma^1, \sigma^2)^2\rangle_{0,0} = 1 - \frac{\beta_c}{\beta} < 1.$$

The non-differentiability of p at $\lambda = 0$ is observed as pointed out by Mukaida [15]. Corollary 3.3 agrees with these results.

3.2 Quantum Heisenberg model without disorder

Here we study spontaneous symmetry breaking of SU(2) invariance in the antiferromagnetic quantum Heisenberg model without disorder. Let V_N be a hyper cubic lattice $V_N := L^d \cap \mathbb{Z}^d$ and bipartite, namely there exist two subsets A and B of V_N such that $V_N = A \cup B$ and $A \cap B = \phi$. The model Hamiltonian is defined by

$$H_V(\mathbf{S}) := \sum_{i \in A, j \in B} \sum_{p=x,y,z} J_{i,j} S_i^p S_j^p, \tag{24}$$

where $J_{i,j} \geq 0$ is short-ranged and translationally invariant, i.e. there exists $c \geq 1$ such that $J_{i,j} = 0$ for any $|i - j| > c$, and $J_{i+v,j+v} = J_{i,j}$ for any $i, j, v \in V_N$. Consider an antiferromagnetic order operator as a perturbation operator

$$h_V(\mathbf{S}) := \frac{1}{N}\Big(\sum_{i \in A} S_i^z - \sum_{j \in B} S_j^z\Big). \tag{25}$$

This operator is bounded by $\|h_V(\mathbf{S})\| \leq S$. Define a perturbed Hamiltonian by

$$H := H_V(\mathbf{S}) - N\lambda h_V(\mathbf{S}).$$

In this model, Assumption 1 is proved in a standard method to show the sequence $p_N(\beta, \lambda)$ for positive integers N becomes Cauchy for any $(\beta, \lambda) \in (0, \infty) \times \mathbb{R}$. Assumption 2 is trivial for the model without disorder and Assumption 3 is obvious for short-range interactions.

Theorem 1.1 $\lim_{N \nearrow \infty} [\langle h_V(\mathbf{S})^2 \rangle_{\lambda,0} - \langle h_V(\mathbf{S}) \rangle_{\lambda,0}^2] = 0$ and SU(2) invariance $\langle h_V(\mathbf{S}) \rangle_{0,0} = 0$ yield the following corollary.

Corollary 3.4 *If the SU(2) invariant Gibbs state of the antiferromagnetic quantum Heisenberg model has a long-range order*

$$\lim_{N \nearrow \infty} \langle h_V(\mathbf{S})^2 \rangle_{0,0} \neq 0,$$

then we have

$$\lim_{\lambda \searrow 0} \lim_{N \nearrow \infty} [\langle h_V(\mathbf{S})^2 \rangle_{\lambda,0} - \langle h_V(\mathbf{S}) \rangle_{\lambda,0}^2] \neq \lim_{N \nearrow \infty} \lim_{\lambda \searrow 0} [\langle h_V(\mathbf{S})^2 \rangle_{\lambda,0} - \langle h_V(\mathbf{S}) \rangle_{\lambda,0}^2].$$

for almost all $\lambda \in \mathbb{R}$.

The non-commutativity of limiting procedures in Corollary 3.4 claims the spontaneous SU(2) symmetry breaking, when a long-range order exists. Koma and Tasaki have shown that the long-range order (equivalent to a finite variance of order operator) in the symmetric Gibbs state implies the spontaneous symmetry breaking in the quantum Heisenberg model with short-range antiferromagnetic interactions [19]. They have proved

$$\sqrt{\lim_{N \nearrow \infty} \langle h_V(\mathbf{S})^2 \rangle_{0,0}} \leq \lim_{\lambda \searrow 0} \lim_{N \nearrow \infty} \langle h_V(\mathbf{S}) \rangle_{\lambda,0}.$$

For ferromagnetic case, the corresponding inequality was proved by Griffiths [9]. Even though the symmetric Gibbs state with the long-range order is unstable and unrealistic, it is mathematically well defined and can detect the symmetry breaking in the evaluation result of the finite variance of order operator. Recently, Tasaki has shown that the variance of order operator vanishes in the symmetry breaking ground state in the infinite volume limit [26]. Theorem 1.1 for quantum spin models is consistent with his result.

3.3 Quantum Edwards-Anderson model

It is quite interesting whether or not, a replica symmetry breaking occurs in short-range disordered spin systems as in the Sherrington-Kirkpatrick model described by the Parisi formula [22, 24, 25]. Here, we discuss a replica symmetry breaking as a spontaneous symmetry breaking phenomenon in the quantum Edwards-Anderson model [6].

Let $(S_j^p)_{j \in V_N, p=x,y,z}$ be spin operators on a d-dimensional hyper cubic lattice $V_N := [1, L]^d \cap \mathbb{Z}^d$, where $N = |V_N| = L^d$. Let A be a bounded subset of V_N, such that $|A| \leq C$, where C is a positive constant independent of N. Define a collection of interaction ranges by

$$C_N := \{X | X = A + v \subset V_N, v \in V_N\}. \tag{26}$$

Define

$$S_X^p := \prod_{j \in X} S_j^p$$

for $X \in C_N$ and for $p = x, y, z$. The Hamiltonian has disordered short-range interaction

$$H_V(\mathbf{S}, \mathbf{J}) := -\sum_{X \in C_N} \sum_{p=x,y,z} J_X K^p S_X^p, \tag{27}$$

where $(J_X)_{X \in C_N}$ are i.i.d standard Gaussian random variables and positive constants $(K^p)_{p=x,y,z}$. The interaction is short-ranged and translationally invariant, where $|C_N| \leq |A|N$. Consider a n-replicated model with spin operators $(S_j^{p,1}, \cdots, S_j^{p,n})_{j \in V_N, p=x,y,z}$ and define a replica symmetric Hamiltonian

$$H_V(\mathbf{S}^1, \cdots, \mathbf{S}^n, \mathbf{J}) := \sum_{a=1}^n H_V(\mathbf{S}^a, \mathbf{J}). \tag{28}$$

Define a spin overlap as a perturbing operator

$$h_V(\mathbf{S}^1, \mathbf{S}^2) := \frac{1}{N} \sum_{i \in V_N} S_i^{z,1} S_i^{z,2}, \tag{29}$$

which breaks the replica symmetry. Note the following bound

$$\|h_V(\mathbf{S}^1, \mathbf{S}^2)\| \leq S^2.$$

Consider the model defined by

$$H := H_V(\mathbf{S}^1, \cdots, \mathbf{S}^n, \mathbf{J}) - N\lambda h_V(\mathbf{S}^1, \mathbf{S}^2). \tag{30}$$

In this model, Assumption 1 is proved in a standard method to show the sequence $p_N(\beta, \lambda, 0)$ for positive integers N becomes Cauchy for any $(\beta, \lambda, 0) \in (0, \infty) \times \mathbb{R}$ as in the previous model. Assumption 3 is obvious for short-range interactions. Assumption 2 is proved in the following lemma.

Lemma 3.5 *The variance of $\psi_N(\beta, \lambda, \mathbf{J})$ defined by the Hamiltonian (30) vanishes in the infinite volume limit for each $(\beta, \lambda) \in (0, \infty) \times \mathbb{R}$.*
Proof. To prove this, we employ the generating function $\gamma(u, 0)$ defined in Lemma 2.4. Namely, regarding the function $\psi_N(\mathbf{J})$ as the i.i.d. standard Gaussian random variables $\mathbf{J} = (J_X)_{X \in C_N}$, and

$$\gamma(u, 0) := \mathbb{E}[\mathbb{E}' \psi_N(\sqrt{u}\mathbf{J} + \sqrt{1-u}\mathbf{J}')]^2,$$

where $\mathbf{J}' = (J_X{}')_{X \in C_N}$ are also i.i.d. standard Gaussian random variables and $\mathbb{E}'$ stands for the expectation over only $\mathbf{J}'$. Its derivative in u is evaluated as

$$\gamma_u(u, 0) = \frac{\beta^2}{N^2} \mathbb{E} \sum_{X \in C_N} \Big(\mathbb{E}' \sum_{p=x,y,z} \sum_{a=1}^{n} K^p \langle S_X^{p,a} \rangle_u \Big)^2 \tag{31}$$

$$\leq \frac{\beta^2 |A| n^2 S^{2|A|}}{N} (\sum_{p=x,y,z} K^p)^2. \tag{32}$$

The variance of ψ_N is given by

$$\mathbb{E}\psi_N(\beta, \lambda)^2 - p_N(\beta, \lambda)^2 = \int_0^1 du \gamma_u(u, 0) \leq \frac{\beta^2 |A| n^2 S^{2|A|}}{N} (\sum_{p=x,y,z} K^p)^2.$$

Then the variance of $\psi_N(\beta, \lambda)$ vanishes in the infinite volume limit for arbitrary $(\beta, \lambda) \in (0, \infty) \times \mathbb{R}$. $\square$

Chatterjee's definition of replica symmetry breaking [2] is the following finite variance of spin overlap in the replica symmetric Gibbs state with $\lambda = \mu = 0$

$$\lim_{N \nearrow \infty} \mathbb{E}\langle (h_V(\mathbf{S}^1, \mathbf{S}^2) - \mathbb{E}\langle h_V(\mathbf{S}^1, \mathbf{S}^2) \rangle_{0,0})^2 \rangle_{0,0} > 0.$$

Theorem 1.1 gives

$$\lim_{\lambda \searrow 0} \lim_{N \nearrow \infty} \mathbb{E}\langle (h_V(\mathbf{S}^1, \mathbf{S}^2) - \mathbb{E}\langle h_V(\mathbf{S}^1, \mathbf{S}^2) \rangle_{\lambda,0})^2 \rangle_{\lambda,0} = 0,$$

then this yields the following corollary.

Corollary 3.6 *If the replica symmetry breaking defined by Chatterjee occurs in the model defined by the Hamiltonian (28), the following limiting procedures do not commute*

$$\lim_{N \nearrow \infty} \lim_{\lambda \searrow 0} \mathbb{E}\langle (h_V(\mathbf{S}^1, \mathbf{S}^2) - \mathbb{E}\langle h_V(\mathbf{S}^1, \mathbf{S}^2) \rangle_{\lambda,0})^2 \rangle_{\lambda,0} \neq \lim_{\lambda \searrow 0} \lim_{N \nearrow \infty} \mathbb{E}\langle (h_V(\mathbf{S}^1, \mathbf{S}^2) - \mathbb{E}\langle h_V(\mathbf{S}^1, \mathbf{S}^2) \rangle_{\lambda,0})^2 \rangle_{\lambda,0}.$$

Ref. [18] indicates that the variance of the order operator (29) vanishes by the disordered replica symmetry breaking perturbation

$$\sum_{i \in V_N} (\nu g_i + \lambda) S_i^{z,1} S_i^{z,2},$$

with Gaussian random variables g_i and constants $(\lambda, \nu) \in \mathbb{R}$. Even for $v = 0$, however, Theorem 1.1 implies that the variance of the order operator (29) vanishes for almost all $\lambda \in \mathbb{R}$.

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